

Dynamics of a classical Hall system driven by a time-dependent Aharonov–Bohm flux

J. Asch*, P. Štoviček†

04.09.2006

Abstract

We study the dynamics of a classical particle moving in a punctured plane under the influence of a strong homogeneous magnetic field, an electrical background, and driven by a time-dependent singular flux tube through the hole.

We exhibit a striking classical (de)localization effect: in the far past the trajectories are spirals around a bound center; the particle moves inward towards the flux tube losing kinetic energy. After hitting the puncture it becomes “conducting”: the motion is a cycloid around a center whose drift is outgoing, orthogonal to the electric field, diffusive, and without energy loss.

PACS numbers: 45.50.Pk Particle orbits classical mechanics, 45.50.-j Dynamics and kinematics of a particle and a system of particles, 73.43.-f Quantum Hall effects, 73.50.Gr Charge carriers: generation, recombination, lifetime, trapping, mean free paths

1 Introduction

The motivation to study the dynamics of this classical system is to sharpen our intuition on its quantum counterpart which is, following Laughlin's [14] and

*CPT-CNRS, Luminy Case 907, F-13288 Marseille Cedex 9, France. e-mail: asch@cpt.univ-mrs.fr

†Department of Mathematics, Faculty of Nuclear Science, Czech Technical University, Trojanova 13, 120 00 Prague, Czech Republic

Halperin's [12] proposals, widely used for an explanation of the Integer Quantum Hall effect. Of special interest is how the topology influences on the dynamics. In the mathematical physics literature Bellissard et al. [5] and Avron, Seiler, Simon [3], [4] used an adiabatic limit of the model to introduce indices. The indices explain the quantization of charge transport observed in the experiments [13]. See [7, 10, 8, 9, 11] for recent developments. We discussed the adiabatics of the quantum system in [2], its quantum and semiclassical dynamics will be treated elsewhere. The dynamics of the classical system without magnetic field were discussed in [1].

We state the model and our main results:

Consider a classical point particle of mass $m > 0$ and charge $e > 0$ moving in the punctured plane $\mathbb{R}^2 \setminus (0)$. Suppose that a magnetic flux line with time varying strength Φ pierces the origin and further the presence of a homogeneous magnetic field of strength $B > 0$ orthogonal to the plane and an interior electric field with smooth bounded potential V .

The equations of motions are Hamiltonian. For a point $(q, p) = ((q_1, q_2), (p_1, p_2))$ in phase space

$$\mathbb{P} = \mathbb{R}^2 \setminus (0) \times \mathbb{R}^2$$

the time dependent Hamiltonian is :

$$\frac{1}{2m} (p - eA(t, q))^2 + eV(t, q); \quad A(t, q) = \left(\frac{B}{2} - \frac{\Phi(t)}{2\pi|q|^2} \right) q^\perp$$

where $q^\perp := (-q_2, q_1)$. We suppose that

$$\Phi : \mathbb{R} \rightarrow \mathbb{R} \text{ and } V : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R} \text{ are smooth functions.}$$

The electric field is $-\partial_t A - \partial_q V$, the force on the particle with velocity \dot{q} :

$$e(\dot{q} \wedge \text{rot}(A) - \partial_t A - \partial_q V) = -e \left(B\dot{q}^\perp - \frac{\partial_t \Phi}{2\pi} \frac{q^\perp}{|q|^2} + \partial_q V \right)$$

Remark that the part of the electric field induced by the flux has circulation $\frac{e\partial_t \Phi}{2\pi}$ but vanishing rotation, and is long range with an $1/r$ singularity at the origin, we call it the circular parts. V is smooth on the entire plane so that the circulation of the corresponding field is zero. This is the topology essential for the dynamics.

Recall that when only the constant magnetic field is present, the particle follows the Landau orbits; these are circles around a fixed center with frequency $\frac{eB}{m}$ whose squared radius is proportional to the energy.

Our result for the case $\Phi \sim t$, B large, V such that the torque $q \wedge \partial_q V$ is small is qualitatively:

- the motion in configuration space is approximately rotation with radius proportional to the square root of the (time-dependent) energy around a drifting center.
- for large enough negative times the center is trapped by the flux line and the energy is linearly decreasing with time, so the particle is spiraling inwards
- from the hitting time on (i.e. the time when the Landau orbit “hits” the singularity) the center starts to drift away from the flux line, the energy remains asymptotically constant in the future. The drift is diffusive. The situation is described by Fig. 1, showing a typical orbit in q -space.

Remark that the corresponding analysis remains true if the sign of B is changed. In this case we may state our observation as: *Hall conducting states are eventually trapped by the flux line and trapped states are energy conducting.*

Here “hall conducting” means that the center follows the lines of the potential diffusively.

We shall discuss the corresponding quantum behavior elsewhere.

In the first section of this paper we state some general remarks on the model and discuss the problem for frozen values of the flux. Next we define appropriate action angle coordinates and use an averaging (adiabatic) method to approximate the dynamics near the hitting time between the particle and the flux line. In the last section we discuss the asymptotic behavior of the solution of the full equations of motion.

Let us remark that our method includes (for the two dimensional case) a simple proof for the guiding center approximation widely used in plasma physics.

2 Dynamics of the frozen system

Denote

$$\omega = \frac{eB}{m}, \lambda = \frac{1}{\sqrt{eB}}.$$

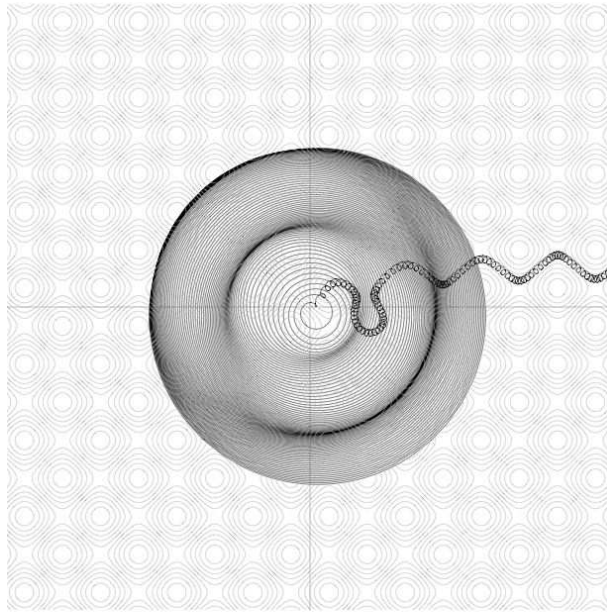


Figure 1: Typical trajectory of the Hamiltonian $\frac{1}{2} \left(p - \left(\frac{1}{2}q^\perp - s\frac{q^\perp}{q^2} + s\partial_q V \right) \right)^2$ with V chosen to be $V(x, y) = 1/3(\sin x + \sin y)$

We use the scaling $(t, q, p) \mapsto (\omega t, q/\lambda, p\lambda)$ and “absorb” V into the time dependent vectorpotential. The scaled variables are called (s, q, p) . The Hamiltonian under consideration then reads

$$H(s; p, q) := \frac{1}{2} (p - a(s; q))^2; \quad a(s; q) := \left(\frac{1}{2} q^\perp + a_E(s; q) \right)$$

where $a_E(s) : \mathbb{R}^2 \setminus (0) \rightarrow \mathbb{R}^2$ is smoothly time dependent with $\text{rot}(a_E)(s) = 0$. $a_E(s)$ and the electric field $E(s) : \mathbb{R}^2 \setminus (0) \rightarrow \mathbb{R}^2$ are defined by:

$$-\partial_s a_E(s) := E(s) := \frac{1}{\omega} \left(\frac{\partial_t \Phi}{2\pi} \left(\frac{s}{\omega} \right) \frac{q^\perp}{|q|^2} - \lambda (\partial_q V) \left(\frac{s}{\omega}, \lambda q \right) \right) \quad (1)$$

We discuss first the solution of the equation of motions for a frozen time $\sigma \in \mathbb{R}$. As $\partial_s a_E(\sigma; q) = 0$, the solution of the frozen equations generated by the Hamiltonian $H(\sigma)$ goes along the lines of the classical Landau problem (which means: the case $\Phi = 0; V(q) = 0$)

For $\sigma \in \mathbb{R}$ define

1. the velocity field: $v(\sigma) : \mathbb{P} \rightarrow \mathbb{R}^2, \quad v(\sigma; q, p) := p - a(\sigma; q);$
2. the center: $c(\sigma) : \mathbb{P} \rightarrow \mathbb{R}^2, \quad c(\sigma; q, p) := q - v^\perp(\sigma; q, p);$
3. the angular momentum: $L : \mathbb{P} \rightarrow \mathbb{R}, \quad L(q, p) := q \wedge p.$

Denote the Poisson bracket: $\{f, g\} = \partial_q f \partial_p g - \partial_p f \partial_q g.$

We list some useful formulas:

Proposition 2.1 *The following identities hold as functions on phase space \mathbb{P} for all $\sigma \in \mathbb{R}$:*

1. $\{v_1, v_2\} = 1, \quad \{c_1, c_2\} = -1, \quad \left\{c, \frac{c^2}{2}\right\} = c^\perp, \quad \{c_i, v_j\} = 0;$
2. $H = \frac{1}{2}v^2, \quad \{v, H\} = -v^\perp, \quad \{c, H\} = 0;$
3.
$$\frac{1}{2}c^2 = \frac{1}{2}v^2 + L - q \wedge a_E = H + L - q \wedge a_E; \quad (2)$$

4. the frozen flow $(q(\sigma; s), p(\sigma; s))$ defined by
 $\partial_s q(\sigma; s) = \partial_p H(\sigma), \quad \partial_s p(\sigma; s) = -\partial_q H(\sigma),$
 $(q(\sigma; 0), p(\sigma; 0)) = (q, p)$ is :

$$q(\sigma; s) = c(\sigma) + \cos(s)v^\perp(\sigma) + \sin(s)v(\sigma)$$

$$p(\sigma; s) = \frac{1}{2} \left(c^\perp(\sigma) + \cos(s)v(\sigma) - \sin(s)v^\perp(\sigma) \right) + a_E(\sigma; q(\sigma; s))$$

Proof. (1),(2),(3): $\{v_1, v_2\} = \{p_1 - a_1(\sigma, q), p_2 - a_2(\sigma, q)\} = \text{rot}(a(\sigma)) = 1$, $\{q_i, v_j\} = \delta_{ij}$. $H = \frac{1}{2}v^2$ so $\{q, H\} = v, \{v, H\} = -v^\perp$. $c^2 = q^2 + v^2 + 2q \wedge v$; on the other hand $L = q \wedge v + \frac{1}{2}q^2 + q \wedge a_E(\sigma; q)$.

(4): The force is $-\dot{q}^\perp$ independently of σ , Newton's equation $\ddot{q} = -\dot{q}^\perp$ is readily verified. On the other hand: $p = v + a = a + c^\perp - q^\perp = c^\perp - \frac{1}{2}q^\perp + a_E(\sigma; q)$. So $p(s)$ follows from $q(s)$ \square .

Remarks 2.2 1. Since the energy $H(\sigma) = \frac{1}{2}v(\sigma)^2$ is conserved under the frozen flow, the projections of the trajectories to q -space are circles around $c(\sigma)$ with radius $\sqrt{2H(\sigma)}$. An orbit encircles the origin (has non-trivial homotopy) in $\mathbb{R}^2 \setminus (0)$ if and only if

$$c^2 < 2H \iff L - q \wedge a_E(\sigma; q) < 0;$$

2. the flow is, strictly speaking, not complete as for $L - q \wedge a_E(\sigma; q) = 0$ the particle reaches the origin in q -space (and infinity in p -space) in finite time; the energy remains, however, finite. This is a mathematical subtlety which can be handled.

3 Action angle coordinates

In order to discuss the full dynamics for large B we introduce action angle coordinates. The frozen dynamics as discussed in Proposition 2.1 suggests to take as coordinates the angles and absolute values of c and v^\perp , i.e. with the

notation: $e(\theta) := (\cos \theta, \sin \theta) :$

$$q = c + v^\perp = |c| \frac{c}{|c|} + |v| \frac{v^\perp}{|v|} =: |c|e(\varphi_1) + |v|e(-\varphi_2)$$

$$p = \frac{1}{2} (c^\perp + v) + a_E(\sigma; q) = \frac{1}{2} (|c|e^\perp(\varphi_1) - |v|e^\perp(-\varphi_2)) + a_E(\sigma; q)$$

Motivated by this we define for $\sigma \in \mathbb{R}$

$$\begin{aligned} q(\sigma; \varphi, I) &:= \sqrt{2I_1}e(\varphi_1) + \sqrt{2I_2}e(-\varphi_2) \\ p(\sigma; \varphi, I) &:= \frac{1}{2} \left(\sqrt{2I_1}e^\perp(\varphi_1) - \sqrt{2I_2}e^\perp(-\varphi_2) \right) + a_E(\sigma; q(\sigma; \varphi, I)) \end{aligned}$$

and, denoting by \mathcal{C} the nullset $\{(\varphi, I); \varphi_1 + \varphi_2 = \pi, I_1 = I_2\}$ where $q(\sigma; \varphi, I) = 0$, by \mathcal{D} the nullset $\{(q, p); v^2 = 0 \text{ or } c^2 = 0\}$. Thus for each frozen time $\sigma \in \mathbb{R}$ the transformation to action angle coordinates $T(\sigma)$ is defined by

$$\begin{aligned} T(\sigma) : S^1 \times S^1 \times \{(I_1, I_2); I_1 \geq 0, I_2 \geq 0\} \setminus \mathcal{C} &\rightarrow \mathbb{P} \setminus \mathcal{D} \\ T(\sigma; \varphi, I) = T(\sigma; \varphi_1, \varphi_2, I_1, I_2) &:= (q(\sigma; \varphi, I), p(\sigma; \varphi, I)) \end{aligned}$$

We have

Lemma 3.1 1. $T(\sigma)$ is a canonical diffeomorphism

2. $T^{-1}(\sigma)$ is determined by

$$\begin{aligned} I_1(\sigma) = \frac{c^2(\sigma)}{2} &= \frac{1}{2} \left(p - \left(-\frac{1}{2}q^\perp + a_E(\sigma; q) \right) \right)^2; \\ I_2(\sigma) = H(\sigma) &= \frac{1}{2} \left(p - \left(\frac{1}{2}q^\perp + a_E(\sigma; q) \right) \right)^2 \\ e(\varphi_1(\sigma)) = \frac{c}{|c|}(\sigma) &= \frac{\frac{1}{2}q - p^\perp - a_E^\perp(\sigma; q)}{\sqrt{2(H(\sigma) + L - q \wedge a_E(\sigma; q))}} \\ e(-\varphi_2(\sigma)) = \frac{v^\perp}{|v|}(\sigma) &= \frac{\frac{1}{2}q + p^\perp + a_E^\perp(\sigma; q)}{\sqrt{2H(\sigma)}} \end{aligned}$$

Proof: These identities follow immediately from Proposition 2.1:

$$\begin{aligned} \{I_1, I_2\} &= 0, \{e(\varphi_1), e(\varphi_2)\} = 0, \{I_1, e(\varphi_2)\} = 0 = \{I_2, e(\varphi_1)\}, \\ \{e(\varphi_1), I_1\} &= \frac{1}{|c|} \left\{ c, \frac{c^2}{2} \right\} = \frac{c^\perp}{|c|} = e^\perp(\varphi_1). \end{aligned}$$

On the other hand, $\{e(\varphi_1), I_1\} = e^\perp(\varphi_1)\{\varphi_1, I_1\}$, so $\{\varphi_1, I_1\} = 1$. Similarly: $\{\varphi_2, I_2\} = 1$. \square

We now investigate the full equations of motion, i.e. those for time-dependent flux, in these action angle coordinates. As $rot(E) = 0$ there exists a (possibly multi-valued) function which we denote by $m = m(s; q)$ such that

$$\partial_q m(s) = E(s) = -\partial_s a_E(s).$$

Then $T(s)$ is generated by m :

$$\partial_s T(s; \varphi, I) = (0, \partial_s a_E(q(s; \varphi, I))) = (\partial_p m, -\partial_q m) \circ T(s; \varphi, I).$$

Denote by $U(s) : \mathbb{P} \rightarrow \mathbb{P}$ the hamiltonian flow of $H(s)$ defined by $U(s) := (q(s), p(s))$

$$\dot{q}(s) = \partial_p H, \quad \dot{p}(s) = -\partial_q H, \quad (q(0), p(0)) = (q, p),$$

then for the flow $\widehat{U}(s) = (\varphi(s), I(s))$ in action angle coordinates defined by

$$T(s) \circ \widehat{U}(s) = U(s) \circ T(s=0)$$

it holds:

$$\dot{\varphi}(s) = \partial_I K \circ \widehat{U}(s), \quad \dot{I}(s) = -\partial_\varphi K \circ \widehat{U}(s), \quad (\varphi(0), I(0)) = (\varphi, I),$$

where the Hamiltonian in action angle coordinates, $K = H \circ T - m \circ T$, is

$$K(s; \varphi, I) = I_2 - m(s; q(s; \varphi, I))$$

and the equations of motion are (with the notation $\langle \cdot, \cdot \rangle$ for the scalar product)

$$\dot{\varphi}(s) = \partial_I K = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \langle E(s; q(s; \varphi, I)), \partial_I q \rangle \quad (3)$$

$$\dot{I}(s) = -\partial_\varphi K = \langle E(s; q(s; \varphi, I)), \partial_\varphi q \rangle \quad (4)$$

Remark 3.2 *Another way to derive these equations is to start from Newton's equation*

$$\ddot{q} = -\dot{q}^\perp + E(s; q).$$

From the very definition of c and v one gets:

$$\dot{c} = -E^\perp(c + v^\perp) \quad \dot{v} = -v^\perp + E(c + v^\perp)$$

which in action angle coordinates gives (3), (4).

4 Averaged dynamics

We apply averaging with respect to the fast angle φ_2 to the system (3), (4) (see [15, 6]). The singularity problem can be overcome by a regularization technique (see [16]). The result is that the solutions of the equations are at a distance of order $1/B$ to the solution of the averaged equations over times of order B .

Remark that at this place we are mainly interested in the (de)localization effect so we did not make use of more involved adiabatic or KAM methods in order to go to longer or even infinite time scales.

We detail this for the case

$$\Phi(t) = \Phi_0 t, \quad V \text{ time independent,}$$

i.e., a flux Φ_0 per unit time is added ad-eternam.

Denote the average of a function f on the phase space by

$$f_{av}(\varphi_1, I) := \frac{1}{2\pi} \int_0^{2\pi} f(\varphi_1, \varphi_2, I) d\varphi_2$$

In particular for a function f defined on the plane thus depending only on the variable q we denote

$$f_{av}(\varphi_1, I) = \frac{1}{2\pi} \int_0^{2\pi} f\left(\sqrt{2I_1}e(\varphi_1) + \sqrt{2I_2}e(-\varphi_2)\right) d\varphi_2$$

The field (1) is

$$E(s; q) = \frac{e}{\omega} \left(\frac{\Phi_0 q^\perp}{2\pi |q|^2} - \lambda(\partial_q V)(\lambda q) \right)$$

Define

$$\mathbf{f} := \frac{e\Phi_0}{2\pi\omega}$$

and choose m and thus K :

$$\begin{aligned} m(q) &= \mathbf{f} \operatorname{arg}(q) - \frac{e}{\omega} V(\lambda q) \\ K(\varphi, I) &= I_2 - m\left(\sqrt{2I_1}e(\varphi_1) + \sqrt{2I_2}e(-\varphi_2)\right) \end{aligned}$$

Making use of the identities

$$\left\langle \frac{q^\perp}{q^2}, \partial_I q \right\rangle = \frac{\sin(\varphi_1 + \varphi_2)}{q^2} \begin{pmatrix} \sqrt{\frac{I_2}{I_1}} \\ -\sqrt{\frac{I_1}{I_2}} \end{pmatrix}, \quad \left\langle \frac{q^\perp}{q^2}, \partial_\varphi q \right\rangle = \begin{pmatrix} \frac{I_1 - I_2}{q^2} + \frac{1}{2} \\ \frac{I_1 - I_2}{q^2} - \frac{1}{2} \end{pmatrix}$$

the system (3), (4) reads

$$\begin{aligned} \dot{\varphi}(s) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{f} \frac{\sin(\varphi_1 + \varphi_2)}{2(I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\varphi_1 + \varphi_2))} \begin{pmatrix} \sqrt{\frac{I_2}{I_1}} \\ -\sqrt{\frac{I_1}{I_2}} \end{pmatrix} + \frac{e}{\omega} \partial_I V(\lambda q) \\ \dot{I}(s) &= \mathbf{f} \frac{I_1 - I_2}{2(I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\varphi_1 + \varphi_2))} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\mathbf{f}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{e}{\omega} \partial_\varphi V(\lambda q) \end{aligned}$$

The averaged quantities are readily calculated: using

$$\left(\frac{1}{q^2} \right)_{av} = \frac{1}{2|I_1 - I_2|}, \quad \left(\frac{\sin(\varphi_1 + \varphi_2)}{q^2} \right)_{av} = 0,$$

one finds for the averaged vectorfield

$$\begin{aligned} (\partial_I K)_{av}(\varphi_1, I) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{e}{\omega} \partial_I V_{av}(\varphi_1, \lambda^2 I) \\ -(\partial_\varphi K)_{av}(\varphi_1, I) &= \mathbf{f} \begin{pmatrix} \chi(I_1 > I_2) \\ -\chi(I_1 < I_2) \end{pmatrix} - \frac{e}{\omega} \begin{pmatrix} \partial_{\varphi_1} V_{av}(\varphi_1, \lambda^2 I) \\ 0 \end{pmatrix} \end{aligned} \quad (5)$$

where we used the binary function χ : $\chi(True) := 1$, $\chi(False) := 0$.

Remark 4.1 *Remark that the averaged vectorfield is the hamiltonian vectorfield derived from the from the “averaged” Hamiltonian K_{av} . Indeed, using the splitting of $\arg(q)$, which is a multi-valued function defined on the covering space of $\mathbb{R}^2 \setminus (0)$, into a linear and oscillating part*

$$\arg(q(\varphi, I)) = \begin{cases} \varphi_1 + \arg\left((1, 0) + \sqrt{\frac{I_2}{I_1}} e(-\varphi_1 - \varphi_2)\right) & \text{if } I_1 > I_2 \\ -\varphi_2 + \arg\left((1, 0) + \sqrt{\frac{I_1}{I_2}} e(\varphi_1 + \varphi_2)\right) & \text{if } I_2 > I_1 \end{cases}.$$

and:

$$\int_0^{2\pi} \arg((1, 0) + a e(s)) ds = 0 \quad \text{for } 0 \leq a < 1;$$

One finds that for

$$K_{av}(\varphi, I) := I_2 - \frac{e}{\omega} \left(\frac{\Phi_0}{2\pi} (\varphi_1 \chi(I_1 > I_2) - \varphi_2 \chi(I_1 < I_2)) - V_{av}(\varphi_1, \lambda^2 I) \right)$$

one has $\partial_\varphi K_{av} = (\partial_\varphi K)_{av}$, $\partial_I K_{av} = (\partial_I K)_{av}$.

The result on the dynamics now is:

Theorem 4.2 Denote by $J = (J_1, J_2)$, $\psi = (\psi_1, \psi_2)$ the solution of the averaged equations (5)

$$\begin{aligned} \dot{\psi}(s) &= \partial_I K_{av}(\psi(s), J(s)), & J(0) &= (J_1^0, J_2^0) \\ \dot{J}(s) &= -\partial_\varphi K_{av}(\psi(s), J(s)), & \psi(0) &= (\psi_1^0, \psi_2^0) \end{aligned}$$

and by $I = (I_1, I_2)$, $\varphi = (\varphi_1, \varphi_2)$ the solution of the full equations (3), (4)

$$\begin{aligned} \dot{\varphi}(s) &= \partial_I K(\varphi(s), I(s)), & I(0) &= (I_1^0, I_2^0) \\ \dot{I}(s) &= -\partial_\varphi K(\varphi(s), I(s)), & \varphi(0) &= (\varphi_1^0, \varphi_2^0) \end{aligned}$$

then it holds

1. Let $V = 0$, denote $\Delta J = J_2^0 - J_1^0$ then:

$$\begin{aligned} J(s) &= \min\{J_1^0, J_2^0\} + (\mathbf{f}s - \Delta J) \begin{pmatrix} \chi(\mathbf{f}s > \Delta J) \\ -\chi(\mathbf{f}s < \Delta J) \end{pmatrix} \\ \psi(s) &= \begin{pmatrix} \psi_1^0 \\ \psi_2^0 + s \end{pmatrix} \end{aligned}$$

2. For any V and any $s_1, s_2 \in \mathbb{R}$

$$|J_2(s_2) - J_2(s_1)| = \mathbf{f} \left| \int_{s_1}^{s_2} \chi(J_1(u) < J_2(u)) du \right|$$

3. Let V be such that the torque of the corresponding field satisfies for a $c \in [0, 1)$:

$$|q \wedge \partial_q V| \leq \frac{\Phi_0}{2\pi} c$$

then for any initial condition it holds:

$I_1 - I_2$ is strictly increasing, furthermore

$$\mathbf{f}(1 - c) \leq \dot{I}_1(s) - \dot{I}_2(s) \leq \mathbf{f}(1 + c) \quad (\forall s \in \mathbb{R}).$$

4. In particular if $q \wedge \partial_q V = 0$ it holds for all $s \in \mathbb{R}$:

$$I_1(s) - I_2(s) = \mathbf{f}(s - s_0) \quad (6)$$

where s_0 is the unique "hitting time" defined by this equation.

Proof. Using that for $V = 0$ it holds $J_1(s) - J_2(s) - \mathbf{f}s = \Delta J$ the first assertion follows by inspection. The second assertion follows from integration of (5). Finally we have from (2):

$$\dot{I}_1 - \dot{I}_2 = \partial_s(I_1 - I_2) = q \wedge E = \frac{e}{\omega} \left(\frac{\Phi_0}{2\pi} - (q \wedge \partial_q V)(\lambda q) \right)$$

from which the last assertion follows. \square

Remarks 4.3 1. The first equation explains the qualitative behavior of the solution exhibited in Fig. 1: J_1 is linear in time in the future and is constant in the past.

2. Loosely speaking the second assertion of the theorem means that, on the average, one has

$$|\text{energychange}| = |\text{fluxchange through the orbit during stay time}|$$

where the stay time means the time where the "orbit surrounds the origin". This should be like this as the the change in energy equals the work of the electric field along the orbit:

$$H(s; q(s)) - H(s_0; q(s_0)) = \int_{s_0}^s \langle a_E(s), ds \rangle.$$

3. The last assertion says that the orbit presented in Fig. 1 in the introduction is generic, i.e.: inward spiraling motion with fixed center followed by the usual Hall cycloids with the center following the lines of the potential. We argue that our condition on the potential is far from optimal and that for large enough magnetic field the situation described in this paper is generic for V smooth and bounded with bounded derivative. This needs further investigation.

5 Large time asymptotics, potential free case

For the case $\Phi(t) = \Phi_0 t$, $V = 0$ we can determine the large time asymptotics of the solution. We keep the notation $\mathbf{f} := \frac{e\Phi_0}{2\pi\omega}$. Observe also that

$$K = K(\varphi, I) = I_2 - \arg\left(\sqrt{2I_1} e(\varphi_1) + \sqrt{2I_2} e(-\varphi_2)\right)$$

is an integral of motion.

Theorem 5.1 Denote by $I = (I_1, I_2)$, $\varphi = (\varphi_1, \varphi_2)$ the solution of the full equations of motion (3), (4)

$$\begin{aligned}\dot{\varphi}(s) &= \partial_I K(\varphi(s), I(s)), & I(0) &= (I_1^0, I_2^0) \\ \dot{I}(s) &= -\partial_\varphi K(\varphi(s), I(s)), & \varphi(0) &= (\varphi_1^0, \varphi_2^0)\end{aligned}$$

then the following asymptotic behavior holds:

in the future, $s \rightarrow \infty$

The following limits exist and define the constants $a_0 > 0$, b_0 :

$$\lim_{s \rightarrow \infty} I_2(s) =: \frac{a_0^2}{4\mathbf{f}}, \quad \lim_{s \rightarrow \infty} (\varphi_1(s) + \varphi_2(s) - s) =: b_0, \quad \lim_{s \rightarrow \infty} (I_2(s) - \mathbf{f}\varphi_1(s)) = K,$$

the asymptotics are

$$\begin{aligned}I_2(s) &= \frac{a_0^2}{4\mathbf{f}} - \left(\frac{a_0}{2} \sin(s + b_0)\right) \frac{1}{\sqrt{s}} + \frac{1}{4} \left(\mathbf{f} + \frac{a_0^2}{2\mathbf{f}} \sin(2(s + b_0))\right) \frac{1}{s} + \mathcal{O}\left(\frac{1}{s^{3/2}}\right) \\ I_1(s) &= I_2(s) + \mathbf{f}(s - s_0) \\ \varphi_1(s) &= \frac{a_0^2}{4\mathbf{f}^2} - \frac{K}{\mathbf{f}} - \frac{1}{4s} + \mathcal{O}\left(\frac{1}{s^{3/2}}\right) \\ \varphi_2(s) &= s + b_0 - \frac{a_0^2}{4\mathbf{f}^2} + \frac{K}{\mathbf{f}} - \frac{\mathbf{f}}{a_0} \cos(s + b_0) \frac{1}{\sqrt{s}} \\ &\quad + \frac{1}{8} \left(-1 + 2 \cos(2(s + b_0)) - \frac{4\mathbf{f}^2}{a_0^2} \sin(2(s + b_0))\right) \frac{1}{s} + \mathcal{O}\left(\frac{1}{s^{3/2}}\right)\end{aligned}$$

with s_0 defined as in (6);

in the past, $s \rightarrow -\infty$

The following limits exist and define the constants $\tilde{a}_0 > 0$, \tilde{b}_0 :

$$\lim_{s \rightarrow -\infty} I_1(s) =: \frac{\tilde{a}_0^2}{4\mathbf{f}}, \quad \lim_{s \rightarrow -\infty} (\varphi_1(s) + \varphi_2(s) - s) =: \tilde{b}_0, \quad \lim_{s \rightarrow -\infty} (I_2(s) + \mathbf{f}\varphi_2(s)) = K,$$

the asymptotics are

$$\begin{aligned} I_1(s) &= \frac{\tilde{a}_0^2}{4\mathbf{f}} + \left(\frac{\tilde{a}_0}{2} \sin(s + \tilde{b}_0) \right) \frac{1}{\sqrt{|s|}} - \frac{1}{4} \left(\mathbf{f} - \frac{\tilde{a}_0^2}{2\mathbf{f}} \sin(2(s + \tilde{b}_0)) \right) \frac{1}{s} + \mathcal{O}\left(\frac{1}{|s|^{3/2}}\right) \\ I_2(s) &= I_1(s) - \mathbf{f}(s - s_0) \\ \varphi_1(s) &= s_0 + \tilde{b}_0 + \frac{\tilde{a}_0^2}{4\mathbf{f}^2} - \frac{K}{\mathbf{f}} + \frac{\mathbf{f}}{\tilde{a}_0} \cos(s + \tilde{b}_0) \frac{1}{\sqrt{|s|}} \\ &\quad - \frac{1}{8} \left(1 - 2 \cos(2(s + \tilde{b}_0)) - \frac{4\mathbf{f}^2}{\tilde{a}_0^2} \sin(2(s + \tilde{b}_0)) \right) \frac{1}{s} + \mathcal{O}\left(\frac{1}{|s|^{3/2}}\right) \\ \varphi_2(s) &= s - s_0 - \frac{\tilde{a}_0^2}{4\mathbf{f}^2} + \frac{K}{\mathbf{f}} - \frac{1}{4s} + \mathcal{O}\left(\frac{1}{|s|^{3/2}}\right). \end{aligned}$$

Proof. We give an outline of the main steps of the proof for the case $t \rightarrow \infty$. Some particular computations in the proof turned out to be quite tedious and thus computer algebra systems were employed to facilitate them.

Suppose $t > 0$

Step 1

From (6) we know $I_1(s) - I_2(s) = \mathbf{f}(s - s_0)$. So the equations of motion only involve $J := I_1 + I_2$ and $\psi := \varphi_1 + \varphi_2$ and transform to

$$\dot{\psi} = 1 + \frac{\mathbf{f}^2 s \sin \psi}{\sqrt{J^2 - \mathbf{f}^2 s^2} (J + \sqrt{J^2 - \mathbf{f}^2 s^2} \cos \psi)}, \quad \dot{J} = \frac{\mathbf{f}^2 s}{J + \sqrt{J^2 - \mathbf{f}^2 s^2} \cos \psi},$$

Step 2

Do a second transformation

$$x_1 := \sqrt{J^2 - \mathbf{f}^2 s^2} \cos \psi, \quad x_2 := \sqrt{J^2 - \mathbf{f}^2 s^2} \sin \psi,$$

the J, ψ equations transform to

$$\dot{x}_1 - \frac{x_1}{s} + x_2 = F(s, x_1, x_2), \quad \dot{x}_2 - x_1 = 0,$$

with

$$F(s, x_1, x_2) := \mathbf{f} - \frac{x_1}{s} - \frac{\mathbf{f}^2 s}{\sqrt{x_1^2 + (x_2 - \mathbf{f})^2 + \mathbf{f}^2 s^2 + x_1}}.$$

The corresponding homogeneous system is equivalent to

$$\ddot{x}_1 - \frac{\dot{x}_1}{s} + \left(1 + \frac{1}{s^2}\right)x_1 = 0 \quad \text{or} \quad s\ddot{y} + \dot{y} + sy = 0$$

with y defined by $x_1 = sy$. The latter is Bessel's equation of order 0 so one has two independent solutions of the homogeneous system:

$$\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \begin{pmatrix} sJ_0(s) \\ sJ_1(s) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \begin{pmatrix} sY_0(s) \\ sY_1(s) \end{pmatrix}$$

with the Bessel functions J_m (Y_m) of the first (second) kind.

Step 3

Transform the x -differential equation to the integral equation

$$\begin{aligned} x_1(s) &= c_1 s J_0(s) + c_2 s Y_0(s) \\ &\quad - \frac{\pi s}{2} \int_s^\infty (Y_0(s) J_1(\tau) - J_0(s) Y_1(\tau)) F(\tau, x_1(\tau), x_2(\tau)) d\tau \\ x_2(s) &= c_1 s J_1(s) + c_2 s Y_1(s) \\ &\quad - \frac{\pi s}{2} \int_s^\infty (Y_1(s) J_1(\tau) - J_1(s) Y_1(\tau)) F(\tau, x_1(\tau), x_2(\tau)) d\tau \end{aligned}$$

where the numbers c_1, c_2 involve the initial conditions.

The equation is of the form $x = \mathcal{K}(x)$, the solution is constructed as the limit of the sequence $x_{n+1} = \mathcal{K}(x_n)$ starting from $x_0 = 0$. To verify the convergence one can apply yet another substitution $x(s) = y(s)/\sqrt{s}$, $G(s, y) = s^{-1/2}F(s, s^{-1/2}y)$. Consequently the integral equation takes the form

$$y(s) = y_0(s) - \int_s^\infty \mathcal{F}(s, \tau) G(\tau, y_1(\tau), y_2(\tau)) d\tau$$

where

$$\begin{aligned} y_{0j}(s) &= c_1 \sqrt{s} J_{j-1}(s) + c_2 \sqrt{s} Y_{j-1}(s), \quad j = 1, 2, \\ \mathcal{F}_j(s, \tau) &= \frac{\pi}{2} \sqrt{s\tau} (Y_{j-1}(s) J_1(\tau) - J_{j-1}(s) Y_1(\tau)), \quad j = 1, 2. \end{aligned}$$

Considering the new integral equation in the Banach space $L^\infty([s_*, \infty]) \otimes \mathbb{R}^2$ one can show that the iteration process is indeed contracting provided $s_* \geq 1$ is sufficiently large. It is then straightforward to derive from the integral equation the asymptotic expansion of the solution $x(s)$. One finds that

$$x(s) = a_0 e(t + b_0) \sqrt{s} + \left(\frac{a_0^3}{8f^2} e(t + b_0) - \frac{5}{8} a_0 e^\perp(t + b_0) \right) \frac{1}{\sqrt{s}} + \mathcal{O}\left(\frac{1}{s}\right)$$

Step 4

Transforming back first to the J, ψ then to $I_1, I_2, \varphi_1, \varphi_2$ variables gives the claimed asymptotic expansion. \square

The asymptotic formulae for the actions and the angles imply the following asymptotic behavior of the solutions and the energy thus defining the transport coefficients:

Denote

$$\mathcal{H} := \frac{1}{2m} \left(p - e \left(\frac{B}{2} - \frac{\Phi(t)}{2\pi|q|^2} \right) q^\perp \right)^2$$

the energy in the original coordinates q, p , and $q_{sc} = q/\lambda$ the scaled coordinate. Rescaling then gives

$$\mathcal{H}(t) = \omega H(\omega t) = \omega I_2(\omega t), \quad q(t) = \lambda q_{sc}(\omega t), \quad q_{sc} = \sqrt{2I_1} e(\varphi_1) + \sqrt{2I_2} e(-\varphi_2).$$

This leads to the following limits valid for any fixed initial condition and any $B > 0, \Phi_0 > 0$:

$$\begin{aligned} \frac{q(t)}{\sqrt{t}} &\rightarrow_{t \rightarrow \infty} \sqrt{\frac{\Phi_0}{2\pi B}} e \left(\frac{a_0^2}{4f^2} - \frac{K}{f} \right) \\ \frac{q(t)}{\sqrt{|t|}} &\sim_{t \rightarrow -\infty} \sqrt{\frac{\Phi_0}{2\pi B}} e(-\omega t) \\ \mathcal{H}(t) &\rightarrow_{t \rightarrow \infty} \frac{\omega a_0^2}{4f} \\ \frac{\mathcal{H}(t)}{t} &\rightarrow_{t \rightarrow -\infty} -\frac{e^2 B}{m} \frac{\dot{\Phi}}{2\pi} = -\frac{e^2 B}{m} \frac{\Phi_0}{2\pi} \end{aligned}$$

Acknowledgments

P. Š. wishes to acknowledge gratefully partial support from the grants No. 201/05/0857 of the Grant Agency of the Czech Republic and No. LC06002 of the Ministry of Education of the Czech Republic.

References

- [1] Asch, J. and Benguria, R. D. and Šťovíček, P., “Asymptotic properties of the differential equation $h^3(h'' + h') = 1$,” *Asymptot. Anal.* **41**, 23–40 (2005).
- [2] Asch, J. and Hradecký, I. and Šťovíček, P., “Propagators weakly associated to a family of Hamiltonians and the adiabatic theorem for the Landau Hamiltonian with a time-dependent Aharonov-Bohm flux,” *J. Math. Phys.* **46**, 053303 ff. (2005).
- [3] Avron, J. E., Seiler, R., and Simon, B., “Quantum Hall Effect and the Relative Index for Projections,” *Phys. Rev. Lett.* **65**, 2185–2188 (1990).
- [4] Avron, J. E., Seiler, R., and Simon, B., “Charge deficiency, charge transport and comparison of dimensions,” *Commun. Math. Phys.* **159**, 399–422 (1994).
- [5] Bellissard, J., van Elst, A., and Schulz-Baldes, H., “The noncommutative geometry of the quantum Hall effect,” *J. Math. Phys.* **35**, 5373–5451 (1994).
- [6] Berglund, N. “Perturbation theory of dynamical systems,” Lecture Notes. <http://arXiv.org/abs/math.HO/0111178> (2001).
- [7] Combes, J.-M., and Germinet, F., “Edge and impurity effects on quantization of Hall currents,” *Comm. Math. Phys.* **256**, 159–180, (2005)
- [8] Combes, J.-M., Germinet, F., and Hislop, P.D. On the Quantization of Hall Currents in Presence of Disorder. In Asch, J. and Joye, A. (eds), *Mathematical Physics of Quantum Mechanics*, Lecture Notes in Physics , Vol. 690, (Springer, New York, 2006).
- [9] Elgart, A., Equality of the Bulk and Edge Hall Conductances in 2D. In Asch, J. and Joye, A. (eds), *Mathematical Physics of Quantum Mechanics*, Lecture Notes in Physics , Vol. 690, (Springer, New York, 2006).
- [10] Elgart, A., and Graf, G. M., and Schenker, J. H., “Equality of the bulk and edge Hall conductances in a mobility gap,” *Comm. Math. Phys.* **259**, 185–221, (2005).
- [11] Graf, G. M., “Aspects of the integer quantum Hall effect,” *Simon Festschrift*, (2006).

- [12] Halperin, B. I., "Quantized Hall Conductance, Current-Carrying Edge States and the Existence of Extended States in a Two-Dimensional Disordered Potential," *Phys. Rev. B* **25**, 2185-2188 (1982).
- [13] von Klitzing, K., Dorda, G., and Pepper, M., "New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance," *Phys. Rev. Lett.* **45**, 494-497 (1980).
- [14] Laughlin, R. B., "Quantized Hall conductivity in two dimensions," *Phys. Rev. B* **23**, 5632-5633 (1981).
- [15] J. A. Sanders and F. Verhulst, *Averaging methods in nonlinear dynamical systems*, (Springer, New York, 1985)
- [16] E. L. Stiefel and G. Scheifele, *Linear and regular celestial mechanics. Perturbed two-body motion, numerical methods, canonical theory*, (Springer, New York, 1971)