Recent progress on nonlinear wave equations via KAM theory*†

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Abstract. *In this note, the present author's recent works on nonlinear wave equations via KAM theory are introduced and reviewed.*

The existence of solutions, periodic in time, for non-linear wave (NLW) equations has been studied by many authors. A wide variety of methods such as bifurcation theory and variational techniques have been brought on this problem. See [11] and the references therein, for example. There are, however, relatively less methods to find the quasi-periodic solutions of NLW or other PDE's. The KAM theory is a very powerful tool in order to construct families of quasi-periodic solutions, which are on an invariant manifold, for some nearly integrable Hamiltonian systems of finite many degrees of freedom. In the 1980's,the celebrated KAM theory has been successfully extended to infinitely dimensional Hamiltonian systems of short range so as to deal with certain class of Hamiltonian networks of weakly coupled oscillators. Vittot & Bellissard [27], Frohlich, Spencer & Wayne [15] showed that there are plenty of almost periodic solutions for some weakly coupled oscillators of short range. In [30], it was also shown that there are plenty of quasi-periodic solutions for some weakly coupled oscillators of short range.

Because of the restrict of short range, those results obtained in [27, 15] does not apply to PDE's. In the 1980-90's, the KAM theory has been significantly generalized, by Kuksin[17, 18, 19], to infinitely dimensional Hamiltonian systems without being of short range so as to show that there is quasi-periodic solution for some class of partial differential equations. Also see Pöschel[24]. Let us focus our attention to the following nonlinear wave equation

$$u_{tt} - u_{xx} + V(x)u + u^3 + h.o.t. = 0, \tag{1}$$

subject to Dirichlet and periodic boundary conditions on the space variable x. *1. Dirichlet boundary condition.* In 1990, Wayne[28] obtained the time-quasi-periodic solutions of (1), when the potential V is lying on the outside of the set of some "bad" potentials. In [28], the set of all potentials is given some Gaussian measure and then the set of "bad" potentials is of small measure. Kuksin[17] assumed the potential V depends on n-parameters, namely, $V = V(x; a_1, ..., a_n)$, and showed that there are many quasi-periodic solutions of (1) for "most" (in the sense of Lebesgue measure) parameters a's. However, their results exclude the constant-value potential $V(x) \equiv m \in \mathbb{R}^+$, in particular, $V(x) \equiv 0$. When the potential V is constant, the parameters required can be extracted from the

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nonlinear term u^3. In order to use the KAM theorem, it is necessary to assume that there are some parameters in the Hamiltonian corresponding to (1). When $V(x) \equiv m > 0$, these parameters can be extracted from the nonlinear term u^3 by Birkhoff normal form[25], or by regarding (1) as a perturbation of sine-Gordon/sinh-Gordon equation[4]. And it was then shown that, for a prescribed potential $V(x) \equiv m > 0$, there are many elliptic invariant tori which are the closure of some quasi-periodic solutions of (1). By Remark 7 in [25], the same result holds also true for the parameter values $-1 < m < 0$. When $m \in (-\infty, -1) \setminus \mathbb{Z}$, it is shown in [29] that there are many invariant tori for (1). In this case, the tori are partially hyperbolic and partially elliptic.

2. periodic boundary condition. In this case, the eigenvalues of the linear operator $-\frac{d^2}{dx^2} + V(x)$ are double (at least, asymptotically double). This results in some additional difficulties in applying KAM technique since the normal frequencies are double. According to our knowledge, the difficulty arising from the multiple normal frequencies (including double ones) was overcome in the book [9] in the year of 1969 when the multiplicity is bounded, although Hamiltonian systems are not considered. A key point is to bound the inverse of some matrix by requiring the determinant of the matrix is nonzero. Using Lyapunov-Schmidt decomposition and Newton's iteration, Craig and Wayne[12] showed that for an open dense set of $V(x)$ there exist time periodic solutions of (1) subject to periodic (also Dirichlet) boundary condition. (The equation considered by them contains more general form than (1). By developing Craig-Wayne's method, in 1994, Bourgain[7] showed there are many quasi-periodic solutions of (1) for "most" parameters $\sigma \in \mathbb{R}^n$ where $V = V(x;\sigma)$. In 2000, a similar result was obtained by KAM technique in [13]. When the potential $V \equiv m \neq 0$, the existence of the quasi-periodic solutions was also obtained in [5] via the renormalization group method.

However, from the works mentioned above one does not know whether there is any invariant tori for prescribed (*not random*) non-constant-value potential $V(x)$. Recently, the present author has shown that there are many invariant tori for any prescribed non-zero potential $V(x)$ such as $\sin x$ and $\cos x$. To give the statement of our results, we need to introduce some notations. We study equation (1) as an infinitely dimensional Hamiltonian system. Following Pöschel[25], the phase space one may take, for example, the product of the usual Sobolev spaces $\mathcal{W} = H_0^1([0,\pi]) \times L^2([0,\pi])$ with coordinates u and $v = u_t$. The Hamiltonian is then

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \frac{1}{4}u^4$$

where $A = d^2/dx^2 - V(x)$ and $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in L^2. The Hamiltonian equation of motions are

$$u_t = \frac{\partial H}{\partial v} = v, \qquad -v_t = \frac{\partial H}{\partial u} = Au + u^3.$$

Our aim is to construct time-quasi-periodic solutions of small amplitude. Such quasi-periodic solutions can be written in the form

$$u(t,x) = U(\omega_1 t, \cdots, \omega_n t, x),$$

where $\omega_1, \cdots, \omega_n$ are rationally independent real numbers which are called the basic frequency of u, and U is an analytic function of period 2π in the first n

arguments. Thus, u admits a Fourier series expansion

$$u(t,x)=\sum_{k\in\mathbb{Z}^n} e^{\sqrt{-1}\langle k,\omega\rangle t}U_k(x),$$

where $\langle k,\omega\rangle=\sum_j k_j\omega_j$ and $U_k\in L^2[0,\pi]$ with $U_k(0)=U_k(\pi)$.

Since the quasi-periodic solutions to be constructed are of small amplitude, Eq.(1) may be considered as the linear equation $u_{tt}=u_{xx}-V(x)u$ with a small nonlinear perturbation u^3. Let $\phi_j(x)$ and λ_j $(j=1,2,...)$ be the eigenfunctions and eigenvalues of the Sturm-Liouville problem $-Ay=\lambda y$ subject to Dirichlet boundary conditions $y(0)=y(\pi)=0$, respectively. Then every solution of the linear system is the superposition of their harmonic oscillations and of the form

$$u(t,x)=\sum_{j\geq 1} q_j(t)\phi_j(x), \qquad q_j(t)=y_j\cos(\sqrt{\lambda_j}t+\phi_j^0)$$

with amplitude $y_j\geq 0$ and initial phase ϕ_j^0. The solution $u(t,x)$ is periodic, quasi-periodic or almost periodic depending on whether one, finitely many or infinitely many modes are excited, respectively. In particular, for the choice

$$N_d=\{j_1,j_2,\cdots,j_d\}\subset\mathbb{N},$$

of finitely many modes there is an invariant $2d$-dimensional linear subspace E_{N_d} that is completely foliated into rational tori with frequencies $\lambda_{j_1},\cdots,\lambda_{j_d}$:

$$\begin{aligned}E_{N_d}&=\{(u,v)=(q_{j_1}\phi_{j_1}+\cdots+q_{j_d}\phi_{j_d},\dot q_{j_1}\phi_{j_1}+\cdots+\dot q_{j_d}\phi_{j_d})\}\\ &=\bigcup_{y\in\bar{\mathbb{P}^d}}\mathcal{T}_j(y),\end{aligned}$$

where $\mathbb{P}^d=\{y\in\mathbb{R}^d: y_j>0 \quad \text{for} \quad 1\leq j\leq d\}$ is the positive quadrant in $\mathbb{R}^d$ and

$$\mathcal{T}_{N_d}(y)=\{(u,v): q_{j_k}^2+\lambda_{j_k}^{-2}\dot q_{j_k}^2=y_k, \quad \text{for} \quad 1\leq k\leq d\}.$$

Upon restoring the nonlinearity u^3 the invariant manifold E_{N_d} with their quasi-periodic solutions will not persist in their entirety due to resonance among the modes and the strong perturbing effect of u^3 for large amplitudes. In a sufficiently small neighborhood of the origin, however, there does persist a large Cantor sub-family of rotational d-tori which are only slightly deformed. More exactly, we have the following theorem:

Theorem 1.([30])*Assume that $V(x)$ is sufficiently smooth in the interval $[0,\pi]$, and $\int_0^\pi V(x)\,dx\neq 0$. Let K and N be positive constants large enough. Let $N_d=\{i_p\in\mathbb{N}: p=1,2,\cdots,d\}$ with*

$$\min N_d>NK, \max N_d\leq C_0 dNK, \text{ and } \mathcal{K}_1\leq|i_p-i_q|\leq\mathcal{K}_2, \quad \text{for} \quad p\neq q,$$

where $C_0>1$ is an absolute constant and $\mathcal{K}_1,\mathcal{K}_2$, positive constants large enough, depending on K instead of N. Then, for given compact set $\mathcal{C}^*$ in $\mathbb{P}^d$ with positive Lebesgue measure, there is a set $\mathcal{C}\subset\mathcal{C}^*$ with meas $\mathcal{C}>0$, a family of d-tori

$$\mathcal{T}_{N_d}(\mathcal{C})=\bigcup_{y\in\mathcal{C}}\mathcal{T}_{N_d}(y)\subset E_{N_d}$$

over $\mathcal{C}$, and a Lipschitz continuous embedding

$$\Phi : \mathcal{T}_{N_d}[\mathcal{C}] \hookrightarrow H_0^1([0,\pi]) \times L^2([0,\pi]) = \mathcal{W},$$

which is a higher order perturbation of the inclusion map $\Phi_0 : E_{N_d} \hookrightarrow \mathcal{W}$ restricted to $\mathcal{T}_{N_d}[\mathcal{C}]$, such that the restriction of Φ to each $\mathcal{T}_{N_d}(y)$ in the family is an embedding of a rotational invariant d-torus for the nonlinear equation (1).

Basic idea of the proof. It is observed that when $\int_0^\pi V(x)\, dx \neq 0$ the eigenvalues λ_i's satisfy

$$|\sqrt{\lambda_i} \pm \sqrt{\lambda_j} \pm \sqrt{\lambda_k} \pm \sqrt{\lambda_l}| \geq C_m \min(i,j,k,l)^{-1}, \text{ if } \min(i,j,k,l) \gg 1,$$

unless trivial relations like

$$\sqrt{\lambda_i} - \sqrt{\lambda_i} + \sqrt{\lambda_k} - \sqrt{\lambda_k},$$

where C_m is a constant depending on m. This estimate implies that in the neighborhood of the origin the equation (1) can be put to the Borkhoff normal form[6] up to terms of the fourth order. Then the application of KAM theorem for PDEs implies that there are many invariant tori for (1) in the neighborhood of the origin.
Remarks *1.* The assumption $\int_0^\pi V(x)\, dx \neq 0$ is not essential. One has

$$\sqrt{\lambda_j} = j + \frac{c_1}{j} + \frac{c_2}{j^2} + \cdots + \frac{c_n}{j^n} + O\left(\frac{1}{j^{n+1}}\right),$$

where c_j's are some constants depending on V, in particular, $c_1 = -\frac{1}{2\pi}\int_0^\pi V(x)\, dx$. Then the assumption $\int_0^\pi V(x)\, dx \neq 0$ is equivalent to $c_1 \neq 0$. The assumption $c_1 \neq 0$ is used just only in making Birkhoff normal form. By overcoming more technical trouble one can still get the normal form true under conditions $c_1 = 0, \cdots, c_{k-1} = 0$ and $c_k \neq 0$ for some $1 \leq k \leq n$. Therefore the assumption $\int_0^\pi V(x)\, dx \neq 0$ can be nearly replaced by $V(x) \neq 0$ in the Theorem 1. *2.* Theorem 1 still holds true for the following equation

$$u_{tt} = u_{xx} - V(x)u \pm u^3 + \sum_{m \geq k \geq 2} a_k u^{2k+1}$$

where m is a positive integer and a_k's are some real numbers. *3.* The method in proving Theorem 1 can be applied to NLS equation:

$$\sqrt{-1}u_t - u_{xx} + V(x)u \pm u^3 = 0$$

subject to Dirichlet boundary conditions. *4.* If $\lambda_1 > 0$, then the obtained invariant tori are elliptic. If $\lambda_1 < 0$, then the tori are hyperbolic-elliptic. *5* We can give the measure estimate of the set $\mathcal{C}$:

$$\text{meas } \mathcal{C} \geq \text{ meas } \mathcal{C}^* \cdot (1 - O(\epsilon^{1/13})).$$

Naturally one can ask: *Is there any invariant torus for (1) when $V(x) \equiv 0$?*

This problem remains open for a relatively long time, which was proposed by many authors, such as Pöschel[25], Craig and Wayne[12], Kuksin[18], and Marmi and Yoccoz[23]. The present author answers this question:

Theorem 2. ([31])*Assume* $v(x) \equiv 0$. *For any* $d \in \mathbb{N}$, *the equation (1) subject to the periodic boundary condition possesses many* $d+1$*-dimensional invariant tori in the neighborhood of the equilibrium* $u \equiv 0$. *The motions on the tori are quasi-periodic.*

Basic idea of the proof. Assume $V(x) \equiv 0$. Let $u_0(t)$ be a non-zero solution of the equation $\ddot{u}_0 + u_0^3 = 0$. We will construct the invariant tori or quasi-periodic solutions in the neighborhood of the solution $u_0(t)$ which is uniform in space and periodic in time. To that end, inserting $u = u_0 + \epsilon u$ into (1) we get

$$u_{tt} - u_{xx} + 3u_0^2(t)u + \epsilon \cdot (h.o.t.) = 0, \quad x \in S^1. \tag{2}$$

In considerably rough speaking, by the averaging method we reduce this equation to

$$u_{tt} - u_{xx} + 3\widehat{u_0^2}(0)u + \epsilon \cdot (h.o.t.) = 0, \quad x \in S^1, \tag{3}$$

where $\widehat{u_0^2}(0) = \frac{1}{2\pi}\int_0^{2\pi} u_0^2(t)\ dt \neq 0$. Then we construct the invariant tori or quasi-periodic solutions of (3) by advantage of $\widehat{u_0^2}(0) \neq 0$. At this time, we should deal with (3) by the same way as in [25]. Unfortunately, one of the frequencies of the Hamiltonian corresponding to (3) is zero (see (1.9)). This causes the "integrable" part of the Hamiltonian serious degenerate, incurring great expanse in using KAM technique.

Firstly one can easily find the periodic solution $u_0(t)$ of $\ddot{u}_0 + u_0^3 = 0$ with its frequency ω and show that $\widehat{u_0^2}(0) \neq 0$. Then consider a family of Hamiltonian functions

$$H_n = \sqrt{\lambda_n} z_n \bar{z}_n + \frac{3u_0^2(t)}{4\sqrt{\lambda_n}}(z_n + \bar{z}_n)^2, \quad \lambda_n \neq 0, n \in \mathbb{N} = \{1, 2, ...\}. \tag{4}$$

Notice that their equation of motion is linear. By the reducing theory from KAM theory [9, 14] we reduce (4) to

$$H_n = \mu_n z_n \bar{z}_n, \quad \mu_n = \sqrt{\lambda_n} + \frac{3}{n\pi}\omega^2 + O(\omega^{23/9}/n), n \in \mathbb{N}, \tag{5}$$

where λ_n is the eigenvalues of the Sturm-Liouville problem $-y'' = \lambda y, \quad x \in S^1$. At the same time by the Floquet theory [22] one can reduce the Hamiltonian

$$H_0 = \frac{1}{2}y_0^2 + \frac{\lambda_0}{2}x_0^2 + \frac{3u_0^2(t)}{2}x_0^2, \quad \lambda_0 = 0, \tag{6}$$

to

$$H_0(q, p) = \frac{1}{2}c_0\omega^2 p^2, \tag{7}$$

where c_0 is a constant. In order to exclude the multiplicity of the eigenvalues λ_n, one can find a solution which is even in the space variable x. That is, one can write $u(t, x) = \sum_{n \geq 0} x_n(t) \cos nx$. From this we get a Hamiltonian corresponding to (2) which reads as

$$H = \frac{1}{2}\sum_{n \geq 0} y_n^2 + \lambda_n x_n^2 + \frac{3}{2}u_0^2(t)x_n^2 + \epsilon G^3 + \epsilon^2 G^4, \tag{8}$$

where G^3 (and G^4, resp.) is a polynomial of order 3 (and 4, resp.) in variables $x_0, x_1, ...$ Introducing the complex variables we re-write (8):

$$\begin{aligned} H = \quad & \tfrac{1}{2}y_0^2 + \tfrac{3}{2}u_0^2(\vartheta)x_0^2 + \textstyle\sum_{n\geq 1}\sqrt{\lambda_n}z_n\bar{z}_n + \tfrac{3}{4}\tfrac{u_0^2(t)}{\sqrt{\lambda_n}}(z_n+\bar{z}_n)^2 \\ & +\epsilon G^3 + \epsilon^2 G^4 \end{aligned} \tag{9}$$

Applying (4-7) we get a symplectic transformation Ψ such that

$$H := H\circ\Psi = \frac{1}{2}(c_0\omega^2)p^2 + \sum_{n>0}\mu_n z_n\bar{z}_n + \epsilon\tilde{G}^3 + \epsilon^2\tilde{G}^4. \tag{10}$$

Notice that $\tilde{G}^3$ and $\tilde{G}^4$ involve the time t. Let $\vartheta = \omega t$ be an angle-variable and J =Const. be an action-variable. Then (10) can reads as

$$H := H\circ\Psi = \frac{1}{2}(c_0\omega^2)p^2 + J\omega + \sum_{n>0}\mu_n z_n\bar{z}_n + \epsilon\tilde{G}^3 + \epsilon^2\tilde{G}^4, \tag{11}$$

which is autonomous. One can now kill the perturbations $\tilde{G}^3$ and the non-resonant part of the perturbation $\tilde{G}^4$ by Birkhoff normal form. The one gets

$$\begin{aligned} H = \quad & \tfrac{1}{2}(c_0\omega^2)p^2 + \epsilon^2 c_1 q^4 + \epsilon^3 O(q^5) + \textstyle\sum_{n\neq 0}\mu_n z_n\bar{z}_n + J\omega \\ & + \textstyle\sum_j \left(c_2\epsilon^2 q^2 + \epsilon^3 O(q^3)\right) z_j\bar{z}_j \\ & + \textstyle\sum_{i\in N_d, j\in\mathbb{N}} \left(\epsilon^2 c_3 + \epsilon^3 O(q)\right) z_i\bar{z}_i z_j\bar{z}_j \\ & + \text{small perturbation }, \end{aligned} \tag{12}$$

After introducing action-angle variables (I_0, ϕ_0) corresponding to (q, p), then (13) reads as

$$\begin{aligned} H = \quad & \epsilon^{2/3}c_4 I_0^{4/3} + \textstyle\sum_{j>0}\mu_j z_j\bar{z}_j + J\omega \\ & +\Gamma(I_0,\phi_0) + \textstyle\sum_{j>0}\Gamma_j(I_0,\phi_0)z_j\bar{z}_j + \sum_{j\in\mathbb{N}, i\in N_d}\Gamma_{ij}z_i\bar{z}_i z_j\bar{z}_j \\ & + \textstyle\sum_{j\in\mathbb{N}, i\in N_d}\epsilon^2 c_5 z_i\bar{z}_i z_j\bar{z}_j + \text{ small perturbation }. \end{aligned} \tag{13}$$

The c_i's are constants. Using the averaging method we remove the dependence of Γ, Γ_j and Γ_{ij} on the angle variable ϕ_0. After this, we get a Hamiltonian $H = H_0$+small perturbation, where H_0 is *integrable* and "*twist*". The "twist" property can provide the parameters which we need in using KAM technique. Finally, one gets the invariant tori for (1) with $V(x) \equiv 0$, by making use of KAM theorem.

Remark. Bourgain[8], Bambusi-Paleari [10], Berti- Bolle[2, 3] and Gentile-Mastropietro-Procesi[16] construct countably many families of periodic solutions for the nonlinear wave equation $u_{tt} - u_{xx} \pm u^3 + \text{h.o.t.} = 0$. See also [3, 11, 21] and the references theirin for the related problems. More recently, Procesi [26] and Baldi[1] constructed quasi-periodic solutions of 2-dimensional frequency and of Lebesgue measure 0 for the completely resonant nonlinear wave equations. Their construction of quasi-periodic solutions is concise and elegant.

Theorem 3. ([31])*For any $d \in \mathbb{N}$, the equation*

$$u_{tt} - u_{xx} - u^3 = 0 \tag{14}$$

subject to the Dirichlet boundary condition

$$u(t, 0) = u(t, \pi) = 0 \tag{15}$$

possesses many d-dimensional hyperbolic-elliptic invariant tori in the neighborhood of the equilibrium $u \equiv 0$. The motions on the tori are quasi-periodic.

Basic idea of the proof. Let $u_0(t,x) \equiv u_0(x)$ solve ODE $u_{xx} + u^3 = 0$ with b.c. (15). let

$$u = u_0 + \epsilon \tilde{u}. \tag{16}$$

Inserting (16) into (14) we get $\tilde{u}$ obeys the following equation and b.c.

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} - 3u_0^2 \tilde{u} - 3\epsilon u_0 \tilde{u}^2 - \epsilon^2 \tilde{u}^3 = 0, \\ \tilde{u}(t,0) = \tilde{u}(t,\pi) = 0. \end{cases} \tag{17}$$

Let $V(x) = -3u_0^2$. It is easy to verify $\int_0^\pi V(x)\, dx \neq 0$. By the method similar to that of Theorem 1 we can show that the existence of invariant tori.

References

[1] Baldi, P., Quasi-periodic solutions of the equation $u_{tt} - u_{xx} + u^3 = f(u)$, preprint

[2] Berti, M., Bolle, P. Periodic solutions of nonlinear wave equations with general nonlinearity, Preprint

[3] Berti, M., Bolle, P. Cantor families of periodic solutions for completely resonant nonlinear wave equations, Preprint

[4] A. I. Bobenko and S. Kuksin, The nonlinear Klein-Gordon equation on an interval as perturbed sine-Gordon equation, Comm. Math. Helv., **70**, 63–112 (1995)

[5] Bricmont, J., Kupiainen, A. and Schenkel, A., Renormalization group and the Melnikov Problems for PDE's, Commun. Math. Phys., **221**, 101–140 (2001)

[6] Birkhoff, G. D., Dynamical systems, Am. Math. Soc. Colloq. Publ. IX. New York: American Mathematical Society. VIII, 1927

[7] Bourgain, J., Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and application to nonlinear pde, Int. Math. Research Notices **11** 475–497(1994)

[8] Bourgain, J., Periodic solutions of nonlinear wave equations, Harmonic analysis and partial equations, Chicago Univ. Press, 1999, pp.69-97

[9] Bogoljubov, N. N., Mitropoliskii, Yu. A. and Samoilenko, A. M.: Methods of Accelerated Convergence in Nonlinear Mechanics. Springer-Verlag, New York (1976) [Russian Original: Naukova Dumka, Kiev, 1969]

[10] Bambusi, D., and Paleari, S., Families of periodic orbits for resonant PDE's, J. Nonlinear Science, **11** (2001), 69-87

[11] Brézis, H., Periodic solutions of nonlinear vibrating strings and duality principles, Bull. AMS **8** 409–426(1983)

[12] Craig, W. and Wayne, C., Newton's method and periodic solutions of nonlinear wave equation, Commun. Pure. Appl. Math.,**46** 1409–1501 (1993)

[13] Chierchia, L. and You, J.: KAM tori for 1D nonlinear wave equations with periodic boundary conditions. Comm. Math. Phys. **211** 497–525 (2000)

[14] Eliasson, L. H., Reducibility and point spectrum for linear quasi-periodic skew-products, Proceedings of ICM 1998, Vol. **II**, Doc. Math. J. DMV, 779-787

[15] Frohlich,J., Spencer,T. and Wayne, C.E.: Localization in Disordered, Nonlinear Dynamical Systems. J. Statistical Physics. **42** 247–274 (1986)

[16] Gentile, G. and Mastropietro, V. and Procesi, M., Periodic solutions for completely resonant nonlinear wave equations, Comm. Math. Phys. **256**, 437–490 (2005)

[17] Kuksin, S. B.: Nearly integrable infinite-dimensional Hamiltonian systems.(Lecture Notes in Math. 1556). Springer-Verlag, New York (1993)

[18] Kuksin, S. B.: Elements of a qualitative theory of Hamiltonian PDEs, Proceedings of ICM 1998, Vol. **II**, Doc. Math. J. DMV, 819–829 (1998)

[19] Kuksin, S. B.: Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum. Funct. Anal. Appl. **21**, 192–205 (1987)

[20] Kuksin, S. B. and Pöschel, J.: Invariant Cantor manifolds of quasiperiodic oscillations for a nonlinear Schrödinger equation, Ann. Math. **143** 149–179 (1996)

[21] Lidskij, B. V. and Shulman, E., Periodic solutions of the equation $u_{tt} - u_{xx} + u^3 = 0$, Funct. Anal. Appl. **22** 332-333 (1988)

[22] Meyer, K. R. and Hall, G. R., Introduction to Hamiltonian dynamical systems and the N-body problem, (Applied Math. Sciences, 90), Springer-Verlag, New York (1992)

[23] Marmi, S. and Yoccoz, J.-C., Some open problems related to small divisors, (Lecture Notes in Math. 1784). Springer-Verlag, New York (2002)

[24] Pöschel, J., A KAM-theorem for some nonlinear PDEs, Ann. Scuola Norm. Sup. Pisa, Cl. Sci., IV Ser. 15, **23** 119–148 (1996)

[25] Pöschel, J., Quasi-periodic solutions for nonlinear wave equation, Commun. Math. Helvetici **71** 269–296 (1996)

[26] Procesi, M., Qasui-periodic solutions for completely resonant nonlinear wave equations in 1D and 2D, Discr. and Cont. Dyn. Syst. **13** 541-552 (2005)

[27] Vittot, M. and Bellissard, J.: Invariant tori for an infinite lattice of coupled classical rotators. Preprint, CPT-Marseille (1985)

[28] Wayne, C. E.: Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. Commun. Math. Phys.**127** 479–528 (1990)

[29] Yuan, X.: Invariant Manifold of Hyperbolic-Elliptic Type for Nonlinear Wave Equation, Int. J. Math. Math. Science, **18** 1111-1136 (2003)

[30] Yuan, X.: Quasi-periodic solutions of nonlinear wave equations with a prescribed potential, Discrete and continuous dynamical systems, (to appear)

[31] Yuan, X.:Quasi-periodic solutions of completely resonant nonlinear wave equations, Preprint, FDIM-2004-12, September, J. Differential Equations, in press