# LONG RANGE ORDER AND GIANT COMPONENTS OF QUANTUM RANDOM GRAPHS

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Abstract. Mean field quantum random graphs give a natural generalization of classical Erdős-Rényi percolation model on complete graph $\mathsf{G}_N$ with $p = \beta/N$. Quantum case incorporates an additional parameter $\lambda \geqslant 0$, and the short-long range order transition should be studied in the $(\beta, \lambda)$-quarter plane. In this work we explicitly compute the corresponding critical curve $\gamma_c$, and derive results on two-point functions and sizes of connected components in both short and long range order regions. In this way the classical case corresponds to the limiting point $(\beta_c, 0) = (1, 0)$ on $\gamma_c$.

## 1. Introduction and Results

1.1. **Classical Erdős-Rényi random graphs.** In the classical Erdős - Rényi model of random graphs each two vertices $i \neq j$ of the complete graph $\mathsf{G}_N = \{1, \dots, N\}$ are connected with probability $p = \beta/N$ independently from all other edges. The phase transition [JLR] occurs at the critical value

$$\beta_c \;=\; 1. \tag{1.1}$$

Namely, for $\beta > \beta_c$ with probabilities of order $1 - \mathrm{o}(1)$ there is a unique giant connected component of size $O(N)$ , whereas for $\beta < \beta_c$ all the connected components of $\mathsf{G}_N$ have sizes of the order $O(\log N)$ or less.

1.2. **Quantum version of Erdős-Rényi random graphs.** Let us formulate now a quantum version of Erdős-Rényi random graphs. As we shall briefly explain in the end of the section, both the motivation and the choice of terminology comes from the stochastic geometric (Fortuin-Kasteleyn type) representation of quantum Curie-Weiss model in transverse magnetic field, which was originally developed in the general ferromagnetic context in [AKN].

There are two parameters $\beta \in (0, \infty]$ - the inverse temperature and $\lambda \in [0, \infty)$ - the strength of the transversal field. The case $\beta = \infty$ corresponds to the ground state, and the case $\lambda = 0$ brings us back to the context of classical random graphs discussed above.

Given $\beta \in [0, \infty]$ let us use $\mathbb{S}_\beta$ to denote the circle of length $\beta$ under the convention $\mathbb{S}_\infty = \mathbb{R}$. The model is built on the space $\mathfrak{G}_N^\beta = \mathsf{G}_N \times \mathbb{S}_\beta$, that is to each site $i \in \mathsf{G}_N$ we attach a copy $\mathbb{S}_\beta^i$ of $\mathbb{S}_\beta$. With a slight abuse of notation we shall also write $\mathbb{S}_\beta^i = i \times \mathbb{S}_\beta$.

Our next step is to make finite $(\beta < \infty)$ or countable $(\beta = \infty)$ random number of holes in each $\mathbb{S}_\beta^i$ and draw finite ( $\beta < \infty$ ) or countable $(\beta = \infty)$ random number of

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links between various points of $\mathfrak{G}_N$ with the same time coordinates, that is between points of the type $(i,t)$ and $(j,t)$ where $i \neq j$ and $t \in \mathbb{S}_\beta$. Both operations are going to be performed with the help of independent Poisson point processes over the time space $\mathbb{S}_\beta$ and, eventually, will lead to a splitting of $\mathfrak{G}_N^\beta$ into a finite ($\beta < \infty$) or countable ($\beta = \infty$) number of disjoint maximal connected components,

$$\mathfrak{G}_N^\beta \setminus \mathcal{H} \;=\; \mathfrak{C}_1 \vee \cdots \vee \mathfrak{C}_n, \tag{1.2}$$

where $\mathcal{H}$ is the set of the holes. An example for $N = 3$ is given on Figure 1.2. For each fixed $x \in \mathfrak{G}_N^\beta$ the probability $\mathbb{P}_N^{\beta,\lambda}(x \in \mathcal{H}) = 0$. Thus, for given $x \in \mathfrak{G}_N^\beta$ the notion $\mathfrak{C}(x)$ of the connected component containing $x$ in the decomposition (1.2) is well defined.

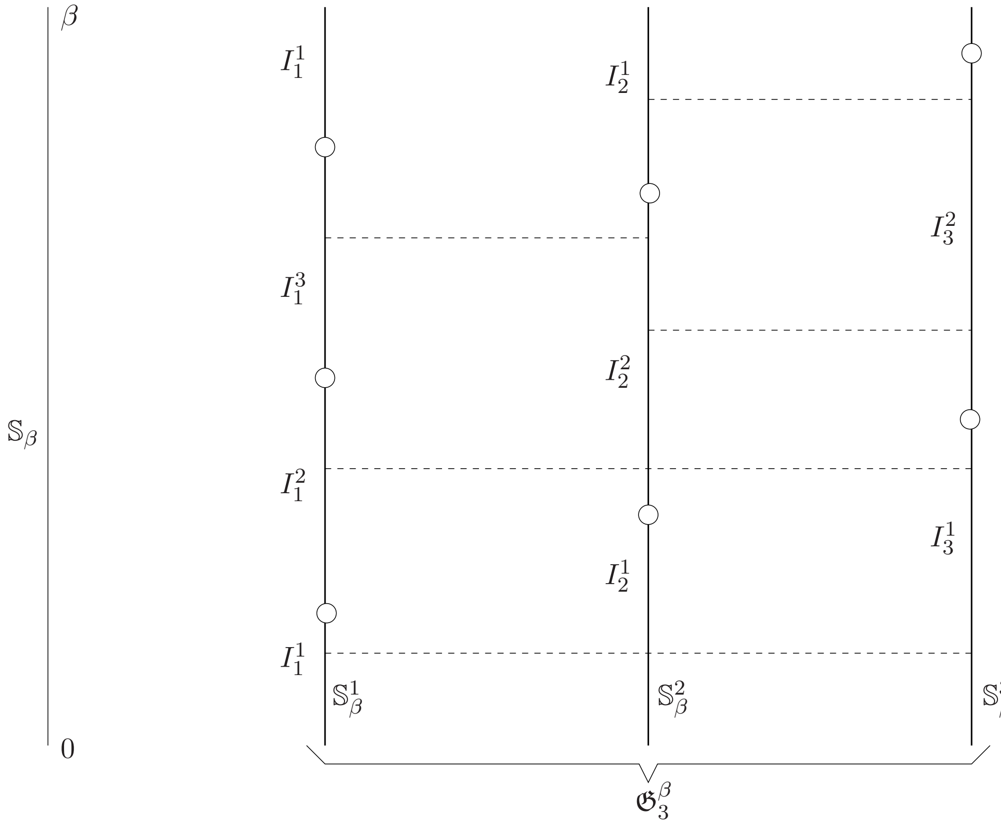


FIGURE 1. An example of the decomposition of $\mathfrak{G}_3^\beta$ after all the holes are punched and all the links are drawn: $\mathfrak{G}_3^\beta \setminus \mathcal{H} = \mathfrak{C}_1 \vee \mathfrak{C}_2$ , where $\mathfrak{C}_1 = I_1^1 \cup I_1^2 \cup I_3^1$ and $\mathfrak{C}_2 = I_1^3 \cup I_2^1 \cup I_2^2 \cup I_3^2$

*Processes of holes $\mathcal{H}_i$.* For each $i \in \mathsf{G}_N$ the process of holes $\mathcal{H}_i$ is the Poisson point process on $\mathbb{S}_\beta^i$ with intensity $\lambda$. $\mathcal{H}_i$-s are assumed to be independent for different $i$-s. For $\beta < \infty$ the punched circle $\mathbb{S}_\beta^i \setminus \mathcal{H}_i$ consists of $n$ disjoint connected intervals,

$$\mathbb{S}_\beta^i \setminus \mathcal{H}_i \;=\; I_i^1 \cup I_i^2 \cup \cdots \cup I_i^n. \tag{1.3}$$

Of course $n = 1$ whenever the cardinality $\#\mathcal{H}_i = 0, 1$.

In the case $\beta = \infty$ the punched real line $\mathbb{S}^i_\beta \setminus \mathcal{H}_i$ is split into a countable disjoint union of connected intervals,

$$\mathbb{S}^i_\infty \setminus \mathcal{H}_i \;=\; \bigcup_{r=-\infty}^{\infty} I^r_i, \tag{1.4}$$

where we label $I^0_i$ the interval which contains $(i, 0)$.

In the sequel we shall use $|I^k_i|$ to denote the length of $I^k_i$, and we shall write $\mathcal{H}$ for the total collection of all the holes,

$$\mathcal{H} \;=\; \cup_i \mathcal{H}_i \;\subset\; \mathfrak{G}^\beta_N.$$

*Processes of links* $\mathcal{L}_{ij}$ *and decomposition* (1.2). With each (unordered) pair of vertices $i, j \in \mathsf{G}_N$ we associate a copy $\mathbb{S}^{ij}_\beta$ of $\mathbb{S}_\beta$ and a Poisson point process $\mathcal{L}_{ij}$ on $\mathbb{S}^{ij}_\beta$ with intensity $1/N$. Processes $\mathcal{L}_{ij} = \mathcal{L}_{ji}$ are assumed to be independent for different $(i, j)$ and also independent of the processes of holes $\mathcal{H}_i$.

Two intervals $I^k_i$ and $I^l_j$ either in the decomposition (1.3) or accordingly , in the case $\beta = \infty$, in the decomposition (1.4) are said to be connected if there exists $t \in \mathbb{S}^{ij}_\beta$ such that $t \in \mathcal{L}_{ij}$, whereas $(i, t) \in I^k_i$ and $(j, t) \in I^l_j$. The decomposition (1.2) of $\mathfrak{G}^\beta_N \setminus \mathcal{H}$ into maximal connected components is, thereby, well defined.

*Relation to the classical Erdős-Rényi random graph.* If $\lambda = 0$ then there are no holes and $\mathbb{S}^i_\beta \setminus \mathcal{H}_i$ always contains only one connected component, which of course equals to $\mathbb{S}^i_\beta$ itself. In the latter case, the probability ($\beta < \infty$) that $\mathbb{S}^i_\beta$ and $\mathbb{S}^j_\beta$ are connected equals to $1 - \mathrm{e}^{-\beta/N}$ and we are back to the original Erdős- Rényi setup.

1.3. **Phase transition in the $(\beta, \lambda)$-plane.** The critical curve $\gamma_c$ in the $(\beta, \lambda)$-coordinate quarter plane is implicitly given by (see Figure 1.3)

$$F(\beta, \lambda) \;\triangleq\; \frac{2}{\lambda}\left(1 - \mathrm{e}^{-\lambda\beta}\right) - \beta \mathrm{e}^{-\lambda\beta} \;=\; 1. \tag{1.5}$$

It is easy to check that $\gamma_c$ is in fact a graph of a function $\lambda_c = \lambda_c(\beta)$ defined on $\beta \in [1, \infty)$. Consider the decomposition (Figure 1.3) of the off-critical region

$$\mathbb{R}^2_+ \setminus \gamma_c \;=\; A_{\mathrm{LRO}} \cup A_{\mathrm{SRO}}$$

where

$$A_{\mathrm{LRO}} \;=\; \left\{(\beta, \lambda) \in \mathbb{R}^2_+ \;:\; F(\beta, \lambda) > 1\right\}. \tag{1.6}$$

LRO and SRO above stand for the long (respectively short ) range order.

Our main result states that for $(\beta, \lambda) \in A_{\mathrm{LRO}}$ there is a long range order in the sense that the probability of two points $(i, j)$ and $(j, s)$ being connected does not vanish when the size of the system tends to infinity. Contrary to this such probability vanishes in the $N \to \infty$ limit whenever $(\beta, \lambda) \in A_{\mathrm{SRO}}$. The survival of probabilities of connections is related to an emergence of an $O(N)$-giant connected component in the disjoint decomposition (1.2) in the *LRO* regime; in particular for $(\beta, \lambda) \in A_{\mathrm{SRO}}$ typical connected component of any point $(i, t) \in \mathfrak{G}^\beta_N$ is of order $O(\log N)$. This is a quantum version of Erdős- Rényi phase transition phenomenon and, since

$$\lim_{\lambda \to 0} F(\beta, \lambda) \;=\; \beta,$$

the classical case is recovered in the limiting $\lambda = 0$ case.

We proceed with several exact alternative statements of this result.

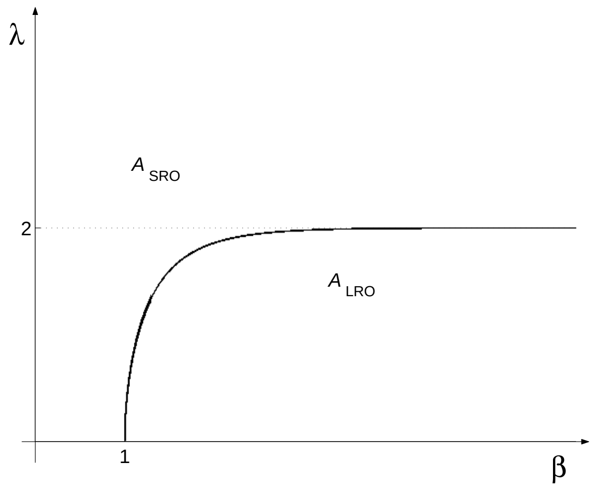


FIGURE 2. Decomposition of the $(\beta,\lambda)$ quarter plane into the short range and long range regions.

1.4. **Long and short range order.** In the sequel we shall use $\mathbb{P}_N^{\beta,\lambda}(\cdot)$ for the joint product measure of all the processes $\mathcal{H}_i$ of holes and all the processes $\mathcal{L}_{ij}$ of links as defined above.

Let us say that two points $(i,t),(j,s) \in \mathfrak{G}_N^\beta$ are connected if they belong to the same connected component in the decomposition (1.2). We shall denote the latter event as $\{(i,t) \longleftrightarrow (j,s)\}$.

**Theorem A.** *If $(\beta,\lambda) \in A_{\text{SRO}}$, then*

$$\mathbb{P}_N^{\beta,\lambda}\left((i,t) \longleftrightarrow (j,s)\right) \;=\; O\left(\frac{\log N}{N}\right) \tag{1.7}$$

*uniformly in $t,s \in \mathbb{S}_\beta$ and $i \neq j$.*
*On the other hand, if $\beta < \infty$ and $(\beta,\lambda) \in A_{\text{LRO}}$, then there exists $p = p(\beta,\lambda) \in (0,1)$, such that*

$$\mathbb{P}_N^{\beta,\lambda}\left((i,t) \longleftrightarrow (j,s)\right) \;=\; p(\beta,\lambda)^2\left(1+\mathrm{o}(1)\right), \tag{1.8}$$

*also uniformly in $t,s \in \mathbb{S}_\beta$ and $i \neq j$.*

1.5. **Emergence of the giant component.** Each connected cluster $\mathfrak{C}_k$ in the decomposition (1.2) consists of disjoint union of intervals

$$\mathfrak{C}_k \;=\; \bigcup_l J_k^l,$$

where each $J_k^l$ coincides with some $I_i^r$ in one of the decompositions (1.3) ((1.4) in the $\beta = \infty$ case). Define,

$$|\mathfrak{C}_k| \;=\; \sum_l |J_k^l|.$$

For any fixed $(i,t) \in \mathfrak{G}_N^\beta$ the probability

$$\mathbb{P}_N^{\beta,\lambda}\left((i,t) \in \mathcal{H}\right) \;=\; 0.$$

Thus, in general position, $(i,t) \in \mathfrak{G}_N^\beta \setminus \mathcal{H}$ and there exists $\mathfrak{C}_k(i,t)$ (which from now on we shall denote as $\mathfrak{C}((i,t))$ such that $(i,t) \in \mathfrak{C}_k$. Evidently the distribution of $|\mathfrak{C}(x)|$ does not depend on a particular $x = (i,t)$.

If $\beta < \infty$ we also define the maximal cluster size

$$\mathcal{M} \;=\; \max_k |\mathfrak{C}_k|, \tag{1.9}$$

and the next to maximal cluster size,

$$\mathcal{M}^{\rm next} \;=\; \max\left\{|\mathfrak{C}_k| \,:\, |\mathfrak{C}_k| \neq \mathcal{M}\right\}. \tag{1.10}$$

These definitions would clearly make little sense if $\beta = \infty$.

**Theorem B.** *If $(\beta,\lambda) \in A_{\rm SRO}$, then for every $\kappa > 0$ there exists $c = c(\beta,\lambda,\kappa) < \infty$, such that*

$$\mathbb{P}_N^{\beta,\lambda}\left(|\mathfrak{C}(x)| > c\log N\right) \;=\; o\left(\frac{1}{N^\kappa}\right). \tag{1.11}$$

*Furthermore, if $\beta < \infty$, then*

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathcal{M} > c\log N\right) \;=\; o\left(\frac{1}{N^{\kappa-1}}\right) \tag{1.12}$$

*If, however, $\beta < \infty$ and $(\beta,\lambda) \in A_{\rm LRO}$ then there exists a sequence of positive numbers $\epsilon_N(\beta,\lambda) \to 0$ such that,*

$$\mathbb{P}_N^{\beta,\lambda}\left(\left|\frac{|\mathfrak{C}(x)|}{N} - \rho\right| < \epsilon_N\right) \;=\; p(\beta,\lambda)(1 - {\rm o}(1)), \tag{1.13}$$

*where $\rho = \rho(\beta,\lambda) = \beta p(\beta,\lambda) > 0$, and $p(\beta,\lambda)$ is the same probability as in* (1.8). *Furthermore, in the $\beta < \infty$ case, there exists a constant $c = c(\beta,\lambda) < \infty$ such that*

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathcal{E}(\rho,\epsilon_N,c)\right) = 1 - {\rm o}(1), \tag{1.14}$$

*where the event $\mathcal{E}(\rho,\epsilon_N,c)$ is defined via*

$$\mathcal{E}(\rho,\epsilon_N,c) \;=\; \left\{\left|\frac{\mathcal{M}}{N} - \rho\right| < \epsilon_N\right\} \cap \left\{\mathcal{M}^{\rm next} < c\log N\right\}. \tag{1.15}$$

In this paper we shall prove the short range order parts of Theorem A and Theorem B for all $\beta \leqslant \infty$, whereas in the long range order case we shall concentrate only on the case of positive temperatures $\beta < \infty$. A treatment of the LRO properties of the ground state case $\beta = \infty$ requires an additional coupling with branching random walks in the sense of [Bi]. Specifically, in the $\beta = \infty$ case in order to show that two large connected clusters intersect one should also control the time spread-off of each of these clusters. The corresponding results may be interesting in their own right and we relegate them to the forthcoming [IL].

1.6. **Relation to Curie-Weiss model in transverse field.** The Hamiltonian of the Curie-Weiss model in transversal field is given by

$$-\frac{1}{2N}\sum_{i\neq j}\sigma_i^{\rm z}\sigma_j^{\rm z} \;-\; \lambda\sum_i \sigma_i^{\rm x}, \tag{1.16}$$

where $\sigma_i^{\rm z}, \sigma_i^{\rm x}$ are Pauli spin 1/2 matrices. As it has been discovered in [AKN] (in a general ferromagnetic context) path-integral type representation of the Curie-Weiss leads to the following modification $\tilde{\mathbb{P}}_N^{\beta,\lambda}$ of our basic product measure of "holes" and "links", which is in fact the Fortuin-Kasteleyn representation of (1.16)

$$\tilde{\mathbb{P}}_N^{\beta,\lambda}(\mathrm{d}\mathcal{H},\mathrm{d}\mathcal{L}) = \frac{1}{\mathcal{Z}_N(\beta,\lambda)} 2^{\#_{\mathrm{c}}(\mathcal{H},\mathcal{L})}\mathbb{P}_N^{\beta,\lambda}(\mathrm{d}\mathcal{H},\mathrm{d}\mathcal{L}), \tag{1.17}$$

where $\#_{\mathrm{c}}(\mathcal{H},\mathcal{L})$ is the number of maximal connected components in the decomposition (1.2). In particular, the two point function could be expressed in terms of this FK-measure as

$$\langle \sigma_i^{\mathrm{z}}\sigma_j^{\mathrm{z}}\rangle_N^{\beta,\lambda} = \tilde{\mathbb{P}}_N^{\beta,\lambda}(i \longleftrightarrow j).$$

A sample path large deviation analysis [IL] of (1.16) indicates that the long/short range order critical curve in the $(\beta,\lambda)$-plane is still given by (1.5). This, in view of the analysis of classical FK models on complete graphs [BGJ], is not very surprising, however so far we did not find a way it deduce it from purely stochastic geometric considerations, which would generalize recolouring techniques of the latter paper.

1.7. **FKG properties of $\mathbb{P}_N^{\beta,\lambda}(\cdot)$.** Many of our arguments rely on the following FKG (Fortuin-Kasteleyn-Ginibre) property of $\mathbb{P}_N^{\beta,\lambda}(\cdot)$:

Let us define the partial order of the probability space, $\Omega$, in which $(\mathcal{H},\mathcal{L})$ takes values in the following way:

$$(\mathcal{H}',\mathcal{L}') \gg (\mathcal{H},\mathcal{L}) \Leftrightarrow \mathcal{H}' \subseteq \mathcal{H} \text{ and } \mathcal{L}' \supseteq \mathcal{L}$$

In other words, in the decomposition of $\mathfrak{G}_N^\beta$ generated by $(\mathcal{H}',\mathcal{L}')$ there are less holes and more links than in the decomposition corresponding to $(\mathcal{H},\mathcal{L})$.

In the sequel we shall say that $A$ is an increasing (decreasing) event if for all $(\mathcal{H},\mathcal{L}) \in A$,

$$\text{if } (\mathcal{H}',\mathcal{L}') \gg (\mathcal{H},\mathcal{L})\big((\mathcal{H}',\mathcal{L}') \ll (\mathcal{H},\mathcal{L})\big) \text{ then } (\mathcal{H}',\mathcal{L}') \in A.$$

As it has been proved in [AKN] a probability measure $\mathbb{P}_N^{\beta,\lambda}(\cdot)$ has the positive association property: if both $A$ and $B$ are increasing (decreasing) events, then

$$\mathbb{P}_N^{\beta,\lambda}(A \cap B) \geqslant \mathbb{P}_N^{\beta,\lambda}(A)\,\mathbb{P}_N^{\beta,\lambda}(B)$$

1.8. **Structure of the paper.** Our proof is built upon the classical treatment (see e.g. [BGJ]). An essential additional complication to be encountered is that in the quantum case two different clusters may share a spatial component without intersecting. In other words it can happen that there exists an index $i \in \mathsf{G}_N$ and two disjoint clusters $\mathfrak{C}_1$ and $\mathfrak{C}_2$ such that $\mathfrak{C}_l \cap \mathbb{S}_\beta^i \neq \emptyset$ for $l = 1, 2$.

In Section 2 we set up most of the relevant notation and develop our basic inductive construction of percolation clusters. In the $\beta < \infty$ case the genealogical structure of percolation clusters could be ignored and, accordingly, our exposition could be slightly simplified. However, the multi-index notation we employ will become indispensable in the $\beta = \infty$ case [IL] and, besides, we feel that it gives a rather natural way to describe connected clusters.

Section 3 is devoted to the proofs of all our main results.

## 2. Construction of connected clusters

2.1. **Underlying probability space.** Let $\mathbf{M}_+$ be the countable set of all finite multi-indices $\underline{\alpha}$,

$$\mathbf{M}_+ = \{\underline{\alpha} = (\alpha_1,\ldots,\alpha_n)\ ;\ \alpha_i \in \mathbb{N}\ ,\ n = 1, 2, \ldots\}.$$

There is a total order on $\mathbf{M}_+$: we shall say that $\underline{\alpha} \prec \underline{\gamma}$ if either $\underline{\alpha}$ has less entries (belongs to an older generation) than $\underline{\gamma}$ or else if $\underline{\alpha}$ is less than $\underline{\gamma}$ in the lexicographical order. The underlying probability measure which we proceed to call $\mathbb{P}_N^{\beta,\lambda}(\cdot)$ is a product measure

$$\mathbb{P}_N^{\beta,\lambda}(\cdot) = \bigotimes_{\underline{\alpha}\in\mathbf{M}_+} \mathbb{Q}^{\beta,\lambda}_{N,\underline{\alpha}}\cdot.$$

Each measure $\mathbb{Q}^{\beta,\lambda}_{N,\underline{\alpha}}(\cdot)$ generates an interval $I[\underline{\alpha}] \subseteq \mathbb{S}_\beta$ and, subsequently, a point process $\mathcal{X}_+[\underline{\alpha}]$ on $\mathfrak{G}_N^\beta = \mathsf{G}_N \times \mathbb{S}_\beta$ according to the following procedure, whose relation to the background $(\mathcal{H}, \mathcal{L})$ process of holes and links should be obvious:

*Construction of* $I[\underline{\alpha}]$. Let $U, V$ be two independent $\mathrm{Exp}(\lambda)$ random variables. If $U+V \geqslant \beta$, then $I = \mathbb{S}_\beta$. Otherwise, $I = (-V, U) \subset \mathbb{S}_\beta$.

In the sequel we shall refer to the distribution of the interval $I[\underline{\alpha}]$, or more precisely to the distribution of its length $|I[\underline{\alpha}]|$, just constructed as to $\Gamma_\beta(2,\lambda)$ distribution. Obviously, in the $\beta = \infty$ case $|I|$ of $I$ is distributed as $\Gamma(2,\lambda)$ variable.

*Construction of the number of offsprings and of the point process* $\mathcal{X}_+[\underline{\alpha}]$. Given $I[\underline{\alpha}]$ and, in particular the length $|I[\underline{\alpha}]|$, sample the number of off-springs $\xi_+[\underline{\alpha}]$ from the Poisson distribution

$$\xi_+[\underline{\alpha}] \sim \text{Poisson}\left(|I[\underline{\alpha}]|\right).$$

We shall denote the (unconditional ) distribution of $\xi_+[\underline{\alpha}]$ constructed above as $\Xi_\beta(2,\lambda)$.

With $I[\underline{\alpha}]$ and $\xi_+[\underline{\alpha}]$ fixed, sample

$$\mathcal{X}_+[\underline{\alpha}] = \left\{x_1[\underline{\alpha}], \dots, x_{\xi_+}[\underline{\alpha}]\right\} = \left\{(d_1[\underline{\alpha}], \tau_1[\underline{\alpha}]), \dots, (d_{\xi_+}[\underline{\alpha}], \tau_{\xi_+}[\underline{\alpha}])\right\},$$

where the departure times $\tau_1, \dots, \tau_{\xi_+}$ are i.i.d. random variables with the uniform distribution on $I[\underline{\alpha}]$, whereas the departure destinations $d_1, \dots d_{\xi_+}$ are i.i.d. uniform $\mathrm{Uni}(\mathsf{G}_N)$ random variables. For the latter use we define the set of all departure destinations from $\underline{\alpha}$,

$$\mathcal{D}_+[\underline{\alpha}] = \mathrm{proj}_{\mathsf{G}_N} \mathcal{X}_+[\underline{\alpha}].$$

2.2. **Construction of $\mathfrak{C}(x)$.** Let $x \in \mathfrak{G}_N^\beta$. The connected cluster $\mathfrak{C}(x)$ (see Subsection 1.2) is a disjoint union,

$$\mathfrak{C}(x) = \bigvee_k i_k \times J_k \tag{2.18}$$

of intervals $J_k \subseteq \mathbb{S}_\beta$ with spatial coordinates $i_k$. These disjoint intervals will be labeled by a subset $\mathbf{M}^x \subset \mathbf{M}_+$ of multi-indices. We shall always record the multi-indices from $\mathbf{M}^x$ in their increasing order, $\mathbf{M}^x = \{\underline{\alpha}_1, \underline{\alpha}_2, \dots\}$. In this way we denote the spatial coordinate $i_k = i[\underline{\alpha}_k]$ and the associated interval $J_k = J[\underline{\alpha}_k]$ and rewrite (2.18) as

$$\mathfrak{C}(x) = \bigvee_{\underline{\alpha}\in\mathbf{M}^x} i[\underline{\alpha}] \times J[\underline{\alpha}]. \tag{2.19}$$

Our construction of $\mathbf{M}^x$ and, accordingly, of $\{i[\underline{\alpha}], J[\underline{\alpha}]\}$ is an inductive one: At each stage we screen a certain multi-index $\underline{\alpha}$ and keep track of

- $\mathbf{M}^x(\underline{\alpha})$ - set of multi-indices which were already saturated into $\mathbf{M}^x$ before $\underline{\alpha}$.
- $\mathbf{R}^x(\underline{\alpha})$ - set of multi-indices (including $\underline{\alpha}$ itself) which are potential candidates for the membership in $\mathbf{M}^x$ and which are yet to be screened.

Both $\mathbf{M}^x$ and $\mathbf{R}^x$ are updated once $\underline{\alpha}$ is screened. The construction is complete whenever we finish an update with $\mathbf{R}^x = \emptyset$.

Notice that at each stage we also keep track both of space coordinates $i[\underline{\alpha}] \in \mathsf{G}_N$ and of time coordinates $t[\underline{\alpha}] \in \mathbb{S}_\beta$ for all multi-indices $\underline{\alpha}$ from $\mathbf{M}^x \cup \mathbf{R}^x$. On the other hand, we sample intervals $J[\underline{\alpha}]$ and the associated point processes $\mathcal{X}[\underline{\alpha}] \subseteq \mathcal{X}_+[\underline{\alpha}]$ only at the moment when $\underline{\alpha}$ is screened.

The fact the construction below indeed reproduces the correct distribution of $\mathfrak{C}(x)$ is straightforward once we try to think about all the Poisson processes involved in terms of the usual approximation by Bernoulli trials.

*Initial stage.* For $x = (j, t)$ set

$$\underline{\alpha}_1 = (1) \quad i[\underline{\alpha}_1] = j \quad t[\underline{\alpha}_1] = t \quad \mathbf{M}^x = \emptyset \quad \mathbf{R}^x = \{\underline{\alpha}_1\}\,.$$

*Screening stage.* If $\mathbf{R}^x$ is empty, then stop. Otherwise, choose $\underline{\alpha}$ to be the minimal element of $\mathbf{R}^x$ (and set $\mathbf{R}^x(\underline{\alpha}) = \mathbf{R}^x$ and $\mathbf{M}^x(\underline{\alpha}) = \mathbf{M}^x$). There are two steps to be performed and several cases to be considered:

STEP 1 Deciding whether $\underline{\alpha}$ is to be included into $\mathbf{M}^x$ (Cases 2 and 3 below) and, if yes, sampling of $J[\underline{\alpha}]$.

CASE 1 If there exists $\underline{\gamma} \in \mathbf{M}^x(\underline{\alpha})$ (and then necessarily satisfying $\underline{\gamma} \prec \underline{\alpha}$) such that

$$i[\underline{\alpha}] = i[\underline{\gamma}] \quad \text{and} \quad t[\underline{\alpha}] \ \in \ J[\underline{\gamma}],$$

then remove $\underline{\alpha}$ from $\mathbf{R}^x$ and proceed to screen the next multi-index of $\mathbf{R}^x$.

CASE 2 If $i[\underline{\alpha}] \neq i[\underline{\gamma}]$ for every $\underline{\gamma} \in \mathbf{M}^x(\underline{\alpha})$, which means that

$$\mathfrak{C}_{\underline{\alpha}}(x) \ \stackrel{\Delta}{=} \bigvee_{\underline{\gamma} \in \mathbf{M}^x(\underline{\alpha})} i[\underline{\gamma}] \times J[\underline{\gamma}] \tag{2.20}$$

does not hit $\mathbb{S}_\beta^{i[\underline{\alpha}]}$, then set $J[\underline{\alpha}] \ = \ t[\underline{\alpha}] + I[\underline{\alpha}]$, where $I[\underline{\alpha}]$ is sampled from $\mathbb{P}_N^{\beta,\lambda}(\cdot)$ as described in Subsection 2.1.

CASE 3 In the remaining case,

$$\mathfrak{C}_{\underline{\alpha}}(x) \cap \mathbb{S}_\beta^{i[\underline{\alpha}]} \neq \emptyset \quad \text{but} \quad (i[\underline{\alpha}], t[\underline{\alpha}]) \not\in \mathfrak{C}_{\underline{\alpha}}(x).$$

In such a situation define $J[\underline{\alpha}]$ as the connected component of $t[\underline{\alpha}]$ of

$$(t[\underline{\alpha}] + I[\underline{\alpha}]) \setminus \left(\mathfrak{C}_{\underline{\alpha}}(x) \cap \mathbb{S}_\beta^{i[\underline{\alpha}]}\right).$$

If either CASE 2 or CASE 3 took place then add $\underline{\alpha}$ to $\mathbf{M}^x$, remove $\underline{\alpha}$ from $\mathbf{R}^x$ and proceed with the second step.

STEP 2 Generating descendants of $\underline{\alpha}$. Sample $\xi_+[\underline{\alpha}]$ and, accordingly, the point process $\mathcal{X}_+[\underline{\alpha}]$ from the underlying distribution $\mathbb{P}_N^{\beta,\lambda}(\cdot)$. Screen all $k = 1, \dots, \xi_+$ departures of $\mathcal{X}_+$ as follows:

CASE 1 If $t[\underline{\alpha}] + \tau_k \not\in J[\underline{\alpha}]$, then ignore this $k$-th departure.

CASE 2 Otherwise register $k$-th departure as follows: Add $(\underline{\alpha}, k)$ to $\mathbf{R}^x$ and set,

$$i[(\underline{\alpha}, k)] \ = \ d_k \quad \text{and} \quad t[(\underline{\alpha}, k)] \ = \ t[\underline{\alpha}] + \tau_k.$$

Return to the beginning of the screening stage. □

In the sequel we shall use the following notation: For each $\underline{\alpha} \in \mathbf{M}^x$ we define $\mathbf{N}^x(\underline{\alpha})$ as the set of all registered descendants of $\underline{\alpha}$. The corresponding point process is $\mathcal{X}[\underline{\alpha}] =$

$\{(i[\underline{\gamma}], \tau[\underline{\gamma}])\ ;\ \underline{\gamma} \in \mathbf{N}^x(\underline{\alpha})\}$. Finally, we shall denote the set of all spatial coordinates of registered descendants of $\underline{\alpha}$ as

$$\mathcal{D}[\underline{\alpha}] \;=\; \mathrm{proj}_{\mathsf{G}_N} \mathcal{X}[\underline{\alpha}].$$

2.3. **The critical curve.** As it becomes clear from the above construction of $\mathfrak{C}(x)$, the size of $\mathbf{M}^x$ is stochastically dominated by the total population size of Galton-Watson process with offspring distribution $\Xi_\beta[2,\lambda]$. Let $\xi \sim \Xi_\beta[2,\lambda]$. Evidently, $\xi$ has finite exponential moments. Furthermore,

$$\mathbb{E}_N^{\beta,\lambda}(\xi) \;=\; N\mathbb{E}_N^{\beta,\lambda}\left(1 - \mathrm{e}^{-|I|/N}\right) \;=\; O\left(\frac{1}{N}\right) \;+\; \mathbb{E}_N^{\beta,\lambda}(|I|).$$

Now (with the usual convention $0 \cdot \infty = 0$ in the $\beta = \infty$ case),

$$\mathbb{E}_N^{\beta,\lambda}(|I|) \;=\; \mathbb{E}\left(U + V; U + V < \beta\right) + \beta\mathbb{P}\left(U + V \geqslant \beta\right),$$

where, as before, $U$ and $V$ are two independent exponential $\mathrm{Exp}(\lambda)$ random variables. Since $U + V \sim \Gamma(2,\lambda)$,

$$\mathbb{P}\left(U + V \geqslant \beta\right) \;=\; \int_\beta^\infty \lambda^2 t \mathrm{e}^{-\lambda t}\mathrm{d}t \;=\; (\lambda\beta + 1)\mathrm{e}^{-\lambda\beta}.$$

In the same fashion,

$$\begin{aligned}\mathbb{E}\left(U + V; U + V \leqslant \beta\right) \;&=\; \int_0^\beta \lambda^2 t^2 \mathrm{e}^{-\lambda t}\mathrm{d}t \\ &=\; \frac{2}{\lambda}\left(1 - \mathrm{e}^{-\lambda\beta}\right) \;-\; \left(\beta^2\lambda + 2\beta\right)\mathrm{e}^{-\lambda\beta}.\end{aligned}$$

Consequently,

$$\mathbb{E}_N^{\beta,\lambda}(|I|) \;=\; \frac{2}{\lambda}\left(1 - \mathrm{e}^{-\lambda\beta}\right) \;-\; \beta\mathrm{e}^{-\lambda\beta} = F(\beta,\lambda), \tag{2.21}$$

which is, of course, precisely the right hand side of (1.5).

2.4. **LRO: size of $\mathcal{S}^x$.** In the *LRO*-case of $F(\beta,\lambda) > 1$ we shall confine the discussion to the case of finite $\beta < \infty$. Define

$$\mathcal{S}^x \;=\; \mathrm{proj}_{\mathsf{G}_N}\mathfrak{C}(x).$$

In other words $\mathcal{S}^x$ is the set of all different spatial coordinates of $\mathfrak{C}(x)$. Obviously,

$$|\mathfrak{C}(x)| \;\leqslant \beta\,\#\mathcal{S}^x.$$

In fact, as we shall see in Section 3 a converse is also true in the sense that large size of $\#\mathcal{S}^x$ necessarily implies that $|\mathfrak{C}(x)|$ is also large. Meanwhile:

**Lemma 2.1.** *In the LRO-case let $\delta > 0$ be such that $(1-\delta)F(\beta,\lambda) > 1$. Then, for every $\kappa > 0$ there exists $c_1 = c_1(\beta,\lambda,\kappa)$, such that*

$$\mathbb{P}_N^{\beta,\lambda}\left(\ \#\mathcal{S}^x \;\in\; [c_1\log N, \delta N]\right) \;=\; o\left(\frac{1}{N^\kappa}\right). \tag{2.22}$$

In the sequel we shall say that $\mathfrak{C}(x)$ is *small* if $\#\mathcal{S}^x < c_1 \log N$ and, accordingly, that it is *large* if $\#\mathcal{S}^x > \delta N$.

*Proof of Lemma 2.1.* Let us go back to STEP 2 of the screening stage of Subsection 2.2. A multi-index descendant $\underline{\gamma} \in \mathbf{N}^x(\underline{\alpha})$ of $\underline{\alpha} \in \mathbf{M}^x$ is called *free* if its colour $i[\underline{\gamma}]$ is encountered for the first time in the course of our construction of $\mathfrak{C}(x)$. Formally, $\underline{\gamma}$ is free if

$$i[\underline{\gamma}] \;\notin\; \left\{i[\underline{\nu}]\ :\ \underline{\nu} \prec \underline{\gamma} \quad \text{and} \quad \underline{\nu} \in \mathbf{M}^x(\underline{\alpha}) \cup \mathbf{R}^x(\underline{\alpha}) \cup \mathbf{N}^x(\underline{\alpha})\right\}.$$

Notice that any free descendant of $\underline{\alpha}$ will be duly included into $\mathbf{M}^x$ at later screening stages. Also notice that the set $\mathbf{F}^x \subseteq \mathbf{M}^x$ of all free multi-indices is in one-to-one correspondence with $\mathcal{S}^x$. Let us define

$$\mathcal{S}^x(\underline{\alpha}) \;=\; \left\{ i[\underline{\gamma}] \;:\; \underline{\gamma} \in \mathbf{M}^x(\underline{\alpha}) \cup \mathbf{R}^x(\underline{\alpha}) \right\}.$$

In other words $\mathcal{S}^x(\underline{\alpha})$ is the set of all different spatial coordinates of $\mathfrak{C}(x)$ which were generated before screening of $\underline{\alpha}$. Then the number $\eta_f(\underline{\alpha})$ of free descendants of $\underline{\alpha}$ is given by

$$\eta_f(\underline{\alpha}) \;=\; \sum_{i \in \mathsf{G}_N \setminus \mathcal{S}^x(\underline{\alpha})} \mathbb{1}_{i \in \mathcal{D}[\underline{\alpha}]}. \tag{2.23}$$

If $\underline{\alpha}$ is itself free and, moreover, $\#\mathcal{S}^x(\underline{\alpha}) < \delta N$, then $\eta_f(\underline{\alpha})$ stochastically dominates and, in fact, could be coupled in a straightforward way with a random variable $\eta[\underline{\alpha}]$ whose conditional on $I[\underline{\alpha}]$ distribution is binomial Bin $\left((1-\delta)N, 1 - \mathrm{e}^{-|I[\underline{\alpha}]|/N}\right)$. Of course, $\mathbb{E}_N^{\beta,\lambda}(\eta) \to (1-\delta)F(\beta,\lambda) > 1$ as $N \to \infty$.

Let us, thereby, consider an i.i.d. family $\{\eta_k\}$ of random variables distributed as above, which is coupled with the collection $\{\eta_f(\underline{\alpha})\}$ in the following way: Write all free multi-indices $\mathbf{F}^x \;=\; \{\underline{\alpha}_1^*, \underline{\alpha}_2^*, \dots\}$ in their increasing order. Then $\eta_k \;\leqslant\; \eta_f[\underline{\alpha}_k^*]$ on the event $\{\#\mathcal{S}^x(\underline{\alpha}_k^*) < \delta N\}$. Since $\#\mathcal{S}^x \;=\; \#\mathbf{F}^x$, and

$$\#\mathcal{S}^x \;\geqslant\; \sum_{\underline{\alpha} \in \mathbf{F}^x} \eta_f(\underline{\alpha}), \tag{2.24}$$

we infer that

$$\mathbb{P}_N^{\beta,\lambda}\left(\,\#\mathcal{S}^x \;\in\; [c_1 \log N, \delta N]\right) \;\leqslant\; \sum_{n=c_1 \log N}^{\delta N} \mathbb{P}_N^{\beta,\lambda}\left(\sum_1^n \eta_k \leqslant n\right),$$

and (2.22) follows from a usual LD upper bound, as $\mathbb{E}_N^{\beta,\lambda}(\eta) > 1$. □

2.5. **Coupling between two clusters.** Recall that we are recording the multi-indices $\mathbf{M}^x = \{\underline{\alpha}_1, \underline{\alpha}_2, \dots\}$ according to the order in which they were saturated or, equivalently, in their increasing order. With each $k = 1, 2, \dots$ we associate a $\sigma$-algebra

$$\Sigma_k^x \;=\; \sigma\left(\mathcal{X}_+[\underline{\alpha}_l], I[\underline{\alpha}_l]\;;\; l \leqslant k\right). \tag{2.25}$$

In order to avoid ambiguities we set $\underline{\alpha}_k \equiv \underline{\alpha}_{\mathfrak{m}^x}$ for $k \geqslant \mathfrak{m}^x$, where $\mathfrak{m}^x \overset{\Delta}{=} \#\mathbf{M}^x$. Clearly, $\Sigma_k^x$ contains all the information on the growth of $\mathfrak{C}(x)$ up to the $k$th (or, in the case $k > \mathfrak{m}^x$, up to the last) saturation. In particular, the eventual number of intervals $\mathfrak{m}^x$ in the decomposition (2.18) is a stopping time with respect to the filtration $\{\Sigma_k^x\}$.

Our proof of various claims in the LRO case is based on the following modification of the growth algorithm described in Subsection 2.2: Let $x, y$ be two points of $\mathfrak{G}_N^\beta$.

First of all we shall grow $\mathfrak{C}(x)$ according to the rules of Subsection 2.2 until a certain stopping time $S$ (with respect to the filtration $\{\Sigma_k^x\}$). We shall define $S$ below. In any case, the relevant notation is (see (2.20)),

$$\mathfrak{C}_k(x) = \mathfrak{C}_{\underline{\alpha}_{k+1}}(x)\;,\; \mathbf{M}_k^x = \mathbf{M}^x(\underline{\alpha}_{k+1}) \quad \text{and} \quad \mathbf{R}_k^x = \mathbf{R}^x(\underline{\alpha}_{k+1}).$$

Accordingly, $\mathfrak{C}_S(x)$ is the shape of the cluster whose growth was suspended at $S$, and $\mathbf{R}_S^x$ is the set of all multi-indices which were registered but not yet saturated (or dropped) by time $S$.

Secondly, we shall grow $\mathfrak{C}(y)$ according to the following modified set of rules, which should necessarily take into account already existing information as recorded in $\Sigma_S^x$.

Growth of $\mathfrak{C}(y)$ will be halted at some random stage and certainly once it will become clear that $\mathfrak{C}(x)$ and $\mathfrak{C}(y)$ are going to merge. In the sequel we shall use $T$ to denote the stage at which we halt the construction of $\mathfrak{C}(y)$. We use $\tilde{\mathbf{M}}^y$, $\tilde{\mathbf{R}}^y$ etc notation to stress that we construct $\mathfrak{C}(y)$ after all the data for $\Sigma_S^x$ has been already sampled. If $\tilde{\mathbf{M}}^y = \left\{\underline{\gamma}_1, \underline{\gamma}_2, \dots\right\}$ is the (ordered) set of all saturated multi-indices of $\mathfrak{C}(y)$, then $T$ will be a the stopping time with respect to the filtration

$$\tilde{\Sigma}_0^y \;=\; \Sigma_S^x \quad \text{and} \quad \tilde{\Sigma}_k^y \;=\; \sigma\left(\Sigma_S^x, \Sigma_k^y\right),$$

where $\Sigma_k^y$ is defined as in (2.25).

*The initial stage.* If $y \in \mathfrak{C}_S(x)$, then set $T = 0$ and halt immediately. Otherwise proceed as in the initial stage of one-cluster construction of Subsection 2.2 (except for setting $(\underline{\gamma}_1) = (2)$, so that $x$ and $y$ related multi-indices are distinguished).

*An update stage.* There are several corrections which are due to the already existing information contained in $\Sigma_S^x$. Let $\underline{\gamma} \in \tilde{\mathbf{R}}^y$ be a multi-index under screening.

CORRECTION 1 In STEP 1 if $(i[\underline{\gamma}], t[\underline{\gamma}]) \in \mathfrak{C}_S(x)$, then remove $\underline{\gamma}$ from $\tilde{\mathbf{R}}^y$ and proceed to screen the next multi-index of $\tilde{\mathbf{R}}^y$.

CORRECTION 2 In CASE 2 and CASE 3 of STEP 1 define the interval $J[\underline{\gamma}]$ as the connected component of $t[\underline{\gamma}]$ of

$$\left((t[\underline{\gamma}] + I[\underline{\gamma}]\right) \setminus \left(\mathfrak{C}_{\underline{\gamma}}(y) \cap \mathfrak{C}_S(x) \cap \mathbb{S}_\beta^{i[\underline{\gamma}]}\right).$$

CORRECTION 3 If, after $J[\underline{\gamma}]$ is sampled, there exists $\underline{\alpha} \in \mathbf{R}_S^x$ such that $i[\underline{\alpha}] = i[\underline{\gamma}]$ and $t[\underline{\alpha}] \in J[\underline{\gamma}]$, then the clusters of $x$ and $y$ are going to merge. If $\underline{\gamma} = \underline{\gamma}_k$, then set $T = k$ and halt the construction of $\mathfrak{C}(y)$.

2.6. **Merging of two large clusters.** Recall that a cluster $\mathfrak{C}(x)$ is called large if $\#(\mathcal{S}^x) > \delta N$, where $\delta > 0$ is some fixed number satisfying $(1-\delta)F(\beta, \lambda) > 1$.

**Lemma 2.2.** *Let $\beta < \infty$. Then for every $\kappa > 0$,*

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \cap \mathfrak{C}(y) = \emptyset \mid \textit{both } \mathfrak{C}(x) \textit{ and } \mathfrak{C}(y) \textit{ are large}\right) \;=\; o\left(\frac{1}{N^\kappa}\right), \tag{2.26}$$

*uniformly in $x, y \in \mathfrak{G}_N^\beta$.*

*Proof of Lemma 2.2* The proof relies on the construction of clusters and, accordingly, on the coupling between $\mathfrak{C}(x)$ and $\mathfrak{C}(y)$ as introduced in Subsection 2.2 and Subsection 2.5.

STEP 1 We start by growing the cluster $\mathfrak{C}(x)$ until the stopping time $S$ which is described as follows: Let $\epsilon > 0$ be small, e.g. $3\epsilon < \delta$ would suffice. Then,

$$S \;=\; \max\left\{k \leqslant \mathfrak{m}^x \;:\; \#\mathcal{S}^x(\underline{\alpha}_k) \;<\; \epsilon N\right\}. \tag{2.27}$$

Evidently, $S$ is a stopping time with respect to the filtration $\{\Sigma_k^x\}$.

Let $\mathbf{F}_S^x$ be the set of all free multi-indices of $\mathfrak{C}(x)$ which were generated by time $S$. In this notation,

$$\mathcal{S}^x(\underline{\alpha}_S) \cup \mathcal{D}(\underline{\alpha}_S) = \bigcup_{\underline{\alpha} \in \mathbf{F}_S^x} i[\underline{\alpha}].$$

We shall distinguish between the free multi-indices $\mathbf{F}^x_{S,-} = \mathbf{F}^x_S \cap \mathbf{M}^x_S$ which were already saturated by time $S$ and the remaining free multi-indices $\mathbf{F}^x_{S,+} = \mathbf{F}^x_S \setminus \mathbf{F}^x_{S,-}$. Let $\nu > 0$ be such that

$$(1-3\nu)(1-\delta)F(\beta,\lambda) > 1. \tag{2.28}$$

We claim that once $\mathfrak{C}(x)$ is large, then up to $o\left(\frac{1}{N^\kappa}\right)$-probabilities the cardinality of unsaturated free multi-indices $\#\mathbf{F}^x_{S,+}$ exceeds $3\nu\epsilon N$, or, more precisely, the cardinality of saturated multi-indices $\#\mathbf{F}^x_{S,+}$ does not exceed $(1-3\nu)\epsilon N$. Indeed, by construction $\#\mathbf{F}^x_S > \epsilon N$, whereas $\#\mathcal{S}^x(\underline{\alpha}_S) < \epsilon N$. On the other hand, as in (2.24),

$$\#\mathcal{S}^x(\underline{\alpha}_S) \;\geqslant \sum_{\underline{\alpha}^*_k \in \mathbf{F}^x_{S,-}\setminus\ \underline{\alpha}_S} \eta_k .$$

Consequently,

$$\begin{aligned}\mathbb{P}^{\beta,\lambda}_N\big(\ \#\mathbf{F}^x_{S,+} < 3\nu\epsilon N\ ;\ \mathfrak{C}(x) \text{ is large}\big) &\leqslant \mathbb{P}^{\beta,\lambda}_N\big(\#\mathbf{F}^x_{S,-} > (1-3\nu)\epsilon N\ ;\ \#\left(\mathcal{S}^x(\underline{\alpha}_S)\right) < \epsilon N\big)\\ &\leqslant \mathbb{P}^{\beta,\lambda}_N\left(\sum_1^{(1-3\nu)\epsilon N}\eta_k < \epsilon N\right) \;=\; o\left(\frac{1}{N^\kappa}\right),\end{aligned} \tag{2.29}$$

since $(1-3\nu)\mathbb{E}^{\beta,\lambda}_N(\eta_k)$ is still larger than 1 whenever $\nu$ complies with (2.28).

STEP 2 We shall now switch to the coupled construction of $\mathfrak{C}(y)$. First of all we shall adjust the notion of the set $\tilde{\mathbf{F}}^y$ of free multi-indices of $\mathfrak{C}(y)$: Recall that free multi-indices are generated at STEP 2 of the screening stage when we consider family of multi-index descendants $\mathbf{N}^y(\underline{\alpha})$ of freshly saturated multi-indices $\underline{\alpha} \in \tilde{\mathbf{M}}^y$. We shall say that $\underline{\gamma} \in \mathbf{N}^y(\underline{\alpha})$ is free if, as before, its colour $i[\gamma]$ is encountered for the first time in the course of construction of $\mathfrak{C}(y)$, but, in addition, $i[\underline{\gamma}]$ was not saturated in $\mathfrak{C}_S(x)$, that is:

$$i[\underline{\gamma}] \;\notin\; \left\{i[\underline{\nu}]\ :\ \underline{\nu} \in \mathbf{M}^x_S\right\}.$$

The latter set is in one-to-one correspondence with $\mathbf{F}^x_{S,-}$. As a result, on the event $\{\mathfrak{C}(y) \text{ is large and } \mathfrak{C}(y)\cap\mathfrak{C}(x) = \emptyset\}$,

$$\#\tilde{\mathbf{F}}^y \;\geqslant \#\mathcal{S}^y \;-\#\mathbf{F}^x_{S,-} \;\geqslant \delta N - (1-3\nu)\epsilon N \geqslant 2\epsilon N,$$

where the first inequality (up to $o\left(\frac{1}{N^\kappa}\right)$-probability) follows from (2.29) and the second inequality follows from our choice of $3\epsilon < \delta$. Define now a $\left\{\tilde{\Sigma}^y_k\right\}$-stopping time

$$T \;=\; \min\left\{k\ :\ \#\tilde{\mathbf{F}}^y(\underline{\gamma}_k) \geqslant \nu\epsilon N\right\},$$

where, $\tilde{\mathbf{F}}^y(\underline{\gamma}_k)$ is the set of all (modified ) free multi-indices which were generated during the first $k$ screening sessions. Since by construction (end of Subsection 2.1) $\{\#\mathcal{D}_+[\underline{\alpha}]\}$ is a family of i.i.d random variables with exponentially decaying tails, we can safely assume that, up to $o\left(\frac{1}{N^\kappa}\right)$-probability

$$\#\tilde{\mathbf{F}}^y(\underline{\gamma}_T) \;\leqslant 2\nu\epsilon N. \tag{2.30}$$

Similarly to (2.29), set $\tilde{\mathbf{F}}^y_{T,-} = \tilde{\mathbf{M}}^y_T \cap \tilde{\mathbf{F}}^y_T$. Then,

$$\mathbb{P}^{\beta,\lambda}_N\left(\#\tilde{\mathbf{F}}^y_{T,-} > (1-3\epsilon)\#\tilde{\mathbf{F}}^y_T\ ;\ \#\tilde{\mathbf{F}}^y_T \in [\nu\epsilon N, 2\nu\epsilon N]\right) \;=\; o\left(\frac{1}{N^\kappa}\right).$$

Consequently, up to $o\left(\frac{1}{N^\kappa}\right)$-probability, at least $3\nu\epsilon^2 N$ modified free multi-indices from $\tilde{\mathbf{F}}^y_T$ were not saturated by time $T$. Let us denote the latter set of multi-indices as $\tilde{\mathbf{F}}^y_{S,T}$,

$$\#\tilde{\mathbf{F}}^y_{S,T} \ \geqslant 3\nu\epsilon^2 N. \tag{2.31}$$

On the other hand, the number of all saturated multi-indices from $\tilde{\mathbf{F}}^y_T$ is, in view of (2.30), bounded above by $2\nu\epsilon N$, which falls short of at least $3\nu\epsilon$ free unsaturated multi-indices of $\mathbf{F}^x_{S,+}$. Thus the cardinality of the set

$$\tilde{\mathbf{F}}^x_{S,T} \ \stackrel{\Delta}{=} \ \left\{\underline{\alpha} \in \mathbf{F}^x_{S,+} \ : \ i[\underline{\alpha}] \not\in \mathrm{proj}_{\mathsf{G}_N} \mathfrak{C}_T(y)\right\}$$

satisfies,

$$\#\tilde{\mathbf{F}}^x_{S,T} \ \geqslant \nu\epsilon N. \tag{2.32}$$

STEP 3 In view of (2.31) and (2.32) let us summarize the above results as follows: Consider the event $\{\mathfrak{C}(x) \cap \mathfrak{C}(y) = \emptyset \ ; \ \text{both } \mathfrak{C}(x) \text{ and } \mathfrak{C}(y) \text{ are large}\}$ and let $S$ and $T$ be stopping times defined as above. Then, up to $o\left(\frac{1}{N^\kappa}\right)$-probability, there exists a set of unsaturated free multi-indices $\tilde{\mathbf{F}}^x_{S,T}$ and, accordingly, a set of unsaturated free multi-indices $\tilde{\mathbf{F}}^y_{S,T}$ such that,

(1) $\#\tilde{\mathbf{F}}^x_{S,T} \geqslant \nu\epsilon N$ and $\#\tilde{\mathbf{F}}^y_{S,T} \geqslant 3\nu\epsilon^2 N$ .
(2) For every $\underline{\alpha} \in \tilde{\mathbf{F}}^x_{S,T} \cup \tilde{\mathbf{F}}^y_{S,T}$ and every $\underline{\gamma} \in \mathbf{M}^x_S \cup \tilde{\mathbf{M}}^y_T$ the colours $i[\underline{\alpha}] \neq i[\underline{\gamma}]$.
(3) Any two different multi-indices of $\tilde{\mathbf{F}}^x_{S,T}$, respectively of $\tilde{\mathbf{F}}^y_{S,T}$, have different colours.

Set

$$\mathcal{S}^{x,y}_{S,T} = \left\{ i[\underline{\alpha}] \ : \ \underline{\alpha} \in \tilde{\mathbf{F}}^x_{S,T} \cup \tilde{\mathbf{F}}^y_{S,T} \right\}.$$

Property (2) above means that none of the processes of holes $\mathcal{H}_i$ was ever sampled for $i \in \mathcal{S}^{x,y}_{S,T}$. Similarly none of the processes $\mathcal{L}_{ij}$ of links was ever sampled for $i,j \in \mathcal{S}^{x,y}_{S,T}$. Since the family of processes

$$\{\mathcal{H}_i \ , \ \mathcal{L}_{ij}\}_{i,j \in \mathcal{S}^{x,y}_{S,T}}$$

is, conditionally on $\mathcal{S}^{x,y}_{S,T}$, independent of $\tilde{\Sigma}^y_T$, we infer that up to a $o\left(\frac{1}{N^\kappa}\right)$-probability,

$$\mathbb{P}^{\beta,\lambda}_N \left( \mathfrak{C}(x) \not\leftrightarrow \mathfrak{C}(y) \mid \tilde{\Sigma}^y_T \right) \ \leqslant \ \max_{\substack{v_1,\dots,v_L \\ u_1,\dots,u_M}} \mathbb{P}^{\beta,\lambda}_N \left( \bigcap_{i,j} \{ I(v_i) \not\leftrightarrow I(u_j) \} \right), \tag{2.33}$$

where $L = \nu\epsilon N$, $M = 3\nu^2\epsilon N$ and the maximum above is over all colour disjoint collections (see property (3) above) of points $\{v_1, \dots, v_L\} \subset \mathfrak{G}^\beta_N$ and $\{u_1, \dots, u_M\} \subset \mathfrak{G}^\beta_N$. We have used the obvious notation in (2.33): For $z = (i,t) \in \mathfrak{G}^\beta_N$, $I(z)$ denotes the interval which contains $t$ in the decomposition (1.3) of $\mathbb{S}^i_\beta$.

STEP 4 It remains to derive an upper bound on the right-hand side of (2.33). Set $K = L - M = \nu(1-3\nu)\epsilon N$. By our choice of $\nu$ in (2.28) the number $K$ is positive and, moreover, proportional to $N$. At least $K$ of colour disjoint $u_j$-s have spatial coordinates different from any of $u_i$ spatial coordinates. There is no loss to assume that this property is enjoyed by the first $K$ points $\{v_1, \dots, v_K\}$. Thus, the right hand side of (2.33) is bounded above by

$$\max_{\substack{v_1,\dots,v_K \\ u_1,\dots,u_M}} \mathbb{P}^{\beta,\lambda}_N \left( \bigcap_{i,j} \{ I(v_i) \not\leftrightarrow I(u_j) \} \right), \tag{2.34}$$

where the maximum is now over all possible $K+L$ *colour disjoint* points of $\mathfrak{G}_N^\beta$. By usual large deviation estimates there is a constant $c = c(\beta\epsilon, \nu) > 0$ such that up to $\mathrm{e}^{-cN}$-probability there are at least $\mathrm{e}^{-\lambda\beta}K/2$ of $v_j$-s lie on $\mathbb{S}_\beta$ circles without holes, and at least $\mathrm{e}^{-\lambda\beta}M/2$ of $u_j$-s which also lie on $\mathbb{S}_\beta$ circles without holes. This is a reduction to the computation employed in the classical case: Indeed, for any two $v_i$ and $u_j$ with spatial coordinates $i \neq j$,

$$\mathbb{P}_N^{\beta,\lambda}\left(I(v_i) \not\leftrightarrow I(u_j) \mid \mathcal{H}_i = \emptyset, \mathcal{H}_j = \emptyset\right) \;=\; \mathrm{e}^{-\beta/N}.$$

Consequently, the expression in (2.34) is bounded above by

$$\mathrm{e}^{-cN} \;+\; \exp\left\{-\frac{\beta \mathrm{e}^{-2\lambda\beta}KM}{4N}\right\}.$$

Since $KM/N = 3\nu^2(1-3\nu)\epsilon^2 N$, the proof of Lemma 2.2 is complete. $\square$

## 3. Proofs of main results

3.1. **Short range order.** We employ the construction and the notation of Subsection 2.1 and Subsection 2.2.

*Proof of* (1.7) *and of* (1.11) *for* $\beta < \infty$. . Let $\mathcal{Z}_+$ be the sub-critical Galton-Watson process with offspring distribution $\Xi_\beta(2,\lambda)$ (Subsection 2.1). The members of $\mathcal{Z}_+$ are naturally labeled by multi-indices from $\mathbf{M}_+$ and, furthermore, it is straightforward to couple the construction of $\mathcal{Z}_+$ with that of $\mathfrak{C}(x)$ in such a way that

$$\mathbf{M}^x \;\subseteq \mathcal{Z}_+ \quad \text{and} \quad |\mathfrak{C}(x)| \;\leqslant \sum_{\underline{\alpha}\in\mathcal{Z}_+} |I[\underline{\alpha}]| \;\stackrel{\Delta}{=}\; |\mathcal{Z}_+|.$$

As a result for every $\kappa > 0$ there exists $c > 0$ such that

$$\mathbb{P}_N^{\beta,\lambda}\left(\#\left(\mathcal{S}^x\right) > c\log N\right) \;\leqslant \mathbb{P}_N^{\beta,\lambda}\left(\#\mathcal{Z}_+ > c\log N\right) \;=\; o\left(\frac{1}{N^\kappa}\right). \tag{3.35}$$

Since, $|\mathfrak{C}(x)| \leqslant \beta\#\mathcal{S}^x$, the bound (1.11) readily follows in the $\beta < \infty$ case.

Furthermore, if $x = (i,t)$ and $j \neq i$, then by the $\mathsf{G}_N$-permutation invariance,

$$\mathbb{P}_N^{\beta,\lambda}\left(j \in \mathcal{S}^x \mid \#\mathcal{S}^x = n\right) \;=\; \frac{n-1}{N-1}, \tag{3.36}$$

which, in view of (3.35), implies the SRO bound (1.7) at any $\beta \leqslant \infty$.

*Proof of* (1.11) *in the* $\beta = \infty$ *case.* First of all notice that in the $\beta = \infty$ case, the underlying distribution $\Xi_\infty(2,\lambda)$ of offsprings is just negative binomial (for the number of failures),

$$\mathbb{P}(\xi_+ = k) \;=\; (k+1)\frac{\lambda^2}{(1+\lambda)^{k+2}} \quad k = 0,1,2,\ldots, \tag{3.37}$$

whereas the conditional distribution of the interval length $|I|$ is

$$|I| \;\sim\; \Gamma\left(k+2, 1+\lambda\right) \quad \text{given } \xi_+ = k \text{ for } k = 0,1,2,\ldots. \tag{3.38}$$

In the $\beta = \infty$ case formula (3.38) implies that the conditional distribution of $|\mathcal{Z}_+|$ given the value of the total size of the population $\#\mathcal{Z}_+$ is precisely that of the sum of $3\#\mathcal{Z}_+ - 1$

independent exponential $\mathrm{Exp}(1+\lambda)$ random variables. Therefore,

$$\begin{aligned}\mathbb{P}_N^{\beta,\lambda}\left(|\mathcal{Z}_+| > c\log N\right) \;&\leqslant\; \mathbb{P}_N^{\beta,\lambda}\left(\#\mathcal{Z}_+ > \frac{1+\lambda}{6} c\log N\right) \\ &+ \max_{n\,\leqslant\,\frac{c(1+\lambda)\log N}{6}} \mathbb{P}_N^{\beta,\lambda}\left(\sum_1^{3n}(\eta_i - \mathbb{E}_N^{\beta,\lambda}(\eta_i)) > \frac{c}{2}\log N\right),\end{aligned} \tag{3.39}$$

where $\eta_1, \eta_2, \ldots$ are i.i.d. $\mathrm{Exp}(1+\lambda)$ random variables. Clearly, for every $\kappa > 0$ one can choose $c$ sufficiently large so that the right-hand side of (3.39) is bounded above by $1/N^\kappa$.

*Proof of* (1.12). Let $\beta < \infty$. By the union bound and $\mathsf{G}_N$-permutation invariance of the distribution of $\mathcal{M}$,

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathcal{M} \geqslant c\log N\right) \;\leqslant\; N\mathbb{P}_N^{\beta,\lambda}\left(\mathcal{M} \geqslant c\log N\,;\, \mathfrak{C}^* \cap \mathbb{S}_\beta^i \neq \emptyset\right),$$

where $\mathfrak{C}^*$ is the maximal cluster in the decomposition (1.2), $|\mathfrak{C}^*| = \mathcal{M}$. At this stage define,

$$\mathcal{M}_i \;=\; \max\left\{|\mathfrak{C}_k| \,:\, \mathfrak{C}_k \cap \mathbb{S}_\beta^i \neq \emptyset\right\}.$$

Obviously,

$$\left\{\mathcal{M} \geqslant c\log N\,;\, \mathfrak{C}^* \cap \mathbb{S}_\beta^i \neq \emptyset\right\} \;\subseteq\; \left\{\mathcal{M}_i \geqslant c\log N\right\}.$$

Since the latter event is already increasing,

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathcal{M}_i > c\log N\right) \;\leqslant\; \mathbb{P}_N^{\beta,\lambda}\left(\mathcal{M}_i > c\log N \,\middle|\, \mathcal{H}_i = \emptyset\right).$$

However,

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathcal{M}_i > c\log N \,\middle|\, \mathcal{H}_i = \emptyset\right) \;\leqslant\; \frac{\mathbb{P}_N^{\beta,\lambda}\left(|\mathfrak{C}((i,0))| > c\log N\right)}{\mathbb{P}_N^{\beta,\lambda}\left(\mathcal{H}_i = \emptyset\right)}.$$

Since $\mathbb{P}_N^{\beta,\lambda}(\mathcal{H}_i = \emptyset) = \mathrm{e}^{-\lambda\beta}$, the target bound (1.12) follows. □

3.2. **LRO: Proof of Theorem B.** For the rest of the paper we shall assume that $\beta < \infty$ and that $F(\beta,\lambda) > 1$.

*Probability of large cluster.* Recall that we say that $\mathfrak{C}(x)$ is large if the number of spatial coordinates satisfies $\#\mathcal{S}^x \geqslant \delta N$, where $\delta$ is a fixed small number, $(1-\delta)F(\beta,\lambda) > 1$.

Both upper and lower bounds on $\mathbb{P}_N^{\beta,\lambda}(\mathfrak{C}(x)$ is large$)$ follow, as in the classical case, from comparison with appropriate Galton-Watson processes.

The upper bound is straightforward: As in Subsection 3.1 let $\mathcal{Z}_+$ be the Galton-Watson process with offspring distribution $\Xi_\beta(2,\lambda)$. Define $p(\beta,\lambda)$ as the survival probability of $\mathcal{Z}_+$, that is $p(\beta,\lambda)$ is the unique non-trivial root of

$$1 - p \;=\; \mathbb{E}_N^{\beta,\lambda}\left((1-p)^{\xi_+}\right), \tag{3.40}$$

where $\xi_+ \sim \Xi_\beta(2,\lambda)$. By the very construction of $\mathfrak{C}(x)$ in Subsection 2.2,

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \text{ is large}\right) \;\leqslant\; p(\beta,\lambda)\left(1 + o(1)\right).$$

The lower bound is slightly more delicate. It relies on a coupling with yet another Galton-Watson process $\mathcal{Z}_+^f$, which lives on free multi-indices $\mathbf{F}^x$ of $\mathfrak{C}(x)$. Specifically, as before set $\mathbf{F}^x = \{\underline{\alpha}_1^*, \underline{\alpha}_2^*, \ldots\}$. Recall the definition of random variables $\eta_f(\underline{\alpha}_k^*)$ in (2.23). On the event

$$\left\{\#\mathcal{S}^x \;<\; c\log N\right\} \;=\; \left\{\#\mathbf{F}^x \;<\; c\log N\right\},$$

the sequence $\{\eta_f(\underline{\alpha}_k^*)\}$ can be coupled with an i.i.d sequence $\{\eta_k^*\}$, such that

$$\forall\ \underline{\alpha}_k^* \in \mathbf{F}^x \quad \eta_f(\underline{\alpha}_k^*) \geqslant \eta_k^*,$$

whereas the distribution of $\eta^*$ is given by: First sample $|I|$ from $\Gamma_\beta(2,\lambda)$, and then sample $\eta^*$ from Bin $\left(N - c\log N, 1 - \mathrm{e}^{-|I|/N}\right)$.

Let $p_N(\beta,\lambda)$ be the survival probability of the Galton-Watson process with the offspring distribution specified by $\eta$ above. Clearly,

$$\lim_{N\to\infty} p_N(\beta,\lambda) \,=\, p(\beta,\lambda). \tag{3.41}$$

On the other hand, by Lemma 2.1,

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \text{ is large}\right) \,=\, 1 - \mathbb{P}_N^{\beta,\lambda}\left(\#\mathcal{S}^x \leqslant c\log N\right) \,\geqslant\, 1 - \mathbb{P}_N^{\beta,\lambda}\left(\mathcal{Z}_+^f \text{ dies out}\right),$$

up to $o(1)$ probabilities. In view of (3.41) this gives a complementary lower bound and, consequently,

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \text{ is large}\right) \,=\, p(\beta,\lambda)\left(1 + o(1)\right). \tag{3.42}$$

*Size of $\mathcal{M}$.* It is a straightforward exercise to deduce from (3.42), Lemma 2.1 and from the FKG properties of $\mathbb{P}_N^{\beta,\lambda}(\cdot)$ that in the LRO regime

$$\mathbb{P}_N^{\beta,\lambda}\left(\exists\, x \in \mathfrak{G}_N^\beta \text{ s.t. } \mathfrak{C}(x) \text{ is large}\right) \,=\, 1 - O\left(\frac{1}{N^\kappa}\right),$$

for every $\kappa > 0$. Furthermore, it is equally straightforward to deduce from Lemma 2.2 that

$$\mathbb{P}_N^{\beta,\lambda}\left(\exists \text{ unique large cluster} \;\middle|\; \exists \text{ large cluster}\right) \,=\, 1 - O\left(\frac{1}{N^\kappa}\right).$$

Let us denote such unique large cluster as $\mathfrak{C}^*$. Of course,

$$\mathbb{P}_N^{\beta,\lambda}\left(x \in \mathfrak{C}^*\right) \,=\, \mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) = \mathfrak{C}^*\right) \,=\, \mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \text{ is large}\right)\left(1 - O\left(\frac{1}{N^\kappa}\right)\right),$$

does not depend on $x \in \mathfrak{G}_N^\beta$. In particular, up to $O\left(\frac{1}{N^\kappa}\right)$-quantities,

$$\begin{aligned}\mathbb{E}_N^{\beta,\lambda}\left(\mathcal{M}\right) \,&=\, \mathbb{E}_N^{\beta,\lambda}\left(\sum_{i=1}^N \int_0^\beta \mathbb{1}_{\{(i.t)\in\mathfrak{C}^*\}}\,\mathrm{d}t\right) \,=\, \mathbb{E}_N^{\beta,\lambda}\left(\sum_{i=1}^N \int_0^\beta \mathbb{1}_{\{\mathfrak{C}((i,t)) \text{ is large}\}}\,\mathrm{d}t\right)\\ &=\, \sum_{i=1}^N \int_0^\beta \mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}((i,t)) \text{ is large}\right)\mathrm{d}t \,=\, N\beta p(\beta,\lambda)\left(1 + o(1)\right),\end{aligned} \tag{3.43}$$

where we have used (3.42) in the last step. Accordingly, define $\rho(\beta,\lambda) = N\beta p(\beta,\lambda)$. Let us compute,

$$\mathbb{E}_N^{\beta,\lambda}\left(\mathcal{M}^2\right) \,=\, \sum_{i=1}^N\sum_{j=1}^N \int_0^\beta\int_0^\beta \mathbb{P}_N^{\beta,\lambda}\left((i,t) \in \mathfrak{C}^*\;;\;(j,s) \in \mathfrak{C}^*\right)\mathrm{d}t\,\mathrm{d}s.$$

However, for $i \neq j$ and any $t, s \in [0,\beta)$,

$$\mathbb{P}_N^{\beta,\lambda}\left((i,t) \in \mathfrak{C}^*\;;\;(j,s) \in \mathfrak{C}^*\right) \,=\, \mathbb{P}_N^{\beta,\lambda}\left((i,t) \longleftrightarrow (j,s)\right)\left(1 + o(1)\right) \,=\, p(\beta,\lambda)^2\left(1 + o(1)\right),$$

as it follows from the proof of Theorem A in Subsection 3.3 below. Therefore,

$$\mathbb{E}_N^{\beta,\lambda}\left(\mathcal{M}^2\right) \,\leqslant\, \beta^2 N \,+\, N^2\beta^2 p(\beta,\lambda)^2\left(1 + o(1)\right).$$

As a result,

$$\mathbb{V}\mathrm{ar}_N^{\beta,\lambda}\left(\frac{\mathcal{M}}{N}\right) = o(1),$$

and both (1.13) and (1.15) follow.

3.3. **LRO: Proof of Theorem A.** Define $x = (i,t)$ and $y = (j,s)$, then for $i \neq j$

$$\mathbb{P}_N^{\beta,\lambda}\left((i,t) \longleftrightarrow (j,s)\right) = \mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \cap \mathfrak{C}(y) \neq \emptyset\right)$$

By (3.36),

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \cap \mathfrak{C}(y) \neq \emptyset \middle| \mathfrak{C}(y) \text{ is small}\right) \leqslant \mathbb{P}_N^{\beta,\lambda}\left(j \in \mathcal{S}^x \middle| \#\mathcal{S}^x \leqslant c\log N\right) = O\left(\frac{\log N}{N}\right)$$

Hence,

$$\begin{aligned}
&\mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \cap \mathfrak{C}(y) \neq \emptyset;\ \text{both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are small}\right) \\
&\leqslant \mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \cap \mathfrak{C}(y) \neq \emptyset \middle|\ \mathfrak{C}(\mathrm{x}) \text{ is small}\right) = O\left(\frac{\log N}{N}\right)
\end{aligned} \tag{3.44}$$

Note that if $\mathfrak{C}(x) \cap \mathfrak{C}(y) \neq \emptyset$, then, up to a $o\left(\frac{1}{N^\kappa}\right)$-probability, either both $\mathfrak{C}(x)$ and $\mathfrak{C}(y)$ are large or both $\mathfrak{C}(x)$ and $\mathfrak{C}(y)$ are small. Thus, in view of (3.44),

$$\begin{aligned}
&\mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \cap \mathfrak{C}(y) \neq \emptyset\right) \\
&\quad = \mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \cap \mathfrak{C}(y) \neq \emptyset;\ \text{both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are large}\right) + O\left(\frac{\log N}{N}\right).
\end{aligned}$$

On the other hand we already know from Lemma 2.2 that two large clusters merge with high $\left(1 - o\left(\frac{1}{N^\kappa}\right)\right)$ probability. Therefore,

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(x) \cap \mathfrak{C}(y) \neq \emptyset\right) = \mathbb{P}_N^{\beta,\lambda}\left(\text{ both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are large}\right) + O\left(\frac{\log N}{N}\right).$$

The asymptotic formula (3.42) implies that up to $o(1)$-terms,

$$\mathbb{P}_N^{\beta,\lambda}\left(\text{ both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are large}\right) = 2p(\beta,\lambda) - 1 + \mathbb{P}_N^{\beta,\lambda}\left(\text{ both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are small}\right).$$

Since both $\{\ \mathfrak{C}(\mathrm{x}) \text{ is small}\}$ and $\{\ \mathfrak{C}(\mathrm{x}) \text{ is small}\}$ are decreasing events, by the FKG properties of $\mathbb{P}_N^{\beta,\lambda}(\cdot)$,

$$\mathbb{P}_N^{\beta,\lambda}\left(\text{ both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are small}\right) \geqslant \left(1 - p(\beta,\lambda)\right)^2\left(1 + o(1)\right).$$

To get a complimentary upper bound, notice that

$$\begin{aligned}
&\mathbb{P}_N^{\beta,\lambda}\left(\mathcal{S}^x \cap \mathcal{S}^y \neq \emptyset \middle|\ \text{both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are small}\right) \\
&\leqslant \mathbb{P}_N^{\beta,\lambda}\left(\mathcal{S}^x \cap \mathcal{S}^y \setminus \{j\} \neq \emptyset \middle| \#\mathcal{S}^x \leqslant c\log N; \#\mathcal{S}^y \leqslant c\log N\right) \\
&\leqslant 2c\log N\, \mathbb{P}_N^{\beta,\lambda}\left(i \in \mathcal{S}^y \middle| \#\mathcal{S}^y \leqslant c\log N\right) = O\left(\frac{(\log N)^2}{N}\right),
\end{aligned}$$

as it follows from (3.36) and $\mathsf{G}_N$-permutation invariance. Thus,

$$\begin{aligned}
&\mathbb{P}_N^{\beta,\lambda}\left(\text{ both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are small}\right) \\
&= \mathbb{P}_N^{\beta,\lambda}\left(\text{ both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are small}; \mathcal{S}^x \cap \mathcal{S}^y = \emptyset\right) + O\left(\frac{(\log N)^2}{N}\right) \\
&\leqslant \mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(\mathrm{x}) \text{ is small} \middle| \mathcal{S}^x \cap \mathcal{S}^y = \emptyset; \mathfrak{C}(\mathrm{y}) \text{ is small}\right) \mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(\mathrm{y}) \text{ is small}\right) + O\left(\frac{(\log N)^2}{N}\right)
\end{aligned}$$

However,

$$\mathbb{P}_N^{\beta,\lambda}\left(\mathfrak{C}(\mathrm{x}) \text{ is small} \middle| \mathcal{S}^x \cap \mathcal{S}^y = \emptyset; \mathfrak{C}(\mathrm{y}) \text{ is small}\right) \leqslant \mathbb{P}_{N-c\log N}^{\tilde{\beta},\lambda}\left(\mathfrak{C}(\mathrm{x}) \text{ is small}\right)$$

where $\tilde{\beta} = \beta \frac{N}{N-c\log N}$.

As $N$ tends to $\infty$, the right hand side above converges to $1 - p(\beta,\lambda)$. Thereby, we are able to conclude,

$$\mathbb{P}_N^{\beta,\lambda}\left(\text{ both } \mathfrak{C}(\mathrm{x}) \text{ and } \mathfrak{C}(\mathrm{y}) \text{ are small}\right) = (1 - p(\beta,\lambda))^2(1 + o(1))$$

and (1.8) follows. □

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