THE GINZBURG-LANDAU EQUATION IN THE HEISENBERG GROUP

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Abstract. We consider a functional related with phase transition models in the Heisenberg group framework. We prove that level sets of local minimizers satisfy some density estimates, that is, they behave as "codimension one" sets. We thus deduce a uniform convergence property of these level sets to interfaces with minimal area.

These results are then applied in the construction of (quasi)periodic, plane-like minimizers, i.e., minimizers of our functional whose level sets are contained in a spacial slab of universal size in a prescribed direction. As a limiting case, we obtain the existence of hypersurfaces contained in such a slab which minimize the surface area with respect to a given periodic metric.

Keywords: Subelliptic operators and minimal surfaces on the Heisenberg group, Allen-Cahn-Ginzburg-Landau-type functionals, geometric properties of minimizers.

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Contents

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Set up and notation

Throughout this paper, χ_A will denote the charachteristic function of a set A, namely $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. Also, the map $\mathbb{R} \ni \tau \mapsto \tau_+$ agrees with τ if $\tau \geq 0$ and with 0 otherwise.

We will set $n \in \mathbb{N}$, $n \geq 1$ and we consider the "space" variables

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n \qquad \text{and} \qquad y = (y_1, \dots, y_n) \in \mathbb{R}^n .$$

Also, we will often write $z = (x, y) \in \mathbb{R}^{2n}$ and we identify it with the vector $z \in \mathbb{C}^n$ with components $z_k = x_k + \sqrt{-1}\, y_k$, for $k = 1, \dots, n$. The vector $\overline{z} \in \mathbb{C}^n$ has components $\overline{z}_k = x_k - \sqrt{-1}\, y_k$, for $k = 1, \dots, n$.

Given z, $w \in \mathbb{C}^n$, the notation zw stands for the product in $\mathbb{C}^n$, that is

$$zw = \sum_{j=1}^{n} z_j w_j .$$

Given $t \in \mathbb{R}$, we will use the notation $\xi = (z, t) = (x, y, t) \in \mathbb{R}^{2n+1}$. We will also consider the radial variables $\rho' = |z|$ and

$$\rho = |\xi|_{\mathbb{H}^n} = \Big(|z|^4 + t^2\Big)^{1/4} . \tag{0.1}$$

As usual, $\mathbb{H}^n$ denotes the Heisenberg group, endowed with the action

$$\xi \circ \xi_0 \;=\; \Big(z + z_0,\, t + t_0 + 2\,\mathrm{Im}\,(\overline{z} z_0)\Big) .$$

By induction, one sees that, if $K^{(1)}, \dots, K^{(\ell)} \in \mathbb{R}^{2n}$, then

$$\begin{aligned} &(K^{(\ell)}, 0) \circ \dots \circ (K^{(1)}, 0) \circ (z, t) = \\ &= \left(z + \sum_{j=1}^{\ell} K^{(j)} ,\; t + 2\,\mathrm{Im}\Big(\sum_{j=1}^{\ell} \overline{K}^{(j)} z + \sum_{1 \leq m < j \leq \ell} \overline{K}^{(j)} K^{(m)} \Big) \right) . \end{aligned} \tag{0.2}$$

We will consider the so-called "Koranyi ball" centered at ξ_0 of radius r, defined by

$$\mathfrak{B}_r(\xi_0) \;=\; \Big\{ \xi \text{ s.t. } |\xi^{-1} \circ \xi_0|_{\mathbb{H}^n} \leq r \Big\} .$$

We also set $Q := 2(n+1)$. If $\mathfrak{L}^k$ denotes the k-dimensional Lebesgue measure, one has that

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0)\Big) \;=\; \mathrm{const}\, r^Q .$$

As usual in the Heisenberg group setting, we consider the vector fields which are left invariant with respect to the group action

$$X_k = \partial_{x_k} + 2y_k\partial_t \qquad \text{and} \qquad Y_k = \partial_{y_k} - 2x_k\partial_t \,,$$

for $k = 1, \ldots, n$ and which generate the whole Lie algebra together with their commutator. We also introduce the so-called "Kohn Laplacian"

$$\Delta_{\mathbb{H}^n} \;=\; \sum_{k=1}^{n}(X_k^2 + Y_k^2)\,.$$

Given a (smooth) function u, we also introduce the intrinsic gradient:

$$\nabla_{\mathbb{H}^n} u \;=\; (X_1 u, \ldots, X_n u, Y_1 u, \ldots, Y_n u)\,.$$

For a radial (with respect to the Koranyi ball) function $v = v(\rho)$, a direct computation shows that

$$|\nabla_{\mathbb{H}^n} v| \;\leq\; \text{const}\,|v'| \tag{0.3}$$

and

$$\Delta_{\mathbb{H}^n} v \;=\; \left(\frac{|z|}{\rho}\right)^2 \left(v'' + \frac{Q-1}{\rho}\,v'\right)\,. \tag{0.4}$$

See, for instance, page 182 of [B03] for further details.

1. Statement of results

Given a domain $\Omega \subseteq \mathbb{R}^{2n+1}$, we define the functional

$$\mathcal{F}_\Omega(u) = \int_\Omega |\nabla_{\mathbb{H}^n} u(\xi)|^2 + F(\xi, u(\xi))\,d\xi\,,$$

under the following structural assumptions on the "double-well potential" F:

- F is non-negative and bounded and $F(\xi, 1) = F(\xi, -1) = 0$;
- for any $\theta \in [0, 1)$, $\inf\limits_{|u|\leq\theta} F(\xi, u) \geq \gamma(\theta)$, where γ is decreasing and strictly positive;
- there exist $d \in [0, 2]$ and $\ell \in (0, 1)$ so that:
  - $F(\xi, t) \geq \text{const}\,(1 - |t|)^d$, if $|t| \in (\ell, 1)$;
  - F is continuous in u, for $|u| < 1$;
  - F is locally Lipschitz continuous in u for $|u| < 1$;
  - if $d > 0$, $F_u(\xi, u)$ is continuous for $|u| < 1$, and, if $s < \ell$, then

$$F_u(\xi, -1+s) \geq \text{const}\, s^{d-1}\,, \qquad F_u(\xi, 1-s) \leq -\text{const}\, s^{d-1}\,;$$

  - if $d = 2$, F is continuous in u for $|u| \leq 1$ and $F_u(\xi, u)$ is increasing for u near ± 1.

Examples of such potentials[1] are given by $F = Q(\xi)\,(1 - u^2)^2$ and $F = Q(\xi)\chi_{(-1,1)}(u)$. The first type of potential, when the Heisenberg group is

[1]We take this opportunity to amend a typo at the end of page 164 of [V04]: the condition "$F_u(x, 1 - s) \geq -\text{const}\, s^{d-1}$" there has to be replaced by "$F_u(x, 1 - s) \leq -\text{const}\, s^{d-1}$".

replaced by standard elliptic integrands, is related to phase transition and superfluid models (see, e.g., [R79], [GL58] and [GP58]). The latter potential is related to flame propagation and fluid jet models (see, e.g., [AC81] and [ACF84]). The minimizers of $\mathcal{F}$ are lead to suitably approach a step function, whose interface is a minimal surface (see [G85] and [M87] for details). Also, functionals of these type have been the subject of a celebrated conjecture of De Giorgi (see, for instance, [DG79], [AAC01], [BL03] and references therein).

We say that a function u is a local minimizer for $\mathcal{F}$ in the domain Ω if $\mathcal{F}_\Omega(u)$ is well-defined and finite and

$$\mathcal{F}_\Omega(u+\varphi) \,\geq\, \mathcal{F}_\Omega(u)\,,$$

for any $\varphi \in C_0^\infty(\overline{\Omega})$.

We prove the following density estimates, which, roughly speaking, state that the level sets of minimizers behave "like sets of dimension $(Q-1)$".

Theorem 1.1. *Fix $\delta > 0$. Let u be a local minimizer for $\mathcal{F}$ in a domain Ω, with $|u| \leq 1$. Then:*

(i) *there exist positive constants c and r_0, depending only on δ and on the structural constants, such that*

$$\mathcal{F}_{\mathfrak{B}_r(\xi)}(u) \leq c\, r^{Q-1}\,,$$

for any $r \geq r_0$, provided that $\mathfrak{B}_{r+\delta}(\xi) \subseteq \Omega$;

(ii) *for any $\theta_0 \in (0,1)$, for any $\theta \in [-\theta_0, \theta_0]$ and for any $\mu_0 > 0$, if*

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_K(\xi) \cap \{u \geq \theta\}\Big) \,\geq\, \mu_0\,, \tag{1.1}$$

then there exist positive constants $c^\star$ and r_0, depending on K, μ_0, θ_0 and on the structural constants, such that

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi) \cap \{u \geq \theta\}\Big) \geq c^\star r^Q\,,$$

for any $r \geq r_0$, provided that $\mathfrak{B}_{r+\delta}(x) \subseteq \Omega$.

Analogously, if

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_K(\xi) \cap \{u \leq \theta\}\Big) \,\geq\, \mu_0\,, \tag{1.2}$$

then

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi) \cap \{u \leq \theta\}\Big) \geq c^\star r^Q\,,$$

for any $r \geq r_0$, provided that $\mathfrak{B}_{r+\delta}(\xi) \subseteq \Omega$.

The original idea of such density estimates goes back to [CC95] and several related techniques have been recently developed in [V04], [PV05a] and [PV05b]. In the Heisenberg group setting, a result analogous to the one in Theorem 1.1[i] was proven in [BL03] for the case in which $F(\xi, u) = (1-u^2)^2$. Also, our Theorem 1.1 may be seen as the extension of analogous estimates for minimal surfaces (see, e.g., [G84]).

As pointed out in [CC95], density estimates of these type easily imply an $L^\infty_{\rm loc}$-convergence of the level sets of the minimizers. Indeed, as proven in [MSC01], if u is a local minimizer and $u_\varepsilon(\xi) = u(z/\varepsilon, t/\varepsilon^2)$, then u_ε converges, up to subsequence, in $L^1_{\rm loc}$ to a step function, whith minimal interface (with respect to the $\mathbb{H}^n$-area). Then, using the argument in Theorem 2 in [CC95] (see also the end of § 6 of [PV04a]), one deduces from Theorem 1.1[ii] the following result:

Theorem 1.2. *Fix $\theta \in (0,1)$. Let $|u| \le 1$ be a local minimizer of $\mathcal{F}$ in a bounded domain Ω and let $u_\varepsilon(\xi) = u(z/\varepsilon, t/\varepsilon^2)$. Assume that, as ε tends to zero, u_ε converges in $L^1_{\rm loc}$ to the step function $\chi_E - \chi_{\Omega\setminus E}$, for a suitable $E \subset \Omega$. Then, $\{|u_\varepsilon| \le \theta\}$ converges locally uniformly to ∂E, that is* $\operatorname{dist}(\xi, \partial E) \to 0$ *uniformly for $\xi \in \{|u_\varepsilon| \le \theta\} \cap K$ for any set K whose closure is contained in Ω.*

As a byproduct, we deduce from Theorem 1.2 that the level sets of the example built in [BL03], though different from being minimal surfaces themselves, approach locally uniformly the hyperplane $\{t = 0\}$, when scaled via the natural $\mathbb{H}^n$-scaling.

Next is a version of Theorem 1.1 which turns out to be convenient for applications:

Theorem 1.3. *Fix $\theta_0 \in (0,1)$ and $\delta > 0$. Let u be a local minimizer for $\mathcal{F}$ in a domain Ω, with $|u| \le 1$. Suppose that $|u(\xi_0)| \le \theta_0$. Then, there exist positive constants c and r_0, depending only on θ_0, δ and on the structural constants, in such a way that*

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0) \cap \{|u| \le \theta_0\}\Big) \ge c r^{Q-1},$$

for any $r \ge r_0$, as long as $\mathfrak{B}_{r+\delta}(\xi_0) \subseteq \Omega$.

Another topic we deal with in this paper is related to the construction of "(quasi)periodic, plane-like" minimizers, that is, local minima of $\mathcal{F}$ in any bounded domain which are either periodic or locally uniformly close to periodic, and whose level sets are contained in a neighborhood of a given hyperplane of universal size. These results extend the ones of [V04] to the Heisenberg group and fit into the theory of plane-like structures for PDEs in periodic media. They also generalize the construction of orbits and minimal measures of prescribed rotation number (which is a classic topic in Aubry–Mather theory) to pseudoelliptic PDEs. We refer to [M86], [B90], [CL98], [CL01], [RS03], [RS04], [CL05], [PV05a] and [PV05b] for related results and further motivations. We also recall that the study of the (quasi)periodic level sets of suitable functions is also a subject of enduring interest in topology and mathematical physics (see, e.g., [DN05]).

We now present two results, namely Theorems 1.4 and 1.6, according to whether we consider rational or irrational frequencies. More precisely, in the rational case, we have the following result:

Theorem 1.4. *Fix $\delta > 0$. Suppose that $\omega \in \mathbb{Q}^{2n} \setminus \{0\}$. Assume also that*

$$F((k,0)\circ\xi,\cdot) = F(\xi,\cdot) \tag{1.3}$$

for any $\xi \in \mathbb{H}^n$ and any $k \in \mathbb{Z}^{2n}$. Then, there exists $M_0 > 0$, depending only on δ and on the structural constants of F, such that the following holds.

There exists a function $u_\omega : \mathbb{H}^n \to [-1,1]$ which is a local minimizer for $\mathcal{F}$ in any bounded domain of $\mathbb{H}^n$ and so that

$$\{\xi \in \mathbb{H}^n\,,\ |u_\omega| \le 1-\delta\} \subseteq \{\xi \in \mathbb{H}^n\,,\ |\xi\cdot(\omega,0)| \le M_0\,|\omega|\}\,. \tag{1.4}$$

Moreover, u_ω enjoys the following periodicity properties: for any $\xi = (z,t) \in \mathbb{H}^n$ and any $k \in \mathbb{Z}^{2n}$ such that $\omega \cdot k = 0$

$$u_\omega((k,0)\circ\xi) = u_\omega(\xi)\,, \tag{1.5}$$

and

$$u_\omega(z,t) = u_\omega(z,t+2)\,. \tag{1.6}$$

Finally, u_ω enjoys the following discrete monotonicity property: for any $k \in \mathbb{Z}^{2n}$ so that

$$\omega\cdot k \ge 0, \text{ we have that } u((k,0)\circ\xi) \ge u(\xi) \text{ for any } \xi \in \mathbb{H}^n. \tag{1.7}$$

Remark 1.5. We would like to observe that the result of Theorem 1.4 cannot hold if one fixes $\omega \in \mathbb{Q}^{2n+1}$ with $\omega_{2n+1} \neq 0$. Indeed, as we will see later, any function satisfying a periodicity condition as in (1.5) turns out to be also periodic in the vertical variable and hence its level sets cannot be contained in a slab orthogonal to ω unless this slab is horizontal, that is, $\omega_{2n+1} = 0$. This may be also rephrased by saying that a function cannot be monotone in a direction where it is periodic.

In the irrational frequency case, an analogous statement holds up to locally uniform approximation. More precisely, the following result holds true:

Theorem 1.6. *Fix $\delta > 0$. Suppose that $\omega \in \mathbb{R}^{2n} \setminus \mathbb{Q}^{2n}$. Assume that* (1.3) *holds. Then, given any sequence of vectors $\omega_j \in \mathbb{Q}^{2n}$ so that*

$$\lim_{j\to+\infty} \omega_j = \omega\,,$$

there exists a sequence of functions $u_j : \mathbb{H}^n \to [-1,1]$, which are local minimizers for $\mathcal{F}$ in any bounded domain of $\mathbb{H}^n$, which satisfy the level set constraint and the periodicity and monotonicity properties in (1.4), (1.5), (1.6) *and* (1.7) *(with ω_j replacing ω there), and which converge to u_ω uniformly on compact subsets of $\mathbb{H}^n$, up to subsequences.*

We derive from Theorems 1.4–1.6 a result on (quasi)periodic minimal surfaces in the Heisenberg group, which can be seen as the natural extension of analogous results poven in the Euclidean case by [M86] and [H32] (for geodesics) and by [CL01] (for general codimension one minimal surfaces).

For this, we consider a bounded function $\alpha : \mathbb{H}^n \to \mathbb{R}$. We suppose that

$$1/\alpha_0 \ge \alpha(\xi) \ge \alpha_0 > 0$$

and that

$$\alpha((k,0)\circ\xi)=\alpha(\xi)$$

for any $\xi\in\mathbb{H}^n$ and $k\in\mathbb{Z}^{2n}$. We consider the surface measure induced by the metric α, that is the surface measure in $\mathbb{H}^n$ induced[2] by the vector field $\sqrt{\alpha}\nabla_{\mathbb{H}^n}$ (see [MSC01] and [FSSC01] for details). Then, given any $\omega\in\mathbb{R}^{2n}\setminus\{0\}$, we construct a minimal surface at a universal distance from the hyperplane $\{(\omega,0)\cdot\xi=0\}$ which is either periodic or close to periodic surfaces, according to whether ω is rational or irrational.

More precisely, we prove the following result:

Theorem 1.7. *For any $\omega\in\mathbb{R}^{2n}\setminus\{0\}$, there exists a set $E_\omega\subset\mathbb{H}^n$ in such a way ∂E_ω is a local minimizer for the surface measure in $\mathbb{H}^n$ induced by the metric α and ∂E_ω is contained in the slab*

$$\{\xi\in\mathbb{H}^n\,,\ |\xi\cdot(\omega,0)|\le M_0\,|\omega|\}\,,$$

for a suitable $M_0>0$, depending only on α_0 and on n, but independent of ω.

2. Proof of Theorem 1.1[i]

We start with a variation of the standard Caccioppoli inequality (as exposed, for instance, in [G03]). Namely, let $\mathfrak{B}$ be a ball (say, centered at 0) of radius 2 and $\widetilde{\mathfrak{B}}$ be a concentric ball of radius 3, and assume that both $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ are well contained in Ω. Let $\tilde\eta\in C_0^\infty([0,3])$ be so that $0\le\tilde\eta\le1$, $\tilde\eta(\tau)=1$ if $\tau\in[0,2]$ and $|\tilde\eta'|\le10$. Let also $\eta(\xi)=\tilde\eta(\rho)$. We observe that, by (0.3),

$$|\nabla_{\mathbb{H}^n}\eta|\ \le\ \text{const}\,.$$

Then, the minimality of u implies that

$$\mathcal{F}_{\widetilde{\mathfrak{B}}}(u)\ \le\ \mathcal{F}_{\widetilde{\mathfrak{B}}}(u-\eta^2u)\,,$$

and therefore

$$\begin{aligned}
0\ &\le\ \int_{\widetilde{\mathfrak{B}}}\Big(|\nabla_{\mathbb{H}^n}(\eta^2u)|^2-2\nabla_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}(\eta^2u)\Big)\,d\xi+\text{const}\ \le\\
&\le\ \int_{\widetilde{\mathfrak{B}}}\Big(4\eta^2u^2|\nabla_{\mathbb{H}^n}\eta|^2+\eta^4|\nabla_{\mathbb{H}^n}u|^2+4\eta^3u\nabla_{\mathbb{H}^n}\eta\cdot\nabla_{\mathbb{H}^n}u-\\
&\qquad -2\eta^2|\nabla_{\mathbb{H}^n}u|^2-4\eta u\nabla_{\mathbb{H}^n}\eta\cdot\nabla_{\mathbb{H}^n}u\Big)\,d\xi+\text{const}\ \le\\
&\le\ -\int_{\widetilde{\mathfrak{B}}}\eta^2\left(2-\eta^2-\frac12\right)|\nabla_{\mathbb{H}^n}u|^2\,d\xi+\text{const}\,,
\end{aligned}$$

where we used a suitably scaled Cauchy Inequality in the last estimate. Thus, we obtain the following Caccioppoli-type estimate:

$$\frac12\int_{\mathfrak{B}}|\nabla_{\mathbb{H}^n}u|^2\le\int_{\widetilde{\mathfrak{B}}}\frac{\eta^2}{2}|\nabla_{\mathbb{H}^n}u|^2\le\ \text{const}\,. \tag{2.1}$$

[2]More general metrics and wider class of vector fields may alo be taken into account, by performing computations similar to the ones we present here.

The proof of Theorem 1.1[i] is now a variation of the one in [CC95] (see also Proposition 3.2 of [BL03] and [AAC01] for related techniques). For short, in the sequel, we will omit in the notation the centers of the balls considered.

Let $\tilde{g} \in C^\infty([0,\, r])$ be so that $|\tilde{g}'| \leq 10$, $\tilde{g}(\tau) = -1$ if $\tau \in [0,\, r-1]$ and $\tilde{g}(r) = 1$ on ∂B_r. We also introduce the radial (with respect to the Koranyi ball) function $g(\xi) = \tilde{g}(\rho)$. By (0.3), $|\nabla_{\mathbb{H}^n} g| \leq$ const.

Define $u^\star = \min\{u, g\}$. Since u is a local minimizer and

$$\mathfrak{L}^{2n+1}(\mathfrak{B}_r \setminus \mathfrak{B}_{r-1}) \leq \text{ const}\, r^{Q-1}\,,$$

we have that

$$\begin{aligned} \mathcal{F}_{\mathfrak{B}_r}(u) &\leq \mathcal{F}_{\mathfrak{B}_r}(u^\star) \leq \text{const}\left(\int_{\mathfrak{B}_r\setminus\mathfrak{B}_{r-1}} |\nabla_{\mathbb{H}^n} u^\star|^2 + r^{Q-1}\right) \leq \\ &\leq \text{const}\left(\int_{\mathfrak{B}_r\setminus\mathfrak{B}_{r-1}} (|\nabla_{\mathbb{H}^n} u|^2 + |\nabla_{\mathbb{H}^n} g|^2) + r^{Q-1}\right) \leq \\ &\leq \text{const}\left(\int_{\mathfrak{B}_r\setminus\mathfrak{B}_{r-1}} |\nabla_{\mathbb{H}^n} u|^2 + r^{Q-1}\right) . \end{aligned} \tag{2.2}$$

Let us now cover $\mathfrak{B}_r \setminus \mathfrak{B}_{r-1}$ with Koranyi balls $\mathfrak{B}_1, \dots \mathfrak{B}_K$ of radius 2; by measure theoretic considerations, one sees that it is possible to take $K \leq \text{const}\, r^{Q-1}$. Let $\widetilde{\mathfrak{B}_j}$ be the ball concentric to $\mathfrak{B}_j$ of radius 3. Then, from (2.2) and (2.1), we infer that

$$\begin{aligned} \mathcal{F}_{\mathfrak{B}_r}(u) &\leq \text{const}\left(\sum_{j=1}^{K} \int_{\mathfrak{B}_j} |\nabla_{\mathbb{H}^n} u|^2 + r^{Q-1}\right) \leq \\ &\leq \text{const}\,(K + r^{Q-1})\,, \end{aligned}$$

which implies the desired result.

3. Proof of Theorem 1.1[ii]

We prove the first claim, the second one being analogous. Furthermore, we observe that, with no loss of generality, we may assume θ as close to -1 as we wish. Indeed: assume the result to be true for $\theta^\star$ (say, close to -1), and let $\theta \in [-\theta_0, \theta_0]$, with $\theta^\star \leq -\theta_0$. Then

$$\mu_0 \leq \mathfrak{L}^{2n+1}\Big(\{u \geq \theta\} \cap \mathfrak{B}_K\Big) \leq \mathfrak{L}^{2n+1}\Big(\{u \geq \theta^\star\} \cap \mathfrak{B}_K\Big)\,,$$

therefore, using the result for $\theta^\star$, we deduce from Theorem 1.1[ii], that

$$\begin{aligned}
&\operatorname{const} r^Q \leq \\
\leq\ & \mathfrak{L}^{2n+1}\Big(\{u \geq \theta^\star\} \cap \mathfrak{B}_r\Big) \leq \\
\leq\ & \mathfrak{L}^{2n+1}\Big(\{u \geq \theta\} \cap \mathfrak{B}_r\Big) + \mathfrak{L}^{2n+1}\Big(\{\theta^\star \leq u < \theta\} \cap \mathfrak{B}_r\Big) \leq \\
\leq\ & \mathfrak{L}^{2n+1}\Big(\{u \geq \theta\} \cap \mathfrak{B}_r\Big) + \frac{1}{\inf\limits_{u\in[\theta^\star,\theta_0]} F} \int_{\mathfrak{B}_r} F(\xi,u)\, d\xi \leq \\
\leq\ & \mathfrak{L}^{2n+1}\Big(\{u \geq \theta\} \cap \mathfrak{B}_r\Big) + \operatorname{const} \mathcal{F}_{\mathfrak{B}_r}(u) \leq \\
\leq\ & \mathfrak{L}^{2n+1}\Big(\{u \geq \theta\} \cap \mathfrak{B}_r\Big) + \operatorname{const} r^{Q-1},
\end{aligned}$$

which gives

$$\mathfrak{L}^{2n+1}\Big(\{u \geq \theta\} \cap \mathfrak{B}_r\Big) \geq \operatorname{const} r^Q .$$

Thus, in the rest of the proof, we may and do assume that θ is appropriately close to -1.

Also, for further use, we recall the following recursive results:

Lemma 3.1. *Fix $\nu \in \mathbb{N}$. Let $v_k \geq 0$ and $a_k \geq 0$ be two nondecreasing sequences such that $v_1 + a_1 \geq c_0$,*

$$v_k^{(\nu-1)/\nu} \leq C_0 \left(v_{k+1} + a_{k+1} - v_k - a_k - c_1 a_k\right) ,$$

for any $k \in \mathbb{N}$ and some positive constants c_0, c_1, C_0. Then, there exists $\kappa > 0$, depending on c_0, c_1, and C_0 such that

$$v_k + a_k \geq \kappa\, k^\nu$$

for any $k \in \mathbb{N}$.

Lemma 3.2. *Fix $\nu \in \mathbb{N}$. Let $a_k \geq 0$ be a sequence such that $a_1 \geq c_0$, $a_k \leq C_0\, L^\nu k^{\nu-1}$,*

$$\Big(\sum_{1\leq j\leq k} a_j\Big)^{(\nu-1)/\nu} \leq C_0 \Big(a_{k+1} + \sum_{1\leq j\leq k} e^{-L(k+1-j)} a_j\Big),$$

for any $k \in \mathbb{N}$ and some positive constants L, c_0, and C_0. Then, if L is suitably large (in dependence of c_0 and C_0), there exists $\kappa > 0$, depending on c_0 and C_0, such that

$$a_k \geq \kappa\, k^{\nu-1}$$

for any $k \in \mathbb{N}$.

For the proof of the above lemmata, see [CC95] or, for further details, Lemmata 2.1 and 2.2 in [PV05b]. Here, we just mention that these iterations may be seen as a "discrete version" of a differential estimate linking volumes and areas in the Isoperimetric Inequality (see again [CC95] for very nice heuristics on this). With this, we define the quantities which will play here the roles of the volume and of the area. Namely, we define

$$\begin{aligned} \mathcal{V}(r) &= \mathfrak{L}^{2n+1}(\mathfrak{B}_r \cap \{u > \theta\}) \,, \\ \mathcal{A}(r) &= \int_{\mathfrak{B}_r \cap \{u \le \theta\}} F(\xi, u)\, d\xi \,. \end{aligned}$$

The proof of Theorem 1.1[ii] follows the ideas of [CC95], as developed in [V04], [PV05a] and [PV05b], and it is divided into three cases, according to whether $d = 0$, $d \in (0, 2)$ or $d = 2$. In any of these cases, A will denote a suitably large free parameter, to be conveniently chosen.

3.1. **The case $d = 0$.** For any $\tau \in [0, r]$, let

$$\tilde{h}(\tau) = 2(\tau - r + 1)_+^2 - 1$$

and define the following radial (with respect to the Koranyi ball) function

$$h(\xi) = \tilde{h}(\rho) \,. \tag{3.1}$$

Let also

$$\sigma = \min\{u,\, h\} \qquad \text{and} \qquad \beta = \min\{u - \sigma,\, 1 + \theta\} \,. \tag{3.2}$$

Exploiting (0.4), one sees that

$$|\Delta_{\mathbb{H}^n} h| \le \text{const} \tag{3.3}$$

in $\mathfrak{B}_r \setminus \mathfrak{B}_{r-1}$. Moreover, by the Cauchy and Sobolev Inequalities (for the latter, applied to β^2, see, e.g., [VSCC92], pages 3 and 59, and the references on page 62), and the minimality of u, we deduce that

$$
\begin{aligned}
& \Big(\int_{\mathfrak{B}_r} \beta^{\frac{2Q}{Q-1}}\Big)^{\frac{Q-1}{Q}} \leq \\
\leq\; & \operatorname{const} \int_{\mathfrak{B}_r \cap \{u-\sigma \leq 1+\theta\}} |\beta|\, |\nabla_{\mathbb{H}^n} \beta| \leq \\
\leq\; & \operatorname{const} A \Big(\int_{\mathfrak{B}_r \cap \{u > \sigma\}} (|\nabla_{\mathbb{H}^n} u|^2 - |\nabla_{\mathbb{H}^n} \sigma|^2 - \\
& -2 \nabla_{\mathbb{H}^n} (u-\sigma) \cdot \nabla_{\mathbb{H}^n} \sigma) \Big) + \\
& + \frac{\operatorname{const}}{A} \int_{\mathfrak{B}_r \cap \{u-\sigma \leq 1+\theta\}} (u-\sigma)^2 \leq \\
\leq\; & \operatorname{const} A \Big(\int_{\mathfrak{B}_r \cap \{u > \sigma\}} (|\nabla_{\mathbb{H}^n} u|^2 - |\nabla_{\mathbb{H}^n} \sigma|^2 + \\
& +2(u-\sigma)\, \Delta_{\mathbb{H}^n} \sigma) \Big) + \frac{\operatorname{const}}{A} \int_{\mathfrak{B}_r \cap \{u-\sigma \leq 1+\theta\}} (u-\sigma)^2 \leq \\
\leq\; & \operatorname{const} A \Big[\int_{\mathfrak{B}_r \cap \{u > \sigma\}} (F(\xi,\sigma) - F(\xi,u)) + \\
& +2 \int_{\mathfrak{B}_r} (u-\sigma)\, \Delta_{\mathbb{H}^n} \sigma \Big] + \frac{\operatorname{const}}{A} \int_{\mathfrak{B}_r \cap \{u-\sigma \leq 1+\theta\}} (u-\sigma)^2 .
\end{aligned}
$$

Note that, up to now, the condition $d = 0$ has not been used yet (this will allow us to use similar procedures also for $0 < d \leq 2$).

We now observe that the left hand side of the above inequality is larger or equal than

$$
\Big(\int_{\mathfrak{B}_r \cap \{u > \theta\}} \beta^{\frac{2Q}{Q-1}}\Big)^{\frac{Q-1}{Q}} \geq (1+\theta)^2\, \mathcal{V}(r-1)^{\frac{Q-1}{Q}} .
$$

By our assumptions on F, the contribution of the right hand side in $\mathfrak{B}_{r-1}$ (where $\sigma = -1$) is bounded by

$$
\begin{aligned}
& -\operatorname{const} A \int_{\mathfrak{B}_{r-1}} F(\xi,u) + \frac{\operatorname{const}}{A} \int_{\mathfrak{B}_{r-1} \cap \{u \leq \theta\}} (u+1)^2 \leq \\
\leq\; & -\int_{\mathfrak{B}_{r-1} \cap \{u \leq \theta\}} \Big[\operatorname{const} A - \frac{\operatorname{const}}{A} \Big] F(\xi,u)\, d\xi \leq \\
\leq\; & -\operatorname{const} \mathcal{A}(r-1) ,
\end{aligned}
$$

if A is conveniently large. Moreover, exploiting (3.3) and the fact that $d = 0$, one bounds the contribution of the right hand side in $\mathfrak{B}_r \setminus \mathfrak{B}_{r-1}$ by

$$\begin{aligned} & \text{const}\,\Big[\int_{(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1})\cap\{u\le\theta\}} \Big(F(\xi,\sigma)-F(\xi,u)+ \\ & +(u-\sigma)+(u-\sigma)^2\Big)\,d\xi+\mathcal{V}(r)-\mathcal{V}(r-1)\Big] \le \\ \le\ & \text{const}\,\left[\int_{(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1})\cap\{u\le\theta\}} F(\xi,u)\,d\xi+\mathcal{V}(r)-\mathcal{V}(r-1)\right] \le \\ \le\ & \text{const}\,\Big[\mathcal{A}(r)-\mathcal{A}(r-1)+\mathcal{V}(r)-\mathcal{V}(r-1)\Big]\,. \end{aligned}$$

Then,

$$\begin{aligned} & \text{const}\,\Big((\mathcal{V}(r-1))^{\frac{Q-1}{Q}}+\mathcal{A}(r-1)\Big) \le \\ \le\ & \mathcal{V}(r)-\mathcal{V}(r-1)+\mathcal{A}(r)-\mathcal{A}(r-1)\,, \end{aligned}$$

which completes the proof of the desired result via Lemma 3.1.

3.2. **The case** $d \in (0,2)$**.** Let $d' = \max\{1, d\}$ and, for $\tau \in [r-1, r]$, we define

$$\tilde{h}(\tau) \,=\, 2(\tau-r+1)_+^{\frac{2}{2-d'}}-1\,.$$

Let h, β and σ be as in (3.1) and (3.2). Recalling (0.4), one obtains that, in $\mathfrak{B}_r \setminus \mathfrak{B}_{r-1}$,

$$|\Delta_{\mathbb{H}^n} h| \le \text{ const}\,(h+1)^{d'-1}\,.$$

Also, in the set $\{u > \sigma\}$ we have $\sigma = h$ and then the last formula implies

$$\begin{aligned} \Delta_{\mathbb{H}^n} h\,(u-\sigma) &\le \text{const}\,(h+1)^{d'-1}\,(u-h) \le \\ &\le \text{const}\,(u+1)^{d'}\,, \end{aligned} \tag{3.4}$$

in $(\mathfrak{B}_r \setminus \mathfrak{B}_{r-1}) \cap \{u > \sigma\}$.

As argued in the case $d = 0$, Sobolev (applied to β^2) and Cauchy Inequalities and the minimality property of u imply that

$$\begin{aligned} & \mathcal{V}(r-1)^{\frac{Q-1}{Q}} \le \\ \le\ & \text{const}\,A\Big[\int_{\mathfrak{B}_r}(F(\xi,\sigma)-F(\xi,u))+ \\ & +2\int_{\mathfrak{B}_r}\Delta_{\mathbb{H}^n}\sigma\,(u-\sigma)\Big]+ \\ & +\frac{\text{const}}{A}\int_{\mathfrak{B}_r\cap\{u-\sigma\le 1+\theta\}}(u-\sigma)^2\,. \end{aligned} \tag{3.5}$$

We now estimate the right hand side in $\mathfrak{B}_{r-1}$, in which $\sigma = -1$. For this, note that, if θ is close to -1 and $u \le \theta$, we have, by our assumptions on F, that

$$F(\xi,u) \ge \text{ const}\,(1+u)^d \ge \frac{\text{const}}{2^{2-d}}(1+u)^2 \tag{3.6}$$

so that

$$\frac{\text{const}}{A}\int_{\mathfrak{B}_{r-1}\cap\{u\le\theta\}}(1+u)^2-\frac{\text{const}\,A}{2}\int_{\mathfrak{B}_{r-1}\cap\{u\le\theta\}}F(\xi,u)\le 0$$

if we choose A suitably big. Therefore, the right hand side of (3.5) in $\mathfrak{B}_{r-1}$ is negative and less than $-\text{const}\,\mathcal{A}(r-1)$.

Note also that, since θ is close to -1 and $\sigma\le u$, our assumptions on F imply that $F(\xi,\sigma)\le F(\xi,u)$ if $u\le\theta$. Thus,

$$\begin{aligned}&\text{const}\,A\int_{\mathfrak{B}_r\setminus\mathfrak{B}_{r-1}}F(\xi,\sigma)-F(\xi,u)\le\\ \le\;&\text{const}\,A\,\mathfrak{L}^{2n+1}(\{u>\theta\}\cap(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1}))\,.\end{aligned}$$

What is more, using again (3.6), one sees that

$$\begin{aligned}&\frac{\text{const}}{A}\int_{\mathfrak{B}_r\setminus\mathfrak{B}_{r-1}}(u-\sigma)^2\le\\ \le\;&\frac{\text{const}}{A}\int_{(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1})\cap\{u\le\theta\}}(u+1)^2+\\ &+\frac{\text{const}}{A}\mathfrak{L}^{2n+1}(\{u>\theta\}\cap(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1}))\le\\ \le\;&\frac{\text{const}}{A}\int_{(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1})\cap\{u\le\theta\}}F(\xi,u)+\\ &+\frac{\text{const}}{A}\mathfrak{L}^{2n+1}(\{u>\theta\}\cap(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1}))\,.\end{aligned}$$

Also, recalling that $\sigma=-1$ in $\mathfrak{B}_{r-1}$, splitting the domain of integration into the sets $\{u\le\theta\}$ and $\{u>\theta\}$, using (3.4) and our hypotheses on F, we get

$$\begin{aligned}&\int_{\mathfrak{B}_r}\Delta_{\mathbb{H}^n}\sigma\,(u-\sigma)=\\ &=\int_{(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1})\cap\{u>\sigma\}}\Delta_{\mathbb{H}^n}\sigma\,(u-\sigma)\le\\ &\le\text{const}\,\mathfrak{L}^{2n+1}\Big((\mathfrak{B}_r\setminus\mathfrak{B}_{r-1})\cap\{u>\theta\}\Big)+\\ &+\text{const}\int_{(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1})\cap\{u\le\theta\}}(u+1)^{d'}\le\\ &\le\text{const}\,\mathfrak{L}^{2n+1}\Big((\mathfrak{B}_r\setminus\mathfrak{B}_{r-1})\cap\{u>\theta\}\Big)+\\ &+\text{const}\int_{(\mathfrak{B}_r\setminus\mathfrak{B}_{r-1})\cap\{u\le\theta\}}F(\xi,u)\,.\end{aligned}$$

Collecting the above estimates and using Lemma 3.1, the result follows.

3.3. **The case $d=2$.** We will use here two further positive free parameters Θ and T: we will fix Θ small enough and then choose T so that ΘT is suitably large.

Let $k \in \mathbb{N}$. On page 183 of [V04], a function $\tilde{h} \in C^2([0,(k+1)T])$ was constructed so that $-1 \le \tilde{h} \le 1$, $\tilde{h}((k+1)T) = 1$, $\tilde{h}'(0) = 0$,

$$\tilde{h}(\tau) + 1 \le \text{const}\, e^{-\Theta T(k+1-j)} \tag{3.7}$$

if $\tau \in [(j-1)T,\, jT]$, for $j = 1, \dots, k+1$,

$$|\tilde{h}'(\tau)| \le \text{const}\, \Theta\tau(\tilde{h}+1)$$

if $\tau \in [0,1]$,

$$|\tilde{h}'(\tau)| \le \text{const}\, \Theta(\tilde{h}+1)$$

if $\tau \in [1,(k+1)T]$, and

$$|\tilde{h}''(\tau)| \le \text{const}\, \Theta(\tilde{h}+1)$$

if $\tau \in [0,(k+1)T]$.

Let h be as in (3.1). In the light of (0.4), we thus have that

$$|\Delta_{\mathbb{H}^n} h| \le \text{const}\,(h+1) \le \text{const}\, e^{-\Theta T(k+1-j)} \tag{3.8}$$

in $\mathfrak{B}_{jT} \setminus \mathfrak{B}_{(j-1)T}$, and

$$|\Delta_{\mathbb{H}^n} h| \le \text{const}\, \Theta(h+1) \tag{3.9}$$

in $\mathfrak{B}_{(k+1)T}$. Consequently, if Θ is suitably small and h is suitably close to -1, then

$$|\Delta_{\mathbb{H}^n} h| \le \sqrt{\Theta} F_u(x,h) \tag{3.10}$$

in $\mathfrak{B}_{(k+1)T}$. We set

$$\theta' = \theta - C_* e^{-\Theta T}\,, \tag{3.11}$$

where C_* denotes a suitably large constant. Note that, if ΘT is large enough, then $\theta' > -1$. Define also

$$\sigma = \min\{u,h\} \quad \text{and} \quad \beta = \min\{u-\sigma,\, 1+\theta'\}\,.$$

By arguing as done on page 10, we gather from the Sobolev and Cauchy Inequalities that

$$\begin{aligned} &\Big(\int_{\mathfrak{B}_{(k+1)T}} \beta^{\frac{2Q}{Q-1}}\Big)^{\frac{Q-1}{Q}} \le \\ &\le \text{const}\, A\left[\int_{\mathfrak{B}_{(k+1)T}\cap\{u>\sigma\}} \Big(F(\xi,\sigma) - F(\xi,u) + \Delta_{\mathbb{H}^n}\sigma(u-\sigma)\Big)\right] + \\ &+ \frac{\text{const}}{A}\int_{\mathfrak{B}_{(k+1)T}\cap\{u-\sigma\le 1+\theta'\}} (u-\sigma)^2\,. \end{aligned} \tag{3.12}$$

We now estimate the left hand side of (3.12). If ΘT is large enough, we gather from (3.7) that $\theta - h \ge (1-\theta_0)/2$ in $\mathfrak{B}_{kT}$. Thus, the left hand side of (3.12) is bigger than

$$\text{const}\, \mathcal{V}(kT)^{\frac{Q-1}{Q}}\,.$$

Let us now estimate the right hand side of (3.12). First of all, we consider the contribution in $\{u \le \theta\}$. Let us observe that, since $-1 \le \sigma \le u \le 1$,

$$\begin{aligned} & (u+1)^2 - (\sigma+1)^2 - \frac{1}{2}(u-\sigma)^2 = \\ = \;& (u-\sigma)\left(\frac{1}{2}u + \frac{3}{2}\sigma + 2\right) \ge 0\,. \end{aligned}$$

Accordingly, in $\{\sigma < u \le \theta\}$,

$$\begin{aligned} & F(\xi,u) - F(\xi,\sigma) = \int_h^u F_u(\xi,\zeta)\,d\zeta \ge \\ & \ge \text{ const } \int_h^u (\zeta+1)\,d\zeta = \text{ const } \Big[(u+1)^2 - (h+1)^2\Big] \ge \\ & \ge \text{ const } (u-h)^2\,. \end{aligned} \tag{3.13}$$

Consequently, recalling again (3.10), we deduce that the contribution of the right hand side of (3.12) in $\{u \le \theta\}$ is controlled by

$$\int_{\mathfrak{B}_{(k+1)T}\cap\{\sigma<u\le\theta\}} \Big(F(\xi,\sigma) - F(\xi,u) + \text{ const } \sqrt{\Theta} F_u(\xi,\sigma)(u-\sigma)\Big)\,, \tag{3.14}$$

as long as A is sufficiently large.

We now show that this quantity is indeed negative. Since we assumed θ to be close to -1 (recall the discussion on page 8), we have that F and F_u are monotone in $\{\sigma < u \le \theta\}$, we have that $F(\xi,\sigma) - F(\xi,u)$ is negative and that

$$|F_u(\xi,\sigma)(u-\sigma)| \le |F(\xi,\sigma) - F(\xi,u)|\,.$$

Since we assumed Θ to be small, the latter inequality yields that the quantity in (3.14) is negative.

Let us now bound the right hand side of (3.12) in $\{u > \theta\}$. The contribution in $\mathfrak{B}_{(k+1)T} \setminus \mathfrak{B}_{kT}$ of such term is bounded by

$$\int_{(\mathfrak{B}_{(k+1)T}\setminus\mathfrak{B}_{kT})\cap\{u>\theta\}} \Big(F(\xi,\sigma) - F(\xi,u) + (\sigma+1)(u-\sigma) + (u-\sigma)^2\Big)\,,$$

due to (3.9), and this term is bounded by

$$\begin{aligned} & \mathfrak{L}^{2n+1}\Big(\{u>\theta\} \cap (\mathfrak{B}_{(k+1)T} \setminus \mathfrak{B}_{kT})\Big) \\ = \;& \mathcal{V}((k+1)T) - \mathcal{V}(kT)\,. \end{aligned}$$

Let us now look at the contribution of the right hand side of (3.12) in $\{u > \theta\} \cap \mathfrak{B}_{kT}$. Notice that, from (3.7),

$$\mathfrak{B}_{kT} \cap \{\sigma < u \le \sigma + 1 + \theta'\} \subseteq \mathfrak{B}_{kT} \cap \{\sigma < u \le \theta\}\,,$$

provided that C_* in (3.11) is large enough. Consequently, the contribution of the right hand side of (3.12) in $\{u > \theta\} \cap \mathfrak{B}_{kT}$ is controlled by

$$\begin{aligned} &\int_{\mathfrak{B}_{kT}\cap\{u>\theta\}} F(\xi,\sigma) - F(\xi,u) + |\Delta_{\mathbb{H}^n} h| \leq \\ &\leq \sum_{j=1}^{k} \int_{\mathfrak{B}_{jT}\setminus\mathfrak{B}_{(j-1)T}\cap\{u>\theta\}} F(\xi,h) + |\Delta_{\mathbb{H}^n} h| \,. \end{aligned} \tag{3.15}$$

By our assumption on F in the case $d = 2$, we have that

$$F(\xi, -1+s) \leq \text{ const}\, s\,,$$

provided that $s > 0$ is small enough. Thus, by means of (3.7), and (3.8), we bound the above term in (3.15) by

$$\sum_{j=1}^{k} e^{-\Theta T(k+1-j)} \Big[\mathcal{V}(jT) - \mathcal{V}((j-1)T)\Big]\,.$$

Collecting all theses estimates, we get that

$$\begin{aligned} &\text{const}\,(\mathcal{V}(kT))^{\frac{Q-1}{Q}} \leq \\ \leq\;& \mathcal{V}((k+1)T) - \mathcal{V}(kT) + \sum_{j=1}^{k} e^{-\Theta T(k+1-j)} \Big[\mathcal{V}(jT) - \mathcal{V}((j-1)T)\Big]\,. \end{aligned}$$

Then, the desired result follows from Lemma 3.2.

4. Proof of Theorem 1.2

This proof is a variation of the one on page 11 of [CC95] (see also page 69 of [PV05a]). Due to some subtleties given by the Heisenberg group and the role played by the Koranyi ball, we provide full details for the facility of the reader.

The proof is based on Theorem 1.1. In fact, we first notice that conditions (1.1) and (1.2) in Theorem 1.1, which are somewhat unpleasant to check, may be replaced by pointwise conditions (that is, conditions (4.1) and (4.2) here below), that are easier to deal with. Namely, we deduce from Theorem 1.1 the following result:

Corollary 4.1. *Let δ, Ω and u be as in the statement of Theorem 1.1. Fix $\theta_0 \in (0,1)$ and let θ_1, $\theta_2 \in (-\theta_0, \theta_0)$. Let $\xi \in \mathbb{R}^{2n+1}$ be so that*

$$u(\xi) \geq \theta_1\,. \tag{4.1}$$

Then, there exists positive constants $c^\star$ and r_0, possibly depending on θ_0 and other structural constants, such that

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi) \cap \{u \geq \theta_2\}\Big) \geq c^\star r^Q\,,$$

for any $r \geq r_0$, provided that $\mathfrak{B}_{r+\delta}(\xi) \subseteq \Omega$. Analogously, if

$$u(\xi) \leq \theta_1\,. \tag{4.2}$$

then

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi) \cap \{u \le \theta_2\}\Big) \ge c^\star r^Q \,.$$

Proof. We prove the first claim, the second one being analogous. We recall that u is uniformly Hölder continuous (see Theorem 2.1 in [M95]), that is, there exist positive universal constants Λ and α so that

$$|u(\zeta) - u(\zeta')| \;\le\; \Lambda\, |(\zeta')^{-1} \circ \zeta|^\alpha \,, \tag{4.3}$$

for any $\zeta,\ \zeta' \in \Omega'$, for Ω' well contained in Ω. We set

$$\hat\theta_1 \;=\; \frac{\theta_1 - 1}{2} \in \left[\frac{-\theta_0 - 1}{2}, \frac{\theta_0 - 1}{2}\right] \subset (-1,1)$$

and

$$K \;=\; \left(\frac{1-\theta_1}{2\Lambda}\right)^{1/\alpha} \,.$$

Thence, we conclude from (4.1) and (4.3) that $u \ge \hat\theta_1$ in $\mathfrak{B}_K(\xi)$. Consequently,

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_K(\xi) \cap \{u \ge \hat\theta_1\}\Big) = \mathfrak{L}^{2n+1}\Big(\mathfrak{B}_K(\xi)\Big) \;=\; \mathrm{const}\, K^Q \,,$$

thus condition (1.1) is fulfilled. So, by Theorem 1.1[ii],

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi) \cap \{u \ge \hat\theta_1\}\Big) \ge \mathrm{const}\, r^Q \,. \tag{4.4}$$

Now: if $\hat\theta_1 \ge \theta_2$, the desired result follows directly from (4.4). If, on the other hand, $\hat\theta_1 < \theta_2$, we combine (4.4) with Theorem 1.1[i], to conclude that

$$\begin{aligned}
&\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi) \cap \{u \ge \theta_2\}\Big) \ge \\
\ge\;& \mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi) \cap \{u \ge \hat\theta_1\}\Big) - \mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi) \cap \{\hat\theta_1 \le u \le \theta_2\}\Big) \ge \\
\ge\;& \mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi) \cap \{u \ge \hat\theta_1\}\Big) - \mathrm{const}\, \mathcal{F}_{\mathfrak{B}_r(\xi)}(u) \ge \\
\ge\;& \mathrm{const}\, r^Q - \mathrm{const}\, r^{Q-1} \ge \\
\ge\;& \mathrm{const}\, r^Q \,,
\end{aligned}$$

if r is large enough. □

With this result, we may now complete the proof of Theorem 1.2, by arguing as follows.

The proof of Theorem 1.2 is by contradiction. If the claim of Theorem 1.2 were false, there would exist a compact set K (say, contained in the Korany ball of radius R), well contained in Ω, an infinitesimal sequence of ε_k and a positive δ so that

$$\sup_{\xi \in \{|u_{\varepsilon_k}| \le \theta\} \cap K} \mathrm{dist}\,(\xi, \partial E) \;\ge\; 8\delta \,,$$

for any $k \in \mathbb{N}$. Consequently, we find a sequence of points $\xi_k \in \{|u_{\varepsilon_k}| \leq \theta\} \cap K$, in such a way that

$$\mathrm{dist}\,(\xi_k, \partial E) \,\geq\, 4\delta^2\,. \tag{4.5}$$

Since, by construction, $\xi_k \in K \subseteq \mathfrak{B}_R(0)$, we also have that

$$|\xi_k|_{\mathbb{H}^n} \,\leq\, R\,. \tag{4.6}$$

Of course, by possibly reducing δ, we may also assume that

$$\delta \,<\, \min\left\{1\,,\ \frac{\mathrm{dist}\,(K, \partial\Omega)}{16}\right\}\,, \tag{4.7}$$

therefore (4.5) implies that either

$$\mathfrak{B}_{2\delta}(\xi_k) \subseteq \Omega \cap E \tag{4.8}$$

or

$$\mathfrak{B}_{2\delta}(\xi_k) \subseteq \Omega \setminus E \tag{4.9}$$

for infinitely many $k \in \mathbb{N}$. We assume that (4.8) holds (the case in which (4.9) holds is indeed analogous). Then, (4.8) implies that

$$\chi_E - \chi_{\Omega\setminus E} \,=\, 1 \qquad \text{in } \mathfrak{B}_{2\delta}(\xi_k) \tag{4.10}$$

for infinitely many $k \in \mathbb{N}$. Also, by construction, if we set $\xi_k = (z_k, t_k)$ and $\zeta_k = (z_k/\varepsilon_k, t_k/\varepsilon_k^2)$ we have that $|u(\zeta_k)| \leq \theta$. Therefore, we conclude from Corollary 4.1 that

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\zeta_k) \cap \{u \leq \theta\}\Big) \geq \mathrm{const}\, r^Q\,,$$

for any $r \geq r_0$, with r_0 universal. Scaling back, this means that

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_{\varepsilon_k r}(\xi_k) \cap \{u_{\varepsilon_k} \leq \theta\}\Big) \geq \mathrm{const}\,(\varepsilon_k r)^Q\,, \tag{4.11}$$

for any $r \geq r_0$. We now set

$$\delta' = \frac{\delta}{2(1+8R)}\,.$$

From (4.10) and (4.11), we conclude that

$$\chi_E - \chi_{\Omega\setminus E} \,=\, 1 \qquad \text{in } \mathfrak{B}_{\delta'}(\xi_k) \tag{4.12}$$

and that, from (4.11) with $r = \frac{\delta'}{\varepsilon_k}$,

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_{\delta'}(\xi_k) \cap \{u_{\varepsilon_k} \leq \theta\}\Big) \geq \mathrm{const}\,(\delta')^Q\,, \tag{4.13}$$

for infinitely many $k \in \mathbb{N}$. Also, if

$$\Omega' \,=\, \bigcup_{k\in\mathbb{N}} \mathfrak{B}_{\delta'}(\xi_k)\,,$$

we deduce from (4.7) that Ω' is well contained in Ω. We utilize this fact, together with (4.12) and (4.13), to infer that

$$\begin{aligned} & \int_{\Omega'} |u_{\varepsilon_k} - (\chi_E - \chi_{\Omega\setminus E})| \geq \\ \geq \; & \int_{\mathfrak{B}_{\delta'}(\xi_k)\cap\{u_{\varepsilon_k}\leq\theta\}} |u_\varepsilon - (\chi_E - \chi_{\Omega\setminus E})| \geq \\ \geq \; & \int_{\mathfrak{B}_{\delta'}(\xi_k)\cap\{u_{\varepsilon_k}\leq\theta\}} (1-\theta) \geq \\ \geq \; & \text{const}\,(\delta')^Q\,(1-\theta)\,, \end{aligned}$$

for infinitely many $k \in \mathbb{N}$. This contradicts the assumption that u_{ε_k} converges to $\chi_E - \chi_{\Omega\setminus E}$ in L^1_{loc}, thus proving Theorem 1.2.

5. Proof of Theorem 1.3

Let $\theta_1 := (\theta_0 + 1)/2 \in (\theta_0, 1)$. Then, since u is uniformly continuous, due to [M95], we get that

$$\mathfrak{B}_{\varepsilon_0}(\xi_0) \subseteq \{|u| \leq \theta_1\}\,,$$

for a universal $\varepsilon_0 > 0$. Consequently,

$$\mathfrak{L}^{2n+1}(\mathfrak{B}_{\varepsilon_0}(\xi_0) \cap \{u \geq -\theta_1\}) \geq \mathfrak{L}^{2n+1}(\mathfrak{B}_{\varepsilon_0}(\xi_0) \cap \{|u| \leq \theta_1\}) \geq \text{const}\,\varepsilon_0^Q$$

and therefore, by Theorem 1.1[ii],

$$\mathfrak{L}^{2n+1}(\mathfrak{B}_r(\xi_0) \cap \{u \geq -\theta_1\}) \geq \text{const}\, r^Q\,, \tag{5.1}$$

for r as in the statement of Theorem 1.3. On the other hand, using Theorem 1.1[i], we have that

$$\mathfrak{L}^{2n+1}(\mathfrak{B}_r(\xi_0) \cap \{-\theta_1 \leq u \leq \theta_0\}) \leq \text{const}\, r^{Q-1}\,. \tag{5.2}$$

By collecting the estimates in (5.1) and (5.2), we deduce that

$$\begin{aligned} \mathfrak{L}^{2n+1}(\mathfrak{B}_r(\xi_0) \cap \{u > \theta_0\}) &\geq \text{const}\, r^Q - \text{const}\, r^{Q-1} \geq \\ &\geq \text{const}\, r^Q\,, \end{aligned} \tag{5.3}$$

as long as r_0 is large enough.

Analogously,

$$\mathfrak{L}^{2n+1}(\mathfrak{B}_r(\xi_0) \cap \{u < -\theta_0\}) \geq \text{const}\, r^Q\,. \tag{5.4}$$

Let now $S^{k,p}$ denote the Sobolev-type space in the Heisenberg group setting (see, for instance, [FS74]). Since u is in $S^{1,2}(\mathfrak{B}_r)$ by construction (and so in $S^{1,1}(\mathfrak{B}_r)$) and it is continuous by [M95], we can find a smooth function v in such a way

$$\int_{\mathfrak{B}_r(\xi_0)} |\nabla_{\mathbb{H}^n} u - \nabla_{\mathbb{H}^n} v| \leq 1 \tag{5.5}$$

and

$$|u(\xi) - v(\xi)| \leq \frac{\theta_0}{4} \text{ for any } \xi \in \mathfrak{B}_r(\xi_0). \tag{5.6}$$

Let also

$$w(\xi) := \begin{cases} v(\xi) & \text{if } |v(\xi)| \le \theta_0/2, \\ \theta_0/2 & \text{if } v(\xi) \ge \theta_0/2, \\ -\theta_0/2 & \text{if } v(\xi) \le -\theta_0/2. \end{cases}$$

Then, w is a Lipschitz function in the Heisenberg group setting; so, by the Coarea Formula in the Heisenberg group (see (1.4) in [MSC01]),

$$\begin{aligned} & \int_{\mathfrak{B}_r(\xi_0)\cap\{|v|\le\theta_0/2\}} |\nabla_{\mathbb{H}^n} v| = \\ = \ & \int_{\mathfrak{B}_r(\xi_0)} |\nabla_{\mathbb{H}^n} w| = \\ = \ & \int_{-\infty}^{+\infty} \mathrm{Per}_{\mathbb{H}^n}\Big(\{w=t\}\cap\mathfrak{B}_r(\xi_0)\Big)\,dt \ge \\ \ge \ & \int_{-\theta_0/2}^{+\theta_0/2} \mathrm{Per}_{\mathbb{H}^n}\Big(\{v=t\}\cap\mathfrak{B}_r(\xi_0)\Big)\,dt\,, \end{aligned}$$

where "$\mathrm{Per}_{\mathbb{H}^n}$" denotes the surface measure in $\mathbb{H}^n$ (see [MSC01] and [FSSC01] for details).

Consequently, using the Isoperimetric Inequality in the Heisenberg group surface theory (see, e.g., Theorem 3.5 in [FSSC01] and references therein),

$$\begin{aligned} & \int_{\mathfrak{B}_r(\xi_0)\cap\{|v|\le\theta_0/2\}} |\nabla_{\mathbb{H}^n} v| \ \ge \\ \ge \ & \mathrm{const}\int_{-\theta_0/2}^{+\theta_0/2}\Big(\min\Big\{\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0)\cap\{v<t\}\Big)\,, \\ & \mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0)\cap\{v>t\}\Big)\Big\}\Big)^{(Q-1)/Q}\,dt \ \ge \\ \ge \ & \mathrm{const}\,\Big(\min\Big\{\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0)\cap\{v<-\theta_0/2\}\Big)\,, \\ & \mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0)\cap\{v>\theta_0/2\}\Big)\Big\}\Big)^{(Q-1)/Q}\,. \end{aligned}$$

Therefore, using (5.5) and (5.6),

$$\begin{aligned} & 1+\int_{\mathfrak{B}_r(\xi_0)\cap\{|u|\le\theta_0\}} |\nabla_{\mathbb{H}^n} u| \ \ge \\ \ge \ & \mathrm{const}\,\Big(\min\Big\{\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0)\cap\{u<-\theta_0\}\Big)\,, \\ & \mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0)\cap\{u>\theta_0\}\Big)\Big\}\Big)^{(Q-1)/Q}\,. \end{aligned}$$

Accordingly,

$$\int_{\mathfrak{B}_r(\xi_0)\cap\{|u|\le\theta_0\}} |\nabla_{\mathbb{H}^n} u| \ \ge \ \mathrm{const}\,r^{Q-1}\,, \tag{5.7}$$

due to (5.3) and (5.4), provided that r_0 is large enough.

We now fix $A > 0$ suitably big and we use the Cauchy Inequality and Theorem 1.1[i], to deduce from (5.7) that

$$\begin{aligned} A\,\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0)\cap\{|u|\le\theta_0\}\Big) + \frac{\mathrm{const}\, r^{Q-1}}{A} \ge \\ \ge A\,\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_r(\xi_0)\cap\{|u|\le\theta_0\}\Big) + \frac{1}{A}\int_{\mathfrak{B}_r(\xi_0)} |\nabla_{\mathbb{H}^n} u|^2 \ge \\ \ge \int_{\mathfrak{B}_r(\xi_0)\cap\{|u|\le\theta_0\}} |\nabla_{\mathbb{H}^n} u| \ge \\ \ge \mathrm{const}\, r^{Q-1}, \end{aligned}$$

from which the claim in Theorem 1.3 follows by taking A large enough.

6. Proof of Theorem 1.4

6.1. **Notions of periodicity in $\mathbb{H}^n$.** From now on, we impose that F is intrinsically periodic, i.e., that it satisfies (1.3), and we fix $\omega = (\omega_1, \dots, \omega_{2n}) \in \mathbb{Q}^{2n}\setminus\{0\}$.

We now see how ω naturally induces a concept of periodicity for functions on the Heisenberg group:

Definition 6.1. $u : \mathbb{H}^n \to \mathbb{R}$ is a function which is ω-periodic if $u(\xi) = u((k,0)\circ\xi)$ for any $k\in\mathbb{Z}^{2n}$ so that $\omega\cdot k = 0$.

We now suppose that

$$n \ge 2\,, \tag{6.1}$$

the case $n = 1$ being dealt with on page 42

We introduce the vectors $k^j \in \mathbb{Z}^{2n}$, for $j = 1, \dots, 2n$, in order to have an integer base of the $2n$-dimensional lattice generated by ω. For this, let us consider a rational orthogonal base of $\mathbb{Q}^{2n}$ given by the vectors ω, v^1, ..., v^{2n-1}.

By (6.1), it is easy to see[3] that there exists some i and j such that $\mathrm{Im}\,(\overline{v}^i v^j) \neq 0$. Let now $Q\in\mathbb{N}$ be a common multiple of the denominators of the coordinates of the vectors $v^1, \dots, v^{2n-1}, \omega$. Then, the vectors

[3]Indeed, we remark that $\mathrm{Im}\,(\overline{v}^i v^j)$ is just the scalar product of v^j with a vector which we will denote $\bar v^i := (-v^i_2, v^i_1, -v^i_4, v^i_3, \cdots - v^i_{2n}, v^i_{2n-1})$. So, fixed $k = 1, \dots, 2n-1$, we use the base property to write $\bar v^k := \sum_{i=1}^{2n-1} \alpha^k_i v^i + b^k\omega$, for suitable $\alpha^k_i, b^k\in\mathbb{R}$. Thus, if $0 = \mathrm{Im}\,(\overline{v}^i v^j) = \bar v^i\cdot v^j$ for all i and j, then

$$\left\{\begin{array}{c} 0 = \bar v^k\cdot v^1 = \alpha^k_1\,|v^1|^2 \\ \vdots \\ 0 = \bar v^k\cdot v^{2n-1} = \alpha^k_{2n-1}\,|v^{2n-1}|^2 \end{array}\right. .$$

We have thus obtained that $\bar v^k = b^k\omega$ and, by the base property, $b^k\neq 0$. But then one gets the contradiction

$$0 \neq b^1 b^2\,|\omega|^2 = \bar v^1\cdot\bar v^2 = v^1\cdot v^2 = 0\,.$$

$k^j := Qv^j$ for $j = 1, \ldots, 2n-1$ and $k^{2n} := Q\omega$ are the required integer base. Furthermore, by construction,

$$\mathrm{Im}\,(\overline{k}^i k^j) \neq 0\,, \tag{6.2}$$

for some i and j not equal to $2n$.

Due to (6.2), and possibly exchanging k^i with k^j, we thus have that

$$\Theta_{ij} := \mathrm{Im}\,(\overline{k}^i k^j) > 0\,. \tag{6.3}$$

If $m \in \mathbb{R}$, we define $k^{ij}_m := mk^i + k^j$. It follows from (0.2) and (6.3) that

$$(k^{ij}_m, 0) \circ (-mk^i, 0) \circ (-k^j, 0) \circ (z, t) = (z, t + 2m\Theta_{ij})\,.$$

Due to the above relation (taken with $m = 1$) and (6.3), an ω-periodic function is also periodic in the vertical variable, that is

$$u(z, t) = u(z, t + 2\Theta_{ij})\,. \tag{6.4}$$

Of course, this holds true for any i and j such that $\Theta_{ij} \neq 0$. Therefore, if u is ω-periodic, it is also Θ-periodic in the vertical direction, with Θ being the greatest common divisors of the $\Theta_{ij} \neq 0$. Observe that, in particular, $F(\cdot, u)$ is 2-periodic in the vertical direction.

6.2. **Constrained minimizers.** We now fix $M \geq 10$, to be chosen suitably large in the sequel. We seek the minimizers of $\mathcal{F}$ among the functions that are ω-periodic and that satisfy a constraint on the hyperplanes $\{\xi \cdot (\omega, 0) = -M|\omega|\}$ and $\{\xi \cdot (\omega, 0) = M|\omega|\}$.

We define

$$U_\omega := \left\{ \sum_{j=1}^{2n-1} s_j k^j\,,\ \ s_j \in \Big[-\frac{1}{2}, \frac{1}{2}\Big) \right\} \times [-\Theta, \Theta)\,.$$

Geometrically, U_ω is a $2n$-dimensional "cube" (or, better to say, rectangular parallelopipedon) with edges of length $|k^1|, \ldots, |k^{2n-1}|, 2\Theta$.

We now flow U_ω in the ω-direction, by using the Heisenberg group action. That is, we define

$$S_\omega := \bigcup_{\alpha \in \mathbb{R}} (\alpha k^{2n}, 0) \circ U_\omega\,. \tag{6.5}$$

We also introduce the horizontal projection of S_ω:

$$H_\omega := \left\{ \sum_{j=1}^{2n-1} s_j k^j + \alpha k^{2n}\,,\ \ s_j \in \Big[-\frac{1}{2}, \frac{1}{2}\Big)\,,\ \ \alpha \in \mathbb{R} \right\}.$$

Given $z = \sum_{1 \leq j \leq 2n} z_j k^j \in \mathbb{R}^{2n}$, we define

$$\zeta(z) := 2 \sum_{1 \leq j \leq 2n} z_j z_{2n}\, \mathrm{Im}\,(\overline{k}^{2n} k^j)\,.$$

It is easily checked that S_ω may be seen as the union of intervals of length 2Θ in the t-direction, centered at $\zeta(z)$, when z varies in H_ω, that is

$$S_\omega = \Big\{(z,\, t+\zeta(z))\,,\ \ z\in H_\omega\,,\ \ t\in[-\Theta,\Theta)\Big\}\,. \tag{6.6}$$

Fixed $\delta\in(0,1/2)$, let

$$\begin{aligned}\mathcal{Y}_M \ := \ \ & \Big\{u\in S^{1,2}_{\rm loc}\,,\ \ \omega-\text{periodic},\\ & u(\xi)\ge 1-\delta \text{ for } \xi\cdot(\omega,0)\ge M\,|\omega|,\\ & u(\xi)\le -1+\delta \text{ for } \xi\cdot(\omega,0)\le -M\,|\omega|\Big\}\,.\end{aligned} \tag{6.7}$$

Then, we have the following existence result for constrained minimizers:

Proposition 6.2. *For any $M\ge 1$ there exists an absolute minimum in $\mathcal{Y}_M$ of*

$$\mathcal{F}_{S_\omega}(u)=\int_{S_\omega}|\nabla_{\mathbb{H}^n}u(\xi)|^2+F(\xi,u(\xi))\,d\xi\,.$$

Moreover, if u^* is any absolute minimizer, then

$$\mathcal{F}_{S_\omega}(u^*)\ \le\ c\,\Theta\,|k^1|\,\ldots\,|k^{2n-1}|\,, \tag{6.8}$$

where $c>0$ depends only on the structural constants of F.

Proof. As usual, we use the notation $\xi=(z,t)$ for points in $\mathbb{H}^n$. Set

$$u_0(\xi)\ :=\ \begin{cases}\frac{4\omega}{|\omega|}\cdot z & \text{if } |\frac{4\omega}{|\omega|}\cdot z|\le 1,\\ \ \ 1 & \text{if } \frac{4\omega}{|\omega|}\cdot z\ge 1,\\ -1 & \text{if } \frac{4\omega}{|\omega|}\cdot z\le -1.\end{cases} \tag{6.9}$$

It is easily seen that $u_0\in\mathcal{Y}_M$.

Moreover, since k^{2n} is parallel to ω, we conclude that

$$\begin{aligned}S_\omega\cap\{|u_0|\neq 1\}\ \ &\subseteq\ \ S_\omega\cap\left\{\Big|\frac{\omega}{|\omega|}\cdot z\Big|\le\frac14\right\}=\\ &=\ \ S_\omega\cap\left\{\Big|\frac{k^{2n}}{|k^{2n}|}\cdot z\Big|\le\frac14\right\}\subseteq\\ &\subseteq\ \ \bigcup_{\alpha\in\left[-\frac{1}{4|k^{2n}|},\frac{1}{4|k^{2n}|}\right]}(\alpha k^{2n},0)\circ U_\omega\,.\end{aligned}$$

Thence,

$$\mathcal{F}_{S_\omega}(u_0)\ \le\ c\,\Theta|k^1|\,\ldots\,|k^{2n-1}|\,,$$

for a suitable $c>0$.

Fix now $R>0$ and let

$$D_R\ :=\ \Big\{\xi\,|\,\omega\cdot z\in[-R|\omega|,R|\omega|]\Big\}\cap S_\omega\,.$$

Let also u_k be a minimizing sequence for $\mathcal{F}_{S_\omega}$.

By possibly cutting u_k at the ± 1-levels, which makes the energy decrease, we may assume that

$$|u_k| \le 1\,, \tag{6.10}$$

for any $k \in \mathbb{N}$.

Also, for k sufficiently large,

$$\mathcal{F}_{S_\omega}(u_k) \le \mathcal{F}_{S_\omega}(u_0) < +\infty\,.$$

Hence,

$$\sup_{k\in\mathbb{N}} \|\nabla_{\mathbb{H}^n} u_k\|_{L^2(S_\omega)} \;<\; +\infty\,.$$

This, (6.10) and standard embedding results (see, e.g., [VSCC92]) imply that there exists a suitable function u so that, up to subsequences, $\nabla_{\mathbb{H}^n} u_k$ weakly converges in $L^2(D_R)$ to $\nabla_{\mathbb{H}^n} u$ and u_k converges to u almost everywhere. Therefore, the above mentioned weak convergence gives that

$$\liminf_{k\to+\infty} \int_{S_\omega} |\nabla_{\mathbb{H}^n} u_k|^2 \ge \liminf_{k\to+\infty} \int_{D_R} |\nabla_{\mathbb{H}^n} u_k|^2 \ge \int_{D_R} |\nabla_{\mathbb{H}^n} u|^2\,.$$

Fix now an arbitrarily small $a > 0$. The pointwise convergence of u_k, Fatou's Lemma and the continuity of F when the second variable is in $(-1, 1)$ imply that

$$\liminf_{k\to+\infty} \int_{S_\omega} F(\xi, u_k) \ge \liminf_{k\to+\infty} \int_{\substack{D_R\\ |u|\le 1-a}} F(\xi, u_k) \ge \int_{\substack{D_R\\ |u|\le 1-a}} F(\xi, u)\,.$$

Since $a > 0$ may be taken as small as we wish, we conclude from the Monotone Convergence Theorem that

$$\liminf_{k\to+\infty} \int_{S_\omega} F(\xi, u_k) \ge \int_{D_R} F(\xi, u)\,.$$

As a consequence,

$$\inf_{\mathcal{Y}_M} \mathcal{F}_{S_\omega} = \liminf_{k\to+\infty} \mathcal{F}_{S_\omega}(u_k) \ge \mathcal{F}_{D_R}(u)\,.$$

Since R may be taken arbitrarily large, we conclude the proof. $\square$

We now investigate further properties of the constrained functions in the space $\mathcal{Y}_M$.

Lemma 6.3. *Let u, $v \in \mathcal{Y}_M$. Then,*

$$\min\{u, v\}\,,\ \max\{u, v\} \in \mathcal{Y}_M$$

and

$$\mathcal{F}_{S_\omega}(u) + \mathcal{F}_{S_\omega}(v) = \mathcal{F}_{S_\omega}(\min\{u, v\}) + \mathcal{F}_{S_\omega}(\max\{u, v\})\,,$$

provided that the above quantities are finite.

Proof. The first claim is obvious. For the proof of the second statement, just split the domain of integration S_ω into $S_\omega \cap \{u \le v\}$ and $S_\omega \cap \{u > v\}$ and compute. $\square$

We denote by $\mathcal{M}_M$ the set of constrained minimizers given by Proposition 6.2 (for a fixed ω, which is omitted in the notation).

Lemma 6.4. *If u, $v \in \mathcal{M}_M$, then so do* $\min\{u, v\}$ *and* $\max\{u, v\}$.

Proof. Since u, $v \in \mathcal{M}_M$,

$$m := \mathcal{F}_{S_\omega}(u) = \mathcal{F}_{S_\omega}(v) \leq \mathcal{F}_{S_\omega}(w)\,,$$

for any $w \in \mathcal{Y}_M$. But, by Lemma 6.3, both $\min\{u, v\}$ and $\max\{u, v\}$ are in $\mathcal{Y}_M$ and therefore

$$\mathcal{F}_{S_\omega}(\min\{u, v\}) \geq m$$

and

$$\mathcal{F}_{S_\omega}(\max\{u, v\}) \geq m\,.$$

If either

$$\mathcal{F}_{S_\omega}(\min\{u, v\}) > m$$

or

$$\mathcal{F}_{S_\omega}(\max\{u, v\}) > m\,,$$

we would have that

$$\mathcal{F}_{S_\omega}(u) + \mathcal{F}_{S_\omega}(v) = 2m < \mathcal{F}_{S_\omega}(\min\{u, v\}) + \mathcal{F}_{S_\omega}(\max\{u, v\})\,,$$

which is in contradiction with Lemma 6.3. Therefore,

$$\mathcal{F}_{S_\omega}(\min\{u, v\}) = \mathcal{F}_{S_\omega}(\max\{u, v\}) = m\,,$$

proving that $\min\{u, v\}$ and $\max\{u, v\}$ are in $\mathcal{M}_M$. □

Lemma 6.5. *Let $v_n \in \mathcal{M}_M$, for any $n \in \mathbb{N}$. Then, there exists a subsequence $n_k \to +\infty$ and a function $v \in \mathcal{M}_M$ such that v_{n_k} converges to v uniformly on compact sets.*

Proof. By [M95], v_n is an equicontinuous family. Also, $|v_n| \leq 1$, and thus v_n converges locally uniformly to some v, up to subsequences, by the Theorem of Ascoli. Then, by Fatou's Lemma,

$$\liminf_{n \to +\infty} \int_{S_\omega} F(\xi, v_n(\xi))\, d\xi \;\geq\; \int_{S_\omega} F(\xi, v(\xi))\, d\xi\,. \tag{6.11}$$

Since $\mathcal{Y}_M$ is obviously closed under pointwise limit and $v_n \in \mathcal{Y}_M$, we also have that

$$v \in \mathcal{Y}_M\,. \tag{6.12}$$

Furthermore, since all the v_n's are minimizers, it follows that $\mathcal{F}_{S_\omega}(v_n)$ takes always the same value for any $n \in \mathbb{N}$, say

$$\mathcal{F}_{S_\omega}(v_n) =: m \geq 0\,. \tag{6.13}$$

In particular,

$$\int_{S_\omega} |\nabla_{\mathbb{H}^n} v_n|^2 \leq m\,,$$

thus, up to subsequences, we may assume that $\nabla_{\mathbb{H}^n} v_n$ converges to $\nabla_{\mathbb{H}^n} v$ weakly in L^2. Therefore,

$$\liminf_{n\to+\infty} \int_{S_\omega} |\nabla_{\mathbb{H}^n} v_n(\xi)|^2 \, d\xi \geq \int_{S_\omega} |\nabla_{\mathbb{H}^n} v(\xi)|^2 \, d\xi \, .$$

Combining this with (6.11) and (6.13), we thus get that $\mathcal{F}_{S_\omega}(v) \leq m$. Therefore, by (6.12), we conclude that $v \in \mathcal{M}_M$. □

We now define the minimal minimizer, that is, the pointwise infimum of all the minimizers in $\mathcal{M}_M$:

$$u_M(\xi) := \inf_{u\in\mathcal{M}_M} u(\xi) \, , \tag{6.14}$$

for any $\xi \in \mathbb{H}^n$. Note that u_M is indeed a minimizer, according to the following observation:

Lemma 6.6. $u_M \in \mathcal{M}_M$.

Proof. For any $\xi \in \mathbb{H}^n$, we consider a sequence $u_{n,\xi} \in \mathcal{M}_M$ in such a way

$$\lim_{n\to+\infty} u_{n,\xi}(\xi) = \inf_{u\in\mathcal{M}_M} u(\xi) = u_M(\xi) \, . \tag{6.15}$$

Let us now write the countable set $\mathbb{Q}^{2n+1}$ as $\{\eta_1, \eta_2, \dots\}$. We define

$$v_n(\xi) := \min\{u_{n,\eta_1}(\xi) \, , \, \dots \, , \, u_{n,\eta_n}(\xi)\} \, . \tag{6.16}$$

From Lemma 6.4, we conclude that $v_n \in \mathcal{M}_M$ for any $n \in \mathbb{N}$. Thus, by Lemma 6.5, perhaps passing to a subsequence, we have that

$$v_n \text{ pointwise converges to a suitable } v \tag{6.17}$$

and that

$$v \in \mathcal{M}_M \, . \tag{6.18}$$

We show that

$$u_M \geq v \, . \tag{6.19}$$

The proof of (6.19) is by contradiction. If (6.19) were false, there would exist $\varepsilon_0 > 0$ and $\tilde{\xi} \in \mathbb{H}^n$ such that

$$u_M(\tilde{\xi}) + 6\varepsilon_0 \leq v(\tilde{\xi}) \, .$$

In the light of (6.14), there exist $\tilde{u} \in \mathcal{M}_M$ such that

$$\tilde{u}(\tilde{\xi}) \leq u_M(\tilde{\xi}) + \varepsilon_0$$

and therefore

$$\tilde{u}(\tilde{\xi}) + 5\varepsilon_0 \leq v(\tilde{\xi}) \, .$$

Since both v and $\tilde{u}$ are in $\mathcal{M}_M$, their modulus of continuity can be bounded uniformly, thanks to [M95]. Thus, since $\mathbb{Q}^{2n+1}$ is dense in $\mathbb{H}^n$, there exists $\tilde{m} \in \mathbb{N}$ in such a way

$$|v(\tilde{\xi}) - v(\eta_{\tilde{m}})| \leq \varepsilon_0 \text{ and}$$
$$|\tilde{u}(\tilde{\xi}) - \tilde{u}(\eta_{\tilde{m}})| \leq \varepsilon_0 \, .$$

Therefore,

$$\tilde{u}(\eta_{\tilde{m}}) + 3\varepsilon_0 \le v(\eta_{\tilde{m}}) \,. \tag{6.20}$$

Recalling (6.15) and (6.17), we now take $\tilde{n} \in \mathbb{N}$ in such a way

$$\tilde{n} \ge \tilde{m} \,, \tag{6.21}$$

$$u_{\tilde{n},\eta_{\tilde{m}}}(\eta_{\tilde{m}}) \le \inf_{u \in \mathcal{M}_M} u(\eta_{\tilde{m}}) + \varepsilon_0 \tag{6.22}$$

and

$$v(\eta_{\tilde{m}}) \le v_{\tilde{n}}(\eta_{\tilde{m}}) + \varepsilon_0 \,. \tag{6.23}$$

In particular, from (6.22),

$$u_{\tilde{n},\eta_{\tilde{m}}}(\eta_{\tilde{m}}) \le \tilde{u}(\eta_{\tilde{m}}) + \varepsilon_0$$

and therefore, by (6.20), (6.23), (6.21) and (6.16),

$$u_{\tilde{n},\eta_{\tilde{m}}}(\eta_{\tilde{m}}) + \varepsilon_0 \le v_{\tilde{n}}(\eta_{\tilde{m}}) \le u_{\tilde{n},\eta_{\tilde{m}}}(\eta_{\tilde{m}}) \,,$$

which is a contradiction and thus gives the proof of (6.19).

In fact, by (6.14) and (6.18), we conclude that $u_M \le v$. Therefore, by (6.18) and (6.19),

$$u_M = v \in \mathcal{M}_M \,,$$

as desired. □

We also point out that, since u_M is a minimizer (due to Lemma 6.6), by cutting u_M at the levels ± 1, it easily follows that $|u_M| \le 1$.

We now observe that the minimal minimizer inherits the vertical periodicity of the functional, thus improving the one in (6.4). This fact will be of crucial importance in the proof of Lemma 6.18 later on.

Lemma 6.7. *u_M is 2-periodic in the vertical variable.*

Proof. If $\Theta = 1$ there is nothing to prove, otherwise let $u_{M,j}(z,t) := u_M(z, t + 2j)$. Since F is 2-periodic in the vertical variable and $\nabla_{\mathbb{H}^n}$ is invariant with respect to vertical translations, then $u_{M,j}$ is also a minimizer for any $j = 0, \dots, \Theta - 1$. Hence $v(\xi) = \min\{u_M(\xi), u_{M,1}(\xi), \dots, u_{M,\Theta-1}(\xi)\}$ is also a minimizer by Lemma 6.4. Hence, by the definitions of u_M and v,

$$u_M \le v \le u_M$$

and so $u_M = v$. But v is 2-periodic in t and thus so is u_M. □

Remark 6.8. Due to Lemma 6.7, without loss of generality, one may suppose that the functions of $\mathcal{Y}_M$ are 2-periodic in the vertical variable, that $\Theta = 1$ and that

$$U_\omega := \left\{ \sum_{j=1}^{2n-1} s_j k^j \,,\ s_j \in \Big[-\frac{1}{2}, \frac{1}{2} \Big) \right\} \times [-1, 1) \,.$$

6.3. **A group action on the space of functions.** Given a function $u : \mathbb{H}^n \longrightarrow \mathbb{R}$ and $k \in \mathbb{Z}^{2n}$, we define

$$T_k u(\xi) := u\Big((k,0)\circ\xi\Big),$$

for any $\xi \in \mathbb{H}^n$.

Lemma 6.9. *Let $k \in \mathbb{Z}^{2n}$. If u is ω-periodic and 2-periodic in the vertical direction, then so is $T_k u$.*

Proof. Let $h \in \mathbb{Z}^{2n}$, with $\omega \cdot h = 0$. Let $a, b \in \mathbb{Z}$ be so that

$$\begin{aligned} (h,0)\circ(k,0) &= (h+k,2a) \qquad \text{and} \\ (k,0)\circ(h,0) &= (h+k,2b)\,. \end{aligned}$$

By the periodicity of u,

$$T_k u\Big((h,0)\circ\xi\Big) = u(h+k+z,t+2b) = u(h+k+z,t+2a) =$$

$$= u\Big((h,0)\circ(k,0)\circ\xi\Big) = u\Big((k,0)\circ\xi\Big) = T_k u(\xi)\,.$$

Thus, u is ω-periodic.

The vertical 2-periodicity is obvious. ☐

We now show that $\mathcal{Y}_M$ behaves nicely with respect to the action T_k:

Lemma 6.10. *Let $k \in \mathbb{Z}^{2n}$ be so that $\omega \cdot k \geq 0$. Let $u \in \mathcal{Y}_M$ be 2-periodic in the vertical direction. Then,*

$$\min\{u,\, T_k u\} \in \mathcal{Y}_M\,.$$

Analogously, if $k \in \mathbb{Z}^{2n}$ is so that $\omega \cdot k \leq 0$ and $u \in \mathcal{Y}_M$, then

$$\max\{u,\, T_k u\} \in \mathcal{Y}_M\,.$$

Proof. We prove the first claim, the second one being analogous. If $\xi\cdot(\omega,0) \leq -M|\omega|$,

$$\min\{u,\, T_k u\}(\xi) \leq u(\xi) \leq -1+\delta\,,$$

because $u \in \mathcal{Y}_M$.

If $\xi\cdot(\omega,0) \geq M|\omega|$, let $\xi' := (k,0)\circ\xi$. Then,

$$\xi'\cdot(\omega,0) = \xi\cdot(\omega,0) + \omega\cdot k \geq M\,|\omega| + 0 = M\,|\omega|\,,$$

and so $u(\xi) \geq 1-\delta$ and $u(\xi') \geq 1-\delta$, since $u \in \mathcal{Y}_M$. Therefore,

$$\min\{u,\, T_k u\}(\xi) = \min\{u(\xi)\,,u(\xi')\} \geq 1-\delta\,.$$

Finally, by Lemma 6.9, $\min\{u,\, T_k u\}$ is ω-periodic. ☐

Given $S \subset \mathbb{H}^n$, we define

$$\begin{aligned} T_k S &:= \{(-k,0)\circ\xi,\ \xi \in S\} \qquad \text{and} \\ T_k\mathcal{Y}_M &:= \{T_k u,\ u \in \mathcal{Y}_M\}\,. \end{aligned}$$

Analogously, we set

$$\mathcal{F}_{T_k S_\omega}(u) := \int_{T_k S_\omega} |\nabla_{\mathbb{H}^n} u(\xi)|^2 + F(\xi, u(\xi))\, d\xi \,.$$

Then, since the functional $\mathcal{F}$ is $\mathbb{Z}^{2n}$-periodic with respect to the Heisenberg group action, then it is invariant under the action of T_k, according to the following observation, whose elementary proof is omitted:

Lemma 6.11. *We have that*

$$\mathcal{F}_{T_k S_\omega}(T_k u) = \mathcal{F}_{S_\omega}(u)\,,$$

for any $k \in \mathbb{Z}^{2n}$.

We will see in Lemma 6.12 here below that Lemma 6.11 may in fact be straightened for ω-periodic functions. For this scope, we need to better investigate the invariance properties of S_ω. First of all, it is obvious from (6.5) that

$$T_{\beta k^{2n}} S_\omega \;=\; S_\omega \,, \tag{6.24}$$

for any $\beta \in \mathbb{R}$. Moreover,

Lemma 6.12. *Let u be ω-periodic. Then, $\mathcal{F}_{S_\omega}(u) = \mathcal{F}_{T_v S_\omega}(u)$, for any $v \in \mathbb{R}^{2n}$.*

Proof. The result is a consequence of the following

Claim: Let $v \in \mathbb{R}^{2n}$ and f be an ω-periodic, integrable function. Then,

$$\int_{T_v S_\omega} f \;=\; \int_{S_\omega} f\,.$$

We observe that, for any fixed $z \in \mathbb{R}^{2n}$, the map $t \mapsto f(z,t)$ is 2Θ-periodic, according to (6.4), and therefore

$$\int_{-\Theta}^{\Theta} f(z, t+\tau)\, dt \;=\; \int_{-\Theta}^{\Theta} f(z,t)\, dt\,, \tag{6.25}$$

for any $\tau \in \mathbb{R}$ and any $z \in \mathbb{R}^{2n}$.

Let also

$$g(z) \;:=\; \int_{-\Theta}^{\Theta} f(z,t)\, dt\,.$$

Exploiting (6.25) and the fact that f is ω-periodic, it follows that

$$\begin{aligned}
g(z+k^j) \;&=\; \int_{-\Theta}^{\Theta} f(z+k^j, t)\, dt = \\
&=\; \int_{-\Theta}^{\Theta} f\Big((k^j, 0) \circ (z,t)\Big)\, dt = \\
&=\; \int_{-\Theta}^{\Theta} f(z,t)\, dt = \\
&=\; g(z)\,,
\end{aligned}$$

for $j=1,\dots,2n-1$.

Consequently,

$$\int_{H_\omega} g(z-v)\,dz = \int_{H_\omega} g(z)\,dz\,,$$

for any $v\in\mathbb{R}^{2n}$.

Therefore, recalling also (6.6), we conclude that

$$\begin{aligned}
\int_{T_v S_\omega} f &= \int_{S_\omega} f\Big((-v,0)\circ\xi\Big)\,d\xi = \\
&= \int_{H_\omega}\int_{-\Theta+\zeta(z)}^{\Theta+\zeta(z)} f\Big(z-v,t+2\,\mathrm{Im}\,(\overline{v}z)\Big)\,dt\,dz = \\
&= \int_{H_\omega}\int_{-\Theta}^{\Theta} f(z-v,t)\,dt\,dz = \\
&= \int_{H_\omega} g(z-v)\,dz = \\
&= \int_{H_\omega} g(z)\,dz = \\
&= \int_{S_\omega} f\,,
\end{aligned}$$

as desired. □

In analogy with Proposition 6.2, we denote by $T_k\mathcal{M}_M$ the set of minimizers of the functional $\mathcal{F}_{T_kS_\omega}$ on the space $T_k\mathcal{Y}_M$.

Lemma 6.13. *We have that $u\in\mathcal{M}_M$ if and only if $T_ku\in T_k\mathcal{M}_M$, for any $k\in\mathbb{Z}^{2n}$.*

Proof. We prove the "only if" part. Let $w\in T_k\mathcal{Y}_M$. Then, $w=T_kv$, for some $v\in\mathcal{Y}_M$. Thus, exploting that $u\in\mathcal{M}_M$ and Lemma 6.11, we conclude that

$$\mathcal{F}_{T_kS_\omega}(T_ku)=\mathcal{F}_{S_\omega}(u)\le\mathcal{F}_{S_\omega}(v)=\mathcal{F}_{T_kS_\omega}(T_kv)=\mathcal{F}_{T_kS_\omega}(w)\,,$$

thence T_ku is a minimizer of $\mathcal{F}_{T_kS_\omega}$ in $T_k\mathcal{Y}_M$. □

Lemma 6.14. *Let $k\in\mathbb{Z}^{2n}$ such that $\omega\cdot k\ge 0$. Let $u\in\mathcal{M}_M$ be 2-periodic in the vertical direction. Then,* $\min\{u,\,T_ku\}\in\mathcal{M}_M$.

Proof. This is a variation of the proof of Lemma 6.4. We provide full details for the facility of the reader. Let $m:=\mathcal{F}_{S_\omega}(u)$. We show that

$$\mathcal{F}_{S_\omega}\Big(\min\{u,\,T_ku\}\Big) = m\,. \tag{6.26}$$

To prove (6.26) suppose, by contradiction, that

$$\mathcal{F}_{S_\omega}\Big(\min\{u,\,T_ku\}\Big) \ne m\,. \tag{6.27}$$

By Lemma 6.10, we know that

$$\min\{u,\, T_k u\} \in \mathcal{Y}_M\,. \tag{6.28}$$

Thus, (6.27) gives that

$$\mathcal{F}_{S_\omega}\Big(\min\{u,\, T_k u\}\Big) \,>\, m\,. \tag{6.29}$$

Let $h := -k$. By Lemma 6.10,

$$\max\{u,\, T_h u\} \in \mathcal{Y}_M$$

and so

$$\max\{T_k u,\, u\} = T_k\Big(\max\{u,\, T_h u\}\Big) \in T_k\mathcal{Y}_M\,.$$

Consequently, from Lemma 6.13,

$$\mathcal{F}_{T_k S_\omega}(T_k u) \,\le\, \mathcal{F}_{T_k S_\omega}\Big(\max\{u,\, T_k u\}\Big)\,.$$

This and Lemma 6.11 give that

$$m \,\le\, \mathcal{F}_{T_k S_\omega}\Big(\max\{u,\, T_k u\}\Big)\,.$$

Therefore, by Lemma 6.12,

$$m \,\le\, \mathcal{F}_{S_\omega}\Big(\max\{u,\, T_k u\}\Big)\,. \tag{6.30}$$

Moreover, using again Lemmata 6.11 and 6.12,

$$\mathcal{F}_{S_\omega}(T_k u) = \mathcal{F}_{T_k S_\omega}(u) = \mathcal{F}_{S_\omega}(u) = m\,. \tag{6.31}$$

Consequently, by (6.29), (6.30) and Lemma 6.3,

$$2m = \mathcal{F}_{S_\omega}(u) + \mathcal{F}_{S_\omega}(T_k u) =$$

$$= \mathcal{F}_{S_\omega}\Big(\min\{u,\, T_k u\}\Big) + \mathcal{F}_{S_\omega}\Big(\max\{u,\, T_k u\}\Big) > 2m\,.$$

This contradiction gives the proof of (6.26).

The desired result thus follows from (6.26) and (6.28). □

6.4. **The Birkhoff property.** In Mather theory, it is quite common to seek orbits which possess some kind of monotonicity, usually referred to as Birkhoff property. Such property has then been extended and used also in [CL98], [CL01] and [RS03].

We now adapt the concept of Birkhoff property, in order to deal with the Heisenberg group.

Definition 6.15. A function $u : \mathbb{H}^n \longrightarrow \mathbb{R}$ is said to satisfy the Birkhoff property with respect to ω if for any $k \in \mathbb{Z}^{2n}$ so that $\omega \cdot k \ge 0$, we have that $T_k u(\xi) \ge u(\xi)$ for any $\xi \in \mathbb{H}^n$.

We observe that, if u satisfies the Birkhoff property and $k \in \mathbb{Z}^{2n}$ is such that $\omega \cdot k \leq 0$, then, writing $k' := -k$ and $\xi' := (k, 0) \circ \xi$, we have $\omega \cdot k' \geq 0$ and thus

$$u(\xi) = T_{k'}(\xi') \geq u(\xi') = T_k u(\xi)\,, \tag{6.32}$$

for any $\xi \in \mathbb{H}^n$.

Lemma 6.16. *Suppose that u verifies the Birkhoff property with respect to ω. Let $\theta \in \mathbb{R}$ and assume that*

$$S \subseteq \{u < \theta\}\,. \tag{6.33}$$

Then,

$$T_k S \subseteq \{u < \theta\}\,,$$

for any $k \in \mathbb{Z}^{2n}$ so that $\omega \cdot k \geq 0$.

Proof. Take k as above and $\xi \in T_k S$. Then, there exists $\eta \in S$ so that $\xi = (h, 0) \circ \eta$, with $h := -k$. Note that $\omega \cdot h \leq 0$, thus, exploiting (6.32) and (6.33), we conclude that

$$\theta > u(\eta) \geq T_h u(\eta) = u((h, 0) \circ \eta) = u(\xi)\,,$$

as desired. □

We show that the minimal minimizer u_M is Birkhoff (with respect to the vector ω in the definition of S_ω):

Lemma 6.17. *u_M satisfies the Birkhoff property.*

Proof. Let $k \in \mathbb{Z}^{2n}$ such that $\omega \cdot k \geq 0$. Then, by Lemmata 6.6, 6.7 and 6.14,

$$\min\{u_M,\, T_k u_M\} \,\in\, \mathcal{M}_M\,.$$

Therefore, by (6.14),

$$u_M \,\leq\, \min\{u_M,\, T_k u_M\} \,\leq\, T_k u_M\,,$$

as desired. □

Here is an interesting geometric property of Birkhoff functions:

Lemma 6.18. *Let $\theta \in \mathbb{R}$, $a, r > 0$. Suppose that u is 2-periodic in the vertical direction, that it satisfies the Birkhoff property with respect to ω and that*

$$\mathfrak{B}_r(\xi_0) \subseteq \{u < \theta\}\,. \tag{6.34}$$

Then, for any $\xi \in \mathbb{H}^n$ such that $(\omega, 0) \cdot (\xi - \xi_0) \leq -a|\omega|$, we have that $u(\xi) < \theta$, as long as r and a are larger than a suitable positive universal constant.

This result somehow guaranties that if, say $u > 1 - \delta$ in a ball, then it is so in a half space.

Proof. Let $\xi = (z,t)$ as requested here above. Denote also $\xi_0 = (z_0, t_0)$. Let us consider the tiling $\mathcal{T}^*$ of $\mathbb{R}^{2n}$ made by standard Euclidean cubes of side 1. More explicitly, $\mathcal{T}^*$ is the collection of the cubes $K + [-1/2, 1/2)^{2n}$, where K varies in $\mathbb{Z}^{2n}$. Let Q^* be the cube of $\mathcal{T}^*$ that contains z_0. Then, there exists $p \in \mathbb{Z}^{2n}$ so that

$$z \in p + Q^* . \tag{6.35}$$

Then,

$$\frac{\omega}{|\omega|} \cdot p \le -a + \sqrt{2n} \le 0 , \tag{6.36}$$

provided that a is large enough.

Also, by (6.35), we get that there exists $w \in Q^*$ such that $z = p + w$. Then, there exists $\tau \in \mathbb{R}$ so that

$$(p, 0) \circ (w, T) = (z, T + \tau) , \tag{6.37}$$

for any $T \in \mathbb{R}$. Thus, let m be the unique integer so that

$$\frac{t - t_0 - \tau - 1}{2} \le m < \frac{t - t_0 - \tau + 1}{2}$$

and define $T := t - \tau - 2m$. Then, $|T - t_0| \le 1$ and so, if r is large enough, $(w, T) \in \mathfrak{B}_r(\xi_0)$.

Moreover, by (6.36), (6.34) and Lemma 6.16,

$$(p, 0) \circ (w, T) \,\in\, T_{-p}\Big(\mathfrak{B}_r(\xi_0)\Big) \,\subseteq\, \{u < \theta\} .$$

Accordingly, by the vertical periodicity and (6.37),

$$\begin{aligned} u(\xi) &= u(z, T + \tau + 2m) = u(z, T + \tau) = \\ &= u\Big((p, 0) \circ (w, T)\Big) < \theta , \end{aligned}$$

as desired. □

6.5. **Koranyi balls outside the interface.** Let us recall that in Theorem 1.1 we introduce a radius r_0 which depends only on the structural data. We now show that we can get a Koranyi ball of universally large radius, say r_0, not touching the interface.

For this, we first need to enlarge our domain S_ω so that the balls of radius r_0 may comfortably fit in. We also need to enlarge our periodicity mesh, in order to obtain local minimizers on page 42.

To these effects, given $p \in \mathbb{N}$, we define

$$\begin{aligned} \mathcal{Y}^p_M \;:=\; & \Big\{ u \in S^{1,2}_{\rm loc} , \quad u\Big((pk, 0) \circ \xi\Big) = u(\xi) = u\Big((0, 2p\Theta) \circ \xi\Big) , \\ & \text{for any } \xi \in \mathbb{H}^n \text{ and } k \in \mathbb{Z}^{2n} \text{ s.t. } \omega \cdot k = 0 , \\ & u(\xi) \ge 1 - \delta \text{ for } \xi \cdot (\omega, 0) \ge M\,|\omega| , \\ & u(\xi) \le -1 + \delta \text{ for } \xi \cdot (\omega, 0) \le -M\,|\omega| \Big\} . \end{aligned} \tag{6.38}$$

Note that $\mathcal{Y}^1_M = \mathcal{Y}_M$, due to (6.4) and (6.7).

Given $v \in \mathbb{R}^{2n+1}$ and $S \subset \mathbb{H}^n$, we define

$$T_v S := \{(-v) \circ \xi\,, \quad \xi \in S\}\,.$$

We also use the following notation. Given $\ell \in \mathbb{Z}^{2n}$ with $\omega \cdot \ell = 0$, we write

$$\ell = \sum_{1 \le j \le 2n-1} \ell^j k^j\,.$$

Then, for $p \ge 1$, we define

$$S^p_\omega := \bigcup_{\substack{\ell \in \mathbb{Z}^{2n},\ \omega\cdot\ell=0 \\ 0 \le \ell^1, \dots \ell^{2n-1} \le p-1 \\ 0 \le m \le p-1}} T_{(0,2m\Theta)} T_{(\ell,0)} S_\omega\,. \tag{6.39}$$

Of course, $S^1_\omega = S_\omega$. Also, if $m \in \mathbb{Z}$ and $\ell \in \mathbb{Z}^{2n}$ in such a way $\omega \cdot \ell = 0$, it can be seen from (6.6) that

$$T_{(0,2m\Theta)} T_{(\ell,0)} S_\omega = \Big\{(z, t - 2m\Theta + \zeta_\ell(z))\,, \quad z \in H_{\omega\ell}\,, t \in [-\Theta, \Theta)\Big\}\,, \tag{6.40}$$

with

$$\begin{aligned} H_{\omega\ell} &:= \Big\{z \in \mathbb{H}^n \mid z + \ell \in H_\omega\Big\} \quad \text{and} \\ \zeta_\ell(z) &:= \zeta(z+\ell) - 2\,\mathrm{Im}\,(\bar{\ell} z)\,. \end{aligned}$$

In particular, it follows that

(6.41) the union in (6.39) is non-overlapping.

We now observe that S^p_ω enjoys a good translation invariance property:

Lemma 6.19. *Let $u \in \mathcal{Y}^p_M$. Then, $\mathcal{F}_{S^p_\omega}(u) = \mathcal{F}_{T_v S^p_\omega}(u)$ for any $v \in \mathbb{R}^{2n+1}$.*

Proof. The proof is a modification of the one of Lemma 6.12. The fact that the Heisenberg group elements do not commute makes the argument quite technical, thence we provide full details for the reader's convenience.

The claim in Lemma 6.19 follows from the following one: if f is an integrable function so that

$$f\Big((pk, 0) \circ \xi\Big) = f(\xi) = f\Big((0, 2p\Theta) \circ \xi\Big) \tag{6.42}$$

for any $\xi \in \mathbb{H}^n$ and any $k \in \mathbb{Z}^{2n}$ so that $\omega \cdot k = 0$, then

$$\int_{T_v S^p_\omega} f = \int_{S^p_\omega} f\,, \tag{6.43}$$

for any $v \in \mathbb{R}^{2n+1}$.

In order to prove (6.43), we deduce from (6.42) that

$$f(z, t + 2\Theta\, p) = f\Big((0, 2\Theta p) \circ (z,t)\Big) = f(z,t)$$

and so, for any fixed $z \in \mathbb{R}^{2n}$, the map $t \mapsto f(z,t)$ is $2\Theta p$-periodic.

Accordingly,

$$\int_{-p\Theta}^{p\Theta} f(z, t+\tau)\, dt = \int_{-p\Theta}^{p\Theta} f(z,t)\, dt\,, \tag{6.44}$$

for any $\tau \in \mathbb{R}$ and any $z \in \mathbb{R}^{2n}$.

We now define

$$g(z) := \int_{-p\Theta}^{p\Theta} f(z,t)\,dt\,.$$

It follows from (6.42) and (6.44) that

$$\begin{aligned} g(z+pk^j) &= \int_{-p\Theta}^{p\Theta} f\Big((pk^j,0)\circ(z,t-2p\,\mathrm{Im}\,(\overline{k}^j z)\Big)\,dt = \\ &= \int_{-p\Theta}^{p\Theta} f\Big(z,t-2p\,\mathrm{Im}\,(\overline{k}^j z)\Big)\,dt = \\ &= \int_{-p\Theta}^{p\Theta} f(z,t)\,dt = \\ &= g(z)\,, \end{aligned}$$

for any $1 \le j \le 2n-1$.

Then, if

$$\begin{aligned} H^p_\omega &:= \bigcup_{\substack{\ell\in\mathbb{Z}^{2n},\ \omega\cdot\ell=0\\ 0\le\ell^j\le p-1}} H_{\omega\ell} = \\ &= \left\{\sum_{j=1}^{2n-1} s_j k^j + \alpha k^{2n}\,,\ \ s_j \in \Big[-\frac{1}{2},\frac{1}{2}+p-1\Big]\,,\ \alpha\in\mathbb{R}\right\}\,, \end{aligned}$$

we conclude that

$$\int_{H^p_\omega} g(z-w)\,dz = \int_{H^p_\omega} g(z)\,dz\,, \tag{6.45}$$

for any $w \in \mathbb{R}^{2n}$.

Take now any $v = (v', v_{2n+1}) \in \mathbb{R}^{2n+1}$. We exploit (6.39), (6.40), (6.41) (6.44) and (6.45) to obtain that

$$\begin{aligned}
&\int_{S^p_\omega} f(v \circ \xi)\, d\xi = \\
&= \sum_{\substack{\ell \in \mathbb{Z}^{2n},\ \omega\cdot\ell=0 \\ 0\le \ell^j \le p-1 \\ 0 \le m \le p-1}} \int_{H_{\omega\ell}} \int_{\zeta_\ell(z)-(2m+1)\Theta}^{\zeta_\ell(z)-(2m-1)\Theta} f(z+v', t+v_{2n+1}+2\,\mathrm{Im}\,(\overline{v}z))\, dt\, dz = \\
&= \sum_{\substack{\ell \in \mathbb{Z}^{2n},\ \omega\cdot\ell=0 \\ 0\le \ell^j \le p-1}} \int_{H_{\omega\ell}} \int_{\zeta_\ell(z)-(2p-1)\Theta}^{\zeta_\ell(z)+\Theta} f(z+v', t+v_{2n+1}+2\,\mathrm{Im}\,(\overline{v}z))\, dt\, dz = \\
&= \sum_{\substack{\ell \in \mathbb{Z}^{2n},\ \omega\cdot\ell=0 \\ 0\le \ell^j \le p-1}} \int_{H_{\omega\ell}} \int_{-p\Theta}^{p\Theta} f(z+v', t)\, dt\, dz = \\
&= \int_{H^p_\omega} \int_{-p\Theta}^{p\Theta} f(z+v', t)\, dz = \\
&= \int_{H^p_\omega} g(z+v')\, dz = \\
&= \int_{H^p_\omega} g(z)\,.
\end{aligned}$$

By applying the above formula to any v and to $v = 0$ we thus conclude that

$$\int_{T_v S^p_\omega} f = \int_{S^p_\omega} f(v \circ \xi)\, d\xi = \int_{H^p_\omega} g(z) = \int_{S^p_\omega} f\,.$$

This concludes the proof of (6.43) (and so the proof of Lemma 6.19). □

We denote by u^p_M the minimal minimizer of $\mathcal{F}_{S^p_\omega}$ under the constraints in $\mathcal{Y}^p_M$. Obviously, $u^1_M = u_M$.

We now show that changing S_ω into S^p_ω does not change the minimal minimizer:

Lemma 6.20. $u^p_M = u_M$, *for any* $p \in \mathbb{N}$, $p \geq 1$.

Proof. Let u be a short hand notation for u^p_M. First, we show that

$$u(z, t + 2\Theta) = u(z, t)\,, \tag{6.46}$$

for any $(z, t) \in \mathbb{H}^n$.

To this effect, given $a \in \mathbb{N}$, we define

$$u_a(\xi) := u((0, 2a\Theta) \circ \xi)\,.$$

It is easily seen that $u_a \in \mathcal{Y}^p_M$. Also, by Lemma 6.19, $\mathcal{F}_{S^p_\omega}(u_a) = \mathcal{F}_{S^p_\omega}(u)$, thence u_a is also a minimizer. Then, as in Lemma 6.3,

$$u^\star := \min\{u\,,\ u_1\,,\ \dots\,,\ u_{p-1}\}$$

is a minimizer too. Since u is the minimal minimizer, $u \le u^\star$. By construction, the converse inequality also holds, thus $u = u^\star$. Since

$$u_{p-1}((2\Theta, 0) \circ \xi) = u((2p\Theta, 0) \circ \xi) = u(\xi)$$

because $u \in \mathcal{Y}^p_M$, we deduce that $u^\star$ (and, then, u) is 2Θ-periodic in t. This proves (6.46).

We now show that

$$u((k, 0) \circ \xi) = u(\xi)\,, \tag{6.47}$$

for any $k \in \mathbb{Z}^{2n}$ so that $\omega \cdot k = 0$. This will be a variation of the proof of (6.46), but some care will be needed due to the fact that the Heisenberg group is non-commutative. For the proof of (6.47), given $\ell \in \mathbb{Z}^{2n}$ so that $\omega \cdot \ell = 0$, we define

$$v_\ell(\xi) \,:=\, u((\ell, 0) \circ \xi)\,.$$

We have that

$$\begin{aligned} v_\ell((0, 2p\Theta) \circ \xi) &= u((\ell, 0) \circ (0, 2p\Theta) \circ \xi) = \\ = u((0, 2p\Theta) \circ (\ell, 0) \circ \xi) &= u((\ell, 0) \circ \xi) = v_\ell(\xi)\,, \end{aligned} \tag{6.48}$$

since $u \in \mathcal{Y}_M$. Moreover, if $K \in \mathbb{Z}^{2n}$ is such that $\omega \cdot K = 0$, we have that

$$\begin{aligned} & (pK, 0) \circ (\ell, 0) = \\ = \;& (0, 4p\,\mathrm{Im}\,(\overline{K}\ell)) \circ (\ell, 0) \circ (pK, 0) = \\ = \;& \Big(0, 4p \sum_{1\le a,b\le 2n-1} K^a \ell^b\, \mathrm{Im}\,(k^a k^b)\Big) \circ (\ell, 0) \circ (pK, 0) = \\ = \;& \Big(0, 4p \sum_{1\le a,b\le 2n-1} K^a \ell^b \Theta_{ab}\Big) \circ (\ell, 0) \circ (pK, 0)\,. \end{aligned} \tag{6.49}$$

Then, since Θ divides any Θ_{ab} (recall the notation on page 21), we gather that

$$(pK, 0) \circ (\ell, 0) = (0, 2p\,\Theta\,\mu) \circ (\ell, 0) \circ (pK, 0)\,,$$

for some $\mu \in \mathbb{Z}$, depending on K and ℓ.

Consequently, using that $u \in \mathcal{Y}^p_M$ and (6.46),

$$\begin{gathered} v_\ell((pK, 0) \circ \xi) = u((\ell, 0) \circ (pK, 0) \circ \xi) = \\ u((pK, 0) \circ (\ell, 0) \circ \xi) = u((\ell, 0) \circ \xi) = v_\ell(\xi)\,. \end{gathered}$$

This and (6.48) imply that $v_\ell \in \mathcal{Y}^p_M$. Moreover, by Lemma 6.19, $\mathcal{F}_{S^p_\omega}(v_\ell) = \mathcal{F}_{S^p_\omega}(u)$, thence v_ℓ is also a minimizer. Then, as pointed out in Lemma 6.3,

$$v^\star := \min\{v_\ell\,,\ 0 \le \ell^1, \dots, \ell^{2n-1} \le p-1\}$$

is a minimizer too. Since u is the minimal minimizer, $u \le v^\star$. By construction, the converse inequality also holds, thus $u = v^\star$.

We now introduce the following notation. Given $r \in \mathbb{Z}$, we set $[r]_p \in p\mathbb{Z}$ and $\{r\}_p \in \{1, \dots, p-1\}$ in such a way $r = [r]_p + \{r\}_p$.

Let now $K \in \mathbb{Z}^{2n}$ be so that $\omega \cdot K = 0$. By arguing as in (6.49), we have that

$$\begin{aligned}&(\ell, 0) \circ (K, 0) = \\ &= (2b\Theta, 0) \circ \Big(\sum_{j=1}^{2n-1} [(\ell^j + K^j)]_p \, k^j, 0 \Big) \circ \Big(\sum_{j=1}^{2n-1} \{(\ell^j + K^j)\}_p \, k^j, 0 \Big)\end{aligned}$$

and so

$$v_\ell((K, 0) \circ \xi) = u\Big(\Big(\sum_{j=1}^{2n-1} \{(\ell^j + K^j)\}_p \, k^j, 0 \Big) \circ \xi \Big),$$

since $u \in \mathcal{Y}_M$ and thanks to (6.47).

Therefore,

$$\begin{aligned} & v^\star((K, 0) \circ \xi) = \\ = \ & \min \Big\{ u\Big(\Big(\sum_{j=1}^{2n-1} \{(\ell^j + K^j)\}_p \, k^j, 0 \Big) \circ \xi \Big), \\ & \quad 0 \le \ell^1, \dots, \ell^{2n-1} \le p - 1 \Big\} = \\ = \ & \min \Big\{ u\Big(\Big(\sum_{j=1}^{2n-1} \tilde{\ell}^j k^j, 0 \Big) \circ \xi \Big), \ 0 \le \tilde{\ell}^1, \dots, \tilde{\ell}^{2n-1} \le p - 1 \Big\} = \\ = \ & v^\star(\xi). \end{aligned}$$

This gives that $v^\star$ (and thence u) satisfies (6.47), as desired.

In particular, from (6.46) and (6.47) it follows that $u = u^p_M \in \mathcal{Y}_\omega$, and so $\mathcal{F}_{S_\omega}(u^p_M) \ge \mathcal{F}_{S_\omega}(u_M)$.

On the other hand, it is obvious that $u_M \in \mathcal{Y}_\omega \subseteq \mathcal{Y}^p_\omega$ and, therefore, $\mathcal{F}_{S^p_\omega}(u_M) \ge \mathcal{F}_{S^p_\omega}(u_M)$.

Moreover, if $\nu \in \mathbb{N}$ is the number of the non-overlapping components of the union in (6.39) (see also (6.41)), it follows from Lemma 6.19 that $\mathcal{F}_{S^p_\omega} = \nu \mathcal{F}_{S_\omega}$.

These observations imply that u_M is a minimizer of $\mathcal{F}_{S^p_\omega}$ in $\mathcal{Y}^p_M$ and that u^p_M is a minimizer of $\mathcal{F}_{S_\omega}$ in $\mathcal{Y}_M$. Since both u_M and u^p_M are minimal minimizers, it follows that $u_M \le u^p_M \le u_M$, as desired. □

Due to Lemma 6.20, by possibly replacing S_ω with S^p_ω, we now assume that, given r_0 as in Theorem 1.3, the Koranyi balls of radius $2r_0$ fit inside S_ω. From now on, this suitably large p will be considered a universal constant, as well as r_0.

In this setting, by a counting argument based on the fact that $\mathcal{F}_{S_\omega}$ is bounded independently of M, we find the desired Koranyi ball outside the interface:

Lemma 6.21. *Fix $\delta > 0$. Fix also r_0 suitably large (according to Theorem 1.3). There exists $M_0 > 0$, depending only on δ and on the structural* *constants of $\mathcal{F}$, so that if $M \geq M_0$, then there exists a ball $\mathfrak{B}_\star$ of radius r_0 such that*

$$\mathfrak{B}_\star \subseteq S_\omega \cap \left\{ |(\omega, 0) \cdot \xi| \leq \frac{M\,|\omega|}{10} \right\} \cap \{|u_M| > 1 - \delta\}\,.$$

Proof. Let

$$\Sigma_M := S_\omega \cap \left\{ |(\omega, 0) \cdot \xi| \leq \frac{M\,|\omega|}{10} \right\}\,.$$

We observe that

$$\mathfrak{L}^{2n+1}(\Sigma_M) \geq c_1\, M\, \Theta\, |k^1| \ldots |k^{2n-1}|\,,$$

for a suitable $c_1 > 0$ which may depend on p (but it does not depend on either ω or M). By measure theoretic considerations, we thus deduce that there are at least N Koranyi balls of radius r_0 contained in Σ_M whose dialations by a constant factor κ (to be chosen appropriately large) overlap at most a finite number of times (which is independent of ω and M), with

$$N \geq c_2\, M\, \Theta\, |k^1| \ldots |k^{2n-1}| \tag{6.50}$$

The quantity c_2 may depend on p and r_0, but not on M and ω.

Let us suppose that $\bar{n}$ of these Koranyi balls intersect the set $\{|u_M| < 1 - \delta\}$ and let us count how big $\bar{n}$ can be. For this, if any of the above Koranyi balls, say $\mathfrak{B}$, intersects $\{|u_M| < 1 - \delta\}$ at some point, say ξ_0, then by Theorem 1.3,

$$\mathfrak{L}^{2n+1}\Big(\mathfrak{B}_{r_0}(\xi_o) \cap \{|u_M| \leq 1 - \delta\}\Big) \geq c_3\,, \tag{6.51}$$

for a suitable c_3 which only depends on r_0.

If κ is sufficiently large, $\mathfrak{B}_{r_0}$ is contained in the dialation by a factor κ of $\mathfrak{B}$, that we denote by $\kappa\mathfrak{B}$. This and (6.51) yield that

$$\mathcal{F}_{\kappa\mathfrak{B}}(u_M) \geq c_4\,,$$

with $c_4 > 0$ independent of ω and M. By summing up above all the Koranyi balls intersecting $\{|u_M| < 1 - \delta\}$ (and by using the finite overlapping properties of these balls), we thus conclude that

$$\mathcal{F}_{\Sigma_M}(u_M) \geq c_5 \bar{n}\,,$$

with $c_5 > 0$ independent of ω and M.

The latter estimate and (6.8) imply that

$$\bar{n} \leq c_6\, \Theta\, |k^1| \ldots |k^{2n-1}|\,,$$

where $c_6 > 0$ is independent of ω and M.

This and (6.50) yield that at least one Koranyi ball of the family must be outside $\{|u_M| < 1 - \delta\}$, as long as M is conveniently large. $\square$

6.6. **Unconstrained minimizer.** We are now going to show that, if M is conveniently large, then the constraints in S_ω do not affect the minimal minimizer.

First of all, we show the existence of a strip of universal size, parallel to the constraints and at a universal distance from them, on which the minimal minimizer takes values very close to ± 1.

Lemma 6.22. *Fix $\delta > 0$ and suppose that $M \geq 100$. Suppose also that the thesis of Lemma 6.21 holds. Then, there exists*

$$\lambda \in \left[-\frac{M}{4}, \frac{M}{4} \right] \tag{6.52}$$

such that either $u_M(\xi) > 1 - \delta$ or $u_M(\xi) < -1 + \delta$ for any $\xi = (z, t) \in \mathbb{H}^n$ so that

$$z \cdot \frac{\omega}{|\omega|} \in [\lambda - 1, \, \lambda + 1] \, . \tag{6.53}$$

Proof. By Lemma 6.21, there exists a Koranyi ball of universally large radius contained in $\{|z \cdot w| \leq M/10\} \cap \{|u_M| > 1 - \delta\}$. Then, the result follows from the fact that u_M is continuous, 2-periodic in t and Birkhoff (recall Lemmata 6.7 and 6.17), by applying Lemma 6.18. □

We are now in the position of showing that u_M does not vary if we enlarge the constraints, provided that M is suitably large:

Proposition 6.23. *Suppose that the hypotheses of Lemma 6.22 hold true. Then, $u_{M+a} = u_M$, for any $a \geq 0$.*

Proof. With no loss of generality, we suppose that $\omega_{2n} > 0$. Let H be the set of points $\xi = (z, t) \in \mathbb{H}^n$ satisfying (6.53) of Lemma 6.22, then either $u < -1 + \delta$ or $u > 1 - \delta$ in H.

We show, in fact that

$$u_M < -1 + \delta \text{ in } H. \tag{6.54}$$

To confirm (6.54), we argue by contradiction and suppose that $u_M > 1 - \delta$ in H. Let $j \in \mathbb{N}$. Then, by Lemma 6.17,

$$T_{(0,\ldots,0,j)} u_M(\eta) \geq u_M(\eta) > 1 - \delta \tag{6.55}$$

for any $\eta \in H$. Take now $\xi = (z, t)$ so that $\omega \cdot z \geq (\lambda + 1)|\omega|$. Let j_0 be the unique integer so that

$$j_0 \in \left[\frac{|\omega|}{\omega_{2n}} \Big(-\lambda - 1 + z \cdot \frac{\omega}{|\omega|} \Big), \, \frac{|\omega|}{\omega_{2n}} \Big(-\lambda - 1 + z \cdot \frac{\omega}{|\omega|} \Big) + 1 \right) .$$

Notice that $j_0 \geq 0$ by construction, thence $j_0 \in \mathbb{N}$. Also, if

$$\eta := (0, \ldots, 0, -j_0, 0) \circ \xi \, ,$$

we have that

$$\eta = \Big(z - (0, \ldots, 0, j_0), \, \tau \Big) \, ,$$

for a suitable $\tau \in \mathbb{R}$ and so

$$\left(\frac{\omega}{|\omega|}, 0\right) \cdot \eta \in [\lambda - 1, \lambda + 1],$$

thence $\eta \in H$.

Exploiting (6.55), we thus conclude that

$$u_M(\xi) = T_{(0,\dots,0,j_0)} u_M(\eta) > 1 - \delta .$$

Summarizing, we have just shown that

$$u_M(\xi) > 1 - \delta \tag{6.56}$$

for any $\xi = (z, t) \in \mathbb{H}^n$ such that $\omega \cdot z \geq (\lambda + 1)|\omega|$.

Let now

$$v := T_{(0,\dots,0,-1)} u_M .$$

Then, if $(\omega, 0) \cdot \xi \geq M|\omega|$, we deduce that

$$(\omega, 0) \cdot \Big((0, \dots, 0, -1, 0) \circ \xi\Big) = -\omega_{2n} + (\omega, 0) \cdot \xi \geq -|\omega| + M\,|\omega| \geq (\lambda + 1)|\omega|,$$

due to (6.52).

Therefore, from (6.56),

$$v(\xi) \geq 1 - \delta \tag{6.57}$$

for any $\xi \in \mathbb{H}^n$ so that $(\omega, 0) \cdot \xi \geq M|\omega|$.

On the other hand, if $(\omega, 0) \cdot \xi \leq -M|\omega|$, we see that

$$(\omega, 0) \cdot \Big((0, \dots, 0, -1, 0) \circ \xi\Big) = -\omega_n + (\omega, 0) \cdot \xi \leq M\,|\omega|,$$

and so, from the fact that $u_M \in \mathcal{Y}_M$, we conclude that

$$v(\xi) \leq -1 + \delta \tag{6.58}$$

for any $\xi \in \mathbb{H}^n$ so that $(\omega, 0) \cdot \xi \leq -M|\omega|$.

Then, (6.57), (6.58) Lemma 6.9 and Lemma 6.7 imply that

$$v \in \mathcal{Y}_M . \tag{6.59}$$

Moreover, $\mathcal{F}_{S_\omega}(u_M) = \mathcal{F}_{S_\omega}(v)$, due to Lemmata 6.11 and 6.12. Consequently, by (6.59), $v \in \mathcal{M}_M$ and so, by (6.14), $u_M \leq v$. Since, on the other hand, $u_M \geq v$ by Lemma 6.17, we deduce that $u_M = v$.

Therefore,

$$u_M(\xi) = u_M\Big((0, \dots, 0, j, 0) \circ \xi\Big),$$

for any $j \in \mathbb{Z}$ and any $\xi \in \mathbb{H}^n$. But then, since $u_M \in \mathcal{Y}_M$,

$$1 - \delta \leq \lim_{j \to +\infty} u_M(0, \dots, 0, j, 0) = \lim_{j \to +\infty} u_M(0, \dots, 0, -j, 0) \leq -1 + \delta ,$$

which is a contradiction since we supposed $\delta < 1/2$ (see page 22).

This proves (6.54).

We now show that if $\xi = (z, t) \in \mathbb{H}^n$ is so that $\omega \cdot z \leq (\lambda - 1)|\omega|$ then

$$u_M(\xi) < -1 + \delta . \tag{6.60}$$

Indeed, if ξ is as above, let j be the unique integer so that

$$j \in \left(\frac{|\omega|}{\omega_{2n}} \Big(\lambda - 1 - \frac{\omega}{|\omega|} \cdot z \Big), \, \frac{|\omega|}{\omega_{2n}} \Big(\lambda - 1 - \frac{\omega}{|\omega|} \cdot z \Big) + 1 \right] .$$

Then, $j \geq 0$ by construction and so, by Lemma 6.17,

$$T_{(0,\dots,0,j)} u_M \geq u_M \, . \tag{6.61}$$

Moreover, if

$$\eta := (0, \dots, 0, j, 0) \circ \xi \, , \tag{6.62}$$

we see that

$$\left(\frac{\omega}{|\omega|}, 0 \right) \cdot \eta \in [\lambda - 1, \, \lambda + 1] \, ,$$

that is, $\eta \in H$.

Hence, by (6.54),

$$T_{(0,\dots,0,j)} u_M(\xi) = u_M(\eta) < -1 + \delta \, .$$

This and (6.61) yield the proof of (6.60).

We now show that the lower constraint is irrelevant for u_M. More precisely, given any $a \geq 0$, we show that if v is ω-periodic, $v(\xi) \leq -1 + \delta$ for any $\xi = (z, t) \in \mathbb{H}^n$ so that $\omega \cdot z \leq -(M + a)|\omega|$ and $v(\xi) \geq 1 - \delta$ for any $\xi = (z, t) \in \mathbb{H}^n$ so that $\omega \cdot z \geq M|\omega|$, then

$$\mathcal{F}_{S_\omega}(u_M) \leq \mathcal{F}_{S_\omega}(v) \, . \tag{6.63}$$

To confirm (6.63), take v as above. Let v_a be the minimal minimizer of $\mathcal{F}_{S_\omega}$ in the space of such v's. Then, by (6.60) (applied here with the lower constraint at level $M + a$ instead of level M), we deduce that $v_a(\xi) \leq -1 + \delta$ for any $\xi \in \mathbb{H}^n$ so that $\omega \cdot z \leq (\lambda - 1)|\omega|$. In particular, by (6.52), we have that $v_a \in \mathcal{Y}_M$. Therefore, since u_M is a minimizer in $\mathcal{Y}_M$,

$$\mathcal{F}_{S_\omega}(u_M) \leq \mathcal{F}_{S_\omega}(v_a) \leq \mathcal{F}_{S_\omega}(v) \, ,$$

yielding the proof of (6.63).

We are now in the position of completing the proof of Proposition 6.23 by arguing as follows. Fix $a \geq 0$ and take

$$w \in \mathcal{Y}_{M+a} \, . \tag{6.64}$$

Take also $j \in \mathbb{N}$ so that

$$j > \frac{a \, |\omega|}{\omega_{2n}} \, . \tag{6.65}$$

Let us define

$$\tilde{w} := T_{(0,\dots,0,j)} w \, .$$

Thus, using (6.64) and (6.65), we have that if $\xi \in \mathbb{H}^n$ is so that $(\omega, 0) \cdot \xi \geq M|\omega|$, then $\tilde{w}(\xi) \geq 1 - \delta$, while if $(\omega, 0) \cdot \xi \leq -(M + j + a)|\omega|$, then $\tilde{w}(\xi) \leq -1 + \delta$.

Therefore, from (6.63) (used here with $j+a$ in the place of a), we deduce that

$$\mathcal{F}_{S_\omega}(u_M) \,\leq\, \mathcal{F}_{S_\omega}(\tilde{w})\,.$$

Then, by Lemmata 6.11 and 6.12,

$$\mathcal{F}_{S_\omega}(u_M) \,\leq\, \mathcal{F}_{S_\omega}(w)\,,$$

which completes the proof of Proposition 6.23. □

We now complete the proof of Theorem 1.4 when $n \geq 2$ (recall (6.1)) by arguing as follows. If $u_\omega := u_M$ is the minimal minimizer constructed above, we know by (6.7) and Lemma 6.7 that the periodicity conditions (1.5)–(1.6) and the level set condition (1.4) hold true. Also, u_ω satisfies the monotonicity condition in (1.7), thanks to Lemma 6.17 and Definition 6.15.

To complete the proof of Theorem 1.4 when $n \geq 2$, we need to show that u_ω is a local minimizer in any given ball B. To this extent, we make the following observation. Let $a \in \mathbb{Z}$ and $K_1, \dots K_j \in \mathbb{Z}^{2n}$, with $\omega \cdot K_1 = \cdots = \omega \cdot K_j = 0$. Since

$$(pK_1, 0) \circ (pK_j, 0) \circ (0, 2ap\Theta) \in p\mathbb{Z}^{2n} \times p\Theta\mathbb{Z}\,,$$

we deduce that the distance between the above points and the origin is large with p (unless it is zero). Consequently, the balls obtained from B via the group actions

$$(pK_1, 0) \circ (pK_j, 0) \circ (0, 2ap\Theta)$$

are far apart one from the other, for large p. Therefore, we can extend ϕ in order to fulfill the periodicity in (6.38), that is, there exists $\phi^\star \in \mathcal{Y}^p_M$ so that $\phi^\star = \phi$ in B. Then, possibly enlarging S_ω to s^p_ω as done in Lemma 6.20 (note that this does not change the minimal minimizer), we may suppose that B lies inside S_ω. Then, using Proposition 6.23, we have that $\mathcal{F}_B(u+\phi) \geq \mathcal{F}_B(u)$, for any $\phi \in C^\infty_0(B)$, as desired.

This ends the proof of Theorem 1.4 when $n \geq 2$.

We now deal with the proof of Theorem 1.4 when $n = 1$.

In this case, the vertical periodicity of the minimal minimizer cannot be recovered from the ω-periodicity, since the integer base constructed on page 20 boils down to the two vectors k^1 and $k^2 = Q\omega$ and so (6.2) cannot hold.

To avoid this inconvenience, the vertical periodicity needs to be imposed by brute force. Namely, we substitute the definition in (6.7) with the following:

$$\begin{aligned}\mathcal{Y}_M \;\; := \;\; & \Big\{ u \in S^{1,2}_{\text{loc}}, \\ & u\Big((k^1,0)\circ(z,t)\Big) = u(z,t) = u(z,t+2) \quad \text{for any} (z,t) \in \mathbb{H}^1\,, \\ & u(\xi) \geq 1-\delta \text{ for } \xi \cdot (\omega, 0) \geq M\,|\omega|, \\ & u(\xi) \leq -1+\delta \text{ for } \xi \cdot (\omega, 0) \leq -M\,|\omega| \Big\}\,.\end{aligned} \tag{6.66}$$

Then, the arguments on pages 21—42 go through verbatim.[4] Remarkably, the minimal minimizer obtained in this way is again a local minimizer in any domain (that is, the imposed vertical periodicity does not affect the minimization). This ends the proof of Theorem 1.4 when $n = 1$.

Remark 6.24. Though we do not make an explicit use of this fact, it is interesting to note that if F is independent of t, then so does the minimal minimizer u_M. To confirm this, fix $s \in \mathbb{R}$ and let $w(z,t) := u_M(z, t+s)$. Observe that the map

$$t \longmapsto |\nabla_{\mathbb{H}^n} u_M(z,t)|^2 + F\Big(z, u_M(z,t)\Big)$$

is Θ-periodic for any fixed $z \in \mathbb{R}^{2n}$, due to (6.4). Then, the fact that F does not depend on t and (6.6) imply that

$$\begin{aligned} \mathcal{F}_{S_\omega}(w) &= \int_{H_\omega} \int_{-\Theta}^{\Theta} |\nabla_{\mathbb{H}^n} u_M(z,t+s)|^2 + F\Big(z, u_M(z,t+s)\Big)\, dt\, dz = \\ &= \int_{H_\omega} \int_{-\Theta}^{\Theta} |\nabla_{\mathbb{H}^n} u_M(z,t)|^2 + F\Big(z, u_M(z,t)\Big)\, dt\, dz = \mathcal{F}_{S_\omega}(u_M)\,. \end{aligned}$$

Thence, since $u_M \in \mathcal{M}_M$ by Lemma 6.6, we have that $w \in \mathcal{M}_M$ too. Then, by Lemma 6.4, $\min\{u_M, w\} \in \mathcal{M}_M$. So, by (6.14), $u_M \le \min\{u_M, w\}$, which says that $u_M(z,t) \le u_M(z,t+s)$ for any $s \in \mathbb{R}$. Then, applying the above observation to both s and $-s$, we get that $u_M(z,t) = u_M(z,t+s)$ for any $s \in \mathbb{R}$, that is, u_M does not depend on t.

Accordingly, if F does not depend on t, the minimal minimizer constructed here agrees with the uniformly elliptic one of [V04].

7. Proof of Theorem 1.6

We consider $\omega \in \mathbb{R}^{2n} \setminus \mathbb{Q}^{2n}$ and $\omega_j \in \mathbb{Q}^{2n}$ be a sequence of rational vectors approaching ω. Exploiting Theorem 1.4, we obtain the existence of a function $u_j : \mathbb{H}^n \to [-1, 1]$ which is a local minimizer for $\mathcal{F}$ in any bounded domain, is ω_j-periodic and has the level sets $\{|u_j| \le 1 - \delta\}$ trapped inside a slab normal to $(\omega_j, 0)$ of size M_0. We stress that M_0 may depend on δ and on the structural constants of the problem, but it is independent of j. Then, to complete the proof of Theorem 1.6, we need to show that, perhaps passing to a subsequence, u_j converges locally uniformly to a suitable u which is a local minimizer in any bounded domain.

For this, we observe that $|u_j| \le 1$ by construction and that the modulus of continuity of u_j is uniformly bounded, thanks to [M95]. Therefore, up to subsequences, u_j converges locally uniformly to a suitable u.

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[4] In fact, in this case, the above arguments simplify, since $\Theta = 1$ and u is ω-periodic if and only if

$$u\Big((k^1, 0) \circ \xi\Big) = u(\xi)$$

for any $\xi \in \mathbb{H}^1$. Also, Lemma 6.7 is, of course, a trivial consequence of (6.66).

Given a bounded domain Ω, we now show that u is a local minimizer of $\mathcal{F}_\Omega$. For this, let $\phi \in C_0^\infty(\Omega)$ and $R > 0$ so large that $\Omega \subset \mathfrak{B}_R$.

Observe that, possibly taking disjoint unions of S_ω as in Lemma 6.20 (which does not affect the minimal minimizer), we may assume that $\mathfrak{B}_{R+2}$ sits inside S_ω.

Then, u_j is a local minimizer in $\mathfrak{B}_{R+2}$. Accordingly, there exists a suitable $C_R > 0$ so that

$$\int_{\mathfrak{B}_{R+1}} |\nabla_{\mathbb{H}^n} u_j|^2 \leq C_R \tag{7.1}$$

for any $j \in \mathbb{N}$, due to the Caccioppoli-type estimate in (2.1). Thus, possibly taking subsequences, we have that $\nabla_{\mathbb{H}^n} u_j$ converges to $\nabla_{\mathbb{H}^n} u$ weakly in $L^2(\mathfrak{B}_{R+1})$, due to the embedding results in [VSCC92].

Therefore,

$$\int_{\mathfrak{B}_{R+\alpha}} |\nabla_{\mathbb{H}^n} u|^2 \leq \liminf_{j\to+\infty} \int_{\mathfrak{B}_{R+\alpha}} |\nabla_{\mathbb{H}^n} u_j|^2 , \tag{7.2}$$

for any $\alpha \in [0, 1]$.

Also, by the continuity of F when the second variable is in $(-1, 1)$, we deduce from the uniform convergence of u_j and Fatou's Lemma that

$$\begin{aligned} &\liminf_{j\to+\infty} \int_{\mathfrak{B}_R} F(\xi, u_j(\xi))\, d\xi \geq \\ &\geq \liminf_{j\to+\infty} \int_{\mathfrak{B}_R \cap \{|u|<1\}} F(\xi, u_j(\xi))\, d\xi \geq \\ &\geq \int_{\mathfrak{B}_R \cap \{|u|<1\}} F(\xi, u(\xi))\, d\xi = \\ &= \int_{\mathfrak{B}_R} F(\xi, u(\xi))\, d\xi . \end{aligned} \tag{7.3}$$

By collecting the estimates in (7.2) and (7.3), we thus conclude that

$$\mathcal{F}_{\mathfrak{B}_R}(u) \leq \liminf_{j\to+\infty} \mathcal{F}_{\mathfrak{B}_R}(u_j) . \tag{7.4}$$

We now define

$$\varepsilon_j := \sup_{\mathfrak{B}_{R+1}} |u - u_j| . \tag{7.5}$$

By the uniform convergence of u_j, we have that ε_j converges to zero. Moreover, by (7.1) and (7.2),

$$\int_{\mathfrak{B}_{R+1}} |\nabla_{\mathbb{H}^n} u|^2 \leq C_R \tag{7.6}$$

and so, by the absolute continuity of the Lebesgue integral,

$$\lim_{j\to+\infty} \int_{\mathfrak{B}_{R+\varepsilon_j} \setminus \mathfrak{B}_R} |\nabla_{\mathbb{H}^n} u|^2 = 0 . \tag{7.7}$$

We also set

$$\mu_j := \mathfrak{L}^{2n+1}(\mathfrak{B}_{R+\varepsilon_j} \setminus \mathfrak{B}_R), \tag{7.8}$$

thence μ_j also converges to zero.

Let now $\tilde{\tau}_j \in C_0^\infty([0, R+\varepsilon_j])$ so that $0 \le \tilde{\tau}_j \le 1$, $\tau_j(t) = 1$ for any $t \in [0, R]$ and $|\tilde{\tau}_j'| \le 10/\varepsilon_j$. Let τ_j be the radial (with respect to the Koranyi ball) function defined by $\tau_j(\xi) := \tilde{\tau}_j(\rho)$ (recall the notation in (0.1)). Then, by (0.3),

$$|\nabla_{\mathbb{H}^n} \tau_j| \le \frac{\text{const}}{\varepsilon_j}. \tag{7.9}$$

Let also

$$w_j := \tau_j u + (1-\tau_j)u_j + \phi.$$

By construction, $w_j(\xi) = u_j(\xi)$ for any $\xi \in \mathbb{H}^n \setminus \mathfrak{B}_{R+\varepsilon_j}$, and so, by the minimizing property of u_j,

$$\begin{aligned} &\mathcal{F}_{\mathfrak{B}_R}(u_j) + \int_{\mathfrak{B}_{R+\varepsilon_j} \setminus \mathfrak{B}_R} |\nabla_{\mathbb{H}^n} u_j|^2 \le \\ \le &\mathcal{F}_{\mathfrak{B}_{R+\varepsilon_j}}(u_j) \le \\ \le &\mathcal{F}_{\mathfrak{B}_{R+\varepsilon_j}}(w_j) = \\ = &\mathcal{F}_{\mathfrak{B}_R}(w_j) + \mathcal{F}_{\mathfrak{B}_{R+\varepsilon_j} \setminus \mathfrak{B}_R}(w_j) = \\ = &\mathcal{F}_{\mathfrak{B}_R}(u+\phi) + \mathcal{F}_{\mathfrak{B}_{R+\varepsilon_j} \setminus \mathfrak{B}_R}(w_j). \end{aligned} \tag{7.10}$$

On the other hand, exploiting (7.9) and (7.5),

$$\begin{aligned} |\nabla_{\mathbb{H}^n} w_j| &= |\nabla_{\mathbb{H}^n} \tau_j (u-u_j) + \tau_j \nabla_{\mathbb{H}^n} u + (1-\tau_j)\nabla_{\mathbb{H}^n} u_j + \nabla_{\mathbb{H}^n} \phi| \le \\ &\le \frac{C}{\varepsilon_j}|u-u_j| + |\nabla_{\mathbb{H}^n} u| + |\nabla_{\mathbb{H}^n} u_j| + |\nabla_{\mathbb{H}^n} \phi| \\ &\le C + |\nabla_{\mathbb{H}^n} u| + |\nabla_{\mathbb{H}^n} u_j| \end{aligned}$$

for a suitable $C > 0$ which may depend on R and ϕ, but it is independent of j.

Using the above estimate, Cauchy Inequality and (7.8), we obtain that

$$\begin{aligned} &\mathcal{F}_{\mathfrak{B}_{R+\varepsilon_j} \setminus \mathfrak{B}_R}(w_j) \le \\ &\le \int_{\mathfrak{B}_{R+\varepsilon_j} \setminus \mathfrak{B}_R} \Big(|\nabla_{\mathbb{H}^n} u|^2 + |\nabla_{\mathbb{H}^n} u_j|^2 \Big) + \\ &+ C\sqrt{\mu_j} \left(\sqrt{\int_{\mathfrak{B}_{R+1}} |\nabla_{\mathbb{H}^n} u|^2} + \sqrt{\int_{\mathfrak{B}_{R+1}} |\nabla_{\mathbb{H}^n} u_j|^2} + 1 \right) + \\ &+ C\sqrt{\int_{\mathfrak{B}_{R+1}} |\nabla_{\mathbb{H}^n} u_j|^2 \int_{\mathfrak{B}_{R+\varepsilon_j} \setminus \mathfrak{B}_R} |\nabla_{\mathbb{H}^n} u|^2}. \end{aligned} \tag{7.11}$$

We deduce from (7.1), (7.6), (7.7) and (7.11) that

$$\mathcal{F}_{\mathfrak{B}_{R+\varepsilon_j}\setminus\mathfrak{B}_R}(w_j) \le \delta_j + \int_{\mathfrak{B}_{R+\varepsilon_j}\setminus\mathfrak{B}_R} |\nabla_{\mathbb{H}^n} u_j|^2 ,$$

for a suitable δ_j converging to zero.

Consequently, from (7.10),

$$\mathcal{F}_{\mathfrak{B}_R}(u_j) \le \mathcal{F}_{\mathfrak{B}_R}(u+\phi) + \delta_j . \tag{7.12}$$

By (7.4), (7.12) and the fact that δ_j converges to zero, we thus conclude that

$$\mathcal{F}_{\mathfrak{B}_R}(u) \le \mathcal{F}_{\mathfrak{B}_R}(u+\phi) .$$

Thence, since ϕ vanishes outside Ω,

$$\mathcal{F}_{\Omega}(u) \le \mathcal{F}_{\Omega}(u+\phi) ,$$

which shows that u is a local minimizer for $\mathcal{F}$ in Ω. This completes the proof of Theorem 1.6.

8. Proof of Theorem 1.7

We suppose that $\omega \in \mathbb{Q}^{2n}\setminus\{0\}$ (the general case then follows by a standard limiting argument, see [CL01]).

Let $Q(\xi) := \alpha(\xi)^2$ and, for $N \in \mathbb{N}$,

$$\mathcal{F}_{\Omega}^{(N)}(u) := \int_{\Omega} \frac{1}{N} |\nabla_{\mathbb{H}^n} u(\xi)|^2 + N\, Q(\xi)(1-u(\xi)^2)^2\, d\xi .$$

Let u_N be the minimal minimizer of $\mathcal{F}^{(N)}$, as defined in (6.14), under the constraint in (6.7). In particular,

$$\{|u| < 1-\delta\} \subseteq \{|(\omega,0)\cdot\xi| \le M\} . \tag{8.1}$$

Then, by Lemma 6.17,

$$u_N \text{ satisfies the Birkhoff property.} \tag{8.2}$$

We also set

$$v_N(z,t) := u_N\left(\frac{z}{N}, \frac{t}{N^2}\right) .$$

Then, v_N minimizes

$$\int |\nabla_{\mathbb{H}^n} v(z,t)|^2 + Q\left(\frac{z}{N}, \frac{t}{N^2}\right)\left(1 - v(z,t)^2\right)^2 dz\, dt \tag{8.3}$$

under the constraint that $v_N(\xi) \ge 1-\delta$ if $\xi\cdot(\omega,0) \ge MN$ and $v_N(\xi) \le -1+\delta$ if $\xi\cdot(\omega,0) \le -MN$. Also, if $k \in \mathbb{Z}^{2n}$ is so that $\omega\cdot k = 0$, then $v_N((Nk,0)\circ\xi) = v_N(\xi)$ for any $\xi \in \mathbb{H}^n$.

Notice that the structural constants of the functional in (8.3) (as they appear on page 3) may be bounded independently of N.

Then, by performing the counting argument in Lemma 6.21, we find a Koranyi ball of radius Nr_0 contained in

$$\{|\xi\cdot(\omega,0)| \le MN\} \cap \{|v_N| > 1-\delta\} ,$$

if $M \geq M_0|\omega|$ and M_0 is large enough.

By scaling back, we find a Koranyi ball of radius r_0 contained in

$$\{|\xi \cdot (\omega, 0)| \leq M\} \cap \{|u_N| > 1 - \delta\}.$$

Then, by using (8.2) and the argument in Proposition 6.23, we deduce that u_N is a free minimizer.

Then, possibly taking subsequences, using the results in [MSC01], we have that u_N converges in $L^1_{\rm loc}$ and almost everywhere to a step function $\chi_E - \chi_{\mathbb{H}^n \setminus E}$ and ∂E has minimal surface area with respect to the metric α. This and (8.1) imply that

$$\partial E \subseteq \{|\xi \cdot (\omega, 0)| \leq M\}$$

as long as $M \geq M_0|\omega|$ and M_0 is universally large, ending the proof of Theorem 1.7.

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