Renormalized Electron Mass in Nonrelativistic QED

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27-June-2008

Abstract

Within the framework of nonrelativistic QED, we prove that, for small values of the coupling constant, the energy function, $E_{\vec{P}}$, of a dressed electron is twice differentiable in the momentum $\vec{P}$ in a neighborhood of $\vec{P} = 0$. Furthermore, $\frac{\partial^2 E_{\vec{P}}}{(\partial |\vec{P}|)^2}$ is bounded from below by a constant larger than zero. Our results are proven with the help of *iterative analytic perturbation theory.*

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I Description of the problem, definition of the model, and outline of the proof

In this paper, we study problems connected with the renormalized electron mass in a model of quantum electrodynamics (QED) with nonrelativistic matter. We are interested in rigorously controlling radiative corrections to the electron mass caused by the interaction of the electron with the *soft* modes of the quantized electromagnetic field. The model describing interactions between nonrelativistic, quantum-mechanical charged matter and the quantized radiation field at low energies (i.e., energies smaller than the rest energy of an electron) is the "standard model", see [6]. In this paper, we consider a system consisting of a single spinless electron, described as a non-relativistic particle that is minimally coupled to the quantized radiation field, and photons. Electron spin can easily be included in our description without substantial complications.

The physical system studied in this paper exhibits space translations invariance. The Hamiltonian, H, generating the time evolution, commutes with the vector operator, $\vec{P}$, representing the total momentum of the system, which generates space translations. If an infrared regularization, e.g., an infrared cutoff σ on the photon frequency, is imposed on the interaction Hamiltonian, there exist single-electron or dressed one-electron states, as long as their momentum is smaller than the bare electron mass, m, of the electron. This means that a notion of mass shell in the energy momentum spectrum is meaningful for velocities $|\vec{P}|/m$ smaller than the speed of light c; (with $c \equiv m \equiv 1$ in our units). Vectors $\{\Psi^\sigma\}$ describing dressed one-electron states are normalizable vectors in the Hilbert space $\mathcal{H}$ of pure states of the system. They are characterized as solutions of the equations

$$H^\sigma \Psi^\sigma = E^\sigma_{\vec{P}} \Psi^\sigma \, , \, |\vec{P}| < 1 \tag{I.1}$$

where H^σ is the Hamiltonian with an infrared cutoff σ in the interaction term and $E^\sigma_{\vec{P}}$, the energy of a dressed electron, is a function of the momentum operator $\vec{P}$. If in the joint spectrum of the components of $\vec{P}$ the support of the vector Ψ^σ is contained in a ball centered at the origin and of radius less than $1 \equiv mc$ then Eq. (I.1) has solutions; see [5]. Since $[H, \vec{P}] = 0$, Eq. (I.1) can be studied for the fiber vectors, $\Psi^\sigma_{\vec{P}}$, corresponding to a value, $\vec{P}$, of the total momentum; (both the total momentum operator and points in its spectrum will henceforth be denoted by $\vec{P}$ – without danger of confusion). Thus we consider the equation

$$H^\sigma_{\vec{P}} \Psi^\sigma_{\vec{P}} = E^\sigma_{\vec{P}} \Psi^\sigma_{\vec{P}} \, , \tag{I.2}$$

where $H^\sigma_{\vec{P}}$ is the fiber Hamiltonian at fixed total momentum $\vec{P}$, and $E^\sigma_{\vec{P}}$ is the value of the function $E^\sigma_{\vec{z}}$ at the point $\vec{z} \equiv \vec{P}$. Physically, states $\{\Psi^\sigma\}$ solving Eq. (I.1) describe a freely moving electron in the absence of asymptotic photons.

It is an essential aspect of the "infrared catastrophe" in QED that Eq. (I.1) does not have any normalizable solution in the limit where the infrared cutoff σ tends to zero, and the underlying dynamical picture of a freely moving electron breaks down; see [4]. Nevertheless, the limiting behavior of the function $E^\sigma_{\vec{P}}$ is of great interest for the following reasons.
As long as $\sigma > 0$, a natural definition of the renormalized electron mass, m_r, is given by the formula

$$m_r(\sigma) := \Big[\frac{\partial^2 E^\sigma_{|\vec{P}|}}{(\partial|\vec{P}|)^2}\Big|_{\vec{P}=0}\Big]^{-1} . \tag{I.3}$$

(Note that $E^\sigma_{\vec{P}} \equiv E^\sigma_{|\vec{P}|}$ is invariant under rotations). Equation (I.3) is expected to remain meaningful in the limit $\sigma \to 0$. In particular, the quantity on the R.H.S. of Eq. (I.3) is expected to be positive and bounded from above uniformly in the infrared cutoff σ.
More importantly, one aims at mathematical control of the function

$$m_r(\sigma, \vec{P}) := \Big[\frac{\partial^2 E^\sigma_{|\vec{P}|}}{(\partial|\vec{P}|)^2}\Big]^{-1} \tag{I.4}$$

in a full neighborhood, $\mathcal{S}$, of $\vec{P} = 0$, corresponding to a slowly moving electron; (i.e., in the nonrelativistic regime). When combined with a number of other spectral properties of the Hamiltonian of nonrelativistic QED the condition

$$\frac{\partial^2 E^\sigma_{|\vec{P}|}}{(\partial|\vec{P}|)^2} > 0 \quad , \quad \vec{P} \in \mathcal{S} , \tag{I.5}$$

uniformly in $\sigma > 0$, suffices to yield a consistent scattering picture in the limit when $\sigma \to 0$ in which the electron exhibits *infraparticle* behavior. In fact, (I.5) is a crucial ingredient in the analysis of Compton scattering presented in [12], [4].

Main results
Assuming the coupling constant, α, small enough, the following results follow.
1) The function

$$\Sigma_{|\vec{P}|} := \lim_{\sigma \to 0} \frac{\partial^2 E^\sigma_{|\vec{P}|}}{(\partial|\vec{P}|)^2} \tag{I.6}$$

is well defined for $\vec{P} \in \mathcal{S} := \{\vec{P} \,|\, |\vec{P}| < \frac{1}{3}\}$; furthermore, it is Hölder-continuous in $\vec{P}$.
2) The function

$$E_{\vec{P}} := \lim_{\sigma \to 0} E^{\sigma}_{\vec{P}} \tag{I.7}$$

is twice differentiable in $\vec{P} \in \mathcal{S}$ and

$$\frac{\partial^2 E_{|\vec{P}|}}{(\partial |\vec{P}|)^2} = \Sigma_{|\vec{P}|} \,. \tag{I.8}$$

3)

$$2 > \frac{\partial^2 E^{\sigma}_{|\vec{P}|}}{\partial^2 |\vec{P}|} > 0 \quad , \quad \vec{P} \in \mathcal{S} \,, \tag{I.9}$$

uniformly in σ.

We wish to mention some related earlier results. Using operator-theoretic renormalization group methods, results (I.6) and (I.9) have been proven in [1] for the special value $\vec{P} = 0$. The point $\vec{P} = 0$ is exceptional, because the Hamiltonian $H_{\vec{P}}$ is *infrared regular* at $\vec{P} = 0$; it has a normalizable ground state. A highly non-trivial extension of the analysis of [1] to arbitrary momenta $\vec{P} \in \mathcal{S}$ has been described in [2].
With regard to *ultraviolet* corrections to the electron mass, we refer the reader to [9], [10], [7], and [8].

I.1 Definition of the model

Hilbert space
The Hilbert space of pure state vectors of a system consisting of one non-relativistic electron interacting with the quantized electromagnetic field is given by

$$\mathcal{H} \, := \, \mathcal{H}_{el} \otimes \mathcal{F} \,, \tag{I.10}$$

where $\mathcal{H}_{el} = L^2(\mathbb{R}^3)$ is the Hilbert space for a single Schrödinger electron; for expository convenience, we neglect the spin of the electron. The Hilbert space, $\mathcal{F}$, used to describe the states of the transverse modes of the quantized electromagnetic field (the *photons*) in the Coulomb gauge is given by the Fock space

$$\mathcal{F} \, := \, \bigoplus_{N=0}^{\infty} \mathcal{F}^{(N)} \,, \qquad \mathcal{F}^{(0)} = \mathbb{C}\,\Omega \,, \tag{I.11}$$

where Ω is the vacuum vector (the state of the electromagnetic field without any excited modes), and

$$\mathcal{F}^{(N)} \;:=\; \mathcal{S}_N \bigotimes_{j=1}^{N} \mathfrak{h}\,, \qquad N \geq 1\,, \tag{I.12}$$

where the Hilbert space $\mathfrak{h}$ of state vectors of a single photon is

$$\mathfrak{h} \;:=\; L^2(\mathbb{R}^3 \times \mathbb{Z}_2)\,. \tag{I.13}$$

Here, $\mathbb{R}^3$ is momentum space, and $\mathbb{Z}_2$ accounts for the two independent transverse polarizations (or helicities) of a photon. In (I.12), $\mathcal{S}_N$ denotes the orthogonal projection onto the subspace of $\bigotimes_{j=1}^{N} \mathfrak{h}$ of totally symmetric N-photon wave functions, which accounts for the fact that photons satisfy Bose-Einstein statistics. Thus, $\mathcal{F}^{(N)}$ is the subspace of $\mathcal{F}$ of state vectors corresponding to configurations of exactly N photons.

Units

In this paper, we employ units such that Planck's constant $\hbar$, the speed of light c, and the mass of the electron m are equal to 1.

Hamiltonian

The dynamics of the system is generated by the Hamiltonian

$$H \;:=\; \frac{\Big(-i\vec{\nabla}_{\vec{x}} \,+\, \alpha^{1/2}\vec{A}(\vec{x})\Big)^2}{2} \,+\, H^f\,. \tag{I.14}$$

The (three-component) multiplication operator $\vec{x} \in \mathbb{R}^3$ represents the position of the electron. The electron momentum operator is given by $\vec{p} = -i\vec{\nabla}_{\vec{x}}$. Furthermore, $\alpha > 0$ is the fine structure constant (which, in this paper, plays the rôle of a small parameter), and $\vec{A}(\vec{x})$ denotes the vector potential of the transverse modes of the quantized electromagnetic field in the *Coulomb gauge*,

$$\vec{\nabla}_{\vec{x}} \cdot \vec{A}(\vec{x}) \;=\; 0\,, \tag{I.15}$$

cutoff at high photon frequencies.

H^f is the Hamiltonian of the quantized, free electromagnetic field. It is given by

$$H^f \;:=\; \sum_{\lambda=\pm} \int d^3k\; |\vec{k}|\, a^*_{\vec{k},\lambda}\, a_{\vec{k},\lambda}\,, \tag{I.16}$$

where $a^*_{\vec{k},\lambda}$ and $a_{\vec{k},\lambda}$ are the usual photon creation- and annihilation operators satisfying the canonical commutation relations

$$[a_{\vec{k},\lambda}\,,\,a^*_{\vec{k}',\lambda'}] \;=\; \delta_{\lambda\lambda'}\,\delta(\vec{k}-\vec{k}')\,, \tag{I.17}$$

$$[a^{\#}_{\vec{k},\lambda}\,,\,a^{\#}_{\vec{k}',\lambda'}] \;=\; 0 \tag{I.18}$$

for $\vec{k},\vec{k}' \in \mathbb{R}^3$ and $\lambda,\lambda' \in \mathbb{Z}_2 \equiv \{\pm\}$, where $a^{\#} = a$ or a^*. The vacuum vector $\Omega \in \mathcal{F}$ is characterized by the condition

$$a_{\vec{k},\lambda}\,\Omega \;=\; 0\,, \tag{I.19}$$

for all $\vec{k} \in \mathbb{R}^3$ and $\lambda \in \mathbb{Z}_2 \equiv \{\pm\}$.

The quantized electromagnetic vector potential is given by

$$\vec{A}(\vec{x}) \;:=\; \sum_{\lambda=\pm}\int_{\mathcal{B}_\Lambda}\frac{d^3k}{\sqrt{|\vec{k}|}}\left\{\vec{\varepsilon}_{\vec{k},\lambda}e^{-i\vec{k}\cdot\vec{x}}a^*_{\vec{k},\lambda} \;+\; \vec{\varepsilon}^{\,*}_{\vec{k},\lambda}e^{i\vec{k}\cdot\vec{x}}a_{\vec{k},\lambda}\right\}, \tag{I.20}$$

where $\vec{\varepsilon}_{\vec{k},-}$, $\vec{\varepsilon}_{\vec{k},+}$ are photon polarization vectors, i.e., two unit vectors in $\mathbb{R}^3 \otimes \mathbb{C}$ satisfying

$$\vec{\varepsilon}^{\,*}_{\vec{k},\lambda}\cdot\vec{\varepsilon}_{\vec{k},\mu} \;=\; \delta_{\lambda\mu}\,, \qquad \vec{k}\cdot\vec{\varepsilon}_{\vec{k},\lambda} \;=\; 0\,, \tag{I.21}$$

for $\lambda,\mu = \pm$. The equation $\vec{k}\cdot\vec{\varepsilon}_{\vec{k},\lambda} = 0$ expresses the Coulomb gauge condition. Moreover, $\mathcal{B}_\Lambda$ is a ball of radius Λ centered at the origin in momentum space. Here, Λ represents an *ultraviolet cutoff* that will be kept fixed throughout our analysis. The vector potential defined in (I.20) is thus cut off in the ultraviolet.

Throughout this paper, it will be assumed that $\Lambda \approx 1$ (the rest energy of an electron), and that α is sufficiently small. Under these assumptions, the Hamitonian H is selfadjoint on $D(H^0)$, i.e., on the domain of definition of the operator

$$H^0 \;:=\; \frac{(-i\vec{\nabla}_{\vec{x}})^2}{2} \;+\; H^f\,. \tag{I.22}$$

The perturbation $H - H^0$ is small in the sense of Kato.

The operator representing the total momentum of the system consisting of the electron and the electromagnetic radiation field is given by

$$\vec{P} \;:=\; \vec{p} + \vec{P}^f\,, \tag{I.23}$$

with $\vec{p} = -i\vec{\nabla}_{\vec{x}}$, and where

$$\vec{P}^f \;:=\; \sum_{\lambda=\pm}\int d^3k\;\vec{k}\,a^*_{\vec{k},\lambda}\,a_{\vec{k},\lambda} \tag{I.24}$$

is the momentum operator associated with the photon field.

The operators H and $\vec{P}$ are essentially selfadjoint on a common domain, and since the dynamics is invariant under translations, they commute, $[H, \vec{P}] = \vec{0}$. The Hilbert space $\mathcal{H}$ can be decomposed into a direct integral over the joint spectrum, $\mathbb{R}^3$, of the three components of the momentum operator $\vec{P}$. Their spectral measure is absolutely continuous with respect to Lebesgue measure, and hence we have that

$$\mathcal{H} := \int^{\oplus} \mathcal{H}_{\vec{P}} \, d^3P \,, \tag{I.25}$$

where each fiber space $\mathcal{H}_{\vec{P}}$ is a copy of Fock space $\mathcal{F}$.

Remark *Throughout this paper, the symbol $\vec{P}$ stands for both a vector in $\mathbb{R}^3$ and the vector operator on $\mathcal{H}$, representing the total momentum, depending on context. Similarly, a double meaning is given to arbitrary functions, $f(\vec{P})$, of the total momentum operator.*

Given any $\vec{P} \in \mathbb{R}^3$, there is an isomorphism, $I_{\vec{P}}$,

$$I_{\vec{P}} \, : \, \mathcal{H}_{\vec{P}} \longrightarrow \mathcal{F}^b \,, \tag{I.26}$$

from the fiber space $\mathcal{H}_{\vec{P}}$ to the Fock space $\mathcal{F}^b$, acted upon by the annihilation- and creation operators $b_{\vec{k},\lambda}$, $b^*_{\vec{k},\lambda}$, where $b_{\vec{k},\lambda}$ corresponds to $e^{i\vec{k}\cdot\vec{x}} a_{\vec{k},\lambda}$, and $b^*_{\vec{k},\lambda}$ to $e^{-i\vec{k}\cdot\vec{x}} a^*_{\vec{k},\lambda}$, and with vacuum $\Omega_f := I_{\vec{P}}(e^{i\vec{P}\cdot\vec{x}})$. To define $I_{\vec{P}}$ more precisely, we consider a vector $\psi_{(f^{(n)};\vec{P})} \in \mathcal{H}_{\vec{P}}$ with a definite total momentum describing an electron and n photons. Its wave function in the variables $(\vec{x}; \vec{k}_1, \lambda_1; \dots, \vec{k}_n, \lambda_n)$ is given by

$$e^{i(\vec{P}-\vec{k}_1-\cdots-\vec{k}_n)\cdot\vec{x}} f^{(n)}(\vec{k}_1, \lambda_1; \dots; \vec{k}_n, \lambda_n) \tag{I.27}$$

where $f^{(n)}$ is totally symmetric in its n arguments. The isomorphism $I_{\vec{P}}$ acts by way of

$$\begin{aligned} & I_{\vec{P}}\Big(e^{i(\vec{P}-\vec{k}_1-\cdots-\vec{k}_n)\cdot\vec{x}} f^{(n)}(\vec{k}_{1,1}\, ; \dots ; \vec{k}_n, \lambda_n)\Big) \qquad\qquad \text{(I.28)} \\ & \quad = \frac{1}{\sqrt{n!}} \sum_{\lambda_1,\dots,\lambda_n} \int d^3k_1 \dots d^3k_n \, f^{(n)}(\vec{k}_1, \lambda_1; \dots; \vec{k}_n, \lambda_n) \, b^*_{\vec{k}_1,\lambda_1} \cdots b^*_{\vec{k}_n,\lambda_n} \, \Omega_f \,. \end{aligned}$$

Because the Hamiltonian H commutes with the total momentum, it preserves the fibers $\mathcal{H}_{\vec{P}}$ for all $\vec{P} \in \mathbb{R}^3$, i.e., it can be written as

$$H = \int^{\oplus} H_{\vec{P}} \, d^3P \,, \tag{I.29}$$

where

$$H_{\vec{P}} \,:\, \mathcal{H}_{\vec{P}} \longrightarrow \mathcal{H}_{\vec{P}}\,. \tag{I.30}$$

Written in terms of the operators $b_{\vec{k},\lambda}$, $b^*_{\vec{k},\lambda}$, and of the variable $\vec{P}$, the fiber Hamiltonian $H_{\vec{P}}$ is given by

$$H_{\vec{P}} \,:=\, \frac{\left(\vec{P} - \vec{P}^f + \alpha^{1/2}\vec{A}\right)^2}{2} \,+\, H^f\,, \tag{I.31}$$

where

$$\vec{P}^f \;=\; \sum_{\lambda} \int d^3k\, \vec{k}\, b^*_{\vec{k},\lambda}\, b_{\vec{k},\lambda}\,, \tag{I.32}$$

$$H^f \;=\; \sum_{\lambda} \int d^3k\, |\vec{k}| b^*_{\vec{k},\lambda} b_{\vec{k},\lambda}\,, \tag{I.33}$$

and

$$\vec{A} := \sum_{\lambda} \int_{\mathcal{B}_\Lambda} \frac{dk}{\sqrt{|\vec{k}|}} \left\{\vec{\varepsilon}_{\vec{k},\lambda} b^*_{\vec{k},\lambda} \,+\, \vec{\varepsilon}^{\,*}_{\vec{k},\lambda} b_{\vec{k},\lambda}\right\}. \tag{I.34}$$

Let

$$\mathcal{S} := \{\, \vec{P} \in \mathbb{R}^3 \,:\, |\vec{P}| < \frac{1}{3}\,\}\,. \tag{I.35}$$

In order to give a mathematically precise meaning to the constructions presented in the following, we introduce an infrared cut-off at a photon frequency $\sigma > 0$ in the vector potential. The calculation of the second derivative of the energy of a dressed electron – in the following called the "ground state energy" – as a function of $\vec{P}$ in the limit where $\sigma \longrightarrow 0$, and for $\vec{P} \in \mathcal{S}$, represent the main problem solved in this paper. Hence we will, in the sequel, study the regularized fiber Hamiltonian

$$H^\sigma_{\vec{P}} \,:=\, \frac{\left(\vec{P} - \vec{P}^f + \alpha^{1/2}\vec{A}^\sigma\right)^2}{2} \,+\, H^f\,, \tag{I.36}$$

acting on the fiber space $\mathcal{H}_{\vec{P}}$, for $\vec{P} \in \mathcal{S}$, where

$$\vec{A}^\sigma \,:=\, \sum_{\lambda} \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\sigma} \frac{dk}{\sqrt{|\vec{k}|}} \left\{\vec{\varepsilon}_{\vec{k},\lambda} b^*_{\vec{k},\lambda} \,+\, \vec{\varepsilon}^{\,*}_{\vec{k},\lambda} b_{\vec{k},\lambda}\right\} \tag{I.37}$$

and where $\mathcal{B}_\sigma$ is a ball of radius σ centered at the origin. In the following, we will consider a sequence of infrared cutoffs

$$\sigma_j \,:=\, \Lambda \epsilon^j \tag{I.38}$$

with $0 < \epsilon < \frac{1}{2}$ and $j \in \mathbb{N}$.

I.2 Outline of the proof

Next, we outline the key ideas used in the proofs of our main results in Eqs. (I.6), (I.8), and (I.9).

For $\vec{P} \in \mathcal{S}$, α small enough, and $\sigma > 0$, $E^{\sigma}_{\vec{P}}$ is an isolated eigenvalue of $H^{\sigma}_{\vec{P}}|_{\mathcal{F}_\sigma}$; see Section II and Eq. (II.4). Because of the analyticity of $H^{\sigma}_{\vec{P}}$ in the variable $\vec{P}$, it follows that

$$\begin{aligned}\frac{\partial^2 E^{\sigma}_{|\vec{P}|}}{(\partial|\vec{P}|)^2} &= \partial_i^2 E^{\sigma}_{|\vec{P}|}|_{\vec{P}=P_i\hat{i}} = \qquad \text{(I.39)}\\ &= 1 - 2\langle \frac{1}{2\pi i}\int_{\gamma_\sigma} \frac{1}{H^{\sigma}_{\vec{P}} - z}[\partial_i H^{\sigma}_{\vec{P}}]\frac{1}{H^{\sigma}_{\vec{P}} - z} dz\, \Psi^{\sigma}_{\vec{P}},\, [\partial_i H^{\sigma}_{\vec{P}}]\Psi^{\sigma}_{\vec{P}}\rangle|_{\vec{P}=P_i\hat{i}}\,,\end{aligned}$$

where $\partial_i = \partial/\partial P_i$, $\hat{i}$ is the unit vector in the direction i, $\Psi^{\sigma}_{\vec{P}}$ is the normalized ground state eigenvector of $H^{\sigma}_{\vec{P}}$ constructed in [5]; γ_σ is a contour path in the complex energy plane enclosing $E^{\sigma}_{\vec{P}}$ and no other point of the spectrum of $H^{\sigma}_{\vec{P}}$, and such that the distance of γ_σ from spec $(H^{\sigma}_{\vec{P}})$ is of order σ.

At first glance, the expression on the R.H. S. of (I.39) might become singular as $\sigma \to 0$, because the spectral gap above $E^{\sigma}_{\vec{P}} = \inf \operatorname{spec}(H^{\sigma}_{\vec{P}}|_{\mathcal{F}_\sigma})$ is of order σ. To prove that the limit $\sigma \to 0$ is, in fact, well defined, we make use of a σ-dependent Bogoliubov transformation, $W_\sigma(\vec{\nabla} E^{\sigma}_{\vec{P}})$; (see Section II, Eq. (II.3)). This transformation has already been employed in [5] to analyze mass shell properties. In fact, conjugation of $H^{\sigma}_{\vec{P}}$ by $W_\sigma(\vec{\nabla} E^{\sigma}_{\vec{P}})$ yields an infrared regularized Hamiltonian

$$K^{\sigma}_{\vec{P}} := W_\sigma(\vec{\nabla} E^{\sigma}_{\vec{P}}) H^{\sigma}_{\vec{P}} W^{*}_\sigma(\vec{\nabla} E^{\sigma}_{\vec{P}}) \qquad \text{(I.40)}$$

with the property that the corresponding ground state, $\Phi^{\sigma}_{\vec{P}}$, has a non-zero limit, as $\sigma \to 0$. The Hamiltonian $K^{\sigma}_{\vec{P}}$ has a "canonical form" derived in [5] (see also [11], where a similar operator has been used in the analysis of the Nelson model):

$$K^{\sigma}_{\vec{P}} = \frac{(\vec{\Gamma}^{\sigma}_{\vec{P}})^2}{2} + \sum_{\lambda} \int_{\mathbb{R}^3} |\vec{k}| \delta^{\sigma}_{\vec{P}}(\hat{k}) b^{*}_{\vec{k},\lambda} b_{\vec{k},\lambda} d^3k + \mathcal{E}^{\sigma}_{\vec{P}}\,, \qquad \text{(I.41)}$$

where $\mathcal{E}^{\sigma}_{\vec{P}}$ is a c-number, and $\vec{\Gamma}^{\sigma}_{\vec{P}}$ is a vector operator defined in Section II.1., Eq. (II.40). By construction,

$$\langle \Phi^{\sigma}_{\vec{P}}\,,\, \vec{\Gamma}^{\sigma}_{\vec{P}}\, \Phi^{\sigma}_{\vec{P}} \rangle = 0\,. \qquad \text{(I.42)}$$

This is a crucial property in the inductive construction of the limit of proof of $\{\Phi^{\sigma}_{\vec{P}}\}$ as $\sigma \to 0$.

Eq. (I.42) is also an important ingredient in the proof of (I.6), because, by applying the unitary operator $W_\sigma(\vec{\nabla} E^\sigma_{\vec{P}})$ to each term of the scalar product on the R.H.S. of (I.39) and using (I.42), one finds that

$$(I.39) \;=\; 1 - 2\langle \frac{1}{2\pi i}\int_{\gamma_\sigma} \frac{1}{K^\sigma_{\vec{P}} - z}\,(\Gamma^\sigma_{\vec{P}})^i\,\frac{1}{K^\sigma_{\vec{P}} - z} dz\,\frac{\Phi^\sigma_{\vec{P}}}{\|\Phi^\sigma_{\vec{P}}\|}\,,\,(\Gamma^\sigma_{\vec{P}})^i\,\frac{\Phi^\sigma_{\vec{P}}}{\|\Phi^\sigma_{\vec{P}}\|}\rangle|_{\vec{P}=P^i\hat{i}} \tag{I.43}$$

remains uniformly bounded in σ.
To see this we use the inequality

$$\Big|\Big\langle(\Gamma^\sigma_{\vec{P}})^i\,\Phi^\sigma_{\vec{P}}\,,\,\Big(\frac{1}{K^\sigma_{\vec{P}} - z}\Big)^2(\Gamma^\sigma_{\vec{P}})^i\,\Phi^\sigma_{\vec{P}}\Big\rangle\Big| \leq \mathcal{O}(\frac{1}{\alpha^{\frac{1}{2}}\,\sigma^{2\delta}})\,, \tag{I.44}$$

for an arbitrarily small $\delta > 0$, with $z \in \gamma_\sigma$. This inequality will be proven inductively and will be combined with an improved (as compared to the result in [5]) estimate of the rate of convergence of $\{\Phi^\sigma_{\vec{P}}\}$ as $\sigma \to 0$.
By telescoping, one can plug these improved estimates into (I.43) to end up with the desired uniform bound. The control of the rate of convergence of the R.H.S. in (I.43), as $\sigma \to 0$, combined with the smoothness in $\vec{P}$, for arbitrary infrared cutoff $\sigma > 0$, finally entails the *Hölder-continuity* in $\vec{P}$ of the limiting quantity

$$\Sigma_{|\vec{P}|} \;:=\; 1 - \lim_{\sigma\to 0} 2\langle \frac{1}{2\pi i}\int_{\gamma_\sigma} \frac{1}{K^\sigma_{\vec{P}} - z}\,(\Gamma^\sigma_{\vec{P}})^i\,\frac{1}{K^\sigma_{\vec{P}} - z} dz\,\frac{\Phi^\sigma_{\vec{P}}}{\|\Phi^\sigma_{\vec{P}}\|}\,,\,(\Gamma^\sigma_{\vec{P}})^i\,\frac{\Phi^\sigma_{\vec{P}}}{\|\Phi^\sigma_{\vec{P}}\|}\rangle|_{\vec{P}=P^i\hat{i}}\,. \tag{I.45}$$

Our paper is organized as follows.
In Section II, we recall how to construct the ground states of the Hamiltonians $H^\sigma_{\vec{P}}$ and $K^\sigma_{\vec{P}}$ by *iterative analytic perturbation theory*. This section contains an explicit derivation of the formula of the transformed Hamiltonians and of related algebraic identities that will be used later on.
In Section III, we first derive inequality (I.44) and the improved convergence rate of $\{\Phi^\sigma_{\vec{P}}\}$ as $\sigma \to 0$. Section III.1 is devoted to an analysis of (I.39) that culminates in the following main results.

Theorem

For α small enough, $\frac{\partial^2 E^\sigma_{|\vec{P}|}}{(\partial|\vec{P}|)^2}$ converges as $\sigma \to 0$. The limiting function $\Sigma_{|\vec{P}|} := \lim_{\sigma\to 0}\frac{\partial E^\sigma_{|\vec{P}|}}{(\partial|\vec{P}|)^2}$ is Hölder continuous in $\vec{P} \in \mathcal{S}$. The bounds

$$2 > \Sigma_{|\vec{P}|} > 0 \tag{I.46}$$

(from above and below) hold true uniformly in $\vec{P} \in \mathcal{S}$.

Corollary

For α small enough, the function $E_{\vec{P}} := \lim_{\sigma\to 0} E^{\sigma}_{\vec{P}}$, $\vec{P}\in\mathcal{S}$, is twice differentiable, and

$$\frac{\partial^2 E_{|\vec{P}|}}{(\partial|\vec{P}|)^2} = \Sigma_{|\vec{P}|} \quad \textit{for all} \quad \vec{P}\in\mathcal{S}\,. \tag{I.47}$$

II Sequences of ground state vectors

In this section, we report on results contained in [5] concerning the ground states of the Hamiltonians $H^{\sigma_j}_{\vec{P}}$, where $\vec{P}\in\mathcal{S}$ and $j\in\mathbb{N}$, and the existence of a limiting vector for the sequence of ground state vectors of the transformed Hamiltonians, $K^{\sigma_j}_{\vec{P}}$, where the Bogoliubov transformation used to obtain $K^{\sigma_j}_{\vec{P}}$ from $H^{\sigma_j}_{\vec{P}}$ (derived in [3]) is determined by

$$b^{*}_{\vec{k},\lambda} \;\to\; W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})b^{*}_{\vec{k},\lambda}W^{*}_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) \;=\; b^{*}_{\vec{k},\lambda} - \alpha^{\frac{1}{2}}\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\hat{k})} \tag{II.1}$$

$$b_{\vec{k},\lambda} \;\to\; W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})b_{\vec{k},\lambda}W^{*}_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) \;=\; b_{\vec{k},\lambda} - \alpha^{\frac{1}{2}}\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\epsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\hat{k})}\,, \tag{II.2}$$

for $\vec{k}\in\mathcal{B}_{\Lambda}\setminus\mathcal{B}_{\sigma_j}$, with

$$W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) \;:=\; \exp\Big(\alpha^{\frac{1}{2}}\sum_{\lambda}\int_{\mathcal{B}_{\Lambda}\setminus\mathcal{B}_{\sigma_j}} d^3k\,\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\hat{k})}\cdot(\vec{\epsilon}_{\vec{k},\lambda}b^{*}_{\vec{k},\lambda} - h.c.)\Big)\,. \tag{II.3}$$

In [5], the first step consists in constructing the ground states of the regularized fiber Hamiltonians $H^{\sigma_j}_{\vec{P}}$. As shown in [5], $H^{\sigma_j}_{\vec{P}}$ has a unique ground state, $\Psi^{\sigma_j}_{\vec{P}}$, that can be constructed by *iterative analytic perturbation theory*, as developed in [11]. To recall how this method works some preliminary definitions and results are needed:

- We introduce the Fock spaces

  $$\mathcal{F}_{\sigma_j} := \mathcal{F}^b(L^2((\mathbb{R}^3\backslash\mathcal{B}_{\sigma_j})\times\mathbb{Z}_2)) \quad , \quad \mathcal{F}^{\sigma_j}_{\sigma_{j+1}} := \mathcal{F}^b(L^2((\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}})\times\mathbb{Z}_2))\,. \tag{II.4}$$

  Note that

  $$\mathcal{F}_{\sigma_{j+1}} \;=\; \mathcal{F}_{\sigma_j}\otimes\mathcal{F}^{\sigma_j}_{\sigma_{j+1}}\,. \tag{II.5}$$

  If not specified otherwise, Ω_f denotes the vacuum state in anyone of these Fock spaces. Any vector ϕ in $\mathcal{F}_{\sigma_j}$ can be identified with the corresponding vector, $\phi\otimes\Omega_f$, in $\mathcal{F}$, where Ω_f is the vacuum in $\mathcal{F}^{\sigma_j}_{0}$.

- Momentum-slice interaction Hamiltonians are defined by

$$\Delta H_{\vec{P}}|_{\sigma_{j+1}}^{\sigma_j} := \alpha^{\frac{1}{2}}\, \vec{\nabla}_{\vec{P}} H_{\vec{P}}^{\sigma_j} \cdot \vec{A}|_{\sigma_{j+1}}^{\sigma_j} + \frac{\alpha}{2}\, (A|_{\sigma_{j+1}}^{\sigma_j})^2 \,, \tag{II.6}$$

  where

$$\vec{A}|_{\sigma_{j+1}}^{\sigma_j} := \sum_{\lambda} \int_{\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}} \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{ \vec{\varepsilon}_{\vec{k},\lambda} b^*_{\vec{k},\lambda} \,+\, \vec{\varepsilon}^{\,*}_{\vec{k},\lambda} b_{\vec{k},\lambda} \right\}; \tag{II.7}$$

- Four real parameters, ϵ, ρ^+, ρ^-, and μ, will appear in our analysis. They have the properties

$$0 < \rho^- < \mu < \rho^+ < 1 - C_\alpha < \frac{2}{3} \tag{II.8}$$

$$0 < \epsilon < \frac{\rho^-}{\rho^+} \tag{II.9}$$

  where C_α, with $\frac{1}{3} < C_\alpha < 1$, for α small enough, is a constant such that the inequality

$$E^{\sigma}_{\vec{P}-\vec{k}} > E^{\sigma}_{\vec{P}} - C_\alpha |\vec{k}| \tag{II.10}$$

  holds for all $\vec{P} \in \mathcal{S}$ and any $\vec{k} \neq 0$. Here $E^{\sigma}_{\vec{P}-\vec{k}} := \inf \mathrm{spec} H^{\sigma}_{\vec{P}-\vec{k}}$. We note that $C_\alpha \to \frac{1}{3}$, as $\alpha \to 0$; (see Statement ($\mathcal{S}4$) of Theorem III.1. in [5]).

By iterative analytic perturbation theory (see [5]), one derives the following results, valid for sufficiently small α (depending on our choice of $\Lambda, \epsilon, \rho^-, \mu$, and ρ^+):

1) $E^{\sigma_j}_{\vec{P}}$ is an isolated simple eigenvalue of $H^{\sigma_j}_{\vec{P}}|_{\mathcal{F}_{\sigma_j}}$ with spectral gap larger or equal to $\rho^-\sigma_j$. Furthermore, $E^{\sigma_j}_{\vec{P}}$ is an isolated simple eigenvalue of $H^{\sigma_j}_{\vec{P}}|_{\mathcal{F}_{\sigma_{j+1}}}$ with spectral gap at least $\rho^+\sigma_{j+1}$.

2) The ground-state energies $E^{\sigma_j}_{\vec{P}}$ and $E^{\sigma_{j+1}}_{\vec{P}}$ of the Hamiltonians $H^{\sigma_j}_{\vec{P}}$ and $H^{\sigma_{j+1}}_{\vec{P}}$, respectively, (acting on the same space $\mathcal{F}_{\sigma_{j+1}}$) satisfy the inequalities

$$0 \le E^{\sigma_j}_{\vec{P}} \le E^{\sigma_{j+1}}_{\vec{P}} + c\,\alpha\,\sigma_j^2 \,, \tag{II.11}$$

   where c is a Λ-dependent, but j- and α-*independent* constant.

3) The ground state vectors, $\Psi^{\sigma_{j+1}}_{\vec{P}}$, of $H^{\sigma_{j+1}}_{\vec{P}}$ can be recursively constructed starting from $\Psi^{\sigma_0}_{\vec{P}} \equiv \Omega_f$ with the help of the spectral projection

$$\frac{1}{2\pi i} \oint_{\gamma_{j+1}} dz_{j+1} \frac{1}{H^{\sigma_{j+1}}_{\vec{P}} - z_{j+1}} \,.$$

More precisely,

$$\Psi_{\vec{P}}^{\sigma_{j+1}} := \frac{1}{2\pi i}\oint_{\gamma_{j+1}} dz_{j+1}\frac{1}{H_{\vec{P}}^{\sigma_{j+1}} - z_{j+1}}\Psi_{\vec{P}}^{\sigma_j}\otimes\Omega_f \tag{II.12}$$

$$= \frac{1}{2\pi i}\sum_{j=0}^{\infty}\oint_{\gamma_{j+1}} dz_{j+1}\frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}}[-\Delta H_{\vec{P}}|_{\sigma_{j+1}}^{\sigma_j}\frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}}]^n\,\Psi_{\vec{P}}^{\sigma_j}\otimes\Omega_f\,, \tag{II.13}$$

where $\gamma_{j+1} := \{z_{j+1} \in \mathbb{C}\,|\,|z_{j+1} - E_{\vec{P}}^{\sigma_j}| = \mu\sigma_{j+1}\}$, with μ as in (II.8).

$\Psi_{\vec{P}}^{\sigma_{j+1}}$ is the ground state of $H_{\vec{P}}^{\sigma_{j+1}}|_{\mathcal{F}_\sigma}$ for any $0 \le \sigma \le \sigma_{j+1}$.

II.1 Transformed Hamiltonians.

In this section, we consider the (Bogoliubov-transformed) Hamiltonians

$$K_{\vec{P}}^{\sigma_j} := W_{\sigma_j}(\vec{\nabla}E_{\vec{P}}^{\sigma_j})H_{\vec{P}}^{\sigma_j}W_{\sigma_j}^*(\vec{\nabla}E_{\vec{P}}^{\sigma_j}) \tag{II.14}$$

with ground state vectors $\Phi_{\vec{P}}^{\sigma_j}$, $j = 0, 1, 2, 3, \ldots$. Some algebraic manipulations to express $K_{\vec{P}}^{\sigma_j}$ in a "canonical form" appear to represent a crucial step before iterative perturbation theory can be applied to the sequence of these transformed Hamiltonians. In addition, some intermediate Hamiltonians, denoted $\hat{K}_{\vec{P}}^{\sigma_j}$, must be introduced to arrive at the right kind of convergence estimates.

The same algebraic relations that are used to obtain the "canonical form" of $K_{\vec{P}}^{\sigma_j}$ also play an important role in the proof of our main result concerning the limiting behavior, as $\sigma \to 0$, of the second derivative of the ground state energy $E_{\vec{P}}^{\sigma}$. It is therefore useful to derive the "canonical form" of $K_{\vec{P}}^{\sigma_j}$ and the relevant algebraic identities in some detail.

The Feynman-Hellman formula (which holds because $(H_{\vec{P}}^{\sigma_j})_{\vec{P}\in\mathcal{S}}$ is an analytic family of type A, and $E_{\vec{P}}^{\sigma_j}$ is an isolated eigenvalue) yields the identity

$$\vec{\nabla}E_{\vec{P}}^{\sigma_j} = \vec{P} - \langle\vec{P}^f - \alpha^{\frac{1}{2}}\vec{A}^{\sigma_j}\rangle_{\psi_{\vec{P}}^{\sigma_j}}\,, \tag{II.15}$$

where, given an operator B and a vector ψ in the domain of B, we use the notation

$$\langle B\rangle_\psi := \frac{\langle\psi\,,\,B\,\psi\rangle}{\langle\psi\,,\,\psi\rangle}\,. \tag{II.16}$$

We define

$$\vec{\beta}^{\sigma_j} := \vec{P}^f - \alpha^{\frac{1}{2}} \vec{A}^{\sigma_j} \tag{II.17}$$

$$\delta_{\vec{P}}^{\sigma_j}(\hat{k}) := 1 - \hat{k} \cdot \vec{\nabla} E_{\vec{P}}^{\sigma_j}, \quad \hat{k} := \frac{\vec{k}}{|\vec{k}|} \tag{II.18}$$

$$c_{\vec{k},\lambda}^{*} := b_{\vec{k},\lambda}^{*} + \alpha^{\frac{1}{2}} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}_{\vec{k},\lambda}^{\,*}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} \tag{II.19}$$

$$c_{\vec{k},\lambda} := b_{\vec{k},\lambda} + \alpha^{\frac{1}{2}} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} . \tag{II.20}$$

We rewrite $H_{\vec{P}}^{\sigma_j}$ as

$$H_{\vec{P}}^{\sigma_j} = \frac{(\vec{P} - \vec{\beta}^{\sigma_j})^2}{2} + H^f , \tag{II.21}$$

and, using (II.15) and (II.17),

$$\vec{P} = \vec{\nabla} E_{\vec{P}}^{\sigma_j} + \langle \vec{\beta}^{\sigma_j} \rangle_{\psi_{\vec{P}}^{\sigma_j}} . \tag{II.22}$$

We then obtain

$$\begin{aligned}
H_{\vec{P}}^{\sigma_j} &= \frac{\vec{P}^2}{2} - (\vec{\nabla} E_{\vec{P}}^{\sigma_j} + \langle \vec{\beta}^{\sigma_j} \rangle_{\psi_{\vec{P}}^{\sigma_j}}) \cdot \vec{\beta}^{\sigma_j} + \frac{\vec{\beta}^{\sigma_j\, 2}}{2} + H^f && \text{(II.23)} \\
&= \frac{\vec{P}^2}{2} + \frac{\vec{\beta}^{\sigma_j\, 2}}{2} - \langle \vec{\beta}^{\sigma_j} \rangle_{\psi_{\vec{P}}^{\sigma_j}} \cdot \vec{\beta}^{\sigma_j} && \text{(II.24)} \\
&\quad + \sum_{\lambda} \int_{\mathbb{R}^3 \setminus (\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_j})} |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) b_{\vec{k},\lambda}^{*} b_{\vec{k},\lambda} d^3k && \text{(II.25)} \\
&\quad + \sum_{\lambda} \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_j}} |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) c_{\vec{k},\lambda}^{*} c_{\vec{k},\lambda} d^3k && \text{(II.26)} \\
&\quad - \alpha \sum_{\lambda} \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_j}} |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}_{\vec{k},\lambda}^{\,*}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}_{\vec{k},\lambda}^{\,*}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} d^3k . && \text{(II.27)}
\end{aligned}$$

Adding and subtracting $1/2\,\langle\vec{\beta}^{\sigma_j}\rangle^2_{\psi^{\sigma_j}_{\vec{P}}}$, one finds that

$$
\begin{aligned}
H^{\sigma_j}_{\vec{P}} &= \frac{\vec{P}^2}{2}-\frac{\langle\vec{\beta}^{\sigma_j}\rangle^2_{\psi^{\sigma_j}_{\vec{P}}}}{2}+\frac{(\vec{\beta}^{\sigma_j}-\langle\vec{\beta}^{\sigma_j}\rangle_{\psi^{\sigma_j}_{\vec{P}}})^2}{2} && \text{(II.28)}\\
&+\sum_{\lambda}\int_{\mathbb{R}^3\setminus(\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j})}|\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\hat{k})b^*_{\vec{k},\lambda}b_{\vec{k},\lambda}d^3k && \text{(II.29)}\\
&+\sum_{\lambda}\int_{\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j}}|\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\hat{k})c^*_{\vec{k},\lambda}c_{\vec{k},\lambda}d^3k && \text{(II.30)}\\
&-\alpha\sum_{\lambda}\int_{\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j}}|\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\hat{k})\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\hat{k})}\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\hat{k})}d^3k\,. && \text{(II.31)}
\end{aligned}
$$

Next, we implement the Bogoliubov transformation

$$
\begin{aligned}
b^*_{\vec{k},\lambda} &\rightarrow W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})b^*_{\vec{k},\lambda}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) = b^*_{\vec{k},\lambda}-\alpha^{\frac{1}{2}}\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\hat{k})}\,, && \text{(II.32)}\\
b_{\vec{k},\lambda} &\rightarrow W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})b_{\vec{k},\lambda}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) = b_{\vec{k},\lambda}-\alpha^{\frac{1}{2}}\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\hat{k})}\,, && \text{(II.33)}
\end{aligned}
$$

for $\vec{k}\in\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j}$, where $W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})$ is defined in (II.3). It is evident that W_{σ_j} acts as the identity on $\mathcal{F}^b(L^2(\mathcal{B}_{\sigma_j}\times\mathbb{Z}_2))$ and on $\mathcal{F}^b(L^2((\mathbb{R}^3\setminus\mathcal{B}_\Lambda)\times\mathbb{Z}_2))$.
We define the vector operators

$$
\vec{\Pi}^{\sigma_j}_{\vec{P}} := W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\vec{\beta}^{\sigma_j}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})-\langle W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\vec{\beta}^{\sigma_j}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\rangle_{\Omega_f}\,, \quad \text{(II.34)}
$$

noting that, by (II.15), (II.17), and (II.34)

$$
\begin{aligned}
\langle\vec{\beta}^{\sigma_j}\rangle_{\psi^{\sigma_j}_{\vec{P}}} &= \vec{P}-\vec{\nabla}E^{\sigma_j}_{\vec{P}} && \text{(II.35)}\\
&= \frac{\langle\Phi^{\sigma_j}_{\vec{P}}\,,\,\vec{\Pi}^{\sigma_j}_{\vec{P}}\Phi^{\sigma_j}_{\vec{P}}\rangle}{\langle\Phi^{\sigma_j}_{\vec{P}}\,,\,\Phi^{\sigma_j}_{\vec{P}}\rangle}+\langle W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\vec{\beta}^{\sigma_j}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\rangle_{\Omega_f}\,, && \text{(II.36)}
\end{aligned}
$$

where $\Phi^{\sigma_j}_{\vec{P}}$ is the ground state of the Bogoliubov-transformed Hamiltonian

$$
K^{\sigma_j}_{\vec{P}} := W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})H^{\sigma_j}_{\vec{P}}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\,. \quad \text{(II.37)}
$$

It is easy to see that

$$
W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\vec{\beta}^{\sigma_j}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})-\langle\vec{\beta}^{\sigma_j}\rangle_{\Psi^{\sigma_j}_{\vec{P}}}=\vec{\Pi}^{\sigma_j}_{\vec{P}}-\langle\vec{\Pi}^{\sigma_j}_{\vec{P}}\rangle_{\Phi^{\sigma_j}_{\vec{P}}}\,. \quad \text{(II.38)}
$$

The "canonical form" of $K^{\sigma_j}_{\vec{P}}$ is given by

$$K^{\sigma_j}_{\vec{P}} \;=\; \frac{(\vec{\Gamma}^{\sigma_j}_{\vec{P}})^2}{2} + \sum_\lambda \int_{\mathbb{R}^3} |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\hat{k}) b^*_{\vec{k},\lambda} b_{\vec{k},\lambda} d^3k + \mathcal{E}^{\sigma_j}_{\vec{P}} \,, \tag{II.39}$$

where

$$\vec{\Gamma}^{\sigma_j}_{\vec{P}} \;:=\; \vec{\Pi}^{\sigma_j}_{\vec{P}} - \langle \vec{\Pi}^{\sigma_j}_{\vec{P}} \rangle_{\Phi^{\sigma_j}_{\vec{P}}} \,, \tag{II.40}$$

so that

$$\langle \vec{\Gamma}^{\sigma_j}_{\vec{P}} \rangle_{\Phi^{\sigma_j}_{\vec{P}}} = 0 \,, \tag{II.41}$$

and

$$\mathcal{E}^{\sigma_j}_{\vec{P}} \;:=\; \frac{\vec{P}^2}{2} - \frac{(\vec{P} - \vec{\nabla} E^{\sigma_j}_{\vec{P}})^2}{2} \tag{II.42}$$

$$-\alpha \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_j}} |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\hat{k}) \frac{\vec{\nabla} E^{\sigma_j}_{\vec{P}} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta^{\sigma_j}_{\vec{P}}(\hat{k})} \frac{\vec{\nabla} E^{\sigma_j}_{\vec{P}} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta^{\sigma_j}_{\vec{P}}(\hat{k})} d^3k \,. \tag{II.43}$$

Eq. (II.38) follows by using that

$$W_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) c^*_{\vec{k},\lambda} W_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) \;=\; b^*_{\vec{k},\lambda} \,, \tag{II.44}$$

$$W_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) c_{\vec{k},\lambda} W_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) \;=\; b_{\vec{k},\lambda} \,, \tag{II.45}$$

for $\vec{k} \in \mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_j}$.
An intermediate Hamiltonian, $\hat{K}^{\sigma_{j+1}}_{\vec{P}}$, is defined by

$$\hat{K}^{\sigma_{j+1}}_{\vec{P}} \;:=\; W_{\sigma_{j+1}}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) H^{\sigma_{j+1}}_{\vec{P}} W^*_{\sigma_{j+1}}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) \,, \tag{II.46}$$

where

$$W_{\sigma_{j+1}}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) \;:=\; \exp\Big(\alpha^{\frac{1}{2}} \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_{j+1}}} d^3k \frac{\vec{\nabla} E^{\sigma_j}_{\vec{P}}}{|\vec{k}|^{\frac{3}{2}} \delta^{\sigma_j}_{\vec{P}}(\hat{k})} \cdot (\vec{\epsilon}_{\vec{k},\lambda} b^*_{\vec{k},\lambda} - h.c.)\Big) \,. \tag{II.47}$$

We decompose $\hat{K}^{\sigma_{j+1}}_{\vec{P}}$ into several different terms, similarly as $K^{\sigma_j}_{\vec{P}}$. We recall that

$$H^{\sigma_{j+1}}_{\vec{P}} \;=\; \frac{(\vec{P} - \vec{\beta}^{\sigma_{j+1}})^2}{2} + H^f \,, \tag{II.48}$$

and, by (II.35),

$$\vec{P} \;=\; \vec{\nabla} E^{\sigma_j}_{\vec{P}} + \langle \vec{\beta}^{\sigma_j} \rangle_{\psi^{\sigma_j}_{\vec{P}}} \,. \tag{II.49}$$

It follows that (see also (II.28)-(II.31))

$$
\begin{aligned}
H_{\vec{P}}^{\sigma_{j+1}} &= \frac{\vec{P}^2}{2} - (\vec{\nabla} E_{\vec{P}}^{\sigma_j} + \langle \vec{\beta}^{\sigma_j} \rangle_{\psi_{\vec{P}}^{\sigma_j}}) \cdot \vec{\beta}^{\sigma_{j+1}} + \frac{\vec{\beta}^{\sigma_{j+1}\,2}}{2} + H^f && \text{(II.50)} \\
&= \frac{\vec{P}^2}{2} + \frac{\vec{\beta}^{\sigma_{j+1}\,2}}{2} - \langle \vec{\beta}^{\sigma_j} \rangle_{\psi_{\vec{P}}^{\sigma_j}} \cdot \vec{\beta}^{\sigma_{j+1}} && \text{(II.51)} \\
&\quad + \sum_\lambda \int_{\mathbb{R}^3 \setminus (\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_{j+1}})} |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) b^*_{\vec{k},\lambda} b_{\vec{k},\lambda} d^3k && \text{(II.52)} \\
&\quad + \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_{j+1}}} |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) c^*_{\vec{k},\lambda} c_{\vec{k},\lambda} d^3k && \text{(II.53)} \\
&\quad - \alpha \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_{j+1}}} |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} d^3k \,. && \text{(II.54)}
\end{aligned}
$$

We now add and subtract $1/2\, \langle \vec{\beta}^{\sigma_j} \rangle^2_{\psi_{\vec{P}}^{\sigma_j}}$ and conjugate the resulting operator with the unitary operator $W_{\sigma_{j+1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})$. After inspecting straightforward operator domain questions (see [5]), we find that

$$
\begin{aligned}
\hat{K}_{\vec{P}}^{\sigma_{j+1}} &= \frac{(\vec{\Gamma}_{\vec{P}}^{\sigma_j} + \vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} + \vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j})^2}{2} && \text{(II.55)} \\
&\quad + \sum_\lambda \int_{\mathbb{R}^3} |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) b^*_{\vec{k},\lambda} b_{\vec{k},\lambda} d^3k + \hat{\mathcal{E}}_{\vec{P}}^{\sigma_{j+1}} \,, && \text{(II.56)}
\end{aligned}
$$

where

$$
\begin{aligned}
\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} &:= -\alpha^{\frac{1}{2}} \sum_{\sigma_j} \int_{\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}} \vec{k}\, \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda} b_{\vec{k},\lambda} + h.c.}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} d^3k \,, && \text{(II.57)} \\
&\quad - \alpha^{\frac{1}{2}} \vec{A}|_{\sigma_{j+1}}^{\sigma_j} && \text{(II.58)} \\
\vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j} &:= \alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}} \vec{k}\, \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} d^3k \,, && \text{(II.59)} \\
&\quad + \alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}} [\vec{\epsilon}_{\vec{k},\lambda} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} + h.c.] \frac{d^3k}{\sqrt{|\vec{k}|}} \\
\hat{\mathcal{E}}_{\vec{P}}^{\sigma_{j+1}} &:= \frac{\vec{P}^2}{2} - \frac{(\vec{P} - \vec{\nabla} E_{\vec{P}}^{\sigma_j})^2}{2} && \text{(II.60)} \\
&\quad - \alpha \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_{j+1}}} |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} d^3k \,.
\end{aligned}
$$

We also define the operators

$$\hat{\vec{\Pi}}^{\sigma_j}_{\vec{P}} := W_{\sigma_j}(\vec{\nabla} E^{\sigma_{j-1}}_{\vec{P}}) W^*_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) \vec{\Pi}^{\sigma_j}_{\vec{P}} W_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) W^*_{\sigma_j}(\vec{\nabla} E^{\sigma_{j-1}}_{\vec{P}}) \,, \tag{II.61}$$

and

$$\hat{\vec{\Gamma}}^{\sigma_j}_{\vec{P}} := \hat{\vec{\Pi}}^{\sigma_j}_{\vec{P}} - \langle \hat{\vec{\Pi}}^{\sigma_j}_{\vec{P}} \rangle_{\hat{\Phi}^{\sigma_j}_{\vec{P}}} \,, \tag{II.62}$$

which are used in the proofs of convergence of the ground state vectors. Here, $\hat{\Phi}^{\sigma_j}_{\vec{P}}$ denotes the ground state vector of the Hamiltonian

$$\hat{K}^{\sigma_j}_{\vec{P}} := W_{\sigma_j}(\vec{\nabla} E^{\sigma_{j-1}}_{\vec{P}}) W^*_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) K^{\sigma_j}_{\vec{P}} W_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) W^*_{\sigma_j}(\vec{\nabla} E^{\sigma_{j-1}}_{\vec{P}})$$

An important identity used in [5] and in the sequel of the present paper is

$$\begin{aligned} \hat{\vec{\Gamma}}^{\sigma_j}_{\vec{P}} - \vec{\Gamma}^{\sigma_{j-1}}_{\vec{P}} &= \vec{\nabla} E^{\sigma_j}_{\vec{P}} - \vec{\nabla} E^{\sigma_{j-1}}_{\vec{P}} + \vec{\mathcal{L}}^{\sigma_{j-1}}_{\sigma_j} \\ &\quad + \alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_{j-1}} \setminus \mathcal{B}_{\sigma_j}} \vec{k} \, \frac{\vec{\nabla} E^{\sigma_{j-1}}_{\vec{P}} \cdot \vec{\epsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta^{\sigma_{j-1}}_{\vec{P}}(\hat{k})} \frac{\vec{\nabla} E^{\sigma_{j-1}}_{\vec{P}} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta^{\sigma_{j-1}}_{\vec{P}}(\hat{k})} d^3k \\ &\quad + \alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_{j-1}} \setminus \mathcal{B}_{\sigma_j}} [\vec{\epsilon}_{\vec{k},\lambda} \frac{\vec{\nabla} E^{\sigma_{j-1}}_{\vec{P}} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta^{\sigma_{j-1}}_{\vec{P}}(\hat{k})} + h.c.] \frac{d^3k}{\sqrt{|\vec{k}|}} \,. \end{aligned} \tag{II.63}$$

Eq. (II.63) can be derived using (II.34), (II.36), (II.38), (II.40), (II.61), and (II.62).

II.2 Convergence of the sequence $\{\Phi^{\sigma_j}_{\vec{P}}\}_{j=0}^\infty$.

To pass from momentum scale j to $j+1$, we proceed in two steps: First, we construct an intermediate vector, $\hat{\Phi}^{\sigma_{j+1}}_{\vec{P}}$, defined by

$$\hat{\Phi}^{\sigma_{j+1}}_{\vec{P}} := \sum_{n=0}^\infty \frac{1}{2\pi i} \int_{\gamma_{j+1}} dz_{j+1} \frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}} [-\Delta K_{\vec{P}}|^{\sigma_j}_{\sigma_{j+1}} \frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}}]^n \Phi^{\sigma_j}_{\vec{P}} \,, \tag{II.64}$$

where

$$\begin{aligned} \Delta K_{\vec{P}}|^{\sigma_j}_{\sigma_{j+1}} &:= \hat{K}^{\sigma_{j+1}}_{\vec{P}} - \hat{\mathcal{E}}^{\sigma_{j+1}}_{\vec{P}} + \mathcal{E}^{\sigma_j}_{\vec{P}} - K^{\sigma_j}_{\vec{P}} && \text{(II.65)} \\ &= \frac{1}{2}\Big(\vec{\Gamma}^{\sigma_j}_{\vec{P}} \cdot (\vec{\mathcal{L}}^{\sigma_j}_{\sigma_{j+1}} + \vec{\mathcal{I}}^{\sigma_j}_{\sigma_{j+1}}) + h.c.\Big) + \frac{1}{2}(\vec{\mathcal{L}}^{\sigma_j}_{\sigma_{j+1}} + \vec{\mathcal{I}}^{\sigma_j}_{\sigma_{j+1}})^2 \,. && \text{(II.66)} \end{aligned}$$

Subsequently, we construct $\Phi^{\sigma_{j+1}}_{\vec{P}}$ by setting

$$\Phi^{\sigma_{j+1}}_{\vec{P}} := W_{\sigma_{j+1}}(\vec{\nabla} E^{\sigma_{j+1}}_{\vec{P}}) W^*_{\sigma_{j+1}}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) \hat{\Phi}^{\sigma_{j+1}}_{\vec{P}} \,. \tag{II.67}$$

The series in (II.64) is termwise well-defined and converges strongly to a non-zero vector, provided α is small enough (*independently* of j). The proof of this claim is based on operator-norm estimates of the type used in controlling the Neumann expansion in (II.13), which requires an estimate of the spectral gap that follows from the unitarity of $W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})$ and Result 1) (after Eq. (II.10)).
A key point in our proof of convergence of the sequence $\{\Phi^{\sigma_j}_{\vec{P}}\}$ is to show that the term

$$\vec{\Gamma}^{\sigma_j}_{\vec{P}} \cdot (\vec{\mathcal{L}}^{\sigma_j}_{\sigma_{j+1}} + \vec{\mathcal{I}}^{\sigma_j}_{\sigma_{j+1}}) + h.c. \tag{II.68}$$

appearing in (II.66), which is superficially "marginal" in the infrared, by power counting, is in fact "irrelevant" (using the terminology of renormalization group theory). This is a consequence of the orthogonality condition

$$\langle \Phi^{\sigma_j}_{\vec{P}} \, , \, \vec{\Gamma}^{\sigma_j}_{\vec{P}} \Phi^{\sigma_j}_{\vec{P}} \rangle = 0 \, , \tag{II.69}$$

which, combined with an inductive argument, implies that

$$\| (\frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}})^{\frac{1}{2}} \, [\vec{\Gamma}^{\sigma_j}_{\vec{P}} \cdot (\vec{\mathcal{L}}^{\sigma_j \, (+)}_{\sigma_{j+1}} + \vec{\mathcal{I}}^{\sigma_j}_{\sigma_{j+1}})] \, (\frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}})^{\frac{1}{2}} \, \Phi^{\sigma_j}_{\vec{P}} \| \tag{II.70}$$

(where $\vec{\mathcal{L}}^{\sigma_j \, (+)}_{\sigma_{j+1}}$ stands for the part which contains only photon creation operators) is of order $\mathcal{O}(\epsilon^{\eta j})$, for some $\eta > 0$ specified in [5]. In particular, this suffices to show that

$$\| \hat{\Phi}^{\sigma_{j+1}}_{\vec{P}} - \Phi^{\sigma_j}_{\vec{P}} \| \leq \mathcal{O}(\epsilon^{\frac{j+1}{2}(1-\delta)}) \, . \tag{II.71}$$

II.2.1 Key ingredients

To prove convergence of the sequence of $\{\Psi^{\sigma_j}_{\vec{P}}\}$ of ground state vectors of the Hamiltonians $K^{\sigma_j}_{\vec{P}}$, some further conditions on α, ϵ, and μ are required, in particular an upper bound on μ and an upper bound on ϵ strictly smaller than the ones imposed by inequalities (II.80), (II.81); (see Lemma A.3 in [5]). We note that the more restrictive conditions on μ and ϵ imply new bounds on ρ^- and ρ^+. Moreover, ϵ must satisfy a lower bound $\epsilon > C\alpha^{\frac{1}{2}}$, with a multiplicative constant $C > 0$ sufficiently large.
Some key inequalities needed in our analysis of the convergence properties of $\{\Phi^{\sigma_j}_{\vec{P}}\}$ are summarized below. They will be marked by the symbol ($\mathcal{B}$). In order to reach some important improvements in our estimates of the convergence rate of $\Phi^{\sigma_j}_{\vec{P}}$, as $j \to \infty$ (discussed in the next section), a refined estimate is needed that is stated in ($\mathcal{B}2$), and a new inequality, see ($\mathcal{B}5$), (analogous to ($\mathcal{B}3$) and ($\mathcal{B}4$)) is required.

- *Estimates on the shift of the ground state energy and its gradient* There are constants C_1, C_2' such that the following inequalities hold. ($\mathcal{B}1$)
$$|E^{\sigma_j}_{\vec{P}} - E^{\sigma_{j+1}}_{\vec{P}}| \leq C_1\,\alpha\,\epsilon^j\,; \tag{II.72}$$
see [5].

- ($\mathcal{B}2$)
$$|\vec{\nabla} E^{\sigma_{j+1}}_{\vec{P}} - \vec{\nabla} E^{\sigma_j}_{\vec{P}}| \leq C_2'\Big(\|\hat{\Phi}^{\sigma_{j+1}}_{\vec{P}} - \Phi^{\sigma_j}_{\vec{P}}\|_{\mathcal{F}} + \alpha^{\frac{1}{4}}\epsilon^{j+1}\Big) \tag{II.73}$$
This is an improvement over a corresponding estimate in [5]: It can be proven *after* the results stated in Theorem III.1 in [5], in particular the uniform bound from below on $\langle \Phi^{\sigma_j}_{\vec{P}}, \Phi^{\sigma_j}_{\vec{P}}\rangle$, $\langle \Phi^{\sigma_j}_{\vec{P}}, \Phi^{\sigma_j}_{\vec{P}}\rangle > \frac{2}{3}$, and following the steps in the proof of Lemma A.2. in [5].

- *Bounds relating expectations of operators to expectations of their absolute values*
There are constants $C_3, C_4, C_5 > 1$ such that the following inequalities hold.
($\mathcal{B}3$) For $z_{j+1} \in \gamma_{j+1}$,
$$\Big\langle (\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\,,\,\Big|\frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}}\Big|(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\Big\rangle \tag{II.74}$$
$$\leq C_3\Big|\Big\langle (\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\,,\,\frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}}(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\Big\rangle\Big|\,, \tag{II.75}$$
where $(\Gamma^{\sigma_j}_{\vec{P}})^i$ is the i^{th} component of $\vec{\Gamma}^{\sigma_j}_{\vec{P}}$.
($\mathcal{B}4$) For $z_{j+1} \in \gamma_{j+1}$,
$$\Big\langle (\mathcal{L}^{\sigma_j\,(+)}_{\sigma_{j+1}})^l\,(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\,,\,\Big|\frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}}\Big|(\mathcal{L}^{\sigma_j\,(+)}_{\sigma_{j+1}})^l\,(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\Big\rangle \tag{II.76}$$
$$\leq C_4\Big|\Big\langle (\mathcal{L}^{\sigma_j\,(+)}_{\sigma_{j+1}})^l\,(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\,,\,\frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}}(\mathcal{L}^{\sigma_j\,(+)}_{\sigma_{j+1}})^l\,(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\Big\rangle\Big|, \tag{II.77}$$
where $(\mathcal{L}^{\sigma_j\,(+)}_{\sigma_{j+1}})^l$ is the l^{th} component of $\vec{\mathcal{L}}^{\sigma_j\,(+)}_{\sigma_{j+1}}$.
($\mathcal{B}5$) For $z_{j+1} \in \gamma_{j+1}$,
$$\Big\langle (\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\,,\,\Big|\frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}}\Big|^2(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\Big\rangle \tag{II.78}$$
$$\leq C_5\Big|\Big\langle (\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\,,\,(\frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}})^2(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\Big\rangle\Big|\,. \tag{II.79}$$

To prove ($\mathcal{B}3$) and ($\mathcal{B}4$), it suffices to exploit the fact that the spectral support (with respect to $K^{\sigma_j}_{\vec{P}}$) of the two vectors $(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}$ and $(\mathcal{L}^{\sigma_j\,(+)}_{\sigma_{j+1}})^l(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}$ is strictly above the ground state energy of $K^{\sigma_j}_{\vec{P}}$, since they are both orthogonal to the ground state $\Phi^{\sigma_j}_{\vec{P}}$ of this operator. In the proof of bound ($\mathcal{B}5$), it is also required that $\rho^- > 3\mu\epsilon$, as will be assumed in the following.

Remark
The constants $C_1, \dots, C_5$ are independent of α, ϵ, μ, and $j \in \mathbb{N}$, provided that α, ϵ, and μ are sufficiently small, and $\epsilon > C\alpha$.

Remark
For the convenience of the reader, we recapitulate the relations between the parameters entering the construction:

$$0 < \rho^- < \mu < \rho^+ < 1 - C_\alpha < \frac{2}{3}\,, \tag{II.80}$$

$$0 < \epsilon < \frac{\rho^-}{\rho^+}\,, \tag{II.81}$$

$$\epsilon > C\alpha\,, \tag{II.82}$$

$$\rho^- > 3\mu\epsilon\,. \tag{II.83}$$

Moreover, we stress that the final result is a small coupling result, i.e., for α small, and that, for technical reasons, small values of the parameters $\epsilon, \mu, \rho^-, \rho^+$ are required within the constraints above.

The crucial estimate for the bound on $\Psi^{\sigma_{j+1}}_{\vec{P}} - \Psi^{\sigma_j}_{\vec{P}}$ obtained in [5] (see (II.71)) is

$$\Big|\Big\langle(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\,,\,\frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}}(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\Big\rangle\Big| \leq \frac{R_0}{\alpha\epsilon^{j\delta}}\,, \tag{II.84}$$

where δ, $0 < \delta < 1$, can be taken arbitrarily small, and R_0 is independent of j. This estimate will be improved in the next section. As a consequence, the estimate of the convergence rate of $\{\Phi^{\sigma_j}_{\vec{P}}\}$ will be improved. As a corollary, the second derivative of $E^{\sigma}_{\vec{P}}$ is proven to converge, as $\sigma \to 0$.

III Improved estimate of the convergence rate of $\{\Phi^{\sigma}_{\vec{P}}\}$, as $\sigma \to 0$, and uniform bound on the second derivative of $E^{\sigma}_{\vec{P}}$.

By repeating some steps of the proof of (II.84), we derive the inequality

$$\Big|\Big\langle(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\,,\,\Big(\frac{1}{K^{\sigma_j}_{\vec{P}}-z_{j+1}}\Big)^2(\Gamma^{\sigma_j}_{\vec{P}})^i\,\Phi^{\sigma_j}_{\vec{P}}\Big\rangle\Big| \le \frac{\mathcal{R}_0}{\alpha^{\frac{1}{2}}\,\epsilon^{2j\delta}}\,, \tag{III.1}$$

where $0 < \delta < 1$ and $\mathcal{R}_0$ is a constant independent of $j \in \mathbb{N}$, for α and ϵ sufficiently small.
For $\mathcal{R}_0$ and α small enough, inequality (III.1) implies (see the paragraph *Proof by induction of the inequality (III.1)* below):

$$\|\hat{\Phi}^{\sigma_j}_{\vec{P}} - \Phi^{\sigma_{j-1}}_{\vec{P}}\| \le \alpha^{\frac{1}{4}}\epsilon^{j(1-\delta)}\,, \tag{III.2}$$

for any $(1 >)\delta > 0$.

From now on, we assume the lower bounds

$$\langle\hat{\Phi}^{\sigma_{j+1}}_{\vec{P}},\hat{\Phi}^{\sigma_{j+1}}_{\vec{P}}\rangle\,,\,\langle\Phi^{\sigma_j}_{\vec{P}},\Phi^{\sigma_j}_{\vec{P}}\rangle > \frac{2}{3} \tag{III.3}$$

uniformly in $j \in \mathbb{N}$, which appear in the proof of Theorem III.1 of ref. [5], (using the *a-priori* results stated as ingredients $\mathcal{A}1,\dots,\mathcal{A}4$).
Proof by induction of the inequality (III.1)

- *Inductive hypothesis* We assume that, at scale $j-1$, the following estimate holds

$$\Big|\Big\langle(\Gamma^{\sigma_{j-1}}_{\vec{P}})^i\,\Phi^{\sigma_{j-1}}_{\vec{P}}\,,\,\Big(\frac{1}{K^{\sigma_{j-1}}_{\vec{P}}-z_j}\Big)^2(\Gamma^{\sigma_{j-1}}_{\vec{P}})^i\,\Phi^{\sigma_{j-1}}_{\vec{P}}\Big\rangle\Big| \le \frac{\mathcal{R}_0}{\alpha^{\frac{1}{2}}\,\epsilon^{2(j-1)\delta}}\,. \tag{III.4}$$

  This estimate readily implies that, for $\mathcal{R}_0$ and α small enough, but uniformly in j,

$$\|\hat{\Phi}^{\sigma_j}_{\vec{P}} - \Phi^{\sigma_{j-1}}_{\vec{P}}\| = \tag{III.5}$$

$$= \|\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{\gamma_j}dz_j\,\frac{1}{K^{\sigma_{j-1}}_{\vec{P}}-z_j}[-\Delta K_{\vec{P}}|^{\sigma_{j-1}}_{\sigma_j}\frac{1}{K^{\sigma_{j-1}}_{\vec{P}}-z_j}]^n\Phi^{\sigma_{j-1}}_{\vec{P}}\|$$

$$\le \alpha^{\frac{1}{4}}\epsilon^{j(1-\delta)}\,. \tag{III.6}$$

  An improved estimate on $\|\hat{\Phi}^{\sigma_{j+1}}_{\vec{P}} - \Phi^{\sigma_j}_{\vec{P}}\|$ is based on the following bounds:

i)

$$\| \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_j} \Delta K_{\vec{P}}|_{\sigma_j}^{\sigma_{j-1}} \Phi_{\vec{P}}^{\sigma_{j-1}} \| \leq \mathcal{O}(\mathcal{R}_0^{\frac{1}{2}} \alpha^{\frac{1}{4}} \epsilon^{j(1-\delta)}) , \tag{III.7}$$

whose proof requires (III.4) and a slightly modified version of Lemma A3 in [5];

ii)

$$\| \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_j} \Delta K_{\vec{P}}|_{\sigma_j}^{\sigma_{j-1}} \|_{\mathcal{F}_{\sigma_j}} \leq \mathcal{O}(\alpha^{\frac{1}{2}}) , \tag{III.8}$$

where we use the notation $\|A\|_{\mathcal{H}} = \|A|_{\mathcal{H}}\|$ for the norm of a bounded operator A acting on Hilbert space $\mathcal{H}$. This estimate can be derived from standard bounds and using the pull-through formula.

- <u>*Induction step from scale $j-1$ to scale j*</u>

By unitarity of $W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}}) W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j})$, we have that

$$\begin{aligned} &\Big| \Big\langle (\Gamma_{\vec{P}}^{\sigma_j})^i \, \Phi_{\vec{P}}^{\sigma_j} \, , \, \Big(\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big)^2 (\Gamma_{\vec{P}}^{\sigma_j})^i \, \Phi_{\vec{P}}^{\sigma_j} \Big\rangle \Big| = \\ &= \Big| \Big\langle (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \, \hat{\Phi}_{\vec{P}}^{\sigma_j} \, , \, \Big(\frac{1}{\hat{K}_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big)^2 (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \, \hat{\Phi}_{\vec{P}}^{\sigma_j} \Big\rangle \Big| . \end{aligned} \tag{III.9}$$

As α is small enough and $\epsilon > C\,\alpha^{\frac{1}{2}}$, where $C > 0$ is large enough, we may use ($\mathcal{B}1$) to re-expand the resolvent and find that

$$\Big| \Big\langle (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \, \hat{\Phi}_{\vec{P}}^{\sigma_j} \, , \, \Big(\frac{1}{\hat{K}_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big)^2 (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \, \hat{\Phi}_{\vec{P}}^{\sigma_j} \Big\rangle \Big| \tag{III.10}$$

$$\leq 2 \Big| \Big\langle (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \, \hat{\Phi}_{\vec{P}}^{\sigma_j} \, , \, \Big| \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}} \Big|^2 (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \, \hat{\Phi}_{\vec{P}}^{\sigma_j} \Big\rangle \Big| . \tag{III.11}$$

It follows that

$$2 \Big| \Big\langle (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \, \hat{\Phi}_{\vec{P}}^{\sigma_j} \, , \, \Big| \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}} \Big|^2 (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \, \hat{\Phi}_{\vec{P}}^{\sigma_j} \Big\rangle \Big| \leq \tag{III.12}$$

$$\leq 4 \Big\| \Big| \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}} \Big| \, ((\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \hat{\Phi}_{\vec{P}}^{\sigma_j} - (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \Phi_{\vec{P}}^{\sigma_{j-1}} \Big\|^2 \tag{III.13}$$

$$+ 4 \Big| \Big\langle (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \, \Phi_{\vec{P}}^{\sigma_{j-1}} \, , \, \Big| \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}} \Big|^2 (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \, \Phi_{\vec{P}}^{\sigma_{j-1}} \Big\rangle \Big| . \tag{III.14}$$

Our recursion relates (III.14) to the initial expression in (III.1), with j replaced by $j-1$, while (III.13) is a remainder term that can be controlled as follows:

$$
\begin{aligned}
& 4\Big\|\Big|\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\Big|\,((\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\hat{\Phi}_{\vec{P}}^{\sigma_j}-(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_j})\Big\|^2 && \text{(III.15)}\\
\leq\ & 8\Big\|\Big|\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\Big|\,((\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\hat{\Phi}_{\vec{P}}^{\sigma_j}-(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\hat{\Phi}_{\vec{P}}^{\sigma_j})\Big\|^2 && \text{(III.16)}\\
& +8\Big\|\Big|\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\Big|\,(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i(\hat{\Phi}_{\vec{P}}^{\sigma_j}-\Phi_{\vec{P}}^{\sigma_{j-1}})\Big\|^2 && \text{(III.17)}\\
\leq\ & \frac{\mathcal{R}_1}{\epsilon^{2j\delta}}+\frac{\mathcal{R}_2}{\epsilon^{2j\delta}}\,. && \text{(III.18)}
\end{aligned}
$$

Here $\mathcal{R}_1 \leq \mathcal{O}(\epsilon^{-2})$ and $\mathcal{R}_2 \leq \mathcal{O}(\epsilon^{-2})$ are constants independent of α, μ, and $j \in \mathbb{N}$, provided that α, μ are sufficiently small, and $\epsilon > C\alpha^{\frac{1}{2}}$. In detail:

- Property ($\mathcal{B}4$) and the two norm-bounds

$$
\Big\|\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Big\|_{\mathcal{F}_{\sigma_j}} \leq \mathcal{O}(\epsilon^{-(j+1)}) \quad , \qquad \|\hat{\Phi}_{\vec{P}}^{\sigma_j}-\Phi_{\vec{P}}^{\sigma_{j-1}}\| \leq \alpha^{\frac{1}{4}}\epsilon^{j(1-\delta)} \qquad \text{(III.19)}
$$

  (see (III.5)) justify the step from (III.17) to (III.18);
- concerning the step from (III.16) to (III.18), it is enough to consider Eq. (II.63) and the two bounds

$$
\|(\mathcal{L}_{\sigma_j}^{\sigma_{j-1}})^i\hat{\Phi}_{\vec{P}}^{\sigma_j}\| \leq \mathcal{O}(\alpha^{\frac{1}{4}}\epsilon^{j+1}) \quad , \quad \|\hat{\Phi}_{\vec{P}}^{\sigma_j}-\Phi_{\vec{P}}^{\sigma_{j-1}}\| \leq \alpha^{\frac{1}{4}}\epsilon^{j(1-\delta)}\,. \qquad \text{(III.20)}
$$

To bound the term (III.14), we use ($\mathcal{B}5$) and the key orthogonality property (II.69). For $z_j \in \gamma_j$ and $z_{j+1} \in \gamma_{j+1}$, we find that

$$
\begin{aligned}
4\ & \Big|\Big\langle(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\,\Phi_{\vec{P}}^{\sigma_{j-1}}\,,\,\Big|\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\Big|^2(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\,\Phi_{\vec{P}}^{\sigma_{j-1}}\Big\rangle\Big| && \text{(III.21)}\\
& \leq 4C_5\Big|\Big\langle(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\,\Phi_{\vec{P}}^{\sigma_{j-1}}\,,\,\Big(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\Big)^2(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\,\Phi_{\vec{P}}^{\sigma_{j-1}}\Big\rangle\Big| && \text{(III.22)}\\
& \leq 8C_5^2\Big|\Big\langle(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\,\Phi_{\vec{P}}^{\sigma_{j-1}}\,,\,\Big(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j}}\Big)^2(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\,\Phi_{\vec{P}}^{\sigma_{j-1}}\Big\rangle\Big|\,. && \text{(III.23)}
\end{aligned}
$$

In passing from (III.22) to (III.23), we again use the constraint on the spectral support (with respect to $K_{\vec{P}}^{\sigma_{j-1}}$) of the vector $(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\,\Phi_{\vec{P}}^{\sigma_{j-1}}$.

Assuming that the parameters ϵ and α are so small that the previous constraints are fulfilled and that

$$0 < \mathcal{R}_1 + \mathcal{R}_2 \leq \, (1 - 8C_5^2 \epsilon^{2\delta}) \frac{\mathcal{R}_0}{\alpha^{\frac{1}{2}}} \,, \tag{III.24}$$

we then conclude that

$$\Big| \Big\langle (\Gamma_{\vec{P}}^{\sigma_j})^i \, \Phi_{\vec{P}}^{\sigma_j} \, , \, \Big(\frac{1}{\hat{K}_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big)^2 (\Gamma_{\vec{P}}^{\sigma_j})^i \, \Phi_{\vec{P}}^{\sigma_j} \Big\rangle \Big| \tag{III.25}$$

$$\leq \; \frac{\mathcal{R}_1}{\epsilon^{2j\delta}} + \frac{\mathcal{R}_2}{\epsilon^{2j\delta}} \tag{III.26}$$

$$+8C_5^2 \Big| \Big\langle (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \, \Phi_{\vec{P}}^{\sigma_{j-1}} \, , \, \Big(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_j} \Big)^2 (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \, \Phi_{\vec{P}}^{\sigma_{j-1}} \Big\rangle \Big| \tag{III.27}$$

$$\leq \; \frac{\mathcal{R}_0}{\alpha^{\frac{1}{2}} \, \epsilon^{2j\delta}} \, . \tag{III.28}$$

- *The zeroth step in the induction*

  Since

$$(\Gamma_{\vec{P}}^{\sigma_0})^i \equiv (\vec{P}^f)^i \quad , \quad \Phi_{\vec{P}}^{\sigma_0} \equiv \Omega_f \, , \tag{III.29}$$

  inequality (III.1) is trivially fulfilled for $j = 0$; thus (III.1) holds for all $j \in \mathbb{N} \cup 0$ and for $\mathcal{R}_0$ *arbitrarily* small, provided α is small enough. □

By standard arguments (see [11]), one obtains similar results for the ground state vectors of the σ-dependent Hamiltonians $K_{\vec{P}}^{\sigma}$, for arbitrary $\sigma > 0$. A precise statement is as follows: For α small enough, the *normalized* ground state vectors (that, with an abuse of notation, we call $\Phi_{\vec{P}}^{\sigma}$)

$$\Phi_{\vec{P}}^{\sigma} := \frac{\frac{1}{2\pi i} \oint_{\gamma_\sigma} dz \frac{1}{K_{\vec{P}}^{\sigma} - z} \Omega_f}{\| \frac{1}{2\pi i} \oint_{\gamma_\sigma} \frac{1}{K_{\vec{P}}^{\sigma} - z} \Omega_f \|} \, , \tag{III.30}$$

where $\gamma_\sigma := \{ z \in \mathbb{C} \, | \, |z - E_{\vec{P}}^{\sigma}| = \frac{\rho^-}{2} \sigma \}$, converges strongly to a nonzero vector $\Phi_{\vec{P}}$, as $\sigma \to 0$, with

$$\| \Phi_{\vec{P}}^{\sigma} - \Phi_{\vec{P}} \| \leq \mathcal{O}(\alpha^{\frac{1}{4}} \Big(\frac{\sigma}{\Lambda} \Big)^{1-\delta}) \tag{III.31}$$

for any $0 < \delta (< 1)$.

III.1 Convergence of the second derivative of the ground state energy $E^{\sigma}_{\vec{P}}$.

Because of rotational symmetry we have that $E^{\sigma}_{\vec{P}} \equiv E^{\sigma}_{|\vec{P}|}$. Moreover, $(H^{\sigma}_{\vec{P}})_{\vec{P}\in\mathcal{S}}$ is an analytic family of type A in $\vec{P} \in \mathcal{S}$, with an isolated eigenvalue $E^{\sigma}_{|\vec{P}|}$. Thus, the second derivative $\frac{\partial^2 E^{\sigma}_{|\vec{P}|}}{(\partial|\vec{P}|)^2}$ is well defined and

$$\frac{\partial^2 E^{\sigma}_{|\vec{P}|}}{(\partial|\vec{P}|)^2} = \partial_i^2 E^{\sigma}_{|\vec{P}|}|_{\vec{P}=P_i\hat{i}}\,, \quad i = 1,2,3\,, \tag{III.32}$$

where $\partial_i := \frac{\partial}{\partial P_i}$.
Without loss of generality, the following results are proven for the standard sequence $(\sigma_j)_{j=0}^{\infty}$ of infrared cutoffs. By simple arguments (see [11]), limiting behavior as $\sigma \to 0$ is shown to be "sequence-independent".
By analytic perturbation theory we have that

$$\begin{aligned}&\partial_i^2 E^{\sigma_j}_{|\vec{P}|}|_{\vec{P}=P_i\hat{i}} = &\text{(III.33)}\\ = \;& 1 - 2\langle \frac{1}{2\pi i}\int_{\gamma_j}\frac{1}{H^{\sigma_j}_{\vec{P}} - z_j}[P^i - (\beta^{\sigma_j})^i]\frac{1}{H^{\sigma_j}_{\vec{P}} - z_j} dz_j\, \Psi^{\sigma_j}_{\vec{P}},\, [P^i - (\beta^{\sigma_j})^i]\Psi^{\sigma_j}_{\vec{P}}\rangle|_{\vec{P}=P_i\hat{i}}\,,\end{aligned}$$

where $\Psi^{\sigma_j}_{\vec{P}}$ is the normalized ground state eigenvector of $H^{\sigma_j}_{\vec{P}}$.
Next, we make use of the Bogoliubov transformation implemented by $W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})$ to show that

$$\langle \frac{1}{2\pi i}\int_{\gamma_j}\frac{1}{H^{\sigma_j}_{\vec{P}} - z_j}[P^i - (\beta^{\sigma_j})^i]\frac{1}{H^{\sigma_j}_{\vec{P}} - z_j} dz_j\, \Psi^{\sigma_j}_{\vec{P}},\, [P^i - (\beta^{\sigma_j})^i]\Psi^{\sigma_j}_{\vec{P}}\rangle \tag{III.34}$$

$$\begin{aligned}= \;& \frac{1}{\|\Phi^{\sigma_j}_{\vec{P}}\|^2}\langle \frac{1}{2\pi i}\int_{\gamma_j}\frac{1}{K^{\sigma_j}_{\vec{P}} - z_j}[P^i - W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})(\beta^{\sigma_j})^i W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})]\frac{1}{K^{\sigma_j}_{\vec{P}} - z_j} dz_j\, \Phi^{\sigma_j}_{\vec{P}}\,,\\ & ,\, [P^i - W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})(\beta^{\sigma_j})^i W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})]\Phi^{\sigma_j}_{\vec{P}}\rangle\,, \qquad \text{(III.35)}\end{aligned}$$

where $\Phi^{\sigma_j}_{\vec{P}}$ is the ground state eigenvector of $K^{\sigma_j}_{\vec{P}}$ (iteratively constructed in Section II).
Recalling the definitions

$$\vec{\Pi}^{\sigma_j}_{\vec{P}} := W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\vec{\beta}^{\sigma_j}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) - \langle W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\vec{\beta}^{\sigma_j}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\rangle_{\Omega_f}\,, \tag{III.36}$$

$$\vec{\Gamma}^{\sigma_j}_{\vec{P}} := \vec{\Pi}^{\sigma_j}_{\vec{P}} - \langle\vec{\Pi}^{\sigma_j}_{\vec{P}}\rangle_{\Phi^{\sigma_j}_{\vec{P}}}\,, \tag{III.37}$$

and because of the identity (Feynman-Hellman, see (II.36))

$$\begin{aligned}\langle\vec{\beta}^{\sigma_j}\rangle_{\psi^{\sigma_j}_{\vec{P}}} &= \vec{P} - \vec{\nabla}E^{\sigma_j}_{\vec{P}} &\text{(III.38)}\\ &= \langle\vec{\Pi}^{\sigma_j}_{\vec{P}}\rangle_{\Phi^{\sigma_j}_{\vec{P}}} + \langle W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\vec{\beta}^{\sigma_j}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\rangle_{\Omega_f}\,, &\text{(III.39)}\end{aligned}$$

we find that

$$P^i - W_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}})(\beta^{\sigma_j})^i W^*_{\sigma_j}(\vec{\nabla} E^{\sigma_j}_{\vec{P}}) = -(\Gamma^{\sigma_j}_{\vec{P}})^i + \partial_i E^{\sigma_j}_{\vec{P}}\,; \tag{III.40}$$

hence,

$$\partial_i^2 E^{\sigma_j}_{|\vec{P}|}|_{\vec{P}=P^i\hat{i}} = \tag{III.41}$$

$$= \quad 1 - 2\frac{1}{\|\Phi^{\sigma_j}_{\vec{P}}\|^2}\langle \frac{1}{2\pi i}\int_{\gamma_j}\frac{1}{K^{\sigma_j}_{\vec{P}} - z_j}[\partial_i E^{\sigma_j}_{\vec{P}} - (\Gamma^{\sigma_j}_{\vec{P}})^i]\frac{1}{K^{\sigma_j}_{\vec{P}} - z_j} dz_j\, \Phi^{\sigma_j}_{\vec{P}}\,,$$

$$, \, [\partial_i E^{\sigma_j}_{\vec{P}} - (\Gamma^{\sigma_j}_{\vec{P}})^i]\Phi^{\sigma_j}_{\vec{P}}\rangle|_{\vec{P}=P^i\hat{i}}\,. \tag{III.42}$$

Using the eigenvalue equation

$$K^{\sigma_j}_{\vec{P}}\Phi^{\sigma_j}_{\vec{P}} = E^{\sigma_j}_{\vec{P}}\Phi^{\sigma_j}_{\vec{P}}\,,$$

the terms proportional to $(\partial_i E^{\sigma_j}_{\vec{P}})^2$ and to the mixed terms – i.e., proportional to the product of $\partial_i E^{\sigma_j}_{\vec{P}}$ and $(\Gamma^{\sigma_j}_{\vec{P}})^i$ – are seen to be identically 0, because the contour integral vanishes for each $i = 1, 2, 3$; e.g.,

$$\int_{\gamma_j}\langle \frac{1}{K^{\sigma_j}_{\vec{P}} - z_j}[\partial_i E^{\sigma_j}_{\vec{P}}]\frac{1}{K^{\sigma_j}_{\vec{P}} - z_j}\,\Phi^{\sigma_j}_{\vec{P}}\,,\,[\partial_i E^{\sigma_j}_{\vec{P}}]\Phi^{\sigma_j}_{\vec{P}}\rangle d\bar{z}_j = \tag{III.43}$$

$$= \quad \int_{\gamma_j}\langle \Phi^{\sigma_j}_{\vec{P}}\,,\,\Phi^{\sigma_j}_{\vec{P}}\rangle \Big(\frac{\partial_i E^{\sigma_j}_{\vec{P}}}{E^{\sigma_j}_{\vec{P}} - \bar{z}_j}\Big)^2 d\bar{z}_j = 0\,.$$

It follows that

$$\partial_i^2 E^{\sigma_j}_{|\vec{P}|}|_{\vec{P}=P^i\hat{i}} = \tag{III.44}$$

$$= \quad 1 + \frac{1}{\pi i}\int_{\gamma_j} d\bar{z}_j\langle \frac{1}{K^{\sigma}_{\vec{P}} - z_j}\,(\Gamma^{\sigma_j}_{\vec{P}})^i\,\frac{1}{K^{\sigma}_{\vec{P}} - z_j}\,\frac{\Phi^{\sigma_j}_{\vec{P}}}{\|\Phi^{\sigma_j}_{\vec{P}}\|},\,(\Gamma^{\sigma_j}_{\vec{P}})^i\,\frac{\Phi^{\sigma_j}_{\vec{P}}}{\|\Phi^{\sigma_j}_{\vec{P}}\|}\rangle|_{\vec{P}=P_i\hat{i}} \tag{III.45}$$

$$= \quad 1 + \frac{1}{\pi i}\int_{\gamma_j} d\bar{z}_j\frac{1}{E^{\sigma_j}_{\vec{P}} - \bar{z}_j}\langle\,(\Gamma^{\sigma_j}_{\vec{P}})^i\,\frac{1}{K^{\sigma_j}_{\vec{P}} - z_j}\,(\Gamma^{\sigma_j}_{\vec{P}})^i\,\frac{\Phi^{\sigma_j}_{\vec{P}}}{\|\Phi^{\sigma_j}_{\vec{P}}\|}\,,\,\frac{\Phi^{\sigma_j}_{\vec{P}}}{\|\Phi^{\sigma_j}_{\vec{P}}\|}\rangle|_{\vec{P}=P_i\hat{i}}\,. \tag{III.46}$$

We are now ready for the key estimate.

Lemma III.1. *For α and ϵ small enough, the estimate below holds true:*

$$\Big|\int_{\gamma_{j-1}}\Big\langle(\Gamma^{\sigma_{j-1}}_{\vec{P}})^i\,\frac{\Phi^{\sigma_{j-1}}_{\vec{P}}}{\|\Phi^{\sigma_{j-1}}_{\vec{P}}\|}\,,\,\frac{1}{K^{\sigma_{j-1}}_{\vec{P}} - \bar{z}_{j-1}}(\Gamma^{\sigma_{j-1}}_{\vec{P}})^i\,\frac{\Phi^{\sigma_{j-1}}_{\vec{P}}}{\|\Phi^{\sigma_{j-1}}_{\vec{P}}\|}\Big\rangle\frac{1}{E^{\sigma_{j-1}}_{\vec{P}} - \bar{z}_{j-1}}d\bar{z}_{j-1}$$

$$-\int_{\gamma_j}\Big\langle(\Gamma^{\sigma_j}_{\vec{P}})^i\,\frac{\Phi^{\sigma_j}_{\vec{P}}}{\|\Phi^{\sigma_j}_{\vec{P}}\|}\,,\,\frac{1}{K^{\sigma_j}_{\vec{P}} - \bar{z}_j}(\Gamma^{\sigma_j}_{\vec{P}})^i\,\frac{\Phi^{\sigma_j}_{\vec{P}}}{\|\Phi^{\sigma_j}_{\vec{P}}\|}\Big\rangle\frac{1}{E^{\sigma_j}_{\vec{P}} - \bar{z}_j}d\bar{z}_j\Big| \le \epsilon^{j(1-2\delta)}\,. \tag{III.47}$$

Proof.
By unitarity of $W_{\sigma_j}(\vec{\nabla}E_{\vec{P}}^{\sigma_{j-1}})W^*_{\sigma_j}(\vec{\nabla}E_{\vec{P}}^{\sigma_j})$,

$$\int_{\gamma_j}\Big\langle(\Gamma_{\vec{P}}^{\sigma_j})^i\,\frac{\Phi_{\vec{P}}^{\sigma_j}}{\|\Phi_{\vec{P}}^{\sigma_j}\|}\,,\,\frac{1}{K_{\vec{P}}^{\sigma_j}-\bar{z}_j}(\Gamma_{\vec{P}}^{\sigma_j})^i\,\frac{\Phi_{\vec{P}}^{\sigma_j}}{\|\Phi_{\vec{P}}^{\sigma_j}\|}\Big\rangle\frac{1}{E_{\vec{P}}^{\sigma_j}-\bar{z}_j}d\bar{z}_j = \tag{III.48}$$

$$= \int_{\gamma_j}\Big\langle(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\,,\,\frac{1}{\hat{K}_{\vec{P}}^{\sigma_j}-\bar{z}_j}(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\Big\rangle\frac{1}{E_{\vec{P}}^{\sigma_j}-\bar{z}_j}d\bar{z}_j\,. \tag{III.49}$$

By assumption, α is so small that the Neumann series expansions of the resolvents below converge:

$$\frac{1}{\hat{K}_{\vec{P}}^{\sigma_j}-\bar{z}_j}=\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-\bar{z}_j}+\Sigma_1^{\infty}(K_{\vec{P}}^{\sigma_{j-1}},\bar{z}_j)\,, \tag{III.50}$$

$$\frac{1}{E_{\vec{P}}^{\sigma_j}-\bar{z}_j}=\frac{1}{E_{\vec{P}}^{\sigma_{j-1}}-\bar{z}_j}+\Delta(E_{\vec{P}}^{\sigma_{j-1}},\bar{z}_j)\,, \tag{III.51}$$

where:

$$\Sigma_1^{\infty}(K_{\vec{P}}^{\sigma_{j-1}},\bar{z}_j):= \tag{III.52}$$
$$=\sum_{l=1}^{\infty}\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-\bar{z}_j}[-(\Delta K_{\vec{P}}|_{\sigma_j}^{\sigma_{j-1}}+\hat{\mathcal{E}}_{\vec{P}}^{\sigma_j}-\mathcal{E}_{\vec{P}}^{\sigma_{j-1}})\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-\bar{z}_j}]^l\,,$$

and $\Delta K_{\vec{P}}|_{\sigma_j}^{\sigma_{j-1}}$ is defined in Eq. (II.66);

$$\Delta(E_{\vec{P}}^{\sigma_{j-1}},\bar{z}_j):=\frac{1}{E_{\vec{P}}^{\sigma_j}-\bar{z}_j}(E_{\vec{P}}^{\sigma_{j-1}}-E_{\vec{P}}^{\sigma_j})\frac{1}{E_{\vec{P}}^{\sigma_{j-1}}-\bar{z}_j}\,. \tag{III.53}$$

We proceed by using the obvious identity:

$$\int_{\gamma_j}\Big\langle(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\,,\,\frac{1}{\hat{K}_{\vec{P}}^{\sigma_j}-\bar{z}_j}(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\Big\rangle\frac{1}{E_{\vec{P}}^{\sigma_j}-\bar{z}_j}d\bar{z}_j \tag{III.54}$$

$$= \int_{\gamma_j}\Big\langle(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\,,\,\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-\bar{z}_j}(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\Big\rangle\frac{1}{E_{\vec{P}}^{\sigma_{j-1}}-\bar{z}_j}d\bar{z}_j \tag{III.55}$$

$$+\int_{\gamma_j}\Big\langle(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\,,\,\Sigma_1^{\infty}(K_{\vec{P}}^{\sigma_{j-1}},\bar{z}_j)(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\Big\rangle\frac{1}{E_{\vec{P}}^{\sigma_{j-1}}-\bar{z}_j}d\bar{z}_j \tag{III.56}$$

$$+\int_{\gamma_j}\Big\langle(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\,,\,\frac{1}{\hat{K}_{\vec{P}}^{\sigma_j}-\bar{z}_j}(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i\,\frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|}\Big\rangle\Delta(E_{\vec{P}}^{\sigma_{j-1}},\bar{z}_j)d\bar{z}_j \tag{III.57}$$

Each of the expressions (III.55) and (III.56) can be rewritten by adding and subtracting $(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|}$. For (III.55) we get

$$
\begin{aligned}
& (III.55) = \\
= & \int_{\gamma_j} \Big\langle (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|} \, , \, \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - \bar{z}_j} (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|} \Big\rangle \frac{1}{E_{\vec{P}}^{\sigma_{j-1}} - \bar{z}_j} d\bar{z}_j \quad \text{(III.58)} \\
& + \int_{\gamma_j} \Big\langle (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|} - (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|} \, , \quad \text{(III.59)} \\
& \qquad , \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - \bar{z}_j} [(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|} - (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|}] \Big\rangle \frac{1}{E_{\vec{P}}^{\sigma_{j-1}} - \bar{z}_j} d\bar{z}_j \\
& + \int_{\gamma_j} \Big\langle (\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|} - (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|} \, , \quad \text{(III.60)} \\
& \qquad , \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - \bar{z}_j} (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|} \Big\rangle \frac{1}{E_{\vec{P}}^{\sigma_{j-1}} - \bar{z}_j} d\bar{z}_j \\
& + \int_{\gamma_j} \Big\langle (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|} \, , \quad \text{(III.61)} \\
& \qquad , \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - \bar{z}_j} [(\hat{\Gamma}_{\vec{P}}^{\sigma_j})^i \frac{\hat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\hat{\Phi}_{\vec{P}}^{\sigma_j}\|} - (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|}] \Big\rangle \frac{1}{E_{\vec{P}}^{\sigma_{j-1}} - \bar{z}_j} d\bar{z}_j \; .
\end{aligned}
$$

The difference in Eq. (III.47) corresponds to the sum of the terms (III.56)-(III.57) and of the terms (III.59)-(III.61). In fact, (III.58) corresponds to the first term in (III.47) after a contour deformation from γ_{j-1} to γ_j.
The sum of the remainder terms (III.56), (III.57), and (III.59)-(III.61) can be bounded by $\epsilon^{j(1-2\delta)}$, for $\mathcal{R}_0$ and α small enough but independent of j, for any $\vec{P} \in \mathcal{S}$. (We recall that $\mathcal{R}_0$ can be taken arbitrarily small, provided α is small enough). The details are as follows.

- For (III.59)-(III.61) use the following inequalities

$$\Big\|\Big(\frac{1}{K^{\sigma_{j-1}}_{\vec{P}}-\bar{z}_j}\Big)(\Gamma^{\sigma_{j-1}}_{\vec{P}})^i\,\Phi^{\sigma_{j-1}}_{\vec{P}}\Big\| \le \mathcal{O}\Big(\frac{\mathcal{R}_0^{\frac{1}{2}}}{\alpha^{\frac{1}{4}}\,\epsilon^{(j-1)\delta}}\Big)\,, \tag{III.62}$$

$$\Big\|\big[(\hat{\Gamma}^{\sigma_j}_{\vec{P}})^i-(\Gamma^{\sigma_{j-1}}_{\vec{P}})^i\big]\,\hat{\Phi}^{\sigma_j}_{\vec{P}}\Big\| \le \mathcal{O}(\alpha^{\frac{1}{4}}\epsilon^{j(1-\delta)})\,, \tag{III.63}$$

$$\Big\|\frac{1}{K^{\sigma_{j-1}}_{\vec{P}}-\bar{z}_j}\Big\|_{\mathcal{F}_{\sigma_j}} \le \mathcal{O}\Big(\frac{1}{\epsilon^{j+1}}\Big)\,, \tag{III.64}$$

$$\Big\|\Big(\frac{1}{K^{\sigma_{j-1}}_{\vec{P}}-\bar{z}_j}\Big)(\Gamma^{\sigma_{j-1}}_{\vec{P}})^i\,(\hat{\Phi}^{\sigma_j}_{\vec{P}}-\Phi^{\sigma_{j-1}}_{\vec{P}})\Big\| \le \mathcal{O}\Big(\frac{\alpha^{\frac{1}{4}}\epsilon^{j(1-\delta)}}{\epsilon^{j+1}}\Big)\,, \tag{III.65}$$

$$\Big\|\hat{\Phi}^{\sigma_j}_{\vec{P}}-\Phi^{\sigma_{j-1}}_{\vec{P}}\Big\| \le \alpha^{\frac{1}{4}}\epsilon^{j(1-\delta)}\,. \tag{III.66}$$

In order to derive the inequality in Eq. (III.63), one uses Eqs. (II.63), (II.73), and (II.57).

- For (III.56), after adding and subtracting $(\Gamma^{\sigma_{j-1}}_{\vec{P}})^i\frac{\Phi^{\sigma_{j-1}}_{\vec{P}}}{\|\Phi^{\sigma_{j-1}}_{\vec{P}}\|}$, one also has to use that

$$\|[-(\Delta K_{\vec{P}}|^{\sigma_{j-1}}_{\sigma_j}+\hat{\mathcal{E}}^{\sigma_j}_{\vec{P}}-\mathcal{E}^{\sigma_{j-1}}_{\vec{P}})]\frac{1}{K^{\sigma_{j-1}}_{\vec{P}}-z_j}(\Gamma^{\sigma_{j-1}}_{\vec{P}})^i\Phi^{\sigma_{j-1}}_{\vec{P}}\| \le \mathcal{O}\Big(\alpha^{\frac{1}{2}}\epsilon^{j-1}\,\frac{\mathcal{R}_0^{\frac{1}{4}}}{\alpha^{\frac{1}{4}}\,\epsilon^{(j-1)\delta}}\Big)\,; \tag{III.67}$$

- To bound (III.57), note that

$$(III.57) \;=\; -2\pi i\Big\langle(\hat{\Gamma}^{\sigma_j}_{\vec{P}})^i\,\frac{\hat{\Phi}^{\sigma_j}_{\vec{P}}}{\|\hat{\Phi}^{\sigma_j}_{\vec{P}}\|}\,,\,\frac{1}{\hat{K}^{\sigma_j}_{\vec{P}}-E^{\sigma_j}_{\vec{P}}}(\hat{\Gamma}^{\sigma_j}_{\vec{P}})^i\,\frac{\hat{\Phi}^{\sigma_j}_{\vec{P}}}{\|\hat{\Phi}^{\sigma_j}_{\vec{P}}\|}\Big\rangle \tag{III.68}$$

$$+2\pi i\Big\langle(\hat{\Gamma}^{\sigma_j}_{\vec{P}})^i\,\frac{\hat{\Phi}^{\sigma_j}_{\vec{P}}}{\|\hat{\Phi}^{\sigma_j}_{\vec{P}}\|}\,,\,\frac{1}{\hat{K}^{\sigma_j}_{\vec{P}}-E^{\sigma_{j-1}}_{\vec{P}}}(\hat{\Gamma}^{\sigma_j}_{\vec{P}})^i\,\frac{\hat{\Phi}^{\sigma_j}_{\vec{P}}}{\|\hat{\Phi}^{\sigma_j}_{\vec{P}}\|}\Big\rangle \tag{III.69}$$

$$= -2\pi i\Big\langle(\hat{\Gamma}^{\sigma_j}_{\vec{P}})^i\,\frac{\hat{\Phi}^{\sigma_j}_{\vec{P}}}{\|\hat{\Phi}^{\sigma_j}_{\vec{P}}\|}\,,\,\frac{(E^{\sigma_{j-1}}_{\vec{P}}-E^{\sigma_j}_{\vec{P}})}{\hat{K}^{\sigma_j}_{\vec{P}}-E^{\sigma_j}_{\vec{P}}}\,\frac{1}{\hat{K}^{\sigma_j}_{\vec{P}}-E^{\sigma_{j-1}}_{\vec{P}}}(\hat{\Gamma}^{\sigma_j}_{\vec{P}})^i\,\frac{\hat{\Phi}^{\sigma_j}_{\vec{P}}}{\|\hat{\Phi}^{\sigma_j}_{\vec{P}}\|}\Big\rangle \tag{III.70}$$

where $|E^{\sigma_{j-1}}_{\vec{P}}-E^{\sigma_j}_{\vec{P}}| \le \mathcal{O}(\alpha\,\epsilon^{j-1})$. Then use the following inequality analogous to (III.62)

$$\Big\|\Big(\frac{1}{\hat{K}^{\sigma_j}_{\vec{P}}-E^{\sigma_j}_{\vec{P}}}\Big)(\hat{\Gamma}^{\sigma_j}_{\vec{P}})^i\,\hat{\Phi}^{\sigma_j}_{\vec{P}}\Big\| \le \mathcal{O}\Big(\frac{\mathcal{R}_0^{\frac{1}{2}}}{\alpha^{\frac{1}{4}}\,\epsilon^{j\delta}}\Big)\,. \tag{III.71}$$

□

Theorem III.2. *For α small enough, $\frac{\partial^2 E^\sigma_{|\vec{P}|}}{(\partial|\vec{P}|)^2}$ converges, as $\sigma \to 0$. The limiting function, $\Sigma_{|\vec{P}|} := \lim_{\sigma\to 0} \frac{(\partial^2 E^\sigma_{|\vec{P}|}}{(\partial|\vec{P}|)^2}$, is Hölder-continuous in $\vec{P} \in \mathcal{S}$ (for an exponent $\eta > 0$). The bounds from above and below*

$$2 > \Sigma_{|\vec{P}|} > 0 \tag{III.72}$$

hold true uniformly in $\vec{P} \in \mathcal{S}$.

Proof
It is enough to prove the result for a fixed choice of a sequence $\{\sigma_j\}_{j=0}^\infty$. The estimate in Lemma III.1 (with $\delta < 1/2$) implies the existence of $\lim_{j\to\infty} \partial_i^2 E^{\sigma_j}_{|\vec{P}|}|_{\vec{P}=P_i\hat{i}}$. We now observe that $\partial_i^2 E^{\sigma_0}_{|\vec{P}|}|_{\vec{P}=P_i\hat{i}} = 1$ (see Eq. (III.46)), because

$$(\Gamma^{\sigma_0}_{\vec{P}})^i \equiv (\vec{P}^f)^i \quad , \quad \Phi^{\sigma_0}_{\vec{P}} \equiv \Omega_f \, . \tag{III.73}$$

For α small enough, we can take ϵ so small that Lemma III.1, combined with (III.73), yields

$$\Big| \frac{1}{\pi i} \int_{\gamma_j} d\bar{z}_j \frac{1}{E^{\sigma_j}_{\vec{P}} - \bar{z}_j} \langle \frac{\Phi^{\sigma_j}_{\vec{P}}}{\|\Phi^{\sigma_j}_{\vec{P}}\|} \, , (\Gamma^{\sigma_j}_{\vec{P}})^i \, \frac{1}{K^{\sigma_j}_{\vec{P}} - \bar{z}_j} \, (\Gamma^{\sigma_j}_{\vec{P}})^i \, \frac{\Phi^{\sigma_j}_{\vec{P}}}{\|\Phi^{\sigma_j}_{\vec{P}}\|} \rangle |_{\vec{P}=P_i\hat{i}} \Big| < 1 \, , \tag{III.74}$$

uniformly in $j \in \mathbb{N}$. Hence the bound (III.72) follows.
The Hölder-continuity in $\vec{P}$ of $\Sigma_{|\vec{P}|}$ is a trivial consequence of the analyticity in $\vec{P} \in \mathcal{S}$ of $E^\sigma_{|\vec{P}|}$, for any $\sigma > 0$, and of Lemma III.1; see [11] for similar results. □

Corollary III.3. *For α small enough, the function $E_{\vec{P}} := \lim_{\sigma\to 0} E^\sigma_{\vec{P}}$, $\vec{P} \in \mathcal{S}$, is twice differentiable, and*

$$\frac{\partial^2 E_{|\vec{P}|}}{(\partial|\vec{P}|)^2} = \Sigma_{|\vec{P}|} \, . \tag{III.75}$$

Proof
The result follows from the Hölder-continuity of $\Sigma_{|\vec{P}|}$, of $\lim_{\sigma\to 0} \frac{\partial E^\sigma_{\vec{P}}}{\partial|\vec{P}|}$, and from the fundamental theorem of calculus applied to the functions $E_{\vec{P}}$ and $\lim_{\sigma\to 0} \frac{\partial E^\sigma_{\vec{P}}}{\partial|\vec{P}|}$, because

- $$\frac{\partial E^\sigma_{\vec{P}}}{\partial|\vec{P}|} \quad \text{and} \quad \frac{\partial^2 E^\sigma_{\vec{P}}}{(\partial|\vec{P}|)^2} \tag{III.76}$$

  converge pointwise, for $\vec{P} \in \mathcal{S}$, as $\sigma \to 0$,

•
$$\left|\frac{\partial E^{\sigma}_{\vec{P}}}{\partial|\vec{P}|}\right| \quad \text{and} \quad \left|\frac{\partial^2 E^{\sigma}_{\vec{P}}}{(\partial|\vec{P}|)^2}\right| \tag{III.77}$$

are uniformly bounded in σ, for all $\vec{P} \in \mathcal{S}$. □

Acknowledgements

The authors thank Thomas Chen for very useful discussions.

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