STOCHASTIC 2D HYDRODYNAMICAL TYPE SYSTEMS: WELL POSEDNESS AND LARGE DEVIATIONS

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Abstract. We deal with a class of abstract nonlinear stochastic models, which covers many 2D hydrodynamical models including 2D Navier-Stokes equations, 2D MHD models and 2D magnetic Bénard problem and also some shell models of turbulence. We first prove the existence and uniqueness theorem for the class considered. Our main result is a Wentzell-Freidlin type large deviation principle for small multiplicative noise which we prove by weak convergence method.

1. Introduction

In recent years there has been a wide-spread interest in the study of qualitative properties of stochastic models which describe cooperative effects in fluids by taking into account macroscopic parameters such as temperature or/and magnetic field. The corresponding mathematical models consists in coupling the stochastic Navier-Stokes equations with some transport or/and Maxwell equations, which are also stochastically perturbed.

Our goal in this paper is to suggest and develop a unified approach which makes it possible to cover a wide class mathematical coupled models from fluid dynamics. Due to well-known reasons we mainly restrict ourselves to spatially two dimensional models. Our unified approach is based on an abstract stochastic evolution equation in some Hilbert space of the form

$$\partial_t u + \mathcal{A}u + \mathcal{B}(u,u) + \mathcal{R}(u) = \sigma(t,u)\,\dot{W}, \tag{1.1}$$

where $\sigma(t,u)\,\dot{W}$ is a multiplicative noise white in time with spatial correlation. The hypotheses which we impose on the linear operator $\mathcal{A}$, the bilinear mapping $\mathcal{B}$ and the operator $\mathcal{R}$ are true in the case of 2D Navier-Stokes equations (where $\mathcal{R}=0$), and also for some other classes of two dimensional hydrodynamical models such as magneto-hydrodynamic equations, the Boussinesq model for the Bénard convection and 2D magnetic Bénard problem. They also cover the case of regular higher dimensional problems such as the 3D Leray α-model for the Navier-Stokes equation and some shell models of turbulence. See a further discussion in Sect.2.1 below.

For general abstract stochastic evolution equation in infinite dimensional spaces we refer to [11]. However the hypotheses in [11] do not cover our hydrodynamical type model. We also note the stochastic Navier-Stokes equations were studied by many authors (see, e.g., [5, 17, 27, 33] and the references therein).

Our first result states existence, uniqueness and provides a priori estimates for a weak (variational) solution to the abstract problem of the form (1.1) where the forcing term also includes a stochastic control term with a multiplicative coefficient (see Theorem 3.1). As a particular case, we deduce well posedness when the Brownian motion W is translated by a random element of its Reproducing Kernel Hilbert Space (RKHS), as well a priori

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bounds of the solution with constants which only depend on an a.s. bound of the RKHS norm of the control. In all the concrete hydrodynamical examples described above, the diffusion coefficient may contain a small multiple of the gradient of the solution. Thus, this result contains the corresponding existence and uniqueness theorems and a priori bounds for 2D Navier-Stokes equations (see, e.g. [27, 31]), for the Boussinesq model of the Bénard convection (see [16], [13]), and also for the GOY shell model of turbulence (see [1] and [26]). Theorem 3.1 generalizes the existence result for MHD equations given in [2] to the case of multiplicative noise and also covers new situations such as the 2D magnetic Bénard problem, the 3D Leray α-model and the Sabra shell model of turbulence.

Our argument mainly follows the local monotonicity idea suggested in [27, 31]. However, since we deal with an abstract hydrodynamical model with a forcing term which contains a stochastic control under a minimal set of hypotheses, the argument requires substantial modifications compared to that of [31] or [26]. It relies on a two-step Gronwall lemma (see Lemma 3.2 below and also [13]).

Our main result (see Theorem 4.2) is a Wentzell-Freidlin type large deviation principle (LDP) for stochastic equations of the form (1.1) with $\sigma := \sqrt{\varepsilon}\sigma$ as $\varepsilon \to 0$, which describes the exponential rate of convergence of the solution $u := u^\varepsilon$ to the deterministic solution u^0. As in the classical case of finite-dimensional diffusions, the rate function is described by an energy minimization problem which involves deterministic controlled equations. The LDP result is that which would hold true if the solution were a continuous functional of the noise W. Our proof consists in transferring the LDP satisfied by the Hilbert-valued Brownian motion $\sqrt{\varepsilon}W$ to that of a Polish-space valued measurable functional of W as established in [3]; see also [4], [12] and [14]. This is related to the Laplace principle. This approach has been already applied in several specific infinite dimensional situations (see, e.g, [31] for 2D Navier-Stokes equations, [13] for 2D Bénard convection, [4] for stochastic reaction-diffusion system, [24] for stochastic p-Laplacian equation, [26] for the GOY shell model of turbulence). Our result in Theorem 4.2 comprehends a wide class of hydrodynamical systems. In particular, in addition to the 2D Navier-Stokes equations and the Boussinesq model mentioned above, Theorem 4.2 also proves LDP for 2D MHD equations, 2D magnetic Bénard convection, 3D Leray α-model, the Sabra shell model and dyadic model of turbulence. Note that unlike [31] and [26], in order to give a complete argument for the weak convergence (Proposition 4.5) and the compactness result (Proposition 4.6), we need to prove a time approximation result (Lemma 4.3). This requires to make stronger assumptions on the diffusion coefficient σ, which should have some Hölder time regularity, and in the explicit hydrodynamical models, no longer can include the gradient of the solution (see also [13]).

Note that recently [24] a LDP has been proved for a class of *abstract* equations with monotone dissipative nonlinearity, and with multiplicative noise. The main PDE model for this class is a reaction-diffusion equation with a nonlinear monotone diffusion term perturbed by globally Lipschitz sub-critical nonlinearity. This class does not contains the hydrodynamical systems considered in this paper. The technique used in [24] to prove both weak convergence and compactness is slightly different from ours; it relies on integration by parts and also requires that the diffusion coefficient does not include the gradient of the solution. See also [6] for large deviation results in the case on non-Lipschitz coefficients.

The paper is organized as follows. In Section 2 we describe our mathematical model with details and provide the corresponding motivations from the theory of (coupled) models of fluid dynamics. In this section we also formulate our abstract hypotheses. In Section 3 we study well posedness of the abstract stochastic equation which also may contain some random control term. We need properties (such as a priori bounds and localized time increment estimates) of this stochastic control system as a preliminary step in order to apply the general LDP results from [3, 4] in our situation. Note that these technical preliminaries will be proved in a more general framework than what is needed to establish the large deviation principle. Indeed, we will need them in a forthcoming paper where we characterize the support of the distribution of the solution to the stochastic hydrodynamical equations. We formulate and prove the large deviations principle by the weak convergence approach in Section 4.

2. Description of the model

Let $(H, |.|)$ denote a separable Hilbert space, A be an (unbounded) self-adjoint positive linear operator on H. Set $V = Dom(A^{\frac{1}{2}})$. For $v \in V$ set $\|v\| = |A^{\frac{1}{2}} v|$. Let V' denote the dual of V (with respect to the inner product $(.,.)$ of H). Thus we have the triple $V \subset H \subset V'$. Let $\langle u, v\rangle$ denote the duality between $u \in V$ and $v \in V'$ such that $\langle u, v\rangle = (u, v)$ for $u \in V$, $v \in H$, and let $B : V \times V \to V'$ be a continuous mapping (satisfying the condition (**C1**) given below).

The goal of this paper is to study stochastic perturbations of the following abstract model in H

$$\partial_t u(t) + Au(t) + B\big(u(t), u(t)\big) + Ru(t) = f, \tag{2.1}$$

where R is a linear bounded operator in H. We assume that the mapping $B : V \times V \to V'$ satisfies the following antisymmetry and bound conditions:

Condition (C1):

- *$B : V \times V \to V'$ is a bilinear continuous mapping.*
- *For $u_i \in V$, $i = 1, 2, 3$,*

$$\langle B(u_1, u_2)\,,\, u_3\rangle = -\,\langle B(u_1, u_3)\,,\, u_2\rangle. \tag{2.2}$$

- *There exists a Banach (interpolation) space $\mathcal{H}$ possessing the properties*
  (i) $V \subset \mathcal{H} \subset H$;
  (ii) there exists a constant $a_0 > 0$ such that

$$\|v\|^2_{\mathcal{H}} \le a_0 |v|\, \|v\| \quad \textit{for any } v \in V; \tag{2.3}$$

  (iii) for every $\eta > 0$ there exists $C_\eta > 0$ such that

$$|\langle B(u_1, u_2)\,,\, u_3\rangle| \le \eta\, \|u_3\|^2 + C_\eta\, \|u_1\|^2_{\mathcal{H}}\, \|u_2\|^2_{\mathcal{H}}, \quad for\ u_i \in V,\ i = 1, 2, 3. \tag{2.4}$$

Remark 2.1. **(1)** The relation in (2.4) obviously implies that

$$|\langle B(u_1, u_2)\,,\, u_3\rangle| \le C_1 \|u_3\|^2 + C_2\, \|u_1\|^2_{\mathcal{H}}\, \|u_2\|^2_{\mathcal{H}}, \quad \text{for } u_i \in V,\ i = 1, 2, 3, \tag{2.5}$$

for some positive constants C_1 and C_2. On the other hand, if we put in (2.5) $\eta C_1^{-1} u_3$ instead of u_3, then we recover (2.4) with $C_\eta = C_1 C_2 \eta^{-1}$ Thus the requirements (2.4) and(2.5) are equivalent. If for $u_3 \neq 0$ we put now $\eta = \|u_1\|_{\mathcal{H}} \|u_2\|_{\mathcal{H}} \|u_3\|^{-1}$ in (2.4) with $C_\eta = C_1 C_2 \eta^{-1}$, then using (2.2) we obtain that

$$|\langle B(u_1, u_2)\,,\, u_3\rangle| \le C\, \|u_1\|_{\mathcal{H}}\, \|u_2\|\, \|u_3\|_{\mathcal{H}}, \quad \text{for } u_i \in V, \tag{2.6}$$

for some $C > 0$. It is also evident that (2.6) and (2.2) imply (2.4). Thus the conditions in (2.4), (2.5) and (2.6) are equivalent to each other.

(2) To lighten notations for $u_1 \in V$, set $B(u_1) := B(u_1, u_1)$; relations (2.2), (2.3) and (2.4) yield for every $\eta > 0$ the existence of $C_\eta > 0$ such that for $u_1, u_2 \in V$,

$$|\langle B(u_1)\,,\, u_2\rangle| \le \eta\, \|u_1\|^2 + C_\eta\, |u_1|^2\, \|u_2\|^4_{\mathcal{H}}. \tag{2.7}$$

Relations (2.2) and (2.7) yield

$$|\langle B(u_1) - B(u_2)\,,\, u_1 - u_2\rangle| \;\le\; \eta \|u_1 - u_2\|^2 + C_\eta\, |u_1 - u_2|^2\, \|u_2\|^4_{\mathcal{H}}. \tag{2.8}$$

2.1. **Motivation.** The main motivation for the condition (**C1**) is that it covers a wide class of 2D hydrodynamical models including the following ones. An element of $\mathbb{R}^2$ is denoted $u = (u^1, u^2)$.

2.1.1. *2D Navier-Stokes equation.* Let D be a bounded, open and simply connected domain of $\mathbb{R}^2$. We consider the Navier-Stokes equation with the Dirichlet (no-slip) boundary conditions:

$$\partial_t u - \nu\Delta u + u\nabla u + \nabla p = f, \quad \operatorname{div} u = 0 \ \text{ in } \ D, \qquad u = 0 \quad \text{on} \quad \partial D, \tag{2.9}$$

where $u = (u^1(x,t), u^2(x,t))$ is the velocity of a fluid, $p(x,t)$ is the pressure, ν the kinematic viscosity and $f(x,t)$ is an external density of force per volume. Let n denote the outward normal to ∂D and let

$$H_{(1)} = \{f \in \left[L^2(D)\right]^2 : \operatorname{div} f = 0 \text{ in } \ D \text{ and } \ f\,.\,n = 0 \text{ on } \ \partial D\}$$

be endowed with the usual L^2 scalar product. Here above we set $\operatorname{div} f = \sum_{i=1,2} \partial_i f_i$, Projecting on the space $H_{(1)}$ of divergence free vector fields, problem (2.9) can be written in the form (2.1) (with $R \equiv 0$) in the space $H_{(1)}$ (see e.g. [32]), where A is the Stokes operator defined by the bilinear form

$$a(u_1, u_2) = \nu \sum_{j=1}^{2} \int_D \nabla u_1^j \cdot \nabla u_2^j \, dx, \tag{2.10}$$

with $u_1, u_2 \in V = V_1 \equiv \left[H_0^1(D)\right]^2 \cap H_{(1)}$. The map $B \equiv B_1 : V_1 \times V_1 \to V_1'$ is defined by

$$\langle B_1(u_1, u_2)\,,\, u_3\rangle = \int_D [u_1(x)\nabla u_2(x)]\, u_3(x) dx \equiv \sum_{i,j=1}^{2} \int_D u_1^j \,\partial_j u_2^i \, u_3^i \, dx, \quad u_i \in V_1. \tag{2.11}$$

Using integration by parts, Schwarz's and Young's inequality, one checks that this map B_1 satisfies the conditions of (**C1**) with $\mathcal{H} = \left[L^4(D)\right]^2 \cap H_{(1)}$. The inequality in (2.3) is the well-known Ladyzhenskaya inequality (see e.g. [9] or [32]).

We can also include in (2.9) Coriolis type force by changing f into $f - Ru$, where $R(u^1, u^2) = c_0(-u^2, u^1)$, c_0 is a constant. In this case we get (2.1) with $R \neq 0$.

The case of unbounded domains D (including $D = \mathbb{R}^2$) can be also considered in our abstract framework. For this we only need make shift from zero spectrum by changing A into $A + Id$ and introducing $R = -Id$.

2.1.2. *2D magneto-hydrodynamic equations.* We consider magneto-hydrodynamic (MHD) equations for a viscous incompressible resistive fluid in a 2D domain D, which have the form (see, e.g., [28]):

$$\partial_t u - \nu_1 \Delta u + u\nabla u = -\nabla\left(p + \frac{s}{2}|b|^2\right) + sb\nabla b + f, \tag{2.12}$$

$$\partial_t b - \nu_2 \Delta b + u\nabla b = b\nabla u + g, \tag{2.13}$$

$$\operatorname{div} u = 0, \quad \operatorname{div} b = 0 \tag{2.14}$$

where $u = (u^1(x,t), u^2(x,t))$ and $b = (b^1(x,t), b^2(x,t))$ denote velocity and magnetic fields, $p(x,t)$ is a scalar pressure. We consider the following boundary conditions

$$u = 0, \quad b\,.\,n = 0, \quad \partial_1 b^2 - \partial_2 b^1 = 0 \quad \text{on } \partial D \tag{2.15}$$

In equations above ν_1 is the kinematic viscosity, ν_2 is the magnetic diffusivity (which is determined from magnetic permeability and conductivity of the fluid), the positive parameter s is defined by the relation $s = Ha^2\nu_1\nu_2$, where Ha is the so-called Hartman number. The given functions $f = f(x,t)$ and $g = g(x,t)$ represent external volume forces and the curl of external current applied to the fluid. We refer to [22], [15] and [30] for mathematical theory for the MHD equations.

Again, the above equations are a particular case of equation (2.1) for the following spaces and operators which satisfy (**C1**). To see this we first note that without loss of generality we can assume that $s = 1$ in (2.12) (indeed, if $s \neq 1$ we can introduce new magnetic field $b := \sqrt{s}b$ and rescale the curl of the current $g := \sqrt{s}g$). For the velocity part on MHD equations, we use the same spaces $H_{(1)}$ and V_1 and the Stokes operator generated by (2.10) with $\nu = \nu_1$. Now we denote this operator by A_1.

As for the magnetic part we set $H_{(2)} = H_{(1)}$ and $V_2 = \left[H^1(D)\right]^2 \cap H_{(2)}$ and define another Stokes operator A_2 as an unbounded operator in $H_{(2)}$ generated by the form (2.10) with $\nu = \nu_2$ considered on the space V_2.

As in the previous case we can write (2.12)–(2.15) in the form (2.1) in the space $H = H_{(1)} \times H_{(2)}$ with $A = A_1 \times A_2$, $R \equiv 0$. We also set $V = V_1 \times V_2$ and define $B : V \times V \to V'$ by the relation

$$\langle B(z_1, z_2), z_3\rangle = \langle B_1(u_1, u_2), u_3\rangle - \langle B_1(b_1, b_2), u_3\rangle + \langle B_1(u_1, b_2), b_3\rangle - \langle B_1(b_1, u_2), b_3\rangle$$

for $z_i = (u_i, b_i) \in V = V_1 \times V_2$, where B_1 is given by (2.11). The conditions in (**C1**) are satisfied with $\mathcal{H} = \left(\left[L^4(D)\right]^2 \times \left[L^4(D)\right]^2 \right) \cap H$.

2.1.3. *2D Boussinesq model for the Bénard convection.* The next example is the following coupled system of Navier-Stokes and heat equations from the Bénard convection problem (see e.g. [18] and the references therein). Let $D = (0, l) \times (0, 1)$ be a rectangular domain in the vertical plane, (e_1, e_2) the standard basis in $\mathbb{R}^2$ and $x = (x^1, x^2)$ an element of $\mathbb{R}^2$. Denote by $p(x, t)$ the pressure field, f, g external forces, $u = (u^1(x, t), u^2(x, t))$ the velocity field and $\theta = \theta(x, t)$ the temperature field satisfying the following system

$$\partial_t u + u\nabla u - \nu\Delta u + \nabla p = \theta e_2 + f, \quad \operatorname{div} u = 0, \tag{2.16}$$

$$\partial_t \theta + u\nabla\theta - u^2 - \kappa\Delta\theta = g, \tag{2.17}$$

with boundary conditions

$$u = 0 \ \ \& \ \ \theta = 0 \ \text{ on } \ x^2 = 0 \text{ and } x^2 = 1,$$
$$u, p, \theta, u_{x^1}, \theta_{x^1} \text{ are periodic in } x^1 \text{ with period } l.^1$$

Here above ν is the kinematic viscosity, κ is the thermal diffusion coefficient. Let

$$H_{(3)} = \left\{u \in \left[L^2(D)\right]^2,\ \operatorname{div} u = 0,\ u^2|_{x^2=0} = u^2|_{x^2=1} = 0,\ u^1|_{x^1=0} = u^1|_{x^1=l}\right\}$$

and $H_{(4)} = L^2(D)$. We also denote

$$V_3 = \left\{u \in H_{(3)} \cap \left[H^1(D)\right]^2,\ u|_{x^2=0} = u|_{x^2=1} = 0,\ u \text{ is } l\text{-periodic in } x^1\right\},$$
$$V_4 = \left\{\theta \in H^1(D),\ \ \theta|_{x^2=0} = \theta|_{x^2=1} = 0,\ \theta \text{ is } l\text{-periodic in } x^1\right\}.$$

Let A_3 be the Stokes operator in $H_{(3)}$ generated by the form (2.10) considered on V_3 and A_4 be the operator in $H_{(4)}$ generated by the Dirichlet form

$$a(\theta_1, \theta_2) = \kappa \int_D \nabla\theta_1 \cdot \nabla\theta_2 \, dx, \quad \theta_1, \theta_2 \in V_4.$$

Again, the above equations are a particular case of equation (2.1) for the following spaces and operators which satisfy (**C1**). Let $H = H_{(3)} \times H_{(4)}$ and $V = V_3 \times V_4$. We set

[1] Here and below this means that $\phi|_{x^1=0} = \phi|_{x^1=l}$ for the corresponding function.

$A(u,\theta) = (A_3 u\,,\, A_4\theta)$, $R(u,\theta) = -(\theta e_2\,,\, u^2)$, and define the mapping $B : V \times V \to V'$ by the relation

$$\langle B(z_1, z_2), z_3\rangle = \langle B_1(u_1, u_2), u_3\rangle + \sum_{i=1,2} \int_D u_1^i\, \partial_i\, \theta_2\, \theta_3\, dx$$

for $z_i = (u_i, \theta_i) \in V = V_3 \times V_4$, where B_1 is given by (2.11). With these notations, the Boussinesq equations for (u, θ) are a particular case of (2.1) with condition (**C1**) for $\mathcal{H} = \big(\big[L^4(D)\big]^2 \times L^4(D)\big) \cap H$.

2.1.4. *2D magnetic Bénard problem.* This is Boussinesq model coupled with magnetic field (see [19]). As above let $D = (0, l) \times (0, 1)$ be a rectangular domain in the vertical plane, (e_1, e_2) the standard basis in $\mathbb{R}^2$. We consider the equations

$$\begin{aligned}
\partial_t u + u\nabla u - \nu_1 \Delta u + \nabla \Big(p + \frac{s}{2}|b|^2\Big) - sb\nabla b &= \theta e_2 + f, \quad \operatorname{div} u = 0,\\
\partial_t \theta + u\nabla\theta - u^2 - \kappa\Delta\theta &= f,\\
\partial_t b - \nu_2 \Delta b + u\nabla b - b\nabla u &= h, \quad \operatorname{div} b = 0,
\end{aligned}$$

with boundary conditions

$$u = 0 \;\;\&\;\; \theta = 0 \;\;\&\;\; b^2 = 0,\; \partial_2 b^1 = 0 \;\text{ on }\; x^2 = 0 \text{ and } x^2 = 1,$$
$$u, p, \theta, b, u_{x^1}, \theta_{x^1}, b_{x^1} \text{ are periodic in } x^1 \text{ with period } l.$$

As for the MHD case we can assume that $s = 1$. In this case we have (2.1) for the variable $z = (u, \theta, b)$ with $H = H_{(3)} \times H_{(4)} \times H_{(5)}$, where $H_{(3)}$ and $H_{(4)}$ are the same as in the previous example and $H_{(5)} = H_{(3)}$. We also set $V = V_3 \times V_4 \times V_5$, where V_3 and V_4 are the same as above and $V_5 = H_{(3)} \cap \big[H^1(D)\big]^2$. The operator A is generated by the form

$$a(z_1, z_2) = \nu_1 \sum_{j=1}^2 \int_D \nabla u_1^j \cdot \nabla u_2^j\, dx + \kappa \int_D \nabla\theta_1 \cdot \nabla\theta_2\, dx + \nu_2 \sum_{j=1}^2 \int_D \nabla b_1^j \cdot \nabla b_2^j\, dx$$

for $z_i = (u_i, \theta_i, b_i) \in V$. The bilinear operator B is defined by

$$\begin{aligned}
\langle B(z_1, z_2), z_3\rangle &= \langle B_1(u_1, u_2), u_3\rangle - \langle B_1(b_1, b_2), u_3\rangle\\
&\quad + \langle B_1(u_1, b_2), b_3\rangle - \langle B_1(b_1, u_2), b_3\rangle + \sum_{i=1,2} \int_D u_1^i\, \partial_i\, \theta_2\, \theta_3\, dx
\end{aligned}$$

for $z_i = (u_i, \theta_i, b_i) \in V$, where B_1 is given by (2.11). We also set $R(u, \theta, b) = -(\theta e_2\,,\, u^2, 0)$. It is easy to check that this model is an example of equation (2.1) with (**C1**), where $\mathcal{H} = \big(\big[L^4(D)\big]^2 \times L^4(D) \times \big[L^4(D)\big]^2 \big) \cap H$.

2.1.5. *3D Leray α-model for Navier-Stokes equations.* The theory developed in this paper can be also applied to some 3D models. As an example we consider 3D Leray α-model (see [23]; for recent development of this model we refer to [7, 8] and to the references therein). In a bounded 3D domain D we consider the following equations:

$$\partial_t u - \nu\Delta u + v\nabla u + \nabla p = f, \tag{2.18}$$

$$(1 - \alpha\Delta)v = u, \quad \operatorname{div} u = 0, \quad \operatorname{div} u = 0 \;\text{ in }\; D, \tag{2.19}$$

$$v = u = 0 \quad \text{on} \quad \partial D. \tag{2.20}$$

where $u = (u^1, u^2, u^3)$ and $v = (v^1, v^2, v^3)$ are unknown fields, $p(x, t)$ is the pressure. In the space

$$H = \{u \in \big[L^2(D)\big]^3 : \operatorname{div} u = 0 \text{ in } \; D \text{ and } \; u\,.\,n = 0 \text{ on } \; \partial D\}$$

problem (2.18)–(2.20) can be written in the form

$$u_t + Au + B(G_\alpha u, u) = \tilde{f},$$

where A is the corresponding 3D Stokes operator (defined as in the 2D case by the form $a(u_1,u_2) = \nu \sum_{j=1}^3 \int_D \nabla u_1^j \, \nabla u_2^j \, dx$ on $V \equiv H \cap \left[H_0^1(D)\right]^3$), $G_\alpha = \left(Id + \alpha\nu^{-1}A\right)^{-1}$ is the Green operator and

$$\langle B(u_1,u_2)\,,\, u_3\rangle = \sum_{i,j=1}^3 \int_D u_1^j \, \partial_j u_2^i \, u_3^i \, dx, \quad u_i \in V = H \cap \left[H_0^1(D)\right]^3.$$

Note that the embedding $H^{1/2}(D) \subset L^3(D)$ implies that the inequality (2.3) holds true for $\mathcal{H} = \left[L^3(D)\right]^3 \cap H$. Furthermore, Hölder's inequality and the embedding $H^1(D) \subset L^6(D)$ imply that for $u_1, u_2, u_3 \in V$,

$$\begin{aligned} |\langle B(G_\alpha u_1, u_2)\,,\, u_3\rangle| &\leq C\|u_2\| \, |G_\alpha u_1|_{L^6(D)} \, |u_3|_{L^3(D)} \leq C\|u_2\| \, \|G_\alpha u_1\| \, |u_3|_{L^3(D)} \\ &\leq C\|u_2\| \, |u_1|_{L^3(D)} \, |u_3|_{L^3(D)}, \end{aligned}$$

where the last inequality comes from the fact that $A^{\frac{1}{2}}G_\alpha$ is a bounded operator on H, so that $\|G_\alpha u_1\| = |A^{\frac{1}{2}}G_\alpha u_1| \leq C|u_1| \leq C|u_1|_{L^3(D)}$. By Remark 2.1(1) this implies condition (**C1**) for $B_\alpha(u_1,u_2) := B(G_\alpha u_1, u_2)$.

2.1.6. *Shell models of turbulence.* Let H be a set of all sequences $u = (u_1, u_2, \ldots)$ of complex numbers such that $\sum_n |u_n|^2 < \infty$. We consider H as a *real* Hilbert space endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$ of the form

$$(u,v) = \mathrm{Re} \sum_{n=1}^\infty u_n v_n^*, \quad |u|^2 = \sum_{n=1}^\infty |u_n|^2,$$

where v_n^* denotes the complex conjugate of v_n. In this space H we consider the evolution equation (2.1) with $R = 0$ and with linear operator A and bilinear mapping B defined by the formulas

$$(Au)_n = \nu k_n^2 u_u, \quad n = 1,2,\ldots, \qquad Dom(A) = \left\{ u \in H \, : \, \sum_{n=1}^\infty k_n^4 |u_n|^2 < \infty \right\},$$

where $\nu > 0$, $k_n = k_0 \mu^n$ with $k_0 > 0$ and $\mu > 1$, and

$$[B(u,v)]_n = -i\left(ak_{n+1}u_{n+1}^* v_{n+2}^* + bk_n u_{n-1}^* v_{n+1}^* - ak_{n-1}u_{n-1}^* v_{n-2}^* - bk_{n-1}u_{n-2}^* v_{n-1}^*\right)$$

for $n = 1,2,\ldots$, where a and b are real numbers (here above we also assume that $u_{-1} = u_0 = v_{-1} = v_0 = 0$). This choice of A and B corresponds to the so-called GOY-model (see, e.g., [29]). If we take

$$[B(u,v)]_n = -i\left(ak_{n+1}u_{n+1}^* v_{n+2} + bk_n u_{n-1}^* v_{n+1} + ak_{n-1}u_{n-1} v_{n-2} + bk_{n-1}u_{n-2} v_{n-1}\right),$$

then we obtain the Sabra shell model introduced in [25]. In both cases the equation (2.1) is an infinite sequence of ODEs.

One can easily show (see [1] for the GOY model and [10] for the Sabra model) that the trilinear form

$$\langle B(u,v), w\rangle \equiv \mathrm{Re} \sum_{n=1}^\infty [B(u,v)]_n \, w_n^*$$

possesses the property (2.2) and also satisfies the inequality

$$|\langle B(u,v), w\rangle| \leq C|u||A^{1/2}v||w|, \quad \forall u, w \in H, \quad \forall v \in Dom(A^{1/2}).$$

Thus by Remark 2.1(1) the condition (**C1**) holds with $\mathcal{H} = Dom(A^s)$ for any choice of $s \in [0, 1/4]$.

We can also consider the so-called dyadic model (see, e.g., [20] and the references therein) which can be written as an infinite system of real ODEs of the form

$$\partial_t u_n + \nu\lambda^{2\alpha n}u_n - \lambda^n u_{n-1}^2 + \lambda^{n+1}u_n u_{n+1} = f_n, \quad n = 1, 2, \dots, \tag{2.21}$$

where $\nu, \alpha > 0$, $\lambda > 1$, $u_0 = 0$. Simple calculations show that under the condition $\alpha \geq 1/2$ the system (2.21) can be written as (2.1) and that condition (**C1**) holds for $[B(u,v)]_n = -\lambda^n u_{n-1}v_{n-1} + \lambda^{n+1}\, u_n\, v_{n+1}$ and $(Au)_n = \nu\,\lambda^{2\alpha n}\, u_n$.

2.2. **Stochastic model.** We will consider a stochastic external random force f of the equation in (2.1) driven by a Wiener process W and whose intensity may depend on the solution u. More precisely, let Q be a linear positive operator in the Hilbert space H which belongs to the trace class, and hence is compact. Let $H_0 = Q^{\frac{1}{2}}H$. Then H_0 is a Hilbert space with the scalar product

$$(\phi, \psi)_0 = (Q^{-\frac{1}{2}}\phi, Q^{-\frac{1}{2}}\psi),\ \forall \phi, \psi \in H_0,$$

together with the induced norm $|\cdot|_0 = \sqrt{(\cdot,\cdot)_0}$. The embedding $i : H_0 \to H$ is Hilbert-Schmidt and hence compact, and moreover, $i\ i^* = Q$. Let $L_Q \equiv L_Q(H_0, H)$ be the space of linear operators $S : H_0 \mapsto H$ such that $SQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator from H to H. The norm in the space L_Q is defined by $|S|_{L_Q}^2 = tr(SQS^*)$, where S^* is the adjoint operator of S. The L_Q-norm can be also written in the form

$$|S|_{L_Q}^2 = tr([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k=1}^{\infty} |SQ^{1/2}\psi_k|^2 = \sum_{k=1}^{\infty} |[SQ^{1/2}]^*\psi_k|^2 \tag{2.22}$$

for any orthonormal basis $\{\psi_k\}$ in H.

Let $W(t)$ be a Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, taking values in H and with covariance operator Q. This means that W is Gaussian, has independent time increments and that for $s, t \geq 0$, $f, g \in H$,

$$\mathbb{E}(W(s), f) = 0 \quad \text{and} \quad \mathbb{E}(W(s), f)(W(t), g) = \big(s \wedge t\big)\,(Qf, g).$$

We also have the representation

$$W(t) = \lim_{n\to\infty} W_n(t) \ \text{ in } \ L^2(\Omega; H) \ \text{ with } W_n(t) = \sum_{j=1}^{n} q_j^{1/2}\beta_j(t)e_j, \tag{2.23}$$

where β_j are standard (scalar) mutually independent Wiener processes, $\{e_j\}$ is an orthonormal basis in H consisting of eigen-elements of Q, with $Qe_j = q_j e_j$. For details concerning this Wiener process we refer to [11], for instance.

The noise intensity $\sigma : [0,T] \times V \to L_Q(H_0, H)$ of the stochastic perturbation which we put in (2.1) is assumed to satisfy the following growth and Lipschitz conditions:

Condition (C2): $\sigma \in C\big([0,T] \times V; L_Q(H_0, H)\big)$, *and there exist non negative constants* K_i *and* L_i *such that for every* $t \in [0,T]$ *and* $u, v \in V$*:*
(i) $|\sigma(t,u)|_{L_Q}^2 \leq K_0 + K_1|u|^2 + K_2\|u\|^2$,
(ii) $|\sigma(t,u) - \sigma(t,v)|_{L_Q}^2 \leq L_1|u-v|^2 + L_2\|u-v\|^2$.

Remark 2.2. Assume that $\sigma \in C\big([0,T] \times Dom(A^s); L_Q(H_0,H)\big)$ for some $s < 1/2$ is such that

$$|\sigma(t,u)|_{L_Q}^2 \leq K_0' + K_1'|A^s u|^2, \quad |\sigma(t,u) - \sigma(t,v)|_{L_Q}^2 \leq L'|A^s(u-v)|^2$$

for every $t \in [0,T]$ and $u, v \in Dom(A^s)$ with some positive constants K_0', K_1' and L'. By interpolation we have that:

$$|A^s u|^2 \leq \eta |A^{1/2}u|^2 + C_\eta |u|^2, \quad \forall\, \eta > 0,\ u \in V.$$

Therefore in this case the conditions in (**C2**) are valid with positive constants K_2 and L_2 which can be taken arbitrary small. This observation is important because in Theorem 3.1 below we impose some restrictions on the values of the parameters K_2 and L_2.

Recall that for $u \in V$, $B(u) = B(u,u)$. Consider the following stochastic equation

$$du(t) + \big[Au(t) + B\big(u(t)\big) + Ru(t)\big]\,dt = \sigma(t,u(t))\,dW(t). \tag{2.24}$$

For technical reasons, in order to prove a large deviation principle for the law the solution to (2.24), we will need some precise estimates on the solution of the equation deduced from (2.24) by shifting the Brownian W by some random element (see e.g. [31] and [13]). This cannot be deduced from similar ones on u by means of a Girsanov transformation since the Girsanov density is not uniformly bounded when the intensity of the noise tends to zero (see [13]). Thus we also need to consider the corresponding shifted problem.

To describe a set of admissible random shifts we introduce the class $\mathcal{A}$ as the set of H_0-valued $(\mathcal{F}_t)-$predictable stochastic processes h such that $\int_0^T |h(s)|_0^2 ds < \infty$, a.s. Let

$$S_M = \Big\{ h \in L^2(0,T;H_0) : \int_0^T |h(s)|_0^2 ds \le M \Big\}.$$

The set S_M endowed with the following weak topology is a Polish space (complete separable metric space) [4]: $d_1(h,k) = \sum_{i=1}^\infty \frac{1}{2^i} \big| \int_0^T \big(h(s)-k(s), \tilde{e}_i(s)\big)_0 ds \big|$, where $\{\tilde{e}_i(s)\}_{i=1}^\infty$ is an orthonormal basis for $L^2(0,T;H_0)$. Define

$$\mathcal{A}_M = \{ h \in \mathcal{A} : h(\omega) \in S_M,\ a.s.\}. \tag{2.25}$$

In order to define the stochastic control equation, we introduce another intensity coefficient $\tilde{\sigma}$ and also nonlinear feedback forcing $\tilde{R}$ (instead of R)[2] which satisfy

Condition (C3):

- **(i)** *$\tilde{\sigma} \in C\big([0,T]\times V; L_Q(H_0,H)\big)$ satisfies Condition (**C2**) with similar constants $\tilde{K}_i$, and $\tilde{L}_j$, for $i=1,2,3$, $j=1,2$ instead of K_i and L_j.*
- **(ii)** *$\tilde{R} : [0,T]\times H \mapsto H$ is a continuous mapping such that*

$$|\tilde{R}(t,0)| \le R_0, \quad |\tilde{R}(t,u) - \tilde{R}(t,v)| \le R_1 |u-v|, \quad \forall u,v \in H,\ \forall t \in [0,T],$$

for non-negative constants R_0 and R_1.

Let $M > 0$, $h \in \mathcal{A}_M$ and $\xi \in H$. Under Conditions (**C2**) and (**C3**) we consider the nonlinear SPDE with initial condition $u_h(0) = \xi$:

$$du_h(t) + \big[Au_h(t) + B\big(u_h(t)\big) + \tilde{R}(t,u_h(t))\big]\,dt = \sigma(t,u_h(t))\,dW(t) + \tilde{\sigma}(t,u_h(t))h(t)\,dt. \tag{2.26}$$

3. Well posedness and a priori bounds

Fix $T > 0$ and let $X := C\big([0,T];H\big) \cap L^2\big(0,T;V\big)$ denote the Banach space with the norm defined by

$$\|u\|_X = \Big\{ \sup_{0\le s\le T} |u(s)|^2 + \int_0^T \|u(s)\|^2 ds \Big\}^{\frac{1}{2}}. \tag{3.1}$$

The aim of this section is to prove that equation (2.24), as well as (2.26), has a unique solution in X, and to provide a priori bounds for the X norm of the solution u_h to (2.26) only depending on M when $h \in \mathcal{A}_M$. Recall that an $(\mathcal{F}_t)$-predictable stochastic process

[2] We need this generalization $\tilde{R}$ of the operator R to make some preparations for further considerations related to the support theorem for the system considered.

$u_h(t,\omega)$ is called a *weak solution* in X for the stochastic equation (2.26) on $[0,T]$ with initial condition ξ if $u \in X = C([0,T];H) \cap L^2((0,T);V)$, a.s., and satisfies

$$(u_h(t),v) - (\xi,v) + \int_0^t [\langle u_h(s), Av\rangle + \big\langle B(u_h(s))\,,\, v\big\rangle + (\tilde R(s,u_h(s)),v)]ds$$
$$= \int_0^t \big(\sigma(s,u_h(s))dW(s),v\big) + \int_0^t \big(\tilde\sigma(s,u_h(s))h(s)\,,\, v\big)\, ds, \text{ a.s.,}$$

for all $v \in Dom(A)$ and all $t \in [0,T]$. Note that this solution is a strong one in the probabilistic meaning, that is written in terms of stochastic integrals with respect to the given Brownian motion W. The main result of this section is the following

Theorem 3.1. *Assume that Conditions* (**C1**)*–*(**C3**) *are satisfied. Then for every $M > 0$ and $T > 0$ there exists $\bar K_2 := K_2(T,M) > 0$, (which also depends on K_i, $\tilde K_i$ and R_i, $i = 0,1$, and on $\tilde K_2$ if $\tilde K_2 \neq K_2$) such that under conditions $\mathbb{E}|\xi|^4 < \infty$, $h \in \mathcal{A}_M$, $K_2 \in [0,\bar K_2[$ and $L_2 < 2$ there exists a weak solution u_h in X of the equation* (2.26) *with initial data $u_h(0) = \xi \in H$. Furthermore, for this solution there exists a constant $C := C(K_i, L_i, \tilde K_i, \tilde L_i, R_i, T, M)$ such that for $h \in \mathcal{A}_M$,*

$$\mathbb{E}\Big(\sup_{0\le t\le T} |u_h(t)|^4 + \int_0^T \|u_h(t)\|^2\, dt + \int_0^T \|u_h(t)\|^4_{\mathcal{H}}\, dt\Big) \le C\,\big(1 + E|\xi|^4\big). \tag{3.2}$$

If the constant L_2 is small enough, the equation (2.26) *has a unique solution in X. If one only requires $L_2 < 2$, then equation* (2.26) *has again a unique solution in X if either $\tilde\sigma = \sigma$, or if the function h possesses a deterministic bound, i.e., there exists a (deterministic) scalar function $\psi(t) \in L^2(0,T)$ such that $|h(t)|_0 \le \psi(t)$ a.s.*

The proof is similar to that given in [13] for 2D Boussinesq model (2.16) and (2.17) (see also [31] for the case of 2D Navier-Stokes equations (2.9)). To lighten notations, we suppress the dependence of σ, $\tilde\sigma$ and $\tilde R$ on t. Let $F : V \to V'$ be defined by

$$F(u) = -Au - B(u,u) - \tilde R(u)\,, \quad \forall u \in V.$$

The inequality (2.8) implies that any $\eta > 0$ there exists $C_\eta > 0$ such that for $u, v \in V$,

$$\langle F(u) - F(v)\,,\, u - v\rangle \le -(1-\eta)\|u-v\|^2 + \big(R_1 + C_\eta\|v\|^4_{\mathcal{H}}\big)\,|u-v|^2. \tag{3.3}$$

Let $\{\varphi_n\}_{n\ge1}$ be an orthonormal basis of the Hilbert space H such that $\varphi_n \in Dom(A)$. For any $n \ge 1$, let $H_n = span(\varphi_1, \cdots, \varphi_n) \subset Dom(A)$ and $P_n : H \to H_n$ denote the orthogonal projection from H onto H_n, and finally let $\sigma_n = P_n\sigma$ and $\tilde\sigma_n = P_n\tilde\sigma$. Since P_n is a contraction of H, from (2.22) we deduce that $|\sigma_n(u)|^2_{L_Q} \le |\sigma(u)|^2_{L_Q}$ and $|\tilde\sigma_n(u)|^2_{L_Q} \le |\tilde\sigma(u)|^2_{L_Q}$.

For $h \in \mathcal{A}_M$, consider the following stochastic ordinary differential equation on the n-dimensional space H_n defined by $u_{n,h}(0) = P_n\xi$, and for $v \in H_n$:

$$d(u_{n,h}(t),v) = \big[\langle F(u_{n,h}(t)),v\rangle + (\tilde\sigma(u_{n,h}(t))h(t),v)\big]dt + (\sigma(u_{n,h}(t))dW_n(t),v), \tag{3.4}$$

where $W_n(t)$ is defined in (2.23). Then for $k = 1, \cdots, n$ we have

$$\begin{aligned} d(u_{n,h}(t),\varphi_k) \;&=\; \big[\langle F(u_{n,h}(t)),\varphi_k\rangle + (\tilde\sigma(u_{n,h}(t))h(t),\varphi_k)\big]dt \\ &\quad + \sum_{j=1}^n q_j^{\frac12}\big(\sigma(u_{n,h}(t))e_j\,,\,\varphi_k\big)d\beta_j(t). \end{aligned}$$

Note that for $v \in H_n$ the map $u \in H_n \mapsto \langle Au + \tilde R(u)\,,\, v\rangle$ is globally Lipschitz, while using (2.7) we deduce that the map $u \in H_n \mapsto \langle B(u)\,,\, v\rangle$ is locally Lipschitz. Furthermore, since there exists some constant $C(n)$ such that $\|v\| \le C(n)|v|$ for $v \in H_n$, Conditions (**C1**) and (**C2**) imply that the map $u \in H_n \mapsto \big((\sigma_n(u)e_j\,,\,\varphi_k) : 1 \le j,k \le n\big)$, respectively

$u\in H_n \mapsto \big((\tilde{\sigma}_n(u)h(t)\,,\,\varphi_k) : 1\leq k\leq n\big)$, is globally Lipschitz from H_n to $n\times n$ matrices, respectively to $\mathbb{R}^n$. Hence by a well-known result about existence and uniqueness of solutions to stochastic differential equations [21], there exists a maximal solution $u_{n,h} = \sum_{k=1}^n (u_{n,h}\,,\,\varphi_k)\,\varphi_k$ to (3.4), i.e., a stopping time $\tau_{n,h}\leq T$ such that (3.4) holds for $t<\tau_{n,h}$ and as $t\uparrow \tau_{n,h}<T$, $|u_{n,h}(t)|\to\infty$.

The following proposition shows that $\tau_{n,h}=T$ a.s. It gives estimates on $u_{n,h}$ depending only on T, M, K_i, L_i and $\mathbb{E}|\xi|^{2p}$, which are valid for all n and all $K_2\in[0,\bar{K}_2]$ for some $\bar{K}_2>0$. Its proof depends on the following version of Gronwall's lemma.

Lemma 3.2. *Let X, Y, I and φ be non-negative processes and Z be a non-negative integrable random variable. Assume that I is non-decreasing and there exist non-negative constants C, $\alpha,\beta,\gamma,\delta$ with the following properties*

$$\int_0^T \varphi(s)\,ds \leq C \text{ a.s.},\quad 2\beta e^C\leq 1,\quad 2\delta e^C\leq \alpha, \tag{3.5}$$

and such that for $0\leq t\leq T$,

$$X(t)+\alpha Y(t) \;\leq\; Z+\int_0^t \varphi(r)\,X(r)\,dr + I(t),\ \text{a.s.}, \tag{3.6}$$

$$\mathbb{E}(I(t)) \;\leq\; \beta\,\mathbb{E}(X(t))+\gamma\int_0^t \mathbb{E}(X(s))\,ds+\delta\,\mathbb{E}(Y(t))+\tilde{C}, \tag{3.7}$$

where $\tilde{C}>0$ is a constant. If $X\in L^\infty([0,T]\times\Omega)$, then we have

$$\mathbb{E}\big[X(t)+\alpha Y(t)\big]\leq 2\,\exp\big(C+2t\gamma e^C\big)\,\big(\mathbb{E}(Z)+\tilde{C}\big),\quad t\in[0,T]. \tag{3.8}$$

Proof. Let $\Phi(t)=\int_0^t\varphi(r)\,X(r)\,dr$. Ignoring Y in (3.6) we get that for almost every t

$$\frac{d}{dt}\Phi(t)-\varphi(t)\Phi(t)\leq \varphi(t)[Z+I(t)].$$

Thus integrating and using the monotonicity of $I(t)$ we obtain that for every $t\in[0,T]$,

$$\Phi(t)\leq \int_0^t \varphi(s)\;[Z+I(s)]\;e^{\int_s^t\varphi(r)dr}\;ds\leq [Z+I(t)]\left[e^{\int_0^t\varphi(r)dr}-1\right].$$

Consequently (3.6) and (3.5) yield

$$X(t)+\alpha Y(t)\leq [Z+I(t)]\;e^{\int_0^t\varphi(r)dr}\leq [Z+I(t)]\;e^C\ \text{ a.s.}$$

Taking expected values and using (3.7) and (3.5), we deduce

$$\mathbb{E}(X(t)+\alpha Y(t))\leq 2\,e^C\,[\mathbb{E}(Z)+\tilde{C}]+2\,\gamma\,e^C\int_0^t\mathbb{E}X(r)\,dr.$$

As above this implies

$$2\,\gamma\,e^C\int_0^t\mathbb{E}X(r)\,dr\leq 2\,e^C\,[\mathbb{E}(Z)+\tilde{C}]\left[e^{2\,\gamma\,e^C t}-1\right]$$

for $t\in[0,T]$, which leads to (3.8). □

Proposition 3.3. *Fix $M>0$, $T>0$ and let Conditions (**C1**)–(**C3**) be in force. For any integer $p\geq 1$ there exists $\bar{K}_2=\bar{K}_2(p,T,M)$ (which also depends on K_i, $\tilde{K}_i$ and R_i, $i=0,1$, and on $\tilde{K}_2$ if $\tilde{K}_2\neq K_2$) such that the following result holds. Let $h\in\mathcal{A}_M$, $0\leq K_2\leq\bar{K}_2$ and $\xi\in L^{2p}(\Omega,H)$. Then equation (3.4) has a unique solution on $[0,T]$ (i.e. $\tau_{n,h}=T$ a.s.) with a modification $u_{n,h}\in C([0,T],H_n)$ and satisfying*

$$\sup_n\mathbb{E}\Big(\sup_{0\leq t\leq T}|u_{n,h}(t)|^{2p}+\int_0^T\|u_{n,h}(s)\|^2\,|u_{n,h}(s)|^{2(p-1)}ds\Big)\leq C\big(\mathbb{E}|\xi|^{2p}+1\big) \tag{3.9}$$

for some positive constant C (depending on $p, K_i, \tilde{K}_i, R_i, T, M$).

Proof. Let $u_{n,h}(t)$ be the approximate maximal solution to (3.4) described above. For every $N>0$, set

$$\tau_N = \inf\{t:\ |u_{n,h}(t)|\geq N\}\wedge T.$$

Let $\Pi_n : H_0 \to H_0$ denote the projection operator defined by $\Pi_n u = \sum_{k=1}^n \big(u\,,\,e_k\big)\,e_k$, where $\{e_k, k\geq 1\}$ is the orthonormal basis of H made by eigen-elements of the covariance operator Q and used in (2.23).

Itô's formula and the antisymmetry relation in (2.2) yield that for $t\in[0,T]$,

$$|u_{n,h}(t\wedge\tau_N)|^2 = |P_n\xi|^2 + 2\int_0^{t\wedge\tau_N}\big(\sigma_n(u_{n,h}(s))dW_n(s), u_{n,h}(s)\big) - 2\int_0^{t\wedge\tau_N}\|u_{n,h}(s)\|^2 ds$$
$$- 2\int_0^{t\wedge\tau_N}\big(\tilde{R}(u_{n,h}(s)) - \tilde{\sigma}_n(u_{n,h}(s))h(s), u_{n,h}(s)\big)\,ds + \int_0^{t\wedge\tau_N}|\sigma_n(u_{n,h}(s))\,\Pi_n|_{L_Q}^2\,ds. \tag{3.10}$$

Apply again Itô's formula for $f(z)=z^p$ when $p\geq 2$ and $z=|u_{n,h}(t\wedge\tau_N)|^2$. This yields for $t\in[0,T]$, and any integer $p\geq 1$ (using the convention $p(p-1)x^{p-2}=0$ if $p=1$)

$$|u_{n,h}(t\wedge\tau_N)|^{2p} + 2p\int_0^{t\wedge\tau_N}|u_{n,h}(r)|^{2(p-1)}\,\|u_{n,h}(r)\|^2\,dr \leq\ |P_n\xi|^{2p} + \sum_{1\leq j\leq 5}T_j(t), \tag{3.11}$$

where

$$\begin{aligned}
T_1(t) &= 2p\int_0^{t\wedge\tau_N}(R_0 + R_1|u_{n,h}(r)|)\,|u_{n,h}(r)|^{2p-1}dr,\\
T_2(t) &= 2p\,\Big|\int_0^{t\wedge\tau_N}\big(\sigma_n(u_{n,h}(r))\,dW_n(r), u_{n,h}(r)\big)\,|u_{n,h}(r)|^{2(p-1)}\Big|,\\
T_3(t) &= 2p\int_0^{t\wedge\tau_N}|(\tilde{\sigma}_n(u_{n,h}(r))\,h(r), u_{n,h}(r))|\,|u_{n,h}(r)|^{2(p-1)}dr,\\
T_4(t) &= p\int_0^{t\wedge\tau_N}|\sigma_n(u_{n,h}(r))\,\Pi_n|_{L_Q}^2\,|u_{n,h}(r)|^{2(p-1)}dr,\\
T_5(t) &= 2p(p-1)\int_0^{t\wedge\tau_N}|\Pi_n\sigma_n^*(u_{n,h}(r))\,u_{n,h}(r)|_0^2\,|u_{n,h}(r)|^{2(p-2)}dr.
\end{aligned}$$

Since $h\in\mathcal{A}_M$, the Cauchy-Schwarz inequality, condition (**C3**) and the fact that

$$\|\sigma(u)\|_{\mathcal{L}(H_0,H)} = \|\sigma^*(u)\|_{\mathcal{L}(H,H_0)} \leq |\sigma(u)|_{L_Q}, \tag{3.12}$$

imply that

$$\begin{aligned}
T_3(t) \leq&\ 2p\int_0^{t\wedge\tau_N}\Big(\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1}\,|u_{n,h}(r)| + \sqrt{\tilde{K}_2}\,\|u_{n,h}(r)\|\Big)\,|h(r)|_0\,|u_{n,h}(r)|^{2p-1}dr\\
\leq&\ \frac{p}{2}\int_0^{t\wedge\tau_N}\|u_{n,h}(r)\|^2\,|u_{n,h}(r)|^{2(p-1)}\,dr + 2p\,\tilde{K}_2\int_0^{t\wedge\tau_N}|h(r)|_0^2\,|u_{n,h}(r)|^{2p}\,dr\\
&+ 2p\sqrt{\tilde{K}_1}\int_0^{t\wedge\tau_N}|h(r)|_0\,|u_{n,h}(r)|^{2p}dr + 2p\sqrt{\tilde{K}_0}\int_0^{t\wedge\tau_N}|h(r)|_0|u_{n,h}(r)|^{2p-1}dr.
\end{aligned}$$

Therefore using the inequality $|u|^{2p-1}\leq 1+|u|^{2p}$ to bound the last term, we obtain

$$\begin{aligned}
T_3(t) \leq&\ \frac{p}{2}\int_0^{t\wedge\tau_N}\|u_{n,h}(r)\|^2\,|u_{n,h}(r)|^{2(p-1)}\,dr + 2p\sqrt{\tilde{K}_0}\int_0^t|h(r)|_0dr\\
&+ 2p\int_0^{t\wedge\tau_N}\Big[\Big(\sqrt{\tilde{K}_0}+\sqrt{\tilde{K}_1}\Big)\,|h(r)|_0 + \tilde{K}_2|h(r)|_0^2\Big]\,|u_{n,h}(r)|^{2p}dr.
\end{aligned}\tag{3.13}$$

Using condition (**C2**) and also (3.12) and (2.22), we deduce that

$$T_4(t)+T_5(t)\le (2p^2-p)\,K_2\int_0^{t\wedge\tau_N}\|u_{n,h}(r)\|^2\,|u_{n,h}(r)|^{2(p-1)}\,dr$$
$$+(2p^2-p)\int_0^{t\wedge\tau_N}\Big(K_1\,|u_{n,h}(r)|^{2p}+K_0|u_{n,h}(r)|^{2(p-1)}\Big)\,dr$$
$$\le (2p^2-p)\,K_2\int_0^{t\wedge\tau_N}\|u_{n,h}(r)\|^2\,|u_{n,h}(r)|^{2(p-1)}\,dr$$
$$+c_p\int_0^{t\wedge\tau_N}\big[K_0+(K_0+K_1)\,|u_{n,h}(r)|^{2p}\big]\,dr.$$

Consequently (3.11) for $K_2\le(4p-2)^{-1}$ yields

$$|u_{n,h}(t\wedge\tau_N)|^{2p}+p\int_0^{t\wedge\tau_N}|u_{n,h}(r)|^{2(p-1)}\,\|u_{n,h}(r)\|^2\,dr$$
$$\le\ |P_n\xi|^{2p}+c_p\left[(K_0+R_0)\,T+\sqrt{\tilde K_0}\int_0^T|h(r)|_0dr\right]+\int_0^{t\wedge\tau_N}\varphi(r)|u_{n,h}(r)|^{2p}\,dr+I(t)$$

for $t\in[0,T]$, where $I(t)=\sup_{0\le s\le t}|T_2(s)|$ and

$$\varphi(r)=c_p\left(R_0+R_1+K_0+K_1+\left[\sqrt{\tilde K_0}+\sqrt{\tilde K_1}\right]|h(r)|_0+\tilde K_2|h(r)|_0^2\right)$$

for some constant $c_p>0$. The Burkholder-Davies-Gundy inequality, (**C2**) and Schwarz's inequality yield that for $t\in[0,T]$ and $\beta>0$,

$$\mathbb{E}I(t)=\mathbb{E}\Big(\sup_{0\le s\le t}|T_2(s)|\Big)\ \le\ 6p\,\mathbb{E}\Big\{\int_0^{t\wedge\tau_N}|u_{n,h}(r)|^{2(2p-1)}\,|\sigma_n(u_{n,h}(r))\,\Pi_n|_{L_Q}^2\,dr\Big\}^{\frac12}$$
$$\le\beta\,\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_N}|u_{n,h}(s)|^{2p}\Big)+\frac{9p^2K_2}{\beta}\,\mathbb{E}\int_0^{t\wedge\tau_N}\|u_{n,h}(r)\|^2\,|u_{n,h}(r)|^{2(p-1)}dr$$
$$+\frac{9p^2(K_0+K_1)}{\beta}\mathbb{E}\int_0^{t\wedge\tau_N}|u_{n,h}(r)|^{2p}dr+\frac{9p^2K_0}{\beta}\,T. \tag{3.14}$$

Thus we can apply Lemma 3.2 for

$$X(t)=\sup_{0\le s\le t\wedge\tau_N}|u_{n,h}(s)|^{2p},\quad Y(t)=\int_0^{t\wedge\tau_N}\|u_{n,h}(r)\|^2\,|u_{n,h}(r)|^{2(p-1)}dr. \tag{3.15}$$

All inequalities for the parameters (see (3.5)) can be achieved by choosing K_2 small enough. Thus there exists $\bar K_2$ such that for $0\le K_2\le\bar K_2$ we have

$$\sup_n\mathbb{E}\Big(\sup_{0\le s\le\tau_N}|u_{n,h}|^{2p}+\int_0^{\tau_N}\|u_{n,h}(s)\|\,|u_{n,h}(s)|^{2(p-1)}\,ds\Big)\le C(p)$$

for all n and p, where the constant $C(p)$ is independent of n.

Note that the previous bound $\bar K_2$ depends on $\tilde K_2$. If one sets $\tilde K_2=K_2$, one may slightly change the proof by replacing the inequality in (3.13) by the following

$$T_3(t)\le\ p\,K_2\int_0^{t\wedge\tau_N}\|u_{n,h}(r)\|^2\,|u_{n,h}(r)|^{2(p-1)}\,dr$$
$$+c_p\int_0^{t\wedge\tau_N}\left[\sqrt{\tilde K_0}|h(r)|_0+\left(\left[\sqrt{\tilde K_0}+\sqrt{\tilde K_1}\right]|h(r)|_0+|h(r)|_0^2\right)|u_{n,h}(r)|^{2p}\right]dr.$$

Now we are in position to conclude the proof of Proposition 3.3. As $N\to\infty$, $\tau_N\uparrow\tau_{n,h}$, and on the set $\{\tau_{n,h}<T\}$, we have $\sup_{0\le s\le\tau_N}|u_{n,h}(s)|\to\infty$. Hence $\mathbb{P}(\tau_{n,h}<T)=0$ and

for almost all ω, for $N(\omega)$ large enough, $\tau_{N(\omega)}(\omega) = T$ and $u_{n,h}(.)(\omega) \in C([0,T], H_n)$. By the Lebesgue monotone convergence theorem, we complete the proof. □

Remark 3.4. If the control (shift) function h admits a deterministic bound $\psi(t)$ from $L^2(0,T)$, then we can improve Proposition 3.3 in the way that the constant $\bar{K}_2$ would only depend on p. The point is that in this case we do not need to apply a *two steps* procedure in the Gronwall type argument (see Lemma 3.2). In this case the function φ is deterministic. Therefore by taking first $\sup_t$ in (3.11) and then taking expected values, we obtain that for X and Y given by (3.15):

$$(1-\beta)\mathbb{E}X(t) + \big(\frac{3p}{2} - c_{p,\beta}K_2\big)\mathbb{E}Y(t) \le \mathbb{E}|P_n\xi|^{2p} + c_{p,\beta}T + c_{p,\beta}\int_0^t \varphi(r)\mathbb{E}X(r)\,dr$$

for $t \in [0,T]$ and for arbitrary $\beta > 0$. Choosing $\beta = 1/2$ and K_2 such that $3p/2 - c_{p,\beta}K_2 > 0$ after application of the standard Gronwall lemma, we obtain the desired result.

We now prove the main result of this section.

Proof of Theorem 3.1:

Let $\Omega_T = [0,T]\times\Omega$ be endowed with the product measure $ds \otimes d\mathbb{P}$ on $\mathcal{B}([0,T])\otimes\mathcal{F}$. Let $\bar{K}_2$ be defined by Proposition 3.3 with $p=2$. The inequalities (3.9) and (2.3) imply that for $K_2 \in [0,\bar{K}_2]$ we have the following additional a priori estimate

$$\sup_n \mathbb{E}\int_0^T \|u_{n,h}(s)\|_{\mathcal{H}}^4 ds \le C_2(1+\mathbb{E}|\xi|^4). \tag{3.16}$$

The proof consists of several steps.

Step 1: The inequalities (3.9) and (3.16) imply the existence of a subsequence of $(u_{n,h})_{n\ge 0}$ (still denoted by the same notation), of processes

$$u_h \in \mathcal{X} := L^2(\Omega_T, V)\cap L^4(\Omega_T,\mathcal{H})\cap L^4(\Omega, L^\infty([0,T],H)),$$

$F_h \in L^2(\Omega_T, V')$ and $S_h, \tilde{S}_h \in L^2(\Omega_T, L_Q)$, and finally of random variables $\tilde{u}_h(T) \in L^2(\Omega, H)$, for which the following properties hold:

(i) $u_{n,h} \to u_h$ weakly in $L^2(\Omega_T, V)$,
(ii) $u_{n,h} \to u_h$ weakly in $L^4(\Omega_T, \mathcal{H})$,
(iii) $u_{n,h}$ is weak star converging to u_h in $L^4(\Omega, L^\infty([0,T],H))$,
(iv) $u_{n,h}(T) \to \tilde{u}_h(T)$ weakly in $L^2(\Omega, H)$,
(v) $F(u_{n,h}) \to F_h$ weakly in $L^2(\Omega_T, V')$,
(vi) $\sigma_n(u_{n,h})\Pi_n \to S_h$ weakly in $L^2(\Omega_T, L_Q)$,
(vii) $\tilde{\sigma}_n(u_{n,h})h \to \tilde{S}_h$ in the $\sigma(L^1(\Omega_T,H), L^\infty(\Omega_T,H))$ topology

Indeed, (i)-(iv) are straightforward consequences of Proposition 3.3, of (3.16), and of uniqueness of the limit of $\mathbb{E}\int_0^T (u_{n,h}(t), v(t))dt$ for appropriate v. Furthermore, given $v \in L^2(\Omega_T, V)$, we have $Av \in L^2(\Omega_T, V')$. Since for $u,v \in L^2(\Omega_T,V)$, $\mathbb{E}\int_0^T \langle Au(t)\,,\, v(t)\rangle\,dt = \mathbb{E}\int_0^T \langle u(t)\,,\, Av(t)\rangle\,dt$,

$$\mathbb{E}\int_0^T \langle Au_{n,h}(t), v(t)\rangle\,dt \to \int_0^T \langle Au_h(t)\,,\, v(t)\rangle\,dt. \tag{3.17}$$

Using (3.9) with $p=2$, (2.4), (3.16), condition **(C3)**, the Poincaré and the Cauchy-Schwarz inequalities, we deduce

$$\sup_n \mathbb{E}\int_0^T \big|\langle B(u_{n,h}(t)), v(t)\rangle + (\tilde{R}(u_{n,h}(t)), v(t))\big|\,dt$$
$$\le\; C_1 \sup_n \Big\{\mathbb{E}\int_0^T \|u_{n,h}(t)\|_{\mathcal{H}}^4 + \mathbb{E}\int_0^T |u_{n,h}(t)|^2 dt\Big\} +\; C_2\mathbb{E}\int_0^T (1+\|v(t)\|^2)dt$$

$$\leq\ C_3\left(1+E|\xi|^4+\mathbb{E}\int_0^T\|v(t)\|^2dt\right)<+\infty.$$

Hence $\{B(u_{n,h}(t))+\tilde{R}(u_{n,h}(t))\,,\,n\geq 1\}$ has a subsequence converging weakly in $L^2(\Omega_T,V')$, which proves (v).

Since Π_n contracts the $|\cdot|$ norm, (**C2**), (2.22) and (3.9) for $p=2$ prove that (vi) is a straightforward of the following

$$\sup_n\mathbb{E}\int_0^T|\sigma_n(u_{n,h}(t))\Pi_n|^2_{L_Q}dt\leq K_0T+\sup_n\mathbb{E}\int_0^T\big(K_1|u_{n,h}(t)|^2+K_2\|u_{n,h}(t)\|^2\big)\,dt<\infty.$$

Let λ denote the Lebesgue measure on $[0,T]$. We finally check that for $h\in\mathcal{A}_M$, the sequence $(\big|\tilde{\sigma}_n(u_{n,h})\big|_{L_Q}|h|_0\,,\,n\geq 1)$ is uniformly integrable on Ω_T with respect to the measure $\lambda\otimes P$, and hence relatively compact for the weak topology on $L^1(\Omega_T)$. Let $\alpha\in]0,1[$ and for any $N>0$ set $C(N)=N^{\frac14}$. Then if $A(n,N)=\{(s,\omega)\in\Omega_T:|h(s)|_0\,\|u_{n,h}(s)\|\geq N\}$, we deduce that

$$T(n,N):=\mathbb{E}\int_0^T 1_{A(n,N)}|h(s)|_0\,\|u_{n,h}(s)\|\,ds\leq T_1(n,N)+T_2(n,N),$$

where

$$\begin{aligned}T_1(n,N)&=\mathbb{E}\int_0^T 1_{\{|h(s)|_0\geq C(N)\}}|h(s)|_0\,\|u_{n,h}(s)\|\,ds,\\ T_2(n,N)&=C(N)^{1-\alpha}\,\mathbb{E}\int_0^T 1_{A(n,N)}|h(s)|_0^\alpha\,\|u_{n,h}(s)\|\,ds.\end{aligned}$$

Since $|h(.)|_0^2\in L^1(\Omega_T)$, it is uniformly integrable. Schwarz's inequality and (3.9) with $p=1$ yield

$$T_1(n,N)\ \leq\ \Big(\mathbb{E}\int_0^T 1_{\{|h(s)|_0\geq C(N)\}}|h(s)|_0^2ds\Big)^{\frac12}\Big(\mathbb{E}\int_0^T\|u_{n,h}(s)\|^2ds\Big)^{\frac12},$$

so that $\lim_{N\to\infty}\sup_{n\geq 1}T_1(n,N)\to 0$. The same computations as in the proof of Step 1 in Theorem 3.1 in [13] prove that

$$T_2(n,N)\leq C\,N^{-\frac{1-\alpha}{4}}\Big(M\,\mathbb{E}\int_0^T\|u_{n,h}(s)\|^2ds\Big)^{\frac{1+\alpha}{4}}.$$

We obtain $\lim_{N\to\infty}\sup_{n\geq 1}T_2(n,N)\to 0$, which completes the proof of the uniform integrability of $(|h|_0\,\|u^\varepsilon\|)_n$. Condition (**C3**) yields that the sequence $\big(\big|\tilde{\sigma}_n(u_{n,h})\big|_{L_Q}|h|_0;n\geq 1\big)$ is uniformly integrable with respect to $\lambda\otimes P$; this completes the proof of (vii).

Step 2: For $\delta>0$, let $f\in H^1(-\delta,T+\delta)$ be such that $\|f\|_\infty=1$, $f(0)=1$ and for any integer $j\geq 1$ set $g_j(t)=f(t)\varphi_j$, where $\{\varphi_j\}_{j\geq 1}$ is the previously chosen orthonormal basis for H. The Itô formula implies that for any $j\geq 1$, and for $0\leq t\leq T$,

$$\big(u_{n,h}(T)\,,\,g_j(T)\big)=\big(u_{n,h}(0)\,,\,g_j(0)\big)+\sum_{i=1}^4 I^i_{n,k},\tag{3.18}$$

where

$$I^1_{n,k}=\int_0^T(u_{n,h}(s),\varphi_j)f'(s)ds,\qquad I^2_{n,k}=\int_0^T\big(\sigma_n(u_{n,h}(s))\Pi_ndW(s),g_j(s)\big),$$

$$I^3_{n,k}=\int_0^T\langle F(u_{n,h}(s)),g_j(s)\rangle ds,\qquad I^4_{n,k}=\int_0^T\big(\tilde{\sigma}_n(u_{n,h}(s))h(s),g_j(s)\big)ds.$$

Since $f' \in L^2([0,T])$ and for every $X \in L^2(\Omega)$, $(t,\omega) \mapsto \varphi_j X(\omega)\, f'(t) \in L^2(\Omega_T, H)$, (i) above implies that as $n\to\infty$, $I^1_{n,k} \to \int_0^T (u_h(s),\varphi_j) f'(s)ds$ weakly in $L^2(\Omega)$. Similarly, (v) implies that as $n\to\infty$, $I^3_{n,k}\to \int_0^T \langle F_h(s), g_j(s)\rangle ds$ weakly in $L^2(\Omega)$, while (vii) implies that $I^4_{n,k} \to \int_0^T \big(\tilde{S}_h(s), g_j(s)\big)ds$ in the $\sigma(L^1(\Omega), L^\infty(\Omega))$ topology.

To prove the convergence of $I^2_{n,k}$, as in [31] (see also [13]), let $\mathcal{P}_T$ denote the class of predictable processes in $L^2(\Omega_T, L_Q(H_0,H))$ with the inner product

$$(G,J)_{\mathcal{P}_T} = \mathbb{E}\int_0^T \big(G(s), J(s)\big)_{L_Q} ds = \mathbb{E}\int_0^T trace(G(s)QJ(s)^*)ds.$$

The map $\mathcal{T} : \mathcal{P}_T \to L^2(\Omega)$ defined by $\mathcal{T}(G)(t) = \int_0^T \big(G(s)dW(s), g_j(s)\big)$ is linear and continuous because of the Itô isometry. Furthermore, (vi) shows that for every $G\in\mathcal{P}_T$, as $n\to\infty$, $\big(\sigma_n(u_{n,h})\Pi_n, G\big)_{\mathcal{P}_T} \to (S_h, G)_{\mathcal{P}_T}$ weakly in $L^2(\Omega)$.

Finally, as $n\to\infty$, $P_n\xi = u^\varepsilon_{n,h}(0)\to\xi$ in H and by (iv), $(u_{n,h}(T), g_j(T))$ converges to $(\tilde{u}_h(T), g_j(T))$ weakly in $L^2(\Omega)$. Therefore, as $n\to\infty$, (3.18) leads to

$$\begin{aligned}(\tilde{u}_h(T),\varphi_j)\, f(T) = \; & \big(\xi,\varphi_j\big) + \int_0^T \big(u_h(s),\varphi_j\big) f'(s)ds + \int_0^T \big(S_h(s)dW(s), g_j(s)\big) \\ & + \int_0^T \langle F_h(s), g_j(s)\rangle ds + \int_0^T \big(\tilde{S}_h(s), g_j(s)\big)ds. \end{aligned}\tag{3.19}$$

For $\delta>0$, $k>\frac{1}{\delta}$, $t\in[0,T]$, let $f_k\in H^1(-\delta, T+\delta)$ be such that $\|f_k\|_\infty = 1$, $f_k=1$ on $(-\delta, t-\frac{1}{k})$ and $f_k = 0$ on $\big(t, T+\delta\big)$. Then $f_k\to 1_{(-\delta,t)}$ in L^2, and $f_k'\to -\delta_t$ in the sense of distributions. Hence as $k\to\infty$, (3.19) written with $f:=f_k$ yields

$$0 = \big(\xi - u_h(t), \varphi_j\big) + \int_0^t \big(S_h(s)dW(s),\varphi_j\big) + \int_0^t \langle F_h(s),\varphi_j\rangle ds + \int_0^t \big(\tilde{S}_h(s),\varphi_j\big)ds$$

for almost all $t\in[0,T]$. This relation makes it possible to suppose (after some modification) that $u_h(t)$ is weakly continuous in H for almost all $\omega\in\Omega$. Now note that j is arbitrary and $\mathbb{E}\int_0^T |S_h(s)|^2_{L_Q} ds<\infty$; we deduce that for $0\le t\le T$,

$$u_h(t) = \xi + \int_0^t S_h(s)dW(s) + \int_0^t F_h(s)ds + \int_0^t \tilde{S}_h(s)ds \in H. \tag{3.20}$$

Moreover $\int_0^t F_h(s)ds\in H$. Let $f = 1_{(-\delta, T+\delta)}$; using again (3.19) we obtain

$$\tilde{u}_h(T) = \xi + \int_0^T S_h(s)dW(s) + \int_0^T F_h(s)ds + \int_0^T \tilde{S}_h(s)ds.$$

This equation and (3.20) yield that $\tilde{u}_h(T) = u_h(T)$ a.s.

Step 3: In (3.20) we still have to prove that $ds\otimes d\mathbb{P}$ a.s. on Ω_T, one has

$$S_h(s) = \sigma(u_h(s)),\; F_h(s) = F(u_h(s)) \text{ and } \; \tilde{S}_h(s) = \tilde{\sigma}(u_h(s))\, h(s).$$

To establish these relations we use the same idea as in [31]. Let

$$v\in\mathcal{X} = L^4(\Omega_T,\mathcal{H})\cap L^4\big(\Omega, L^\infty([0,T],H)\big)\cap L^2(\Omega_T, V)\,.$$

Suppose that $L_2<2$ and let $0<\eta<\frac{2-L_2}{3}$; for every $t\in[0,T]$, set

$$r(t) = \int_0^t \Big[\, 2\,R_1 + 2\,C_\eta\,\|v(s)\|^4_{\mathcal{H}} + L_1 + 2\sqrt{\tilde{L}_1}|h(s)|_0 + \frac{\tilde{L}_2}{\eta}|h(s)|_0^2\Big]\, ds, \tag{3.21}$$

where C_η is the same function of η as in (3.3). Then almost surely, $r(t) < \infty$ for all $t \in [0,T]$. Moreover, we also have that

$$r \in L^1(\Omega, L^\infty(0,T)), \quad r' \in L^1(\Omega_T), \quad r' e^{-r} \in L^\infty(\Omega, L^1((0,T)). \tag{3.22}$$

Weak convergence in (iv) and the property $P_n\xi \to \xi$ in H imply that

$$\mathbb{E}\big(|u_h(T)|^2 e^{-r(T)}\big) - \mathbb{E}|\xi|^2 \le \liminf_n \Big[\mathbb{E}\big(|u_{n,h}(T)|^2 e^{-r(T)}\big) - \mathbb{E}|P_n\xi|^2\Big]. \tag{3.23}$$

We now apply Itô's formula to $|u(t)|^2 e^{-r(t)}$ for $u = u_h$ and $u = u_{n,h}$. This gives the relation

$$\mathbb{E}\big(|u(T)|^2 e^{-r(T)}\big) - \mathbb{E}|u(0)|^2 = \mathbb{E}\int_0^T e^{-r(s)} d\left\{|u(s)|^2\right\} - \mathbb{E}\int_0^T r'(s) e^{-r(s)} |u(s)|^2 ds,$$

which can be justified due to (3.22) and the property $|u|^2 \in L^1(\Omega, L^\infty((0,T))$. Using (3.20), (3.4) and letting $u = v + (u-v)$ after simplification, from (3.23) we obtain

$$\begin{aligned}\mathbb{E}\int_0^T & e^{-r(s)} \big[- r'(s)\big\{\big|u_h(s) - v(s)\big|^2 + 2\big(u_h(s) - v(s)\,,\, v(s)\big)\big\} + 2\langle F_h(s), u_h(s)\rangle \\ & + |S_h(s)|^2_{L_Q} + 2\big(\tilde S_h(s)\,,\, u_h(s)\big)\big]\, ds \le \liminf_n X_n,\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}X_n = \mathbb{E}\int_0^T & e^{-r(s)}\big[- r'(s)\big\{\big|u_{n,h}(s) - v(s)\big|^2 + 2\big(u_{n,h}(s) - v(s)\,,\, v(s)\big)\big\} \\ & + 2\langle F(u_{n,h}(s)), u_{n,h}(s)\rangle + |\sigma_n(u_{n,h}(s))\Pi_n|^2_{L_Q} + 2\big(\tilde\sigma(u_{n,h}(s))h(s)\,,\, u_{n,h}(s)\big)\big]\, ds.\end{aligned}$$

The inequalities in (3.3), (**C2**), (**C3**), and also (3.21) and Schwarz's inequality imply that

$$\begin{aligned}Y_n \;:=\; \mathbb{E}\int_0^T & e^{-r(s)}\big[- r'(s)|u_{n,h}(s) - v(s)|^2 \\ & + 2\langle F(u_{n,h}(s)) - F(v(s)), u_{n,h}(s) - v(s)\rangle \; + |\sigma_n(u_{n,h}(s))\,\Pi_n - \sigma_n(v(s))\,\Pi_n|^2_{L_Q} \\ & + 2\big(\big\{\tilde\sigma_n(u_{n,h}(s)) - \tilde\sigma_n(v(s))\big\}\; h(s), u_{n,h}(s) - v(s)\big)\Big] ds \le 0.\end{aligned} \tag{3.25}$$

Furthermore, $X_n = Y_n + \sum_{i=1}^2 Z_n^i$, with

$$\begin{aligned}Z_n^1 \;=\; \mathbb{E}\int_0^T & e^{-r(s)}\Big[- 2r'(s)(u_{n,h}(s)) - v(s), v(s)) + 2\langle F(u_{n,h}(s)), v(s)\rangle \\ & + 2\langle F(v(s)), u_{n,h}(s)\rangle - 2\langle F(v(s)), v(s)\rangle + 2\big(\sigma_n(u_{n,h}(s))\Pi_n\,,\, \sigma(v(s)\big)_{L_Q} \\ & + 2\big(\tilde\sigma_n(u_{n,h}(s)\big)h(s), v(s)\big) + 2\big(\tilde\sigma(v(s))\; h(s), u_{n,h}(s)) - 2(P_n\tilde\sigma(v(s))h(s), v(s)\big)\Big]\, ds, \\ Z_n^2 \;=\mathbb{E}\int_0^T & e^{-r(s)}\Big[2\Big(\sigma_n(u_{n,h}(s))\Pi_n, \big([\sigma(v(s))\Pi_n - \sigma(v(s))\big)\Big)_{L_Q} - |P_n\sigma(v(s))\Pi_n|^2_{L_Q}\Big] ds.\end{aligned}$$

The weak convergence properties (i)-(vii) imply that, as $n \to \infty$, $Z_n^1 \to Z^1$ where

$$\begin{aligned}Z^1 \;=\; \mathbb{E}\int_0^T & e^{-r(s)}\big[- 2r'(s)\big(u_h(s) - v(s), v(s)\big) + 2\langle F_h(s), v(s)\rangle + 2\langle F(v(s)), u_h(s)\rangle \\ & - 2\langle F(v(s)), v(s)\rangle + 2\big(S_h(s)\,,\, \sigma(v(s))\big)_{L_Q} + 2(\tilde S_h(s), v(s)) \\ & + 2\big(\tilde\sigma(v(s))\; h(s), u_h(s)\big) - 2\big(\tilde\sigma(v(s))h(s), v(s)\big)\,\big] ds.\end{aligned} \tag{3.26}$$

As for Z_n^2 we note that the Lebesgue dominated convergence theorem implies that

$$\mathbb{E}\int_0^T e^{-r(s)} |\sigma(v(s))(\Pi_n - Id_{H_0})|^2_{L_Q} ds \to 0 \;\text{ as }\; n \to \infty.$$

Using once more the dominated Lebesgue convergence theorem, we deduce that

$$Z_n^2 \to -\mathbb{E}\int_0^T e^{-r(s)}|\sigma(v(s))|_{L_Q}^2 ds \quad \text{as} \quad n\to\infty. \tag{3.27}$$

Thus, (3.24)-(3.27) imply that for any $v\in\mathcal{X}$,

$$\mathbb{E}\int_0^T e^{-r(s)}\Big\{ -r'(s)|u_h(s)-v(s)|^2 + 2\langle F_h(s)-F(v(s)),u_h(s)-v(s)\rangle \\ + |S_h(s)-\sigma(v(s))|_{L_Q}^2 + 2\Big(\tilde S_h(s)-\tilde\sigma(v(s))h(s)\,,\,u_h(s)-v(s)\Big)\Big\}ds\le 0. \tag{3.28}$$

Let $v = u_h \in \mathcal{X}$; we conclude that $S_h(s)=\sigma(u_h(s))$, $ds\otimes d\mathbb{P}$ a.e. For $\lambda\in\mathbb{R}$, $\tilde v\in L^\infty(\Omega_T,V)$, set $v_\lambda = u_h-\lambda\tilde v$; then it is clear that $v_\lambda\in\mathcal{X}$. Applying (3.28) to $v:=v_\lambda$ and neglecting $|\sigma(u_h(s))-\sigma(v_\lambda(s))|_{L_Q}^2$, yields

$$\mathbb{E}\int_0^T e^{-r_\lambda(s)}\Big[-\lambda^2 r_\lambda'(s)|\tilde v(s)|^2 + 2\lambda\Big\{\langle F_h(s)-F(v_\lambda(s)),\tilde v(s)\rangle \\ + \Big(\tilde S_h(s)-\tilde\sigma(v_\lambda(s))h(s),\tilde v(s)\Big)\Big\}\Big]\,ds\le 0, \tag{3.29}$$

where $r_\lambda(s)$ is given by (3.21) with v_λ instead of v. Using (**C3**) we obtain

$$\mathbb{E}\int_0^T e^{-r_\lambda(s)}\big|\big(\left[\tilde\sigma(v_\lambda(s))-\tilde\sigma(u_h(s))\right]h(s)\,,\,\tilde v(s)\big)\big|ds \\ \le |\lambda|\,\mathbb{E}\int_0^T |h(s)|_0\,|\tilde v(s)|\,\Big(\sqrt{\tilde L_1}|\tilde v(s)|+\sqrt{\tilde L_2}\,\|\tilde v(s)\|\Big)\,ds\to 0$$

as $\lambda\to 0$. Hence, by the dominated convergence theorem,

$$\lim_{\lambda\to 0}\mathbb{E}\int_0^T e^{-r_\lambda(s)}\Big(\tilde S_h(s)-\tilde\sigma(v_\lambda(s))h(s)\,,\,\tilde v(s)\Big)ds \\ = \mathbb{E}\int_0^T e^{-r_0(s)}\Big(\tilde S_h(s)-\tilde\sigma(u_h(s))h(s)\,,\,\tilde v(s)\Big)ds.$$

Furthermore, (3.3) yields for $\lambda\neq 0$ and $s\in[0,T]$

$$\big|\langle F(v_\lambda(s))-F(u_h(s)),\tilde v(s)\rangle\big|\le C\,|\lambda|\,\left[|\tilde v(s)|^2+\|\tilde v(s)\|^2+|\tilde v(s)|^2\,\|u_h(s)\|_{\mathcal{H}}^4\right].$$

Thus we deduce as $\lambda\to 0$,

$$\mathbb{E}\int_0^T e^{-r_\lambda(s)}\langle F_h(s)-F(v_\lambda(s)),\tilde v(s)\rangle ds\to\mathbb{E}\int_0^T e^{-r_0(s)}\langle F_h(s)-F(u_h(s)),\tilde v(s)\rangle ds.$$

Thus, dividing (3.29) by $\lambda>0$ (resp. $\lambda<0$) and letting $\lambda\to 0$ we obtain that for every $\tilde v\in L^\infty(\Omega_T,V)$, which is a dense subset of $L^2(\Omega_T,V)$,

$$\mathbb{E}\int_0^T e^{-r_0(s)}\Big[\big\langle F_h(s)-F(u_h(s))\,,\,\tilde v(s)\big\rangle + \big(\tilde S_h(s)-\tilde\sigma(u_h(s))h(s)\,,\,\tilde v(s)\big)\Big]\,ds=0.$$

Hence a.e. for $t\in[0,T]$, (3.20) can be rewritten as

$$u_h(t)=\xi+\int_0^t\sigma(u_h(s))dW(s)+\int_0^t\big[F(u_h(s))+\tilde\sigma(u_h(s))h(s)\big]ds. \tag{3.30}$$

Furthermore, (i)-(iii) imply that

$$\begin{aligned}\mathbb{E}\Big(\int_0^T\|u_h(t)\|^2\,dt\Big) &\le \sup_n\mathbb{E}\int_0^T\|u_{n,h}(t)\|^2dt\le C\big(1+E|\xi|^4\big),\\ \mathbb{E}\Big(\sup_{0\le t\le T}|u_h(t)|^4\Big) &\le \sup_n\mathbb{E}\Big(\sup_{0\le t\le T}|u_{n,h}(t)|^4\Big)\le C\left(1+E|\xi|^4\right),\end{aligned}$$

$$\mathbb{E}\Big(\int_0^T \|u_h(t)\|_{\mathcal{H}}^4\,dt\Big) \;\leq\; \sup_n \mathbb{E}\int_0^T \|u_{n,h}(t)\|_{\mathcal{H}}^4\,dt \leq C\big(1+E|\xi|^4\big). \tag{3.31}$$

This completes the proof of (3.2).

Step 4: Now we prove that $u_h \in C([0,T],H)$ almost surely. We first note that (3.30) yields that $e^{-\delta A}u_h \in C([0,T],H)$ a.s. for any $\delta > 0$. Indeed, since for $\delta > 0$ the operator $e^{-\delta A}$ maps H to V and V' to H, we deduce that $e^{-\delta A}\int_0^\cdot F(u_h(s))\,ds$ belongs to $C([0,T],H)$. Condition (**C3**) implies that $e^{-\delta A}\int_0^\cdot \tilde\sigma(u_h(s))\,h(s)\,ds$ also belongs to $C([0,T],H)$. Finally, condition (**C2**) implies $\mathbb{E}\int_0^T |e^{-\delta A}\sigma(u_h)(s)|_{L_Q}^2\,ds < +\infty$. Thus $\int_0^\cdot e^{-\delta A}\sigma(u_h(s))\,dW(s)$ belongs to $C([0,T],H)$ a.s. (see e.g. [11], Theorem 4.12). Therefore it is sufficient to prove that

$$\lim_{\delta\to 0}\mathbb{E}\left\{\sup_{0\leq t\leq T}|u_h(t)-e^{-\delta A}u_h(t)|^2\right\} = 0. \tag{3.32}$$

Let $G_\delta = Id - e^{-\delta A}$ and apply Itô's formula to $|G_\delta u_h(t)|^2$. This yields

$$\begin{aligned}|G_\delta u_h(t)|^2 \;=\; & |G_\delta\xi|^2 - 2\int_0^t \|G_\delta u_{n,h}(s)\|^2 ds + 2I(t) + \int_0^t |G_\delta\sigma(u_h(s))|_{L_Q}^2\,ds \\ & +2\int_0^t \big\langle B(u_h(s)) + \tilde R(u_h(s)) + \tilde\sigma(u_h(s))h(s), G_\delta^2 u_h(s)\big\rangle\,ds, \end{aligned}\tag{3.33}$$

where $I(t) = \int_0^t \big(G_\delta\sigma(u_h(s))dW(s), G_\delta u_h(s)\big)$. By the Burkholder-Davies-Gundy and Schwarz inequalities we have

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}|I(t)| \;\leq\; & C\mathbb{E}\left(\int_0^T |G_\delta u_h(s)|^2 |G_\delta\,\sigma(u_h(s))|_{L_Q}^2 ds\right)^{1/2} \\ \leq\; & \frac{1}{2}\,\mathbb{E}\sup_{0\leq t\leq T}|G_\delta u_h(t)|^2 + \frac{C^2}{2}\mathbb{E}\int_0^T |G_\delta\sigma(u_h(s))|_{L_Q}^2\,ds.\end{aligned}$$

Hence for some constant C, (3.33) yields

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}|G_\delta u_h(t)|^2 \leq\; & 2\,|G_\delta\xi|^2 + C\,\mathbb{E}\int_0^T |G_\delta\sigma(u_h(s))|_{L_Q}^2 ds \\ & +\;4\,\mathbb{E}\int_0^T \Big|\big\langle B(u_h(s)) + \tilde R(u_h(s)) + \tilde\sigma(u_h(s))h(s), G_\delta^2 u_h(s)\big\rangle\Big|\,ds.\end{aligned}$$

Since for every $u \in H$, $|G_\delta u| \to 0$ as $\delta \to 0$ and $\sup_{\delta>0}|G_\delta|_{L(H,H)} \leq 2$, we deduce that if $\{\varphi_k\}$ denotes an orthonormal basis in H, then $|G_\delta\sigma(u_h(s))Q^{1/2}\varphi_k|^2 \to 0$ for every k. Therefore

$$|G_\delta\sigma(u_h(s))|_{L_Q}^2 \equiv \sum_k |G_\delta\sigma(u_h(s))Q^{1/2}\varphi_k|^2 \to 0 \;\text{ as }\; \delta\to 0$$

for almost all $(\omega,s) \in \Omega\times[0,T]$. Since

$$\sup_{\delta>0}|G_\delta\sigma(u_h)|_{L_Q}^2 = \sup_{\delta>0}\sum_k |G_\delta\sigma(u_h)Q^{1/2}\varphi_k|^2 \leq C|\sigma(u_h)|_{L_Q}^2 \in L^1(\Omega\times[0,T]),$$

the Lebesgue dominated convergence theorem implies that $\mathbb{E}\int_0^T |G_\delta\sigma(u_h(s))|_{L_Q}^2 ds \to 0$. Furthermore, given $u \in V$ we have $\|G_\delta^2 u\| \to 0$ as $\delta \to 0$. Hence $\big\langle B(u_h(s)) + \tilde R(u_h(s)) + \tilde\sigma(u_h(s))h(s), G_\delta^2 u_h(s)\big\rangle \to 0$ for almost every (ω,s). Therefore, as above, the Lebesgue dominated convergence theorem concludes the proof of (3.32).

Step 5: To complete the proof of Theorem 3.1, we show that the solution u_h to (3.30) is unique in $X := C([0,T],H)\cap L^2([0,T],V)$. Let $v\in X$ be another solution to (3.30) and

$$\tau_N = \inf\{t\geq 0 : |u_h(t)|\geq N\}\wedge \inf\{t\geq 0 : |v(t)|\geq N\}\wedge T.$$

Since $|u_h(.)|$ and $|v(.)|$ are a.s. bounded on $[0,T]$, we have $\tau_N\to T$ a.s. as $N\to\infty$.

Let $U=u_h-v$. By Itô's formula we have

$$e^{-a\int_0^{t\wedge\tau_N}\|u_h(r)\|_{\mathcal{H}}^4dr}|U(t\wedge\tau_N)|^2 \;=\int_0^{t\wedge\tau_N}\Psi(s)ds+\Phi(t\wedge\tau_N), \tag{3.34}$$

where

$$\begin{aligned}\Psi(s)=e^{-a\int_0^s\|u_h(r)\|_{\mathcal{H}}^4dr}\Big[&-a\|u_h(s)\|_{\mathcal{H}}^4|U(s)|^2\\ &-2\|U(s)\|^2-2\langle B(u_h(s))-B(v(s)),U(s)\rangle+|\sigma(u_h(s))-\sigma(v(s))|_{L_Q}^2\\ &+2\big([\tilde\sigma(u_h(s))-\tilde\sigma(v(s))]h(s),U(s)\big)-2\big(\tilde R(u_h(s))-\tilde R(v(s))\,,U(s)\big)\Big]\end{aligned}$$

and

$$\Phi(\tau)=2\int_0^\tau e^{-a\int_0^s\|u_h(r)\|_{\mathcal{H}}^4dr}\big(U(s),[\sigma(u_h(s))-\sigma(v(s))]\,dW(s)\big).$$

Now we set $a=2C_\eta$ where C_η is defined by (2.8). Then using (2.8) and Conditions (**C2**) and (**C3**) we obtain that for some non negative constant $C(\eta)$ which depends on η and is independent of L_2,

$$\begin{aligned}\Psi(s)\leq e^{-a\int_0^s\|u_h(r)\|_{\mathcal{H}}^4dr}\Big[&-(2-3\eta-L_2)\|U(s)\|^2\\ &+\Big(2R_1+L_1+\frac{\tilde L_2}{\eta}|h(s)|_0^2+2\sqrt{\tilde L_1}|h(s)|_0\Big)|U(s)|^2\Big]\\ \leq e^{-a\int_0^s\|u_h(r)\|_{\mathcal{H}}^4dr}\Big[&-(2-3\eta-L_2)\|U(s)\|^2+C(\eta)\left(1+|h(s)|_0^2\right)|U(s)|^2\Big].\end{aligned}\tag{3.35}$$

First consider the case of a general (random) control function h. Below we use the notations

$$X(t)=\sup_{0\leq s\leq t}\Big\{e^{-a\int_0^{s\wedge\tau_N}\|u_h(r)\|_{\mathcal{H}}^4dr}|U(s\wedge\tau_N)|^2\Big\},\;Y(t)=\int_0^{t\wedge\tau_N}e^{-a\int_0^s\|u_h(r)\|_{\mathcal{H}}^4dr}\|U(s)\|^2ds.$$

Then it follows from (3.34) and (3.35) that

$$X(t)+(2-3\eta-L_2)Y(t)\leq C(\eta)\int_0^t\left(1+|h(s)|_0^2\right)X(s)ds+I(t),$$

where $I(t)=\sup_{0\leq s\leq t}|\Phi(s\wedge\tau_N)|$. An argument similar to that used to prove (3.14), based on the Burkholder-Davies-Gundy inequality, (**C2**) and Schwarz's inequality, yields that for $t\in[0,T]$ and $\beta>0$,

$$\begin{aligned}\mathbb{E}I(t)&\leq 6\,\mathbb{E}\left[\int_0^{t\wedge\tau_N}e^{-2a\int_0^s\|u_h(r)\|_{\mathcal{H}}^4dr}|U(s)|^2|\sigma(u_h(s))-\sigma(v(s))|_{L_Q}^2ds\right]^{1/2}\\ &\leq\beta\,\mathbb{E}X(t)+\frac{9L_1}{\beta}\int_0^t\mathbb{E}X(s)ds+\frac{9L_2}{\beta}\,\mathbb{E}Y(t).\end{aligned}$$

Now we are in position to apply Lemma 3.2. If we choose $\eta=1/3$, $2\beta=\exp\{-C(1/3)(T+M)\}$, then (3.5) holds under the condition $L_2\left(1+36\exp\{2C(1/3)(T+M)\}\right)\leq 1$. Therefore, since $\sup_{0\leq s\leq T}\Big\{e^{-a\int_0^{s\wedge\tau_N}\|u_h(r)\|_{\mathcal{H}}^4dr}|U(s\wedge\tau_N)|^2\Big\}\leq 2N$, relation (3.8) implies that $\mathbb{E}X(t)=0$ and, hence,

$$\sup_{0\leq s\leq T}\;\mathbb{E}\;\Big\{e^{-a\int_0^{s\wedge\tau_N}\|u_h(r)\|_{\mathcal{H}}^4dr}\;|U(s\wedge\tau_N)|^2\Big\}=0. \tag{3.36}$$

Since $\lim_{N\to\infty} \tau_N = T$ a.s., and by (3.31) we have a.s. $\int_0^T \|u_h(s)\|_{\mathcal{H}}^4 ds < \infty$, we deduce that $|U(s,\omega)| = 0$ a.s. on Ω_T. Thus, we conclude that $u_h(t) = v(t)$, a.s., for every $t \in [0,T]$ which yields the uniqueness statement in Theorem 3.1 for a general control function.

Suppose now that we only have $L_2 < 2$ and that h possesses a deterministic bound $\psi(t) \in L^2(0,T)$; let $\eta \in]0, \frac{2-L_2}{3}]$. Then it follows from (3.34) and (3.35) that

$$V_N(t) \le C(\eta) \int_0^t \Big[1 + |\psi(s)|^2\Big] V_N(s) ds \quad \text{with} \quad V_N(t) = \mathbb{E} e^{-a \int_0^{t\wedge\tau_N} \|u_h(r)\|_{\mathcal{H}}^4 dr} |U(t\wedge\tau_N)|^2.$$

Since the function $s \mapsto |\psi|^2$ belongs to $L^1(0,T)$, we can apply the Gronwall lemma to obtain (3.36) and to conclude the proof for the case considered.

Finally, let $L_2 < 2$ and suppose $\tilde{\sigma} = \sigma$. For $h \neq 0$ let $\tilde{W}_t = W_t + \int_0^t h(s)\, ds$ and let $\tilde{\mathbb{P}}$ be the probability defined on $(\Omega, \mathcal{F}_t)$ by $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\Big(- \int_0^t h(s)\, dW_s - \frac{1}{2}\int_0^t |h(s)|_0^2\, ds\Big)$. The Girsanov theorem implies that $\tilde{W}$ is a $\tilde{\mathbb{P}}$ Brownian motion with the same covariance operator Q. Furthermore, under $\tilde{\mathbb{P}}$, one has

$$u_h(t) = \xi + \int_0^t F(u_h(s))\, ds + \int_0^t \sigma(u_h(s))\, d\tilde{W}_s.$$

Thus the previous argument (with $h = 0$) implies that $|U(s,\omega)| = 0$ $\tilde{\mathbb{P}}$ a.s. on Ω_T, and since $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, this completes the proof of Theorem 3.1. □

Remark 3.5. If the control (shift) function h admits a deterministic bound and σ and $\tilde{\sigma}$ satisfies conditions in Remark 2.2, then Theorem 3.1 holds without any restrictions on the bounds of σ and $\tilde{\sigma}$ (see Remark 3.4).

4. Large deviations

We consider large deviations using a weak convergence approach [3, 4], based on variational representations of infinite dimensional Wiener processes. Let $\varepsilon > 0$ and let u^ε denote the solution to the following equation

$$du^\varepsilon(t) + [Au^\varepsilon(t) + B(u^\varepsilon(t)) + \tilde{R}(t, u^\varepsilon(t))]\, dt = \sqrt{\varepsilon}\, \sigma(t, u^\varepsilon(t))\, dW_t\,, u^\varepsilon(0) = \xi \in H. \quad (4.1)$$

Theorem 3.1 shows that for a any choice of K_2 and L_2, for ε small enough the solution of (4.1) exists and is unique in $X := C([0,T],H) \cap L^2([0,T],V)$; it is denoted by $u^\varepsilon = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}W)$ for a Borel measurable function $\mathcal{G}^\varepsilon : C([0,T],H) \to X$.

Let $\mathcal{B}(X)$ denote the Borel $\sigma-$field of the Polish space X endowed with the metric associated with the norm defined by (3.1). We recall some classical definitions; by convention the infimum over an empty set is $+\infty$.

Definition 4.1. *The random family (u^ε) is said to satisfy a large deviation principle on X with the good rate function I if the following conditions hold:*

I is a good rate function. *The function function $I : X \to [0,\infty]$ is such that for each $M \in [0,\infty[$ the level set $\{\phi \in X : I(\phi) \le M\}$ is a compact subset of X.*
For $A \in \mathcal{B}(X)$, set $I(A) = \inf_{u\in A} I(u)$.

Large deviation upper bound. *For each closed subset F of X:*

$$\limsup_{\varepsilon\to 0}\ \varepsilon \log \mathbb{P}(u^\varepsilon \in F) \le -I(F).$$

Large deviation lower bound. *For each open subset G of X:*

$$\liminf_{\varepsilon\to 0}\ \varepsilon \log \mathbb{P}(u^\varepsilon \in G) \ge -I(G).$$

For all $h \in L^2([0,T],H_0)$, let u_h be the solution of the corresponding control equation (4.2) with initial condition $u_h(0)=\xi$:

$$du_h(t) + [Au_h(t) + B(u_h(t)) + \tilde{R}(t,u_h(t))]dt = \sigma(t,u_h(t))h(t)dt. \tag{4.2}$$

Let $\mathcal{C}_0 = \{\int_0^\cdot h(s)ds \,:\, h \in L^2([0,T],H_0)\} \subset C([0,T],H_0)$. Define $\mathcal{G}^0 : C([0,T],H_0) \to X$ by $\mathcal{G}^0(g) = u_h$ for $g = \int_0^\cdot h(s)ds \in \mathcal{C}_0$ and $\mathcal{G}^0(g) = 0$ otherwise. Since the argument below requires some information about the difference of the solution at two different times, we need an additional assumption about the regularity of the map $\sigma(.,u)$.

Condition (C4) (*Time Hölder regularity of* σ): There exist constants $\gamma > 0$ and $C \ge 0$ such that for $t_1,t_2 \in [0,T]$ and $u \in V$:

$$|\sigma(t_1,u) - \sigma(t_2,u)|_{L_Q} \le C\ (1 + \|u\|)\,|t_1 - t_2|^\gamma.$$

The following theorem is the main result of this section.

Theorem 4.2. *Suppose the conditions (***C1***) and (***C2***) with $K_2 = L_2 = 0$ are satisfied. Suppose furthermore that the conditions (***C3 (ii)***) and (***C4***) hold. Then the solution (u^ε) to (4.1) satisfies the large deviation principle in $X = C([0,T];H) \cap L^2((0,T);V)$, with the good rate function*

$$I_\xi(u) = \inf_{\{h \in L^2(0,T;H_0):\, u = \mathcal{G}^0(\int_0^\cdot h(s)ds)\}} \Big\{ \frac{1}{2} \int_0^T |h(s)|_0^2\, ds \Big\}. \tag{4.3}$$

We at first prove the following technical lemma, which studies time increments of the solution to a stochastic control problem extending both (4.1) and (4.2). When σ, $\tilde\sigma$ and $\tilde R$ satisfy (**C2**) and (**C3**), $h \in \mathcal{A}_M$, the stochastic control problem is defined as in (2.26): $u_h^\varepsilon(0) = \xi$ and

$$du_h^\varepsilon(t) + [Au_h^\varepsilon(t) + B(u_h^\varepsilon(t)) + \tilde R(t,u_h^\varepsilon(t))]\,dt = \sqrt{\varepsilon}\,\sigma(t,u_h^\varepsilon(t))\,dW_t + \tilde\sigma(t,u_h^\varepsilon(t))\,h(t)\,dt.$$

To state the lemma mentioned above, we need the following notations. For every integer n, let $\psi_n : [0,T] \to [0,T]$ denote a measurable map such that for every $s \in [0,T]$, $s \le \psi_n(s) \le \big(s + c2^{-n}\big) \wedge T$ for some positive constant c. Given $N > 0$, $h \in \mathcal{A}_M$, and for $t \in [0,T]$, let

$$G_N(t) = \Big\{\omega \,:\, \Big(\sup_{0\le s\le t} |u_h^\varepsilon(s)(\omega)|^2\Big) \vee \Big(\int_0^t \|u_h^\varepsilon(s)(\omega)\|^2 ds\Big) \le N\Big\}.$$

Lemma 4.3. *Let $\varepsilon_0, M, N > 0$, σ and $\tilde\sigma$ satisfy condition (***C2***) and (***C3***), $\xi \in L^4(H)$. Then there exists a positive constant C (depending on $K_i, \tilde K_i, L_i, \tilde L_i, T, M, N, \varepsilon_0$) such that for any $h \in \mathcal{A}_M$, $\varepsilon \in [0,\varepsilon_0]$,*

$$I_n(h,\varepsilon) := \mathbb{E}\Big[1_{G_N(T)} \int_0^T |u_h^\varepsilon(s) - u_h^\varepsilon(\psi_n(s))|^2\, ds\Big] \le C\, 2^{-\frac{n}{2}}. \tag{4.4}$$

Proof. The proof is close to that of Lemma 4.2 in [13]. However we deal with a class of more general functions $\psi_n(s)$ and do not assume that $K_2 = L_2 = \tilde K_2 = \tilde L_2 = 0$. As above, to lighten the notation we skip the time dependence of σ, $\tilde\sigma$ and $\tilde R$. Let $h \in \mathcal{A}_M$, $\varepsilon \ge 0$; for any $s \in [0,T]$, Itô's formula yields

$$|u_h^\varepsilon(\psi_n(s)) - u_h^\varepsilon(s)|^2 = 2\int_s^{\psi_n(s)} (u_h^\varepsilon(r) - u_h^\varepsilon(s), du_h^\varepsilon(r)) + \varepsilon \int_s^{\psi_n(s)} |\sigma(u_h^\varepsilon(r))|_{L_Q}^2 dr.$$

Therefore $I_n(h,\varepsilon) = \sum_{1\le i\le 6} I_{n,i}$, where

$$I_{n,1} \;=\; 2\sqrt{\varepsilon}\,\mathbb{E}\Big(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} \big(\sigma(u_h^\varepsilon(r))dW_r\,,\, u_h^\varepsilon(r) - u_h^\varepsilon(s)\big)\Big),$$

$$\begin{aligned}
I_{n,2} &= \varepsilon\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\psi_n(s)} |\sigma(u_h^\varepsilon(r))|_{L_Q}^2\,dr\Big),\\
I_{n,3} &= 2\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\psi_n(s)} \big(\tilde\sigma(u_h^\varepsilon(r))\,h(r)\,,\,u_h^\varepsilon(r)-u_h^\varepsilon(s)\big)\,dr\Big),\\
I_{n,4} &= -2\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\psi_n(s)} \big(A\,u_h^\varepsilon(r)\,,\,u_h^\varepsilon(r)-u_h^\varepsilon(s)\big)\,dr\Big),\\
I_{n,5} &= -2\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\psi_n(s)} \big\langle B(u_h^\varepsilon(r))\,,\,u_h^\varepsilon(r)-u_h^\varepsilon(s)\big\rangle\,dr\Big),\\
I_{n,6} &= -2\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\psi_n(s)} \big(\tilde R(u_h^\varepsilon(r))\,,\,u_h^\varepsilon(r)-u_h^\varepsilon(s)\big)\,dr\Big).
\end{aligned}$$

Clearly $G_N(T)\subset G_N(r)$ for $r\in[0,T]$. In particular this means that $|u_h^\varepsilon(r)|+|u_h^\varepsilon(s)|\le N$ in each integral $I_{n,j}$. We use this observation in the considerations below.

The Burkholder-Davis-Gundy inequality and (**C2**) yield for $0\le\varepsilon\le\varepsilon_0$

$$\begin{aligned}
|I_{n,1}| &\le 6\sqrt{\varepsilon}\int_0^T ds\,\mathbb{E}\Big(\int_s^{\psi_n(s)} |\sigma(u_h^\varepsilon(r))|_{L_Q}^2 1_{G_N(r)}\,|u_h^\varepsilon(r)-u_h^\varepsilon(s)|^2\,dr\Big)^{\frac12}\\
&\le 6\sqrt{2\varepsilon_0 N}\int_0^T ds\,\mathbb{E}\Big(\int_s^{\psi_n(s)} [K_0+K_1\,|u_h^\varepsilon(r)|^2+K_2\,\|u_h^\varepsilon(r)\|^2]\,\,dr\Big)^{\frac12}.
\end{aligned}$$

Schwarz's inequality and Fubini's theorem as well as (3.2), which holds uniformly in $\varepsilon\in\,]0,\varepsilon_0]$ for fixed $\varepsilon_0>0$ since the constants K_i and L_i are multiplied by at most ε_0, imply

$$\begin{aligned}
|I_{n,1}| &\le 6\sqrt{2\varepsilon_0 NT}\,\Big[\mathbb{E}\int_0^T\big(K_0+K_1\,|u_h^\varepsilon(r)|^2+K_2\,\|u_h^\varepsilon(r)\|^2\big)\,\Big(\int_{(r-c2^{-n})\vee 0}^r ds\Big)\,dr\Big]^{\frac12}\\
&\le C2^{-\frac{n}{2}} \qquad (4.5)
\end{aligned}$$

for some constant C depending only on K_i, $\tilde K_i$, L_i, $\tilde L_i$, M, ε_0, N and T. The property (**C2**) and Fubini's theorem imply that for $0\le\varepsilon\le\varepsilon_0$,

$$\begin{aligned}
|I_{n,2}| &\le \varepsilon\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\psi_n(s)}\big(K_0+K_1|u_h^\varepsilon(r)|^2+K_2\|u_h^\varepsilon(r)\|^2\big)dr\Big)\\
&\le \varepsilon_0\mathbb{E}\int_0^T 1_{G_N(T)}\,(K_0+K_1\,N+K_2\|u_h^\varepsilon(r)\|^2)\,c2^{-n}\,dr\;\le\;C2^{-n} \qquad (4.6)
\end{aligned}$$

for some constant C depending on K_i, ε_0, N and T. Schwarz's inequality, Fubini's theorem, (**C2**) and the definition (2.25) of $\mathcal{A}_M$ yield

$$\begin{aligned}
|I_{n,3}| &\le 2\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\\
&\quad\times\int_s^{\psi_n(s)}\big(\tilde K_0+\tilde K_1|u_h^\varepsilon(r)|^2+\tilde K_2\|u_h^\varepsilon(r)\|^2\big)^{\frac12}\,|h(r)|_0|\,u_h^\varepsilon(r)-u_h^\varepsilon(s)|\,dr\Big)\\
&\le 4\sqrt{N}\,\mathbb{E}\int_0^T 1_{G_N(T)}|h(r)|_0(\tilde K_0+\tilde K_1N+\tilde K_2\|u_h^\varepsilon(r)\|^2)^{\frac12}\,\Big(\int_{(r-c2^{-n})\vee 0}^r ds\Big)\,dr\\
&\le 4\sqrt{N}\,c2^{-n}\sqrt{M}\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T(\tilde K_0+\tilde K_1N+\tilde K_2\|u_h^\varepsilon(r)\|^2)\,dr\Big)^{\frac12}\le C\,2^{-n}. \qquad (4.7)
\end{aligned}$$

Using Schwarz's inequality we deduce that

$$I_{n,4}\;\le\;2\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\psi_n(s)} dr\big[-\|u_h^\varepsilon(r)\|^2+\|u_h^\varepsilon(r)\|\|u_h^\varepsilon(s)\|\|\big]\Big)$$

$$\leq \frac{1}{2}\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\,\|u_h^\varepsilon(s)\|^2\int_s^{\psi_n(s)} dr\Big) \leq c\,N\,2^{-(n+1)}. \tag{4.8}$$

The antisymmetry relation (2.2) and inequality (2.7) yields

$$\big|\big\langle B(u_h^\varepsilon(r))\,,\,u_h^\varepsilon(r)-u_h^\varepsilon(s)\big\rangle\big| = \big|\big\langle B(u_h^\varepsilon(r))\,,\,u_h^\varepsilon(s)\big\rangle\big| \leq \|u_h^\varepsilon(r)\|^2 + C|u_h^\varepsilon(r)|^2\|u_h^\varepsilon(s)\|_{\mathcal{H}}^4.$$

Therefore,

$$\begin{aligned}|I_{n,5}| &\leq 2\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\psi_n(s)} dr\;\|u_h^\varepsilon(r)\|^2\Big)\\ &\quad + 2C\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\|u_h^\varepsilon(s)\|_{\mathcal{H}}^4\int_s^{\psi_n(s)} dr|u_h^\varepsilon(r)|^2\Big) \equiv I_{n,5}^{(1)} + I_{n,5}^{(2)}.\end{aligned} \tag{4.9}$$

Fubini's theorem implies

$$\begin{aligned}I_{n,5}^{(1)} &\leq 2\mathbb{E}\Big(1_{G_N(T)}\int_0^T dr\;\|u_h^\varepsilon(r)\|^2\int_{(r-c2^{-n})\vee 0}^{r} ds\Big)\\ &\leq 2c\,2^{-n}\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T dr\;\|u_h^\varepsilon(r)\|^2\Big) \leq CN2^{-n}.\end{aligned} \tag{4.10}$$

Using (2.3), we deduce that on $G_N(T)$ we have

$$\int_0^T \|u_h^\varepsilon(s)\|_{\mathcal{H}}^4 ds \leq a_0 \sup_{s\in[0,T]} |u_h^\varepsilon(s)|^2 \int_0^T \|u_h^\varepsilon(s)\|^2 ds \leq a_0 N^2.$$

Thus

$$I_{n,5}^{(2)} \leq 2CN\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\|u_h^\varepsilon(s)\|_{\mathcal{H}}^4\Big)c2^{-n} \leq 2a_0CN^3c2^{-n}. \tag{4.11}$$

Finally, Schwarz's inequality implies that

$$|I_{n,6}| \leq 2\,\mathbb{E}\Big[1_{G_N(T)}\int_0^T ds\int_s^{\psi_n(s)} (R_0 + R_1|u_h^\varepsilon(r)|)\,(|u_h^\varepsilon(r)| + |u_h^\varepsilon(s)|)\;dr\Big] \leq C2^{-n}. \tag{4.12}$$

Collecting the upper estimates from (4.5)-(4.12), we conclude the proof of (4.4). □

Remark 4.4. The preceding lemma has been formulated in a general framework to be used in a forthcoming paper about the support characterization of the solution to the stochastic equation (2.24). In the setting of large deviations, we will use it in the following particular case. For any integer n define a step function $s \mapsto \bar{s}_n$ on $[0,T]$ by the formula

$$\bar{s}_n = t_{k+1} \equiv (k+1)T2^{-n} \quad \text{for} \quad s \in [kT2^{-n},(k+1)T2^{-n}[. \tag{4.13}$$

Then the map $\psi_n(s) = \bar{s}_n$ clearly satisfies the previous requirements with $c = T$.

Now we return to the setting of Theorem 4.2.

Let $\varepsilon_0 > 0$, $(h_\varepsilon, 0 < \varepsilon \leq \varepsilon_0)$ be a family of random elements taking values in the set $\mathcal{A}_M$ given by (2.25). Let u_{h_ε}, or strictly speaking, $u_{h_\varepsilon}^\varepsilon$, be the solution of the corresponding stochastic control equation with initial condition $u_{h_\varepsilon}(0) = \xi \in H$:

$$du_{h_\varepsilon} + [Au_{h_\varepsilon} + B(u_{h_\varepsilon}) + \tilde{R}(t,u_{h_\varepsilon})]dt = \sigma(t,u_{h_\varepsilon})h_\varepsilon(t)dt + \sqrt{\varepsilon}\,\sigma(t,u_{h_\varepsilon})dW(t). \tag{4.14}$$

Note that $u_{h_\varepsilon} = \mathcal{G}^\varepsilon\Big(\sqrt{\varepsilon}\big(W_\cdot + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h_\varepsilon(s)ds\big)\Big)$ due to the uniqueness of the solution. The following proposition establishes the weak convergence of the family (u_{h_ε}) as $\varepsilon \to 0$. Its proof is similar to that of Proposition 4.3 in [13], but allows time dependent coefficients $\tilde{R}$ and σ.

Proposition 4.5. *Suppose that the conditions* (**C1**) *and* (**C2**) *are satisfied with* $K_2 = L_2 = 0$. *Suppose furthermore that* $\tilde{R}$ *and* σ *satisfy the conditions* **(C3)(ii)** *and* **(C4)**. *Let* ξ *be* $\mathcal{F}_0$*-measurable such that* $E|\xi|_H^4 < +\infty$, *and let* h_ε *converge to* h *in distribution as random elements taking values in* $\mathcal{A}_M$, *where this set is defined by* (2.25) *and endowed with the weak topology of the space* $L_2(0,T;H_0)$. *Then as* $\varepsilon \to 0$, *the solution* u_{h_ε} *of* (4.14) *converges in distribution to the solution* u_h *of* (4.2) *in* $X = C([0,T];H) \cap L^2((0,T);V)$ *endowed with the norm (3.1). That is, as* $\varepsilon \to 0$, $\mathcal{G}^\varepsilon\Big(\sqrt{\varepsilon}\big(W_. + \frac{1}{\sqrt{\varepsilon}}\int_0^. h_\varepsilon(s)ds\big)\Big)$ *converges in distribution to* $\mathcal{G}^0\big(\int_0^. h(s)ds\big)$ *in* X.

Proof. Since $\mathcal{A}_M$ is a Polish space (complete separable metric space), by the Skorokhod representation theorem, we can construct processes $(\tilde{h}_\varepsilon, \tilde{h}, \tilde{W})$ such that the joint distribution of $(\tilde{h}_\varepsilon, \tilde{W})$ is the same as that of (h_ε, W), the distribution of $\tilde{h}$ coincides with that of h, and $\tilde{h}_\varepsilon \to \tilde{h}$, a.s., in the (weak) topology of S_M. Hence a.s. for every $t \in [0,T]$, $\int_0^t \tilde{h}_\varepsilon(s)ds - \int_0^t \tilde{h}(s)ds \to 0$ weakly in H_0. To lighten notations, we will write $(\tilde{h}_\varepsilon, \tilde{h}, \tilde{W}) = (h_\varepsilon, h, W)$. Let $U_\varepsilon = u_{h_\varepsilon} - u_h$; then $U_\varepsilon(0) = 0$ and

$$\begin{aligned} dU_\varepsilon + \big[AU_\varepsilon + B(u_{h_\varepsilon}) - B(u_h) + \tilde{R}(t,u_{h_\varepsilon}) - \tilde{R}(t,u_h)\big]dt \\ = \big[\sigma(t,u_{h_\varepsilon})h_\varepsilon - \sigma(t,u_h)h\big]dt + \sqrt{\varepsilon}\,\sigma(t,u_{h_\varepsilon})dW(t). \end{aligned} \tag{4.15}$$

On any finite time interval $[0,t]$ with $t \le T$, Itô's formula, (2.8) with $\eta = \frac{1}{2}$ and condition (**C2**) yield for $\varepsilon \ge 0$:

$$\begin{aligned} |U_\varepsilon(t)|^2 &+ 2\int_0^t \|U_\varepsilon(s)\|^2\,ds = -2\int_0^t \big\langle B(u_{h_\varepsilon}(s)) - B(u_h(s))\,,\,U_\varepsilon(s)\big\rangle ds \\ &- 2\int_0^t \big(\tilde{R}(s,u_{h_\varepsilon}(s)) - \tilde{R}(s,u_h(s))\,,\,U_\varepsilon(s)\big)\,ds \\ &+ 2\int_0^t \big(\sigma(s,u_{h_\varepsilon}(s))h_\varepsilon(s) - \sigma(s,u_h(s))\,h(s), U_\varepsilon(s)\big)ds \\ &+ 2\sqrt{\varepsilon}\int_0^t \big(U_\varepsilon(s), \sigma(s,u_{h_\varepsilon}(s))dW(s)\big) + \varepsilon\int_0^t |\sigma(s,u_{h_\varepsilon}(s))|_{L_Q}^2\,ds \\ &\le \int_0^t \|U_\varepsilon(s)\|^2\,ds + \sum_{i=1}^3 T_i(t,\varepsilon) + 2\int_0^t \big(C_{\frac{1}{2}}\,\|u_h(s)\|_{\mathcal{H}}^4 + R_1 + \sqrt{L_1}|h_\varepsilon(s)|_0\big)|U_\varepsilon(s)|^2 ds, \end{aligned}$$

where

$$\begin{aligned} T_1(t,\varepsilon) &= 2\sqrt{\varepsilon}\int_0^t \big(U_\varepsilon(s), \sigma(s,u_{h_\varepsilon}(s))\,dW(s)\big), \\ T_2(t,\varepsilon) &= \varepsilon\int_0^t (K_0 + K_1|u_{h_\varepsilon}(s)|^2)ds, \\ T_3(t,\varepsilon) &= 2\int_0^t \Big(\sigma(s,u_h(s))\,\big(h_\varepsilon(s) - h(s)\big),\,U_\varepsilon(s)\Big)\,ds. \end{aligned}$$

This yields the following inequality

$$|U_\varepsilon(t)|^2 + \int_0^t \|U_\varepsilon(s)\|^2 ds \le \sum_{i=1}^3 T_i(t,\varepsilon) + 2\int_0^t \Big[C_{\frac{1}{2}}\,\|u_h(s)\|_{\mathcal{H}}^4 + R_1 + \sqrt{L_1}|h_\varepsilon(s)|_0\Big]|U_\varepsilon(s)|^2\,ds. \tag{4.16}$$

We want to show that as $\varepsilon \to 0$, $\|U_\varepsilon\|_X \to 0$ in probability, which implies that $u_{h_\varepsilon} \to u_h$ in distribution in X. Fix $N>0$ and for $t\in[0,T]$ let

$$G_N(t) = \Big\{ \sup_{0\le s\le t} |u_h(s)|^2 \le N\Big\} \cap \Big\{ \int_0^t \|u_h(s)\|^2 ds \le N \Big\},$$

$$G_{N,\varepsilon}(t) = G_N(t) \cap \Big\{ \sup_{0\le s\le t} |u_{h_\varepsilon}(s)|^2 \le N\Big\} \cap \Big\{ \int_0^t \|u_{h_\varepsilon}(s)\|^2 ds \le N \Big\}.$$

The proof consists in two steps.

Step 1: For any $\varepsilon_0 \in]0,1]0$, $\sup_{0<\varepsilon\le\varepsilon_0} \sup_{h,h_\varepsilon\in\mathcal{A}_M} \mathbb{P}(G_{N,\varepsilon}(T)^c) \to 0$ as $N\to\infty$.

Indeed, for $\varepsilon\in]0,\varepsilon_0]$, $h,h_\varepsilon \in \mathcal{A}_M$, the Markov inequality and the a priori estimate (3.2), which holds uniformy in $\varepsilon\in]0,\varepsilon_0]$, imply

$$\begin{aligned}
\mathbb{P}(G_{N,\varepsilon}(T)^c) &\le \mathbb{P}\Big(\sup_{0\le s\le T}|u_h(s)|^2 > N\Big) + \mathbb{P}\Big(\sup_{0\le s\le T}|u_{h_\varepsilon}(s)|^2 > N\Big) \\
&\quad + \mathbb{P}\,\Big(\int_0^T \|u_h(s)\|^2 ds > N\Big) + \mathbb{P}\Big(\int_0^T \|u_{h_\varepsilon}(s)\|^2) ds > N\Big) \\
&\le \frac{1}{N} \sup_{h,h_\varepsilon\in\mathcal{A}_M} \mathbb{E}\Big(\sup_{0\le s\le T}|u_h(s)|^2 + \sup_{0\le s\le T}|u_{h_\varepsilon}(s)|^2 + \int_0^T (\|u_h(s)\|^2 + \|u_{h_\varepsilon}(s)\|^2) ds\Big) \\
&\le C\,\big(1+E|\xi|^4\big) N^{-1}, \qquad (4.17)
\end{aligned}$$

for some constant C depending on T and M.

Step 2: For fixed $N>0$, $h,h_\varepsilon\in\mathcal{A}_M$ such that as $\varepsilon\to 0$, $h_\varepsilon\to h$ a.s. in the weak topology of $L^2(0,T;H_0)$; then one has as $\varepsilon\to 0$:

$$\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \Big(\sup_{0\le t\le T} |U_\varepsilon(t)|^2 + \int_0^T \|U_\varepsilon(t)\|^2\,dt\Big)\Big] \to 0. \qquad (4.18)$$

Indeed, (4.16) and Gronwall's lemma imply that on $G_{N,\varepsilon}(T)$,

$$\sup_{0\le t\le T}|U_\varepsilon(t)|^2 \le \Big[\sup_{0\le t\le T} \big(T_1(t,\varepsilon)+T_3(t,\varepsilon)\big) + \varepsilon C_*\Big] \exp\Big(2a_0 C_{\frac12} N^2 + 2R_1 T + 2\sqrt{L_1 M T}\Big),$$

where $C_* = T(K_0+K_1N)$. We also use here the fact that by (2.3)

$$\int_0^T \|u_h(s)\|_{\mathcal{H}}^4 ds \le a_0 \sup_{s\in[0,T]} |u_h(s)|^2 \int_0^T \|u_h(s)\|^2 ds \le a_0 N^2 \quad \text{on } G_{N,\varepsilon}(T).$$

Using again (4.16) we deduce that for some constant $\tilde{C} = C(T,M,N)$, one has for every $\varepsilon>0$:

$$\mathbb{E}\big(1_{G_{N,\varepsilon}(T)}\,\|U_\varepsilon\|_X^2\big) \le \tilde{C}\Big(\varepsilon + \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \sup_{0\le t\le T}\big(T_1(t,\varepsilon)+T_3(t,\varepsilon)\big)\Big]\Big). \qquad (4.19)$$

Since the sets $G_{N,\varepsilon}(.)$ decrease, $\mathbb{E}\big(1_{G_{N,\varepsilon}(T)} \sup_{0\le t\le T}|T_1(t,\varepsilon)|\big) \le \mathbb{E}(\lambda_\varepsilon)$, where

$$\lambda_\varepsilon := 2\sqrt{\varepsilon} \sup_{0\le t\le T} \Big| \int_0^t 1_{G_{N,\varepsilon}(s)} \Big(U_\varepsilon(s), \sigma(s,u_{h_\varepsilon}(s)) dW(s)\Big)\Big|.$$

The scalar-valued random variables λ_ε converge to 0 in L^1 as $\varepsilon\to 0$. Indeed, by the Burkholder-Davis-Gundy inequality, (**C2**) and the definition of $G_{N,\varepsilon}(s)$, we have

$$\begin{aligned}
\mathbb{E}(\lambda_\varepsilon) &\le 6\sqrt{\varepsilon}\,\mathbb{E}\Big\{ \int_0^T 1_{G_{N,\varepsilon}(s)}\,|U_\varepsilon(s)|^2\,|\sigma(s,u_{h_\varepsilon}(s))|_{L_Q}^2 ds\Big\}^{\frac12} \\
&\le 6\sqrt{\varepsilon}\,\mathbb{E}\Big[\Big\{4N \int_0^T 1_{G_{N,\varepsilon}(s)}\,(K_0 + K_1|u_{h_\varepsilon}(s)|^2) ds\Big\}^{\frac12}\Big] \le C(T,N)\,\sqrt{\varepsilon}. \qquad (4.20)
\end{aligned}$$

In further estimates we use Lemma 4.3 with $\psi_n = \bar{s}_n$, where $\bar{s}_n$ is the step function defined in Remark 4.4. For any $n, N \geq 1$, if we set $t_k = kT2^{-n}$ for $0 \leq k \leq 2^n$, we obviously have

$$\mathbb{E}\Big(1_{G_{N,\varepsilon}(T)} \sup_{0\leq t\leq T} |T_3(t,\varepsilon)|\Big) \leq 2 \sum_{i=1}^{4} \tilde{T}_i(N,n,\varepsilon) + 2\,\mathbb{E}\big(\bar{T}_5(N,n,\varepsilon)\big), \tag{4.21}$$

where

$$\begin{aligned}
\tilde{T}_1(N,n,\varepsilon) =& \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \sup_{0\leq t\leq T} \Big| \int_0^t \Big(\sigma(s,u_h(s))\big(h_\varepsilon(s)-h(s)\big)\,,\, \big[U_\varepsilon(s)-U_\varepsilon(\bar{s}_n)\big]\Big) ds\Big|\Big],\\
\tilde{T}_2(N,n,\varepsilon) =& \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\\
&\times \sup_{0\leq t\leq T} \Big| \int_0^t \Big([\sigma(s,u_h(s))-\sigma(\bar{s}_n,u_h(s))](h_\varepsilon(s)-h(s))\,,\, U_\varepsilon(\bar{s}_n)\Big) ds\Big|\Big],\\
\tilde{T}_3(N,n,\varepsilon) =& \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\\
&\times \sup_{0\leq t\leq T} \Big| \int_0^t \Big(\big[\sigma(\bar{s}_n,u_h(s))-\sigma(\bar{s}_n,u_h(\bar{s}_n))\big]\big(h_\varepsilon(s)-h(s)\big)\,,\, U_\varepsilon(\bar{s}_n)\Big) ds\Big|\Big],\\
\tilde{T}_4(N,n,\varepsilon) =& \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \sup_{1\leq k\leq 2^n} \sup_{t_{k-1}\leq t\leq t_k} \Big|\Big(\sigma(t_k,u_h(t_k)) \int_{t_{k-1}}^t (h_\varepsilon(s)-h(s))\,ds\,,\, U_\varepsilon(t_k)\Big)\Big|\Big]\\
\bar{T}_5(N,n,\varepsilon) =& 1_{G_{N,\varepsilon}(T)} \sum_{k=1}^{2^n} \Big|\Big(\sigma(t_k,u_h(t_k)) \int_{t_{k-1}}^{t_k} \big(h_\varepsilon(s)-h(s)\big)\,ds\,,\, U_\varepsilon(t_k)\Big)\Big|.
\end{aligned}$$

Using Schwarz's inequality, (**C2**) and Lemma 4.3 with $\psi_n = \bar{s}_n$, we deduce that for some constant $\bar{C}_1 := C(T,M,N)$ and any $\varepsilon \in]0,\varepsilon_0]$,

$$\begin{aligned}
\tilde{T}_1(N,n,\varepsilon) &\leq \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \int_0^T \big(K_0+K_1|u_h(s)|^2\big)^{\frac12} |h_\varepsilon(s)-h(s)|_0\, \big|U_\varepsilon(s)-U_\varepsilon(\bar{s}_n)\big|\,ds\Big]\\
&\leq \Big(\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \int_0^T \big\{|u_{h_\varepsilon}(s)-u_{h_\varepsilon}(\bar{s}_n)|^2 + |u_h(s)-u_h(\bar{s}_n)|^2\big\}\,ds\Big]\Big)^{\frac12}\\
&\quad\times \sqrt{2(K_0+K_1N)}\, \Big(\mathbb{E}\int_0^T |h_\varepsilon(s)-h(s)|_0^2\,ds\Big)^{\frac12} \leq \bar{C}_1\, 2^{-\frac{n}{4}}.
\end{aligned} \tag{4.22}$$

A similar computation based on (**C2**) and Lemma 4.3 yields for some constant $\bar{C}_3 := C(T,M,N)$ and any $\varepsilon \in]0,\varepsilon_0]$

$$\begin{aligned}
\tilde{T}_3(N,n,\varepsilon) &\leq \sqrt{2NL_1}\Big(\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \int_0^T |u_h(s)-u_h(\bar{s}_n)|^2\,ds\Big]\Big)^{\frac12} \Big(\mathbb{E}\int_0^T |h_\varepsilon(s)-h(s)|_0^2 ds\Big)^{\frac12}\\
&\leq \bar{C}_3\, 2^{-\frac{n}{4}}.
\end{aligned} \tag{4.23}$$

The Hölder regularity **(C4)** imposed on $\sigma(.,u)$ and Schwarz's inequality imply that

$$\tilde{T}_2(N,n,\varepsilon) \leq C\sqrt{N}\, 2^{-n\gamma}\, \mathbb{E}\Big(1_{G_{N,\varepsilon}(T)} \int_0^T (1+\|u_h(s)\|)\, |h_\varepsilon(s)-h(s)|\,ds\Big) \leq \bar{C}_2 2^{-n\gamma} \tag{4.24}$$

for some constant $\bar{C}_2 = C(T,M,N)$. Using Schwarz's inequality and (**C2**) we deduce for $\bar{C}_4 = C(N,M)$ and any $\varepsilon \in]0,\varepsilon_0]$

$$\tilde{T}_4(N,n,\varepsilon) \leq \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \sup_{1\leq k\leq 2^n} \big(K_0+K_1|u_h(t_k)|^2\big)^{\frac12} \int_{t_{k-1}}^{t_k} |h_\varepsilon(s)-h(s)|_0\,ds\, |U_\varepsilon(t_k)|\Big]$$

$$\leq 2\sqrt{N(K_0+K_1N)}\,\mathbb{E}\Big(\sup_{1\leq k\leq 2^n}\int_{t_{k-1}}^{t_k}|h_\varepsilon(s)-h(s)|_0\,ds\Big)\leq 4\bar{C}_4\,2^{-\frac{n}{2}}. \quad (4.25)$$

Finally, note that the weak convergence of h_ε to h implies that for any $a,b\in[0,T]$, $a<b$, the integral $\int_a^b h_\varepsilon(s)ds\to\int_a^b h(s)ds$ in the weak topology of H_0. Therefore, since for the operator $\sigma(t_k,u_h(t_k))$ is compact from H_0 to H, we deduce that for every k,

$$\Big|\sigma(t_k,u_h(t_k))\Big(\int_{t_{k-1}}^{t_k}h_\varepsilon(s)ds-\int_{t_{k-1}}^{t_k}h(s)ds\Big)\Big|_H\to 0 \;\text{ as }\; \varepsilon\to 0.$$

Hence a.s. for fixed n as $\varepsilon\to 0$, $\bar{T}_5(N,n,\varepsilon,\omega)\to 0$. Furthermore, $\bar{T}_5(N,n,\varepsilon,\omega)\leq C(K_0,K_1,N,M)$ and hence the dominated convergence theorem proves that for any fixed n,N, $\mathbb{E}(\bar{T}_4(N,n,\varepsilon))\to 0$ as $\varepsilon\to 0$.

Thus, (4.21)–(4.25) imply that for any fixed $N\geq 1$ and any integer $n\geq 1$

$$\limsup_{\varepsilon\to 0}\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\sup_{0\leq t\leq T}|T_3(t,\varepsilon)|\Big]\leq C_{N,T,M}\;2^{-n(\gamma\wedge\frac{1}{4})}.$$

Since n is arbitrary, this yields for any integer $N\geq 1$:

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\sup_{0\leq t\leq T}|T_3(t,\varepsilon)|\Big]=0.$$

Therefore from (4.19) and (4.20) we obtain (4.18). By the Markov inequality

$$\mathbb{P}(\|U_\varepsilon\|_X>\delta)\leq\mathbb{P}(G_{N,\varepsilon}(T)^c)+\frac{1}{\delta^2}\mathbb{E}\Big(1_{G_{N,\varepsilon}(T)}\|U_\varepsilon\|_X^2\Big)\;\text{ for any }\;\delta>0.$$

Finally, (4.17) and (4.18) yield that for any integer $N\geq 1$,

$$\limsup_{\varepsilon\to 0}\mathbb{P}(\|U_\varepsilon\|_X>\delta)\leq C(T,M)N^{-1},$$

for some constant $C(T,M)$ which does not depend on N. This implies $\lim_{\varepsilon\to 0}\mathbb{P}(\|U_\varepsilon\|_X>\delta)=0$ for any $\delta>0$, which concludes the proof of the proposition. □

The following compactness result is the second ingredient which allows to transfer the LDP from $\sqrt{\varepsilon}W$ to u^ε. Its proof is similar to that of Proposition 4.5 and easier; it will be sketched (see also [13], Proposition 4.4).

Proposition 4.6. *Suppose that* (**C1**) *and* (**C2**) *hold with* $K_2=L_2=0$ *and that conditions* **(C3)(ii)** *and* **(C4)** *hold. Fix* $M>0$, $\xi\in H$ *and let* $K_M=\{u_h\in X: h\in S_M\}$, *where* u_h *is the unique solution of the deterministic control equation* (4.2) *and* $X=C([0,T];H)\cap L^2(0,T;V)$. *Then* K_M *is a compact subset of* X.

Proof. Let $\{u_n\}$ be a sequence in K_M, corresponding to solutions of (4.2) with controls $\{h_n\}$ in S_M:

$$du_n(t)+\big[Au_n(t)+B(u_n(t))+\tilde{R}(t,u_n(t))\big]dt=\sigma(t,u_n(t))h_n(t)dt,\;\; u_n(0)=\xi.$$

Since S_M is a bounded closed subset in the Hilbert space $L^2(0,T;H_0)$, it is weakly compact. So there exists a subsequence of $\{h_n\}$, still denoted as $\{h_n\}$, which converges weakly to a limit h in $L^2(0,T;H_0)$. Note that in fact $h\in S_M$ as S_M is closed. We now show that the corresponding subsequence of solutions, still denoted as $\{u_n\}$, converges in X to u which is the solution of the following "limit" equation

$$du(t)+[Au(t)+B(u(t))+\tilde{R}(t,u(t))]dt=\sigma(t,u(t))h(t)dt,\;\; u(0)=\xi.$$

This will complete the proof of the compactness of K_M. To ease notation we will often drop the time parameters s, t, ... in the equations and integrals.

Let $U_n = u_n - u$; using (2.8) with $\eta = \frac{1}{2}$, (**C2**) and Young's inequality, we deduce that for $t \in [0,T]$,

$$\begin{aligned}
&|U_n(t)|^2 + 2\int_0^t \|U_n(s)\|^2 ds = \\
&\quad - 2\int_0^t \big\langle B(u_n(s)) - B(u(s)), U_n(s)\big\rangle\, ds - 2\int_0^t \big(\tilde{R}(s,u_{h_\varepsilon}(s)) - \tilde{R}(s,u_h(s)), U_n(s)\big) ds \\
&\quad + 2\int_0^t \Big\{ \Big(\big[\sigma(s,u_n(s)) - \sigma(s,u(s))\big] h_n(s), U_n(s)\Big) \\
&\qquad\qquad + \big(\sigma(s,u(s))\big(h_n(s)-h(s)\big)\,,\, U_n(s)\big)\Big\} ds \\
&\leq \int_0^t \|U_n(s)\|^2 ds + 2\int_0^t |U_n(s)|^2 \big(C_{\frac{1}{2}} \|u(s)\|_{\mathcal{H}}^4 + R_1 + \sqrt{L_1}\, |h_n(s)|_0\big)\, ds \\
&\quad + 2\int_0^t \Big(\sigma(s,u(s))\,[h_n(s)-h(s)]\,,\, U_n(s)\Big)\, ds.
\end{aligned} \tag{4.26}$$

The inequality (3.2) implies that there exists a finite positive constant $\bar{C}$ such that

$$\sup_n \Big[\sup_{0\leq t\leq T} \big(|u(t)|^2 + |u_n(t)|^2\big) + \int_0^T \big(\|u(s)\|^2 + \|u(s)\|_{\mathcal{H}}^4 + \|u_n(s)\|^2\big) ds\Big] = \bar{C}. \tag{4.27}$$

Thus Gronwall's lemma implies that

$$\sup_{t\leq T} |U_n(t)|^2 + \int_0^T \|U_n(t)\|^2\, dt \leq \exp\Big(2\big(C_{\frac{1}{2}}\bar{C} + R_1\, T + \sqrt{L_1 M T}\big)\Big) \sum_{i=1}^5 I_{n,N}^i, \tag{4.28}$$

where, as in the proof of Proposition 4.5, we have:

$$\begin{aligned}
I_{n,N}^1 &= \int_0^T \big|\big(\sigma(s,u(s))\,[h_n(s)-h(s)]\,,\, U_n(s) - U_n(\bar{s}_N)\big)\big|\, ds, \\
I_{n,N}^2 &= \int_0^T \Big|\Big(\big[\sigma(s,u(s)) - \sigma(\bar{s}_N,u(s))\big]\,[h_n(s)-h(s)]\,,\, U_n(\bar{s}_N)\Big)\Big|\, ds, \\
I_{n,N}^3 &= \int_0^T \Big|\Big(\big[\sigma(\bar{s}_N,u(s)) - \sigma(\bar{s}_N,u(\bar{s}_N))\big]\,[h_n(s)-h(s)]\,,\, U_n(\bar{s}_N)\Big)\Big|\, ds, \\
I_{n,N}^4 &= \sup_{1\leq k\leq 2^N} \sup_{t_{k-1}\leq t\leq t_k} \Big|\Big(\sigma(t_k,u(t_k)) \int_{t_{k-1}}^t (h_\varepsilon(s) - h(s)) ds\ ,\ U_n(t_k)\Big)\Big|, \\
I_{n,N}^5 &= \sum_{k=1}^{2^N} \Big(\sigma(t_k,u(t_k)) \int_{t_{k-1}}^{t_k} [h_n(s) - h(s)]\, ds\ ,\ U_n(t_k)\Big).
\end{aligned}$$

Schwarz's inequality, (**C2**) and Lemma 4.3 imply that for some constants C_1 and C_2 which do not depend on n and N,

$$\begin{aligned}
I_{n,N}^1 &\leq C_1 \Big(\int_0^T |h_n(s) - h(s)|_0^2 ds\Big)^{\frac{1}{2}} \Big(\int_0^T \big(|u_n(s) - u_n(\bar{s}_N)|^2 + |u(s) - u(\bar{s}_N)|^2\big) ds\Big)^{\frac{1}{2}} \\
&\leq C_2\, 2^{-\frac{N}{4}}\,,
\end{aligned} \tag{4.29}$$

$$I_{n,N}^3 \leq C_1 \Big(\int_0^T |u(s) - u(\bar{s}_N)|^2 ds\Big)^{\frac{1}{2}} \Big(\int_0^T |h_n(s) - h(s)|_0^2\, ds\Big)^{\frac{1}{2}} \leq C_3\, 2^{-\frac{N}{4}}\,, \tag{4.30}$$

$$I_{n,N}^4 \leq C_1 \Big[1 + \Big(\sup_{0\leq t\leq T} |u(t)|\Big) \sup_{0\leq t\leq T} \big(|u(t)| + |u_n(t)|\big)\Big]\, 2^{-\frac{N}{2}} \leq C_4\, 2^{-\frac{N}{2}}\,. \tag{4.31}$$

Condition **(C4)** implies that

$$I_{n,N}^2 \le C2^{-n\gamma} \sup_{0\le t\le T} \big(|u(t)|+|u_n(t)|\big) \int_0^T (1+\|u(s)\|)(|h(s)|_0+|h_n(s)|_0)\,ds \le C_2\,2^{-N\gamma}. \tag{4.32}$$

For fixed N and $k=1,\cdots,2^N$, as $n\to\infty$, the weak convergence of h_n to h implies that of $\int_{t_{k-1}}^{t_k}(h_n(s)-h(s))ds$ to 0 weakly in H_0. Since $\sigma(u(t_k))$ is a compact operator, we deduce that for fixed k the sequence $\sigma(u(t_k))\int_{t_{k-1}}^{t_k}(h_n(s)-h(s))ds$ converges to 0 strongly in H as $n\to\infty$. Since $\sup_{n,k}|U_n(t_k)|\le 2\sqrt{\bar{C}}$, we have $\lim_n I_{n,N}^5=0$. Thus (4.28)–(4.32) yield for every integer $N\ge 1$

$$\limsup_{n\to\infty}\left\{\sup_{t\le T}|U_n(t)|^2+\int_0^T\|U_n(t)\|^2\,dt\right\}\le C2^{-N(\gamma\wedge\frac{1}{4})}.$$

Since N is arbitrary, we deduce that $\|U_n\|_X\to 0$ as $n\to\infty$. This shows that every sequence in K_M has a convergent subsequence. Hence K_M is a sequentially relatively compact subset of X. Finally, let $\{u_n\}$ be a sequence of elements of K_M which converges to v in X. The above argument shows that there exists a subsequence $\{u_{n_k}, k\ge 1\}$ which converges to some element $u_h\in K_M$ for the same topology of X. Hence $v=u_h$, K_M is a closed subset of X, and this completes the proof of the proposition. □

Proof of Theorem 4.2: Propositions 4.6 and 4.5 imply that the family $\{u^\varepsilon\}$ satisfies the Laplace principle, which is equivalent to the large deviation principle, in X with the rate function defined by (4.3); see Theorem 4.4 in [3] or Theorem 5 in [4]. This concludes the proof of Theorem 4.2. □

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