On distribution of energy and vorticity for solutions of 2D Navier-Stokes equation with small viscosity

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Abstract

We study distributions of some functionals of space-periodic solutions for the randomly perturbed 2D Navier-Stokes equation, and of their limits when the viscosity goes to zero. The results obtained give explicit information on distribution of the velocity field of space-periodic turbulent 2D flows.

0 Introduction

We consider the 2D Navier-Stokes equation (NSE) under periodic boundary conditions, perturbed by a random force:

$$\begin{aligned}&v'_\tau - \varepsilon\Delta v + (v\cdot\nabla)v + \nabla\tilde p = \varepsilon^a\,\tilde\eta(\tau,x),\\ &\operatorname{div} v = 0,\quad v = v(\tau,x)\in\mathbb{R}^2,\quad \tilde p = \tilde p(\tau,x),\quad x\in\mathbb{T}^2 = \mathbb{R}^2/(l_1\mathbb{Z}\times l_2\mathbb{Z}).\end{aligned}\tag{0.1}$$

Here $0<\varepsilon\ll 1$, the scaling exponent a is a real number and $l_1,l_2>0$. We assume that $a<\frac32$ since $a\geq\frac32$ corresponds to non-interesting equations with small solutions(see [Kuk06a], Section 10.3). It is also assumed that $\int v\,dx\equiv\int\tilde\eta\,dx\equiv 0$ and that the force $\tilde\eta$ is a divergence-free Gaussian random field, white in time and smooth in x. Under mild non-degeneracy assumption on $\tilde\eta$ (see in Section 1) the Markov process which the equation defines in the function space $\mathcal{H}$,

$$\mathcal{H} = \{u(x)\in L^2(\mathbb{T}^2;\mathbb{R}^2)\mid \operatorname{div} u = 0,\ \int_{\mathbb{T}^2} u\,dx = 0\}\,,$$

has a unique stationary measure. We are interested in asymptotic (as $\varepsilon \to 0$) properties of this measure and of the corresponding stationary solution. The substitution

$$v = \varepsilon^b u\,, \quad \tau = \varepsilon^{-b} t\,, \quad \nu = \varepsilon^{3/2-a}\,,$$

where $b = a - 1/2$, reduces eq. (0.1) to

$$\dot{u} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \sqrt{\nu}\,\eta(t,x), \quad \operatorname{div} u = 0, \tag{0.2}$$

where $\dot{u} = u'_t$ and $\eta(t) = \varepsilon^{b/2}\tilde{\eta}(\varepsilon^{-b}t)$ is a new random field, distributed as $\tilde{\eta}$ (see [Kuk06a]). Below we study eq. (0.2).

Let μ_ν be the unique stationary measure for (0.2) and $u_\nu(t) \in \mathcal{H}$ be the corresponding stationary solution, i.e., $\mathcal{D}u_\nu(t) \equiv \mu_\nu$ (here and below $\mathcal{D}$ signifies the distribution of a random variable). Comparing to other equations (0.1), the equation (0.2) has the special advantage: when $\nu \to 0$ along a subsequence $\{\nu_j\}$, stationary solution u_{ν_j} converges in distribution to a stationary process $U(t) \in \mathcal{H}$, formed by solutions of the Euler equation

$$\dot{u}(t,x) + (u \cdot \nabla)u + \nabla p = 0\,, \quad \operatorname{div} u = 0\,. \tag{0.3}$$

Accordingly, $\mu_{\nu_j} \rightharpoonup \mu_0$, where $\mu_0 = \mathcal{D}U(0)$ is an invariant measure for (0.3) (see below Theorem 1.1). The solution U is called the *Eulerian limit*. This is a random process of order one since $\mathbf{E}|\nabla_x U(t,\cdot)|^2_{\mathcal{H}}$ equals to an explicit non-zero constant. The goal of this paper is to study properties of the measure μ_0 since they are responsible for asymptotic properties of solutions for equation (0.1).

The first main difficulty in this study is to rule out the possibility that with a positive probability the energy $E(u_\nu)$ of the process u_ν, equal to $\frac{1}{2}\int |u_\nu(t,x)|^2\,dx$, becomes very small with ν (and that the energy of the Eulerian limit vanishes with a positive probability). In Section 2 we show that this is not the case and that

$$\mathbf{P}\{E(u_\nu) < \delta\} \le C\delta^{1/4}, \quad \forall\, \delta > 0, \tag{0.4}$$

for each ν. To prove the estimate we develop further some ideas, exploited in [KP08] in a similar situation. Namely, we construct a new process $\tilde{u}_\nu \in \mathcal{H}$, coupled to the process u_ν, such that $E(\tilde{u}_\nu(\tau)) = E(u_\nu(\tau\nu^{-1}))$ and $\tilde{u}_\nu$ satisfies an Ito equation, independent from ν. Next we use Krylov's result [Kry87] on distribution of Ito integrals to estimate $\mathcal{D}\tilde{u}_\nu(\tau)$ and recover (0.4).

In Section 3 we use (0.4) to prove that the distribution of energy of the Eulerian limit U has a density against the Lebesgue measure, i.e.

$$\mathcal{D}E(U) = e(x)\,dx, \quad e \in L_1(\mathbb{R}_+).$$

The functionals $\Phi_f(u(\cdot)) = \int f(\operatorname{rot} u(x))\,dx$ are integrals of motion for the Euler equation. An analogy with the averaging theory for finite-dimensional stochastic equations (e.g., see [FW03]) suggests that their distributions behave well when $\nu \to 0$. Accordingly, in Section 4 we study the distributions of vector-valued random variables

$$\Phi_{\mathbf{f}}(u_\nu(t)) = \big(\Phi_{f_1}(u_\nu(t),\dots,\Phi_{f_m}(u_\nu(t))\big) \in \mathbb{R}^m\,,$$

and of $\Phi_{\mathbf{f}}(U(t))$. Assuming that the functions f_j are analytic, linearly independent and satisfy certain restriction on growth, we show that the distribution of $\Phi_{\mathbf{f}}(U(t))$ has a density against the Lebesgue measure:

$$\mathcal{D}\big(\Phi_{\mathbf{f}}U(t)\big) = p_{\mathbf{f}}(x)\,dx'\,, \quad p_{\mathbf{f}} \in L_1(\mathbb{R}^m).$$

To prove this result we show that the measures $\mathcal{D}\Phi_{\mathbf{f}}u_\nu(t)$ are absolutely continuous with respect to the Lebesgue measure, uniformly in ν. The proof crucially uses (0.4) as well as obtained in [Kuk06b] uniform in ν bounds on exponential moments of the random variables $\operatorname{rot}(u_\nu(t,x))$.

Since m is arbitrary, then this result implies that the measure μ_0 is genuinely infinite dimensional in the sense that any compact set of finite Hausdorff dimension has zero μ_0-measure.

Other equations. The results and the methods of this work apply to other PDE of the form

$$\langle\text{Hamiltonian equation}\rangle + \nu\langle\text{dissipation}\rangle = \sqrt{\nu}\,\langle\text{random force}\rangle\,, \qquad (0.5)$$

provided that the corresponding Hamiltonian PDE has at least two 'good' integrals of motion. In particular, they apply to the randomly forced complex Ginzburg-Landau equation

$$\dot u - (\nu + i)\Delta u + i|u|^2 u = \sqrt{\nu}\,\eta(t,x), \quad \dim x \le 4, \qquad (0.6)$$

supplemented with the odd periodic boundary conditions. The corresponding Hamiltonian PDE is the NLS equation, having two 'good' integrals: the

Hamiltonian H and the total number of particles $E = \frac{1}{2}\int |u|^2\,dx$. Eq. (0.6) was considered in [KS04], where it was proved that for stationary in time solutions u_ν of (0.6) an inviscid limit $V(t)$ (as $\nu \to 0$ along a subsequence) exists and possesses properties, similar to those, stated in Theorem 1.1. The methods of this work allow to prove that the random variable $E(u_\nu(t))$ satisfies (0.4) uniformly in $\nu > 0$, that $H(u_\nu(t))$ meets similar estimates and that V is distributed in such a way that $\mathcal{D}\big(H(V(t))\big)$ and $\mathcal{D}\big(E(V(t))\big)$ are absolutely continuous with respect to the Lebesgue measure.

If $\dim x = 1$, then the NLS equation is integrable and the inviscid limit V may be analysed further, using the methods, developed in [KP08] to study the damped/driven KdV equation (which is another example of the system (0.5)).

Certainly our methods as well apply to some finite-dimensional systems of the form (0.5). In particular – to Galerkin approximations for the 3D NSE under periodic boundary conditions, perturbed by a random force, similar to (1.2). It is easy to establish for that system analogies of results in Sections 1-3. More interesting example is given by system (0.5), where the Hamiltonian equation is the Euler equation for a rotating solid body [Arn89]. This system can be cautiously regarded as a finite-dimensional model for (0.1); see Appendix.[1]

1 Preliminaries

Using the Leray projector $\Pi : L^2(\mathbb{T}^2;\mathbb{R}^2) \to \mathcal{H}$ we rewrite eq. (0.2) as the equation for $u(t) = u(t,\cdot) \in \mathcal{H}$:

$$\dot{u} + \nu A(u) + B(u) = \sqrt{\nu}\,\eta(t). \tag{1.1}$$

Here $A(u) = -\Pi\Delta u$ and $B(u) = \Pi(u\cdot\nabla)u$. We denote by $\|\cdot\|$ and by $(\cdot,\cdot)$ the L_2-norm and scalar product in $\mathcal{H}$. Let $(e_s, s \in \mathbb{Z}^2 \setminus \{0\})$ be the standard trigonometric basis of this space:

$$e_s(x) = \frac{f\left(s_1\frac{2\pi}{l_1}x_1 + s_2\frac{2\pi}{l_2}x_2\right)}{\sqrt{\frac{1}{2}\left(\frac{l_2}{l_1}s_1^2 + \frac{l_1}{l_2}s_2^2\right)}}\begin{bmatrix} -s_2/l_2 \\ s_1/l_1 \end{bmatrix},$$

[1]We are thankful to V. V. Kozlov and members of his seminar in MSU for drawing our attention to this equation.

where $f = \sin$ or $f = \cos$, depending whether $s_1 + s_2\delta_{s_1,0} > 0$ or $s_1 + s_2\delta_{s_1,0} < 0$. Then $\|e_s\| = 1$ and

$$Ae_s = \lambda_s e_s, \quad \lambda_s = (2\pi)^2 \left((s_1/l_1)^2 + (s_2/l_2)^2\right), \qquad \forall\, s\,.$$

The force η is assumed to be a Gaussian random field, white in time and smooth in x:

$$\eta = \frac{d}{dt}\zeta(t,x), \quad \zeta = \sum_{s\in\mathbb{Z}^2\setminus\{0\}} b_s\beta_s(t)e_s(x)\,, \tag{1.2}$$

where $\{b_s\}$ is a set of real constants, satisfying

$$b_s = b_{-s} \neq 0 \quad \forall\, s, \qquad \sum |s|^2 b_s^2 < \infty\,,$$

and $\{\beta_s(t)\}$ are standard independent Wiener processes.

This equation is known to have a unique stationary measure μ_ν.[2] This is a probability Borel measure in the space $\mathcal{H}$ which attracts distributions of all solutions for (1.1) as $t \to \infty$ (e.g., see in [Kuk06a]). Let $u_\nu(t,x)$ be a corresponding stationary solution, i.e.

$$\mathcal{D}u_\nu(t) \equiv \mu_\nu.$$

Apart from being stationary in t, this solution is known to be stationary (=homogeneous) in x.

For any $l \geq 0$ we denote by $\mathcal{H}^l, l \geq 0$, the Sobolev space $\mathcal{H} \cap H^l(\mathbb{T}^2;\mathbb{R}^2)$, given the norm

$$\|u\|_l = \left(\int \left((-\Delta)^{l/2}u(x)\right)^2 dx\right)^{1/2} \tag{1.3}$$

(so $\|u\|_0 = \|u\|$). A straightforward application of Ito's formula to $\|u_\nu(t)\|^2$ and $\|u_\nu(t)\|_1^2$ implies that

$$\mathbf{E}\,\|u_\nu(t)\|_1^2 \equiv \frac{1}{2}\,B_0\,, \quad \mathbf{E}\|u_\nu(t)\|_2^2 \equiv \frac{1}{2}\,B_1\,, \tag{1.4}$$

where for $l \in \mathbb{R}$ we denote $B_l = \sum |s|^{2l} b_s^2$ (note that $B_0, B_1 < \infty$ by assumption); e.g. see in [Kuk06a].

[2] If $\mathbb{T}^2$ is a square torus $\mathbb{R}^2/(l\mathbb{Z}^2)$, then by the results of [HM06] the stationary measure μ_ν is unique if $b_s \neq 0$ for $|s| \leq N$, where N is a ν-independent constant. Accordingly, if $\mathbb{T}^2$ is a square torus, then Theorems 1.1 and 2.1 below remain true under this weaker assumption on the numbers b_s. But our arguments in Sections 3, 4 use essentially that all coefficients b_s are non-zero.

The theorem below describes what happen to the stationary solutions $u_\nu(t,x)$ as $\nu \to 0$. For the theorem's proof see [Kuk06a] (there the result is stated for the square torus $\mathbb{R}^2/(2\pi\mathbb{Z}^2)$, but the proof does not use this assumption).

Theorem 1.1. *Any sequence $\tilde\nu_j \to 0$ contains a subsequence $\nu_j \to 0$ such that*

$$\mathcal{D}u_{\nu_j}(\cdot) \rightharpoonup \mathcal{D}U(\cdot) \quad in \quad \mathcal{P}\big(C(0,\infty;\mathcal{H}^1)\big)\,. \tag{1.5}$$

The limiting process $U(t)\in\mathcal{H}^1$, $U(t)=U(t,x)$, is stationary in t and in x. Moreover,

1)a) every its trajectory $U(t,x)$ is such that

$$U(\cdot)\in L_{2\,loc}(0,\infty;\mathcal{H}^2),\quad \dot U(\cdot)\in L_{1\,loc}(0\,\infty;\mathcal{H}^1)\,.$$

b) It satisfies the free Euler equation (0.3)*, so $\mu_0=\mathcal{D}(U(0))$ is an invariant measure for* (0.3)*,*

c) $\|U(t)\|_0$ and $\|U(t)\|_1$ are time-independent quantities. If g is a bounded continuous function, then $\int_{\mathbb{T}^2} g(\operatorname{rot} U(t,x))\,dx$ also is a time-independent quantity.

2) For each $t\ge 0$ we have $\mathbf{E}\|U(t)\|_1^2=\frac12 B_0$, $\mathbf{E}\|U(t)\|_2^2\le\frac12 B_1$ and $\mathbf{E}\exp\big(\sigma\|U(t)\|_1^2\big)\le C$ for some $\sigma>0, C\ge 1$.

Amplification. If $B_2<\infty$, then the convergence (1.5) holds in the space $\mathcal{P}(C(0,\infty;\mathcal{H}^\varkappa))$, for any $\varkappa<2$.

See [Kuk06a], Remark 10.4.

Due to 1b), the measure $\mu_0=\mathcal{D}U(0)$ is invariant for the Euler equation. By 2) it is supported by the space $\mathcal{H}^2$ and is not a δ-measure at the origin. The process U is called the *Eulerian limit* for the stationary solutions u_ν of (1.1). Note that apriori the process U and the measure μ_0 depend on the sequence ν_j.

Since $\|u\|_1^2\le\|u\|_0\|u\|_2$ and $\mathbf{E}\,\|u\|_1^2\le(\mathbf{E}\|u\|_0^2)^{1/2}(\mathbf{E}\|u\|_1^2)^{1/2}$, then (1.4) implies that

$$\frac12 B_0^2B_1^{-1}\le\mathbf{E}\|u_\nu(t)\|_0^2\le\frac12 B_1 \tag{1.6}$$

for all ν. That is, the characteristic size of the solution u_ν remains ~ 1 when $\nu\to 0$. Since the characteristic space-scale also is ~ 1, then the Reynolds number of u_ν grows as ν^{-1} when ν decays to zero. Hence, Theorem 1.1 describes a transition to turbulence for space-periodic 2D flows, stationary in time. Recall that eq. (0.2) is the only 2D NSE (0.1), having a limit of order one as $\nu \to 0$ (cf. [Kuk06a], Section 10.3). Thus the various Eulerian limits as in Theorem 1.1 with different coefficients $\{b_s\}$ (corresponding to different spectra of the applied random forces) describe all possible 2D space-periodic stationary turbulent flows.

Our goal is to study further properties of the Eulerian limits.

2 Estimate for energy of solutions

2.1 The result

The energy $E_\nu(t) = \frac{1}{2}\|u_\nu(t)\|_0^2$ of a stationary solution u_ν, $\nu \in (0,1]$, is a stationary process. It satisfies the relations

$$\frac{1}{4}B_0^2 B_1^{-1} \le \mathbf{E} E_\nu(t) = \frac{1}{4}B_0\,, \quad \mathbf{E}\exp(\sigma E_\nu(t)) \le C\,, \tag{2.1}$$

where $\sigma, C > 0$ are independent from ν (see (1.6) and [Kuk06a], Section 4.3). The energy $E_0(t)$ of the Eulerian limit U also meets (2.1).

Let $\{|\tilde{b}_j|, j \in \mathbb{N}\}$ be the rearrangement of the numbers $\{|b_s|, s \in \mathbb{Z}^2 \setminus 0\}$ in decreasing order: $|\tilde{b}_1| \ge |\tilde{b}_2| \ge \dots$.

Theorem 2.1. *Assume that $B_2 < \infty$. Then there exists a constant $C > 0$, depending only on B_1 and $|\tilde{b}_2|$, such that*

$$\mathbf{P}\{E_\nu(t) < \delta\} \le C\delta^{1/4}, \tag{2.2}$$

uniformly in $\nu \in (0,1]$.

Due to the convergence (1.5), the energy $E_0(t) = \frac{1}{2}\|U(t)\|^2$ of the Eulerian limit also satisfies (2.2).

Introducing the fast time $\tau = t\nu^{-1}$ we get for $u(\tau) = u(\tau, x)$ the equation

$$du(\tau) = (-Au - \nu^{-1}B(u))d\tau + \sum_s b_s e_s\, d\beta_s(\tau)\,, \tag{2.3}$$

where $\{\beta_s(\tau) = \sqrt{\nu}\,\beta_s(\nu\tau),\ s \in \mathbb{Z}^2 \setminus 0\}$, are new standard independent Wiener processes.

2.2 Beginning of proof

The proof goes in five steps. We start with a geometrical lemma which is used below in the heart of the construction.

Let us denote by S the sphere $\{u \in \mathcal{H} \mid \|u\|_0 = 1\}$. Let $\{e_j, j \geq 1\}$, be the basis $\{e_s, s \in \mathbb{Z}^2 \setminus \{0\}\}$, re-parameterised by natural numbers in such a way that $e_j = e_{s(j)}$, where $\lambda_{s(j)} \geq \lambda_{s(i)}$ if $j \geq i$.

Lemma 2.2. *There exists $\delta > 0$ with the following property. Let v_0, $\tilde{v}_0$ be any two points in S. Then for $(v, \tilde{v}) \in S \times S$ such that*

$$\|v - v_0\|_0 < \delta, \quad \|\tilde{v} - \tilde{v}_0\|_0 < \delta \tag{2.4}$$

there exists an unitary operator $U_{(v,\tilde{v})} = U_{(v,\tilde{v})}^{(v_0,\tilde{v}_0)}$ of the space $\mathcal{H}$, satisfying

i) U is an operator-valued Lipschitz function of v and $\tilde{v}$ with a Lipschitz constant ≤ 2;

ii) $U_{(v,\tilde{v})}(\tilde{v}) = v$;

iii) there exists a unitary vector $\eta = \eta(v, \tilde{v})$ in the plane span$\{e_1, e_2\}$ such that the vector $U_{(v,\tilde{v})}(\eta)$ makes with this plane an angle $\leq \pi/4$. Accordingly,

$$\max_{i,j \in \{1,2\}} \left|(U_{(v,\tilde{v})} e_i, e_j)\right| \geq c_*, \tag{2.5}$$

where $c_* > 0$ is an absolute constant.

Proof. Let us start with the following observation:

There exists $\delta > 0$ such that for any $v_0 \in S$ and $v_1 \in \{v \in S \mid \|v - v_0\|_0 < \delta\}$ there exists an unitary transformation W_{v_1,v_0} of the space $\mathcal{H}$ with the following property: $W_{v_0,v_0} = id$, $W_{v_1,v_0}(v_0) = v_1$ and W is a Lipschitz function of v_1 and v_0 with a Lipschitz constant ≤ 2.

To prove the assertion let us denote by $\mathcal{A}$ the linear space of bounded anti self-adjoint operators in $\mathcal{H}$ (given the operator norm), and consider the map

$$\mathcal{A} \times S \to S, \quad (A, v) \mapsto e^A v\,.$$

Note that the differential of this map in A, evaluated at $A = 0, v = v_0$, is the map $A' \mapsto A'v_0$, which sends $\mathcal{A}$ to the space $T_{v_0}S = \{v \in \mathcal{H} \mid (v, v_0) = 0\}$ and admits a right inverse operator of unit norm. So the assertion with $W = e^A$, where A satisfies the equation $e^A v_0 = v_1$, follows from the implicit function theorem.

To prove the lemma we choose unit vectors $\eta_0, \tilde\eta_0 \in \mathrm{span}\,\{e_1, e_2\}$ such that $(v_0, \eta_0) = 0$ and $(\tilde v_0, \tilde\eta_0) = 0$. Next we choose an unitary transformation U, such that $U(\tilde v_0) = v_0$ and $U(\tilde\eta_0) = \eta_0$. For vectors $v, \tilde v$, satisfying (2.4), denote $U(\tilde v) = \tilde\xi$. Then $\|\tilde\xi - v_0\|_0 < \delta$. Let $W_{v,\tilde\xi}$ be the operator from the assertion above. We set $U_{v,\tilde v} = W_{v,\tilde\xi} \circ U$. This operator obviously satisfies i) and ii). Since $\|U_{v,\tilde v}(\tilde\eta_0) - \eta_0\|_0 \le C\delta$, then choosing $\delta < C^{-1}2^{-1/2}$ we achieve iii) with $\eta = \tilde\eta_0$. $\square$

Remark. Let j_1 and j_2 be any two different natural numbers. The same arguments as above prove existence of an unitary operator U, satisfying i), ii) and such that $\max_{i\in\{1,2\},\, j\in\{j_1,j_2\}} \big|(Ue_i, e_j)\big| \ge c_*$.

For any $(v_0, \tilde v_0) \in S \times S$ let $\mathcal{O}_\delta(v_0, \tilde v_0) \subset S \times S$ be the open domain, formed by all pairs $(v, \tilde v)$, satisfying (2.4). Let $\mathcal{O}^1, \mathcal{O}^2, \dots$ be a countable system of domains $\mathcal{O}_{\delta/2}(v_j, \tilde v_j) =: \mathcal{O}^j$, $j \ge 1$, which cover $S \times S$. We call $(v_j, \tilde v_j)$ the *centre* of a domain $\mathcal{O}^j$.

Consider the mapping

$$S \times S \to \mathbb{N}, \quad (v, \tilde v) \mapsto n(v, \tilde v) = \min\{j \mid (v, \tilde v) \in \mathcal{O}^j\}\,. \tag{2.6}$$

It is measurable with respect to the Borel sigma-algebras. Finally, for $j = 1, 2, \dots$ and $(v, \tilde v) \in \mathcal{O}^j$ we define the operators

$$U^j_{v,\tilde v} = U^{(v_j,\tilde v_j)}_{v,\tilde v}.$$

2.3 Step 1: equation for $\tilde u(t)$

Till the end of Section 2 for any $u \in \mathcal{H}$ we will denote

$$v = v(u) = u/\|u\|_0 \ \text{ if } u \ne 0 \text{ and } v = e_1 \text{ if } u = 0. \tag{2.7}$$

Let us fix any $T_0 > 0$. We start to construct a process $\tilde u(\tau)$, $0 \le \tau \le T_0$, with continuous trajectories, satisfying $\|\tilde u(\tau)\|_0 \equiv \|u(\tau)\|_0$. The process will be constructed as a solution of a stochastic equation, in terms of some stopping times $0 = \tau_0 \le \tau_1 < \tau_2 < \dots$.

We set $\tau_0 = 0$ and define a random variable $n_0 = n(v(0), v(0)) \in \mathbb{N}$ (see (2.6)). Let us consider the following stochastic equation for $\mathbf{u}(\tau) = (u(\tau), \tilde u(\tau)) \in \mathcal{H} \times \mathcal{H}$:

$$du(\tau) = (-Au - \nu^{-1}B(u))d\tau + \sum_s b_s e_s\, d\beta_s(\tau), \tag{2.8}$$

$$d\tilde u(\tau) = -U^*_{\mathbf u} Au\, d\tau + \sum_s U^*_{\mathbf u} b_s e_s\, d\beta_s(\tau). \qquad (2.9)$$

Here $U^*_{\mathbf u}$ is the adjoint to the unitary operator $U_{\mathbf u} = U^{n_0(\omega)}_{v,\tilde v}$, where $v = v(u)$ and $\tilde v = v(\tilde u)$, see (2.7). Let us fix any $\gamma \in (0,1]$ and define the stopping times

$$T_\gamma = \inf\{\tau \in [0,T_0] \mid \|u(\tau)\|_0 \wedge \|\tilde u(\tau)\|_0 \le \gamma \ \text{ or } \ \|u(\tau)\|_2 \ge \gamma^{-1}\}\,,$$
$$\tau_1 = \inf\{\tau \in [0,T_0] \mid \mathbf u(\tau) \notin \mathcal O_\delta(v_{n_0}, \tilde v_{n_0})\} \wedge T_\gamma\,.$$

Here and in similar situations below $\inf \emptyset = T_0$, and $(v_{n_0}, \tilde v_{n_0})$ is the centre of the domain $\mathcal O^{n_0}$.

For $0 \le \tau \le \tau_1$ the operator $U_{\mathbf u}$ is a Lipschitz function of $\mathbf u$ since $\|u\|_0 \ge \gamma$ and $\|\tilde u\|_0 \ge \gamma$. As $\|u(\tau)\|_2 \le \gamma^{-1}$ for $\tau \le T_\gamma$, then it is not hard to see that the system (2.8),(2.9), supplemented with the initial condition

$$\mathbf u(0) = (u(0), u(0)), \qquad (2.10)$$

has a unique strong solution $\mathbf u(\tau)$, $0 \le \tau \le \tau_1$, satisfying

$$\mathbf E \sup_{0\le\tau\le\tau_1} \|\tilde u(\tau)\|_0^2 \le C(T_0, \nu, \gamma). \qquad (2.11)$$

Next we set $n_1 = n(v(\tau_1), \tilde v(\tau_1))$ and for $\tau \ge \tau_1$ re-define the operator $U_{\mathbf u}$ in (2.9) as $U^{n_1(\omega)}_{v,\tilde v}$ (as before, $v = v(u(\tau))$ and $\tilde v = v(\tilde u(\tau))$). We set

$$\tau_2 = \inf\{\tau \in [\tau_1, T_0] \mid \mathbf u(\tau) \notin \mathcal O_\delta(v_{n_1}, \tilde v_{n_1})\} \wedge T_\gamma,$$

where $(v_{n_1}, \tilde v_{n_1})$ is the centre of $\mathcal O^{n_1}$, and consider the system (2.8), (2.9) for $\tau_1 \le \tau \le \tau_2$ with the initial condition at τ_1, obtained by continuity. The system has a unique strong solution and (2.11) holds with τ_1 replaced by τ_2. Iterating this construction we obtain stopping times $\tau_0 \le \tau_1 \le \tau_2 \le \dots$, the operator $U_{\mathbf u}(\tau)$, piecewise constant in τ and discontinuous at points $\tau = \tau_j$, as well as a strong solution $\mathbf u(\tau)$ of (2.8)-(2.10), defined for $0 \le \tau < \lim_{j\to\infty} \tau_j \le T_\gamma$, and satisfying (2.11) with τ_1 replaced by any τ_j. Clearly $\tau_j < \tau_{j+1}$, unless $\tau_j = \tau_{j+1} = T_\gamma$.

2.4 Step 2: growth of stopping times τ_j

For any $\tau \ge 0$ and $N \ge 1$ let us write $\tilde u(\tau \wedge T_N)$ as

$$\tilde u(\tau \wedge T_N) = \Big(u(0) - \int_0^{\tau\wedge T_N} U^* A(u)\, d\theta\Big) + \int_0^{\tau\wedge T_N} \sum_s b_s U^* e_s\, d\beta_s$$
$$=: \tilde u_1(\tau) + \tilde u_2(\tau)\,.$$

Since $\|u\|_2 \le \gamma^{-1}$, then the process $\tilde u_1(\tau) \in \mathcal{H}$ is Lipschitz in τ. A straightforward application of the Kolmogorov criterion implies that the process $\tilde u_2(\tau) \in \mathcal{H}$ a.s. satisfies the Hölder condition with the exponent 1/3. So the process $\tilde u(\tau \wedge T_N)$ is a.s. Hölder. The process $u(\tau \wedge T_N)$ is Hölder as well, so

$$\|\mathbf{u}\big((\tau_j + \Delta) \wedge T_N; \omega) - \mathbf{u}(\tau_j; \omega)\|_0 \le K(\omega)\Delta^{1/3}$$

($K(\omega)$ is independent from N). Since $\|\mathbf{u}(\tau_{j+1}) - \mathbf{u}(\tau_j)\|_0 \ge \frac{\delta}{2}$ unless $\tau_{j+1} = T_\gamma$, then $|\tau_{j+1} - \tau_j| \ge (\delta/2K(\omega))^3$ or $\tau_{j+1} = T_\gamma$. As $\tau_j \le T_\gamma \le T_0$, then

$$\tau_j = T_\gamma \quad \text{for} \quad j \ge j(\gamma;\omega)\,, \tag{2.12}$$

where $j(\gamma) < \infty$ a.s.

For any $0 < \gamma \le 1$ we have constructed a process $\mathbf{u}(\tau) = (u(\tau), \tilde u(\tau))$, $\tau \in [0, T_\gamma]$, satisfying (2.8)-(2.10), where the operator $U_{\mathbf{u}}$ is a piecewise constant function of τ.

2.5 Step 3: $\|\tilde u(\tau)\|_0 \equiv \|u(\tau)\|_0$ for $\tau \le T_\gamma$

For $j = 0, 1, \dots$ we will prove the following assertion:

$$\begin{aligned}&\text{if } \|\tilde u(\tau_j)\|_0 = \| u(\tau_j)\|_0 \text{ a.s., then}\\ &\qquad\qquad \|\tilde u(\tau)\|_0 = \| u(\tau)\|_0 \text{ for } \tau_j \le \tau \le \tau_{j+1}, \text{ a.s.}\end{aligned} \tag{2.13}$$

Since $\tilde u(\tau_0) = u(\tau_0)$, then (2.12) and (2.13) would imply that

$$\|\tilde u(\tau)\|_0 = \| u(\tau)\|_0 \quad \forall\, 0 \le \tau \le T_\gamma\,, \tag{2.14}$$

for any $\gamma > 0$.

To prove (2.13) we consider (following Lemma 7.1 in [KP08]) the quantities $E(\tau) = \frac{1}{2}\|u(\tau)\|_0^2$ and $\tilde E(\tau) = \frac{1}{2}\|\tilde u(\tau)\|_0^2$. Due to Ito's formula we have

$$dE = (u, -Au)\, d\tau + \frac{1}{2} B_0 d\tau + (u, \sum_s b_s e_s\, d\beta_s(\tau))$$

and

$$\begin{aligned} d\tilde E =& (\tilde u, -U^* A u)\, d\tau + \frac{1}{2}\sum b_s^2 |U^* e_s|^2 d\tau + (\tilde u, \sum_s b_s (U^* e_s)\, d\beta_s(\tau)) \\ =& \frac{\|\tilde u\|_0}{\|u\|_0}\,(u, -Au)\, d\tau + \frac{1}{2} B_0 d\tau + \frac{\|\tilde u\|_0}{\|u\|_0}\,(u, \sum_s b_s e_s\, d\beta_s(\tau))\,. \end{aligned}$$

Therefore,

$$
\begin{aligned}
d(E-\tilde E)^2 =& 2(E-\tilde E)\frac{\|u\|_0-\|\tilde u\|_0}{\|u\|_0}(u,-Au)\,d\tau\\
&\Big(\frac{\|u\|_0-\|\tilde u\|_0}{\|u\|_0}\Big)^2\sum_s b_s^2(u,e_s)^2 d\tau+\mathcal{M}_\tau\,,
\end{aligned}
$$

where $\mathcal{M}_\tau$ stands for the corresponding stochastic integral.

For $0\le\tau\le T_\gamma$ let us denote $J(\tau)=(E-\tilde E)^2\big((\tau\vee\tau_i)\wedge\tau_{i+1}\big)$. Then

$$
\begin{aligned}
\frac{d}{d\tau}\mathbf{E}J(\tau) =& 2\,\mathbf{E}\Big((E-\tilde E)\frac{\|u\|_0-\|\tilde u\|_0}{\|u\|_0}(u-Au)I_{\tau_i\le\tau\le\tau_{i+1}}\Big)\\
&+\mathbf{E}\Big(\Big(\frac{\|u\|_0-\|\tilde u\|_0}{\|u\|_0}\Big)^2\sum b_s^2(u,e_s)^2 I_{\tau_i\le\tau\le\tau_{i+1}}\Big)\,.
\end{aligned}
$$

Since $\|u\|_0-\|\tilde u\|_0=\frac{2(E-\tilde E)}{\|u\|_0+\|\tilde u\|_0}$ and $|(u,-Au)|\le\gamma^{-2}$, $\|u\|_0,\|\tilde u\|_0\ge\gamma$, then $\frac{d}{d\tau}\mathbf{E}J(\tau)\le C_\gamma\mathbf{E}J(\tau)$. As $J(0)=0$, then $\mathbf{E}J(\tau)\equiv0$ and (2.13) is established. Accordingly (2.14) also is proved.

2.6 Step 4: limit $\gamma\to0$

Since $B_2<\infty$, then $u(\tau)$ satisfies the γ-independent estimate

$$
\mathbf{E}\sup_{0\le\tau\le T_0}\|u(\tau)\|_2\le C(T_0,\nu)
$$

(see [Kuk06a], Section 4.3). Accordingly

$$
\mathbf{P}\big\{\sup_{0\le\tau\le T_0}\|u(\tau)\|_2\le\gamma^{-1}\big\}\to1\quad\text{as}\quad\gamma\to0. \tag{2.15}
$$

Denote by $\hat u(\tau)$ the two-vector $(u_1(\tau),u_2(\tau))$, where $u(\tau)=\sum u_j(\tau)e_j$ (we recall that $e_1,e_2,\dots$ are the basis vectors e_s, re-parameterised by natural numbers). Then

$$
\hat u_j(\tau)=u_j(0)+\int_0^\tau F_j ds+b_j\beta_j(s),\quad j=1,2,
$$

where F_j is the j-th component of the drift in (2.3). Since $\hat u$ is a stationary process, then $\mathbf{P}\{\hat u(0)=0\}=0$ (this follows, say, from Krylov's result, used in the next subsection). Setting $F_j^R = F_j \wedge R$, we denote by $\hat u^R(\tau) \in \mathbb{R}^4$ the process

$$\hat u_j^R(\tau) = u_j(0) + \int_0^\tau F_j^R ds + b_j\beta_j(s), \quad j = 1, 2.$$

By the Girsanov theorem, distribution of the process $\hat u^R(\tau), 0 \le \tau \le T_0$, is absolutely continuous with respect to the process $(b_1\beta_1, b_2\beta_2) + \hat u(0)$. Therefore

$$\mathbf{P}\{\min_{0\le\tau\le T_0} |\hat u^R(\tau)| = 0\} = 0\,, \tag{2.16}$$

for any R. Since $\max_{0\le\tau\le T_0} |\hat u^R(\tau) - \hat u(\tau)| \to 0$ as $R \to \infty$ in probability, then the process $\hat u(\tau)$ also satisfies (2.16). Jointly with (2.15) this implies that

$$\mathbf{P}\{T_\gamma = T_0\} \to 1 \quad \text{as} \quad \gamma \to 0\,,$$

and we derive from (2.14) the relation

$$\|\tilde u(\tau)\|_0 = \|u(\tau)\|_0 \quad \forall\, 0 \le \tau \le T_0, \ \text{ a.s.}$$

2.7 Step 5: end of proof

The advantage of the process $\tilde u$ compare to u is that it satisfies the ν-independent Ito equation (2.9). Let us consider the first two components of the process:

$$d\tilde u_j = -\big(U^*_{u,\tilde u}(\tau)A(u)\big)_j\, d\tau + \sum_{l=1}^{\infty} \big(U^*_{u,\tilde u}(\tau)\big)_{jl} b_l\, d\beta_l(\tau)\,, \tag{2.17}$$

where $j = 1, 2$. Denoting $a_j(\tau) = \sum_{l=1}^\infty \big(U^*_{jl} b_l\big)^2 = \sum_{l=1}^\infty \big(U_{lj} b_l\big)^2$ and using (2.5) we find that a.s.

$$C \ge a_1(\tau) + a_2(\tau) \ge c > 0 \quad \forall\, \tau\,, \tag{2.18}$$

where $C = 2\sqrt{B_0}$ and c depends only on $|b_1| \wedge |b_2|$. Due to (1.4) for each $\tau \ge 0$ we have $\mathbf{E}|U^*A(u(\tau))|_j \le \sqrt{B_1/2}$. This bound and the first estimate in (2.18) imply that Lemma 5.1 from [Kry87] applies to the Ito equation (2.17) uniformly in ν if we choose the lemma's parameters as follows:

$$d = 1,\ \ \gamma = 1,\ \ A_s = s,\ \ r_s = 1,\ \ c_s = 1,\ \ y_t = t,\ \ \varphi_t = t. \tag{2.19}$$

Taking in the lemma for $f(t,x)$ the characteristic function of the segment $[-\delta,\delta]$, we get

$$\mathbf{E}\int_0^{\gamma_R} e^{-t} a_j(\tau)^{1/2} I_{\{|\tilde u_j(\tau)|\le\delta\}}\,d\tau \le C\sqrt{\delta}\,,\quad j=1,2,$$

where $\gamma_R\le 1$ is the first exit time ≤ 1 of the process $\tilde u_j$ from the segment $[-R,R]$. Sending R to ∞ we get that

$$\mathbf{E}\int_0^1 a_j(\tau)^{1/2} I_{\{|\tilde u_j(\tau)|\le\delta\}}\,d\tau \le C_1\sqrt{\delta}\,,\quad j=1,2\,, \tag{2.20}$$

uniformly in ν.

For c as in (2.18) and any fixed τ let us consider the event $Q_1^\tau=\{a_1(\tau)\ge \frac12 c\}$. Denote by Q_2^τ its complement. Then

$$a_1(\tau)\ge\frac12\,c \text{ on } Q_1^\tau \text{ and } a_2(\tau)\ge\frac12\,c \text{ on } Q_2^\tau. \tag{2.21}$$

Let us set

$$Q^\tau=\{|\tilde u_1(\tau)|+|\tilde u_2(\tau)|\le\delta\}.$$

Then

$$\mathbf{P}(Q^\tau)=\mathbf{E}(I_{Q^\tau}I_{Q_1^\tau}+I_{Q^\tau}I_{Q_2^\tau})\le \mathbf{E}(I_{\{|\tilde u_1(\tau)|\le\delta\}}I_{Q_1^\tau}+I_{\{|\tilde u_2(\tau)|\le\delta\}}I_{Q_2^\tau}).$$

By (2.21) the r.h.s. is bounded by

$$\sqrt{\frac{2}{c}}\,\mathbf{E}\big(I_{\{|\tilde u_1(\tau)|\le\delta\}}\sqrt{a_1}+I_{\{|\tilde u_2(\tau)|\le\delta\}}\sqrt{a_2}\big).$$

Jointly with (2.20) the obtained inequality shows that

$$\int_0^1 \mathbf{P}(Q^\tau)\,d\tau\le C_2\sqrt{\delta}.$$

Since

$$\mathbf{P}\{\|u(\tau)\|_0\le\frac{\delta}{2}\}=\mathbf{P}\{\|\tilde u(\tau)\|_0\le\frac{\delta}{2}\}\le\mathbf{P}(Q^\tau)\,,$$

where the l.h.s. is independent from τ, then

$$\mathbf{P}\{\|u(\tau)\|_0\le\frac{\delta}{2}\}\le C_2\sqrt{\delta}$$

for any $\delta>0$. This relation implies (2.2).

The constant C in (2.2), as well as all other constants in this section, depend only on B_1 and $|b_1|\wedge|b_2|$. Using the Remark in Section 2.2 we may replace $|b_1|\wedge|b_2|$ by $|\tilde b_1|\wedge|\tilde b_2|$. This completes the theorem's proof.

3 Distribution of energy

Again, let $u_\nu(\tau)$ be a stationary solution of (1.1), written in the form (2.3), let $E_\nu(\tau)$ be its energy and $E_0(\tau) = \frac{1}{2}\,\|U(\tau)\|_0^2$ be the energy of the Eulerian limit.

Theorem 3.1. *For any $R > 0$ let $Q \subset [-R,\, R]$ be a Borel set. Then*

$$\mathbf{P}\{E_\nu(\tau) \in Q\} \le p_R(|Q|) \tag{3.1}$$

uniformly in $\nu \in (0, 1]$, where $p_R(t) \to 0$ as $t \to 0$

In particular, the measures $\mathcal{D}(E_\nu(\tau))$ are absolutely continuous with respect to the Lebesgue measure. Since $\mathcal{D}(E_{\nu_j}) \rightharpoonup \mathcal{D}(E_0(\tau))$, then $E_0(\tau)$ satisfies (3.1) for any open set $Q \subset [-R, R]$. Accordingly, $\mathbf{P}\{E_0(\tau) \in Q\} = 0$ if $|Q| = 0$ since the Lebesgue measure is regular. We got

Corollary 3.2. *The measure $\mathcal{D}(E_0(\tau))$ is absolutely continuous with respect to the Lebesgue measure.*

Proof of the theorem. For any $\delta > 0$ let us consider the set

$$\mathcal{O} = \mathcal{O}(\delta) = \{u \in \mathcal{H}^2 \mid \|u\|_2 \le \delta^{-\frac{1}{4}},\ \|u\|_0 \ge \delta\}$$

Writing $u = u_\nu$ as $u = \sum u_s e_s$, we set $u^I = \sum_{|s| \le N} u_s e_s$ and $u^{II} = u - u^I$. For any $u \in \mathcal{O}$ we have $\|u^{II}\|_0^2 \le N^{-4}\|u^{II}\|_2^2 \le \delta^{-\frac{1}{2}} N^{-2}$. So $\|u^I\|_0^2 \ge \delta^2 - \delta^{-\frac{1}{2}} N^{-4}$. Choosing $N = N(\delta) = \left[2^{1/4}\delta^{-5/8}\right]$ we achieve

$$\|u^I\|_0^2 \ge \frac{1}{2}\,\delta^2 \quad \forall\, u \in \mathcal{O}.$$

The stationary process $E(u_\nu(\tau))$ satisfies the Ito equation

$$dE = \big(-\|u(\tau)\|_1^2 + \frac{1}{2} B_0 \big)\, d\tau + \sum b_s u_s(\tau)\, d\beta_s(\tau)$$

(see in Section (2.5)). The diffusion coefficient $a(\tau)$ satisfies

$$a(\tau) = \sum b_s^2 |u_s(\tau)|^2 \ge \underline{b}_N^2 \|u^I(\tau)\|_0^2,$$

where $\underline{b}_N = \min_{|s| \le N} |b_s| > 0$. So,

$$a(\tau) \ge \frac{1}{2}\, \underline{b}_N^2 \delta^2 \quad \text{if} \quad u(\tau) \in \mathcal{O}. \tag{3.2}$$

Besides,

$$\mathbf{E}|a(\tau)| \le \frac{\max_s b_s^2}{2}\, B_0\,, \quad \mathbf{E}\big| - \|u(\tau)\|_1^2 + \frac{1}{2}\, B_0\big| \le B_0\,,$$

(see (1.4)).

Let $Q \subset [-R,\, R]$ be a Borel set and f be its indicator function. Applying the Krylov lemma with the same choices of parameters as in (2.19), passing to the limit as $R \to \infty$ as in Section 2.7 and taking into account that $E(\tau)$ is a stationary process, we get that

$$\mathbf{E}\big(a(\tau)^{1/2} f(E(\tau)\big) \le C|Q|^{1/2}, \tag{3.3}$$

uniformly in $\nu > 0$. Due to (1.4) and (2.2),

$$\mathbf{P}\{u(\tau) \notin \mathcal{O}\} \le \frac{1}{2} B_1\sqrt{\delta} + C\sqrt{\delta}.$$

Jointly with (3.2) and (3.3) this estimate implies that

$$\mathbf{P}(E_\nu(\tau) \in Q) = \mathbf{E} f(E(\tau)) \le C(|Q|^{1/2} \underline{b}_N^{-1} \delta^{-1}) + C_1\sqrt{\delta} \quad \forall\; 0 < \delta \le 1,$$

where $N = N(\delta)$. Now (3.1) follows.

4 Distributions of functionals of vorticity

In his section we assume that $B_6 < \infty$. The vorticity $\zeta = \operatorname{rot} u(t,\, x)$ of a solution u for (1.1), written in the fast time $\tau = \nu t$, satisfies the equation

$$\zeta'_\tau - \Delta\zeta + \nu^{-1}(u \cdot \nabla)\zeta = \xi(\tau,\, x). \tag{4.1}$$

Here $\xi = \frac{d}{dt} \sum_{s \in \mathbb{Z}^2 \setminus \{0\}} \beta_s(\tau)\varphi_s(x)$ and

$$\varphi_s = \operatorname{rot} e_s = \frac{\lambda_s f\left(s_1 \frac{2\pi}{l_1} x_1 + s_2 \frac{2\pi}{l_2} x_2\right)}{\pi \cdot \sqrt{2\left(\frac{l_2}{l_1} s_1^2 + \frac{l_1}{l_2} s_2^2\right)}}$$

for any s, where $f = \cos$ or $f = -\sin$, depending whether $s_1 + s_2\delta_{s_1,0} > 0$ or not. We will study eq. (4.1) in Sobolev spaces

$$H^l = \{\zeta \in H^l(\mathbb{T}^2) \mid \int \zeta\, dx = 0\}, \quad l \ge 0,$$

given the norms $\|\cdot\|_l$, defined as in (1.3).

Let us fix $m \in \mathbb{N}$ and choose any m analytic functions $f_1(\zeta), \dots, f_m(\zeta)$, linear independent modulo constant functions.[3] We assume that the functions $f_j(\zeta), \dots, f_j'''(\zeta)$, $1 \le j \le m$, have at most a polynomial growth as $|\zeta| \to \infty$ and that

$$f_j''(\zeta) \ge -C \quad \forall\, j, \quad \forall \zeta$$

(for example, each $f_j(\zeta)$ is a trigonometric polynomial, or a polynomial of an even degree with a positive leading coefficient). Consider the map

$$F : H^l \to \mathbb{R}^m,\ \zeta \mapsto (F_1(\zeta), \dots, F_m(\zeta)),$$
$$F_j = \int_{\mathbb{T}^2} f_j(\zeta(x))\, dx,$$

where $0 < l < 1$. Since for any $P < \infty$ we have $H^l \subset L_P(\mathbb{T}^2)$ if l is sufficiently close to 1, then choosing a suitable $l = l(F)$ we achieve that the map F is C^2-smooth. Let us fix this l. We have

$$dF(\zeta)(\xi) = \Big(\int f_1'(\zeta(x))\xi(x)\, dx, \dots, \int f_m'(\zeta(x))\xi(x)\, dx \Big).$$

Lemma 4.1. *If $\zeta \not\equiv 0$, then the rank of $dF(\zeta)$ is m.*

Proof. Assume that the rank is $< m$. Then there exists number $C_1, \dots, C_m$, not all equal to zero, such that

$$\int (C_1 f_1'(\zeta) + \cdots + C_m f_m'(\zeta))\xi\, dx = 0 \quad \forall\, \xi \in H^l. \tag{4.2}$$

Denote $P(\zeta) = C_1 f_1'(\zeta) + \cdots + C_m f_m'(\zeta)$. This is a non-constant analytic function. Due to (4.2), $P(\zeta(x)) = \text{const}$. Denote this constant C_*. Then the connected set $\zeta(\mathbb{T}^2)$ lies in the discrete set $P^{-1}(C_*)$. So $\zeta(\mathbb{T}^2)$ is a point, i.e. $\zeta(x) \equiv \text{const}$. Since $\int \zeta\, dx = 0$, then $\zeta(x) \equiv 0$. □

Now let $\zeta(t) = \operatorname{rot} u_\nu(t)$, where u_ν is a stationary solution of (1.1). Applying Ito's formula to the process $F(\zeta(\tau)) \in \mathbb{R}^m$ and using that F_j is an integral of motion for the Euler equation, we get that

$$dF_j(\tau) = \Big(\int f_j'(\zeta(\tau, x))\Delta\zeta(\tau, x)\, dx + \frac{1}{2}\sum_s b_s^2 \int f''{}_j(\zeta(\tau, x))\varphi_s^2(x)\, dx) \Big) d\tau$$
$$+ \sum_s b_s \Big(\int f'{}_j(\zeta(\tau, x))\varphi_s(x)\, dx \Big)\, d\beta_s(\tau).$$

[3] I.e., $C_1 f_1(\zeta) + \cdots + C_m f_m(\zeta) \not\equiv \text{const}$, unless $C_1 = \cdots = C_m = 0$.

Since $b_s \equiv b_{-s}$ and $\varphi_s^2 + \varphi_{-s}^2 \equiv |s|^2/2\pi^2$, then

$$\begin{aligned} dF_j(\tau) = &\Big(\int {f''}_j(\zeta)(-|\nabla_x \zeta|^2 + \frac{1}{4\pi} B_1)\, dx \Big) d\tau \\ &+ \sum_s b_s \Big(\int {f'}_j(\zeta(\tau,\, x)) \varphi_s(x)\, dx \Big)\, d\beta_s(\tau) \\ &:= H_j(\zeta(\tau))\, d\tau + \sum_s h_{js}(\zeta(\tau))\, d\beta_s(\tau)\, . \end{aligned}$$

Ito's formula applies since under our assumptions all moments of the random variables $\zeta(\tau,\, x)$ and $|\nabla_x \zeta(\tau,\, x)|$ are finite (see [Kuk06a], Section 4.3). Using that $F_j(\tau)$ is a stationary process, we get from the last relation that $\mathbf{E} H_j = 0$, i.e.

$$\mathbf{E} \int {f''}_j(\zeta(\tau, x)) |\nabla_x \zeta(\tau, x)|^2\, dx = \frac{B_1}{4\pi}\, \mathbf{E} \int {f''}_j(\zeta(\tau, x))\, dx. \tag{4.3}$$

Since $B_6 < \infty$ then all moments of random variables $|\zeta(\tau, x)|$ are bounded uniformly in $\nu \in (0,\, 1]$, see [Kuk06b] and (10.11) in [Kuk06a]. As the random field ζ is stationary in τ and in x, then the r.h.s. of (4.3) is bounded uniformly in ν. Using the assumption $f_j'' \geq -C$ we find that

$$\big| f_j'' |\nabla_x \zeta|^2 \big| \leq f_j'' |\nabla_x \zeta|^2 + 2C |\nabla_x \zeta|^2.$$

Since

$$\mathbf{E} \int |\nabla_x \zeta(\tau, x)|^2 dx = \mathbf{E} \|u_\nu(\tau)\|_2^2 = \frac{1}{2}\, B_1,$$

then we get that

$$\mathbf{E} |H_j(\zeta(\tau))| \leq C_j < \infty \tag{4.4}$$

uniformly in ν (and for all τ).

Consider the diffusion matrix $a(\zeta(\tau))$, $a_{jl}(\zeta) = \sum_s h_{js}(\zeta) h_{ls}(\zeta)$. Clearly

$$\mathbf{E}\, tr(a_{jl})(\zeta(\tau)) \leq C, \tag{4.5}$$

uniformly in ν. Let us denote $D(\zeta) = \det a_{jl}(\zeta) \geq 0$. Noting that $h_{js}(\zeta) = b_s(dF(\zeta))_{js}$, we obtain from Lemma 4.1

Lemma 4.2. *The function D is continuous on H^l and $D > 0$ outside the origin.*

Now we regard (4.1) as an equation in H^1 and set

$$\mathcal{O}_\delta = \{\zeta \in H^1 \mid \|\zeta\|_1 \le \delta^{-1},\ \|\zeta\|_l \ge \delta\}\,.$$

Since $H^1 \Subset H^l$, then $D \ge c(\delta) > 0$ everywhere in $\mathcal{O}_\delta$.

Estimates (4.4), (4.5) allow to apply Krylov's lemma with $p = d = m$ to the stationary process $F(\zeta_\nu(\tau)) \in \mathbb{R}^m$, uniformly in ν. Choosing there for f the characteristic function of a Borel set $Q \subset \{|z| \le R\}$, we find that

$$\mathbf{P}\{F(\zeta_\nu(\tau)) \in Q\} \le \mathbf{P}\{\zeta_\nu(\tau) \notin \mathcal{O}_\delta\} + c(\delta)^{-1/(m+1)} C_R |Q|^{1/(m+1)} \tag{4.6}$$

(cf. the arguments in Section 3). Since $\|\zeta\|_1 = \|u\|_2$ and $\|\zeta\|_l \ge \|\zeta\|_0 \ge \|u\|_0$ for $\zeta = \operatorname{rot} u$, then due to (1.4) and (2.2) the first term in the r.h.s. of (4.6) goes to zero with δ uniformly in ν, and we get that

$$\mathbf{P}\{F(\zeta_\nu(\tau)) \in Q\} \le p_R(|Q|)\,, \quad p_R(t) \to 0 \ \text{ as } \ t \to 0\,, \tag{4.7}$$

uniformly in ν. Evoking Amplification to Theorem 1.1 we derive from (4.7) that the vorticity ζ_0 of the Eulerian limit U satisfies (4.7), if Q is an open subset of B_R. We have got

Theorem 4.3. *If $B_6 < \infty$, then the distribution of the stationary solution for the 2D NSE, written in terms of vorticity* (4.1)*, satisfies* (4.7) *uniformly in ν. The vorticity ζ_0 of the Eulerian limit U is distributed in such a way that the law of $F(\zeta_0(\tau))$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^m$.*

Corollary 4.4. *Let $X \Subset \mathcal{H} \cap C^1(\mathbb{T}^2; \mathbb{R}^2)$ be a compact set of finite Hausdorff dimension. Then $\mu_0(X) = 0$.*

Proof. Denote the Hausdorff dimension of X by d and choose any $m > d$. Then $(F \circ \mathrm{rot})(X)$ is a subset of $\mathbb{R}^m$ of positive codimension. So its measure with respect to $\mathcal{D}(f(\zeta_0(t))$ equals zero. Since $\mathcal{D}(f(\zeta_0(t)) = (F \circ \mathrm{rot}) \circ \mu_0$, then $\mu_0(X) = 0$.

□

5 Appendix: rotation of solid body

The Euler equation for a freely rotating solid body, written in terms of its momentum $M \in \mathbb{R}^3$, is

$$\dot{M} + [M, A^{-1}M] = 0, \tag{5.1}$$

where A is the operator of inertia and $[\cdot,\cdot]$ is the vector product. The corresponding damped/driven equation (0.5) is

$$\dot{M} + [M, A^{-1}M] + \nu M = \sqrt{\nu}\,\eta(t)\,, \tag{5.2}$$

where the random force is $\eta(t) = \frac{d}{dt}\sum_{j=1}^{3} b_j\beta_j(t)e_j$ with non-zero b_j's, and $\{e_1, e_2, e_3\}$ is the eigenbasis of the operator A^{-1}. Let us denote by $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ the eigenvalues, corresponding to the eigenvectors e_j's.

Eq. (5.2) has a unique stationary measure. Let $M_\nu(t)$ be a corresponding stationary solution. An inviscid limit, similar to that in Theorem 1.1, holds:

$$\mathcal{D}M_{\nu_j}(\cdot) \rightharpoonup \mathcal{D}M_0(\cdot) \quad \text{as } \nu_j \to 0\,, \tag{5.3}$$

where $M_0(t) \in \mathbb{R}^3$ is a stationary process, formed by solutions of (5.1). The Euler equation has two quadratic integrals of motion: $H_1(M) = \frac{1}{2}|M|^2$ and $H_2(M) = \frac{1}{2}(A^{-1}M, M)$. Distributions of the random variables $H_1(M_\nu(t))$ and $H_2(M_\nu(t))$, $0 \leq \nu \leq 1$, satisfy direct analogies of the assertions in Sections 2, 3.

To analyse further the processes M_ν with $\nu \ll 1$ and the inviscid limit M_0, we note that a.e. non-empty level set of the vector-integral $H = (H_1, H_2)$ is formed by two periodic trajectories of (5.1) (see [Arn89]). Denote them $S^{\pm}_{(H_1,H_2)}$. It is easy to see that the conditional probabilities for $M_\nu(t)$ to belong to $S^{+}_{(H_1,H_2)}$ or to $S^{-}_{(H_1,H_2)}$ are equal. Since the dynamics, defined by (5.1) on each set $S^{\pm}_{(H_1,H_2)}$ obviously is ergodic with respect to a corresponding invariant measure $\nu^{\pm}_{(H_1,H_2)}$,[4] then the methods of [FW98, FW03, KP08] apply to the process

$$H^{\nu_j}(\tau) = H(M_{\nu_j}(\tau)) \in K = \{(h_1, h_2) \in \mathbb{R}^2 \mid 0 \leq \lambda_1 h_1 \leq h_2 \leq \lambda_3 h_1\},$$

where $\tau = \nu_j t$. They allow to prove that a limiting (as $\nu_j \to 0$) process $H^0(\tau)$ exists and is a stationary solution of an SDE, obtained from the Ito equation for $H(M(\tau))$ by the usual stochastic averaging with respect to the ergodic measures $\nu^{\pm}_{(H_1,H_2)}$ on the curves $S^{\pm}_{(H_1,H_2)}$. This is a stochastic equation in K. Assume that the matrix A^{-1} is non-degenerate:

$$0 < \lambda_1 < \lambda_2 < \lambda_3. \tag{5.4}$$

[4] the density of the measure $\nu^{\pm}_{(H_1,H_2)}$ against the Lebesgue measure on the curve $S^{\pm}_{(H_1,H_2)}$ is inverse-proportional to velocity of the trajectory.

Then the diffusion matrix for the averaged equation is non-degenerate outside ∂K and the ray $\{h_2 = \lambda_2 h_1, h_1 \geq 0\}$. The process H^0 is a stationary solution of the averaged equation such that

- it has finite quadratic exponential moments (cf. (2.1),
- its marginal distribution $\mathcal{D}(H^0(0))$ is absolutely continuous with respect to the Lebesgue measure on K.

We claim that under the non-degeneracy assumption (5.4) the averaged equation has a unique stationary measure θ, satisfying the two properties above. Accordingly

$$\mathcal{D}(H(M_0(0)) = \theta$$

and

$$\mathcal{D}(M_0(0)) = \sum_{\alpha \in \{+,-\}} \int_{\mathbb{R}^2} \pi_\alpha \nu^\alpha_{(H_1,H_2)}\, \theta(dH_1\, dH_2), \tag{5.5}$$

where $\pi_+ = \pi_- = 1/2$. Cf. Theorem 6.6 in [KP08]. In particular, the convergence (5.3) holds as $\nu \to 0$ (i.e., the limit does not depend on a sequence $\nu_j \to 0$).

The representation (5.5) for the measure $\mathcal{D}(M_0(0))$ is called its *disintegration* with respect to the map $H : \mathbb{R}^3 \to \mathbb{R}^2$. It may be obtained independently from the arguments above (see references in [Kuk07]). The role of the arguments is to represent the measure θ in terms of the averaged equation.

The measure $\mu_0 = \mathcal{D}U(0)$, corresponding to the Eulerian limit U (Theorem 1.1) also admits a disintegration, similar to (5.5), but with much more complicated ingredients, see [Kuk07]. The main difficulty to study this disintegration (and the measure μ_0 itself) comes from the fact that, in difference with the sets $\{H = \text{const}\}$, the iso-integral sets for the Euler equation

$$\{U \mid E(U) = \text{const}, \int f\big(\text{rot}(U(x))\big)\, dx = \text{const} \quad \forall f\}, \tag{5.6}$$

and the Eulerian dynamics on them are understood very poorly. In particular, nothing is known about the measures on the sets (5.6) which enter the disintegration for the Eulerian limit. Still an analogy with eq. (5.2) and with the damped/driven KdV equation allows us in [Kuk07] to conjecture an averaging procedure of the Whitham type to find the measures, involved in the disintegration of μ_0.

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