# Central Limit Theorem and recurrence for random walks in bistochastic random environments

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**Abstract**

We prove the annealed Central Limit Theorem for random walks in bistochastic random environments on $\mathbb{Z}^d$ with zero local drift. The proof is based on a "dynamicist's interpretation" of the system, and requires a much weaker condition than the customary uniform ellipticity. Moreover, recurrence is derived for $d \leq 2$.

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## 1 Introduction

We study the Central Limit Theorem (CLT) and the recurrence properties of a certain class of random walks in random environments (RWREs), namely the random walks in bistochastic environments with zero local drift. Although this class is fairly general, it is not as general as we can prove theorems for, and certainly not new, having been previously investigated at least by Koslov [K] and Komorowski and Olla [KO].

In fact, the purpose of this note is not to give our most original results for the amplest class of RWREs (we will take this point of view in another paper [L3]; see also Section 5 below), but rather to present a technique, at the frontier of dynamical systems and probability theory, that can deliver known results more easily than the current methods, and often improve on them. Perhaps more importantly, it provides a unifying view on the relation between diffusive behavior and recurrence for random as well as deterministic dynamics [L1, L2]. Our method hinges in part on

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a beautiful theorem by Schmidt [S] (a somewhat weaker version of which has been independently achieved by Conze [C]) on the recurrence of commutative cocycles over ergodic dynamical systems.

Let us describe the system: We deal with random walks on $\mathbb{Z}^d$ defined by a bistochastic matrix $p := \{p_{xy}\}_{x,y\in\mathbb{Z}^d}$ of transition probabilites. This means that

$$\forall y \in \mathbb{Z}^d, \qquad \sum_{x\in\mathbb{Z}^d} p_{xy} = 1, \tag{1.1}$$

together of course with the trasposed condition, customarily called *normalization*, which ensures that, $\forall x \in \mathbb{Z}^d$, $y \mapsto p_{xy}$ is indeed a probability distribution on $\mathbb{Z}^d$. For the sake of simplicity we assume that there exists a finite $\Lambda \subset \mathbb{Z}^d$ such that

$$p_{xy} = 0 \qquad \text{if } y - x \notin \Lambda. \tag{1.2}$$

Our random walks will have *zero local drift*, in the sense that

$$\forall x \in \mathbb{Z}^d, \qquad \sum_{y\in\mathbb{Z}^d} (y-x)\, p_{xy} = 0. \tag{1.3}$$

We assume that the transition matrix $p$ is itself random: $p = p(\omega)$, where $\omega$ ranges in the probability space $(\Omega, \Pi)$. We will liberally call both $p(\omega)$ and $\omega$ the *random environment*, or simply the environment. It is not important what $\Omega$ actually is (although the interested reader may look at Section A.1 of the Appendix) but we make the fundamental hypothesis that it is acted upon by the group $\{\tau_z\}_{z\in\mathbb{Z}^d}$ of automorphisms w.r.t. $\Pi$. This action is such that

$$p_{xy}(\tau_z\omega) = p_{x+z,y+z}(\omega) \tag{1.4}$$

and it is ergodic (which is the minimal assumption for the random law on the environment to have something to do with the dynamics of the walker). Because of this, it is no loss of generality to require that the walk always starts at 0.

Our last hypothesis is the *almost sure irreducibility* of $p$: For $\Pi$-a.e. $\omega \in \Omega$, for every $y \in \mathbb{Z}^d$, there exists $n = n(\omega, y)$ such that

$$p_{0y}^{(n)} := \sum_{x_1,\ldots x_{n-1}} p_{0x_1}\, p_{x_1x_2} \cdots p_{x_{n-1}y} > 0. \tag{1.5}$$

(It is easy to see, via (1.4), that (1.5) guarantees mutual accessibility of any two points $x, y$ of $\mathbb{Z}^d$, at least for a.a. environments, whence the name 'irreducibility'.) An example of $(\Omega, \Pi)$ is given in Section A.1 of the Appendix.

**Remark 1.1** Notice how condition (1.5) is much weaker than the *uniform ellipticity* assumed in most results on RWREs, namely the existence of a constant $\varepsilon > 0$ such that, for a.e. $\omega$,

$$p_{0e} \geq \varepsilon, \qquad \forall e \in \mathbb{Z}^d, |e| = 1. \tag{1.6}$$

To the author's knowledge, within the scope of the diffusive or recurrence properties of RWREs [AKS, La, KO], only the papers by Berger and Biskup [Be, BB] do not (and cannot) require uniform ellipticity (cf. also [Z1, Z2]). Moreover, (1.5) can be further relaxed if stronger ergodic properties hold for $(\Omega, \Pi, \{\tau_z\})$, cf. Section 5.

For the sake of mathematical rigor, we now give a formal definition of our system in terms of standard objects of probability theory, warning the reader that the following construction is not the one that we will work with in the rest of the paper. Indeed, from Section 2 onward, we will represent the above RWRE in terms of a suitable measure-preserving dynamical system.

At any rate, fixed $\omega \in \Omega$, the random walk in the environment $\omega$ is the Markov chain $P_\omega$ on $\mathbb{Z}^d$ defined by

$$P_\omega(X_0 = 0) = 1; \tag{1.7}$$

$$P_\omega(X_{n+1} = y \,|\, X_n = x) = p_{xy}(\omega). \tag{1.8}$$

We take into account the complete randomness of the problem by studying the stochastic process $\{X_n\}_{n\in\mathbb{N}}$ w.r.t. the *annealed law*, , which is defined on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ via

$$\mathbb{P}(B \times E) := \int_B \Pi(d\omega)\, P_\omega(E), \tag{1.9}$$

where $B$ is a Borel set of $\Omega$ and $E$ a Borel set of $(\mathbb{Z}^d)^{\mathbb{N}}$ (the latter being the space of the trajectories, where $P_\omega$ is defined).

The paper's main results are the annealed CLT, i.e., the CLT relative to $\mathbb{P}$ (Theorem 3.6), and the almost sure recurrence in dimension $d \le 2$, namely, the property that the random walk is recurrent in a.a. environments (Theorem 4.5).

The exposition is organized as follows: In Section 2 we introduce the dynamical system that we use to represent our RWRE, whose ergodic properties we study in Section 3. In Section 4 we present Schmidt's result on recurrent cocycles and apply it to our system. Finally, in Section 5, we draw some brief conclusions about the present work and discuss how to generalize it in a number of ways.

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# 2 The point of view of the particle

Let us enumerate the elements of $\Lambda$, cf. (1.2), as $d_1, d_2, \ldots, d_N$. Now let us fix $\omega \in \Omega$. For $i = 1, \ldots, N$, we define

$$q_i = q_i(\omega) := p_{0d_i}(\omega) \tag{2.1}$$

$$q'_i = q'_i(\omega) := p_{-d_i 0}(\omega) = q_i(\tau_{-d_i}\omega), \tag{2.2}$$

the last equality coming from (1.4). By (1.1)-(1.2),

$$\sum_{i=1}^{N} q_i = \sum_{i=1}^{N} q_i' = 1. \tag{2.3}$$

We then set $a_0 := 0$ and, recursively for $i = 1, \ldots, N$,

$$\begin{aligned} a_i = a_i(\omega) &:= a_{i-1} + q_i & (2.4)\\ I_i = I_i(\omega) &:= [a_{i-1}, a_i). & (2.5) \end{aligned}$$

By the first of the (2.3), $\{I_i\}$ is a partition of $I := [0,1)$. For $(s,\omega) \in I \times \Omega$, let $i(s,\omega)$ be the unique $i$ such that $s \in I_i(\omega)$. Setting

$$\begin{aligned} D(s,\omega) &:= d_{i(s,\omega)}, & (2.6)\\ \phi(s,\omega) &:= q_{i(s,\omega)}^{-1} \left( s - a_{i(s,\omega)} \right), & (2.7) \end{aligned}$$

defines the functions $D : I \times \Omega \longrightarrow \Lambda$ and $\phi : I \times \Omega \longrightarrow I$. For reasons that will be clear momentarily, $D$ is called the *displacement function* and $\phi$ is called the *internal dynamics*, or the *map on the fibers*. It is apparent that $\phi(\cdot,\omega)$ is a piecewise-linear, at most $N$-to-1 map of $I$ onto itself. More precisely, it is the perfect Markov map $I \longrightarrow I$ relative to the partition $\{I_i(\omega)\}$.

The dynamical system we study for the rest of the paper is the triple $(\mathcal{M}, \mu, T)$, where $\mathcal{M} := I \times \Omega$, $\mu := m \times \Pi$ (having denoted by $m$ the Lebesgue measure on $I$), and $T : \mathcal{M} \longrightarrow \mathcal{M}$ is given by

$$T(s,\omega) := \left( \phi(s,\omega), \tau_{D(s,\omega)}(\omega) \right). \tag{2.8}$$

This system is called the *point-of-view-of-the-particle dynamical system* and the reason can be explained as follows.

Fix $\omega \in \Omega$ and a random $s \in I$ w.r.t. $m$. The probability that $s \in I_i(\omega)$ is $m(I_i) = q_i$, which, in terms of our random walk, is exactly the probability that a particle placed in the origin of $\mathbb{Z}^d$, endowed with the environment $p(\omega)$, jumps by a quantity $d_i$. Then, back to the dynamical system, condition the measure $m$ to $I_i$. Calling $(s_1, \omega_1) := T(s,\omega)$, we see that, upon conditioning, $s_1$ ranges in $I$ with law $m$. Therefore, in a sense, the variable $s$ (which we may call the *internal variable*) has "refreshed" itself. Furthermore, $\omega_1$ is the translation of $\omega$ in the opposite direction to $d_i = D(s,\omega)$, cf. (1.4). Hence we can imagine that we have reset the system to a new initial condition $(s_1,\omega_1)$, corresponding to the particle sitting in $0 \in \mathbb{Z}^d$ and subject to the environment $p(\omega_1)$. Applying the same reasoning to $(s_2,\omega_2) := T(s_1,\omega_1)$, and so on, shows that we are following the motion of the particle in the reference system of the particle itself, whence the 'point of view of the particle'.

In any case, it should be clear that the stochastic process $\{X_n\}$, with $X_0 := 0$ and, for $n \geq 1$,

$$X_n(s,\omega) := \sum_{k=0}^{n-1} D_k(s,\omega) := \sum_{k=0}^{n-1} D \circ T^k(s,\omega), \tag{2.9}$$

is precisely the random walk in the environment $p(\omega)$. The definition of $\mu$ entails that $(\mathcal{M}, \mu, T)$ describes all the realizations of the random walk in *all* the environments, w.r.t. a measure that, in the language of Section 1, is expressed by (1.9). This is our RWRE.

# 3 Ergodic properties and Central Limit Theorem

In this section we study the stochastic properties of the dynamical system defined above, starting with the most basic, the invariance of the measure.

**Lemma 3.1** *$T$ preserves $\mu$.*

PROOF. Without loss of generality, it is sufficient to prove that $\mu(T^{-1}A) = \mu(A)$ for sets of the type $A = [b, c] \times B$, where $B$ is a measurable subset of $\Omega$. For added simplicity, we may assume that every $(s, \omega) \in A$ has exactly $N$ counterimages (the other cases can be considered degenerate versions of the one we are assuming).

By direct inspection of the map (2.8), we can write $T^{-1}A = \bigcup_{i=1}^{N} A'_i$, where

$$A'_i := \{(s', \omega') \mid \omega' \in \tau_{-d_i}(B),\ s' \in [a_i(\omega') + q_i(\omega')b\,,\ a_i(\omega') + q_i(\omega')c]\}\,, \tag{3.1}$$

cf. (2.4), (2.7). These sets are pairwise disjoint because, by construction, they belong to different level sets of the function $D$. From (3.1),

$$\mu(A'_i) = \int_{\tau_{-d_i}(B)} q_i(\omega')(c-b)\,\Pi(d\omega') = (c-b)\int_B q'_i(\omega)\Pi(d\omega), \tag{3.2}$$

having used, in the second equality, (2.2) and the $\tau$-invariance of $\Pi$. Summing the above over $i = 1, \ldots, N$, with the help of the second of the (2.3), yields $(c-b)\Pi(B) = \mu(A)$. Q.E.D.

Let us call *horizontal fiber* of $\mathcal{M}$ any segment of the type $I_\omega := I \times \{\omega\}$, and indicate by $m_\omega$ the Lebesgue measure on it. Also, given a positive integer $n$ and a multi-index $\mathbf{i} := (i_0, i_1, ..., i_{n-1}) \in \mathcal{I}^n := \{1, 2, \ldots, N\}^n$, we set

$$I_{\omega,\mathbf{i}} := \{(s, \omega) \in \mathcal{M} \mid D_k(s, \omega) = d_{i_k},\ \forall k = 0, \ldots, n-1\}\,, \tag{3.3}$$

where $D_k$ is defined in (2.9). Finally, we denote by $I_\mathbf{i} = I_\mathbf{i}(\omega)$ the interval of $I$ corresponding to $I_{\omega,\mathbf{i}}$ via the natural isomorphism $I_\omega \longrightarrow I$.

It is easy to ascertain that $\{I_\mathbf{i}\}_{\mathbf{i}\in\mathcal{I}^n}$ partitions $I$ into $N^n$ intervals (some of which may be empty), each corresponding to one of the realizations of the random walk $\{X_k\}_{k=0}^N$ in the environment $\omega$, in such a way that $m(I_\mathbf{i})$ is the probability of the corresponding realization. In analogy with the previous notation, we indicate with $I_{\mathbf{i}_n(s,\omega)}$ the element of said partition that contains $s$.

**Lemma 3.2** *For a.a. $(s, \omega) \in \mathcal{M}$, $m(I_{\mathbf{i}_n(s,\omega)}(\omega))$ vanishes exponentially fast, as $n \to \infty$.*

Proof. First of all, we introduce a notation that will be convenient for this and other proofs. For $(s,\omega) \in \mathcal{M}$ and $k \in \mathbb{N}$, we write

$$(s_k, \omega_k) := T^k(s,\omega). \tag{3.4}$$

Now define

$$f(s,\omega) := \log q^{-1}_{i(s,\omega)}(\omega) = -\log m(I_{i(s,\omega)}(\omega)) \tag{3.5}$$

(with the convention that $\log 0 = -\infty$). Then $f(s,\omega) \geq 0$, the equality holding only when $\{I_i(\omega)\}$ is the trivial partition of $I$, mod $m$, i.e., when $p_{0y}(\omega) = \delta_{yy_0}$, for some $y_0$. By (1.3) it must be $y_0 = 0$. But this can only happen for a negligible set of $\omega$, due to the almost sure irreducibility (1.5).

Thus, $f > 0$ a.e. A well-known corollary of the Birkhoff Theorem ensures that

$$f^+(s,\omega) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(s_k, \omega_k) > 0 \tag{3.6}$$

as well, for a.a. $(s,\omega) \in \mathcal{M}$. On the other hand, from what we have discussed earlier, it is easy to verify that, for $n \geq 1$,

$$m\left(I_{\mathbf{i}_n(s,\omega)}(\omega)\right) = \exp\left(-\sum_{k=0}^{n-1} f(s_k,\omega_k)\right). \tag{3.7}$$

The combination of (3.6) and (3.7) yields the assertion. Q.E.D.

**Lemma 3.3** *The ergodic components of $(\mathcal{M},\mu,T)$ contain whole horizontal fibers, that is, every invariant set is of the form $I \times B$, mod $\mu$, where $B$ is a measurable subset of $\Omega$.*

Proof. Suppose the assertion is false. There exists an invariant set $A$ whose intersection with many horizontal fibers is neither the full fiber nor empty, mod $m_\omega$. That is, for some $\varepsilon > 0$, the $\Pi$-measure of

$$B_\varepsilon := \{\omega \in \Omega \mid m_\omega(A \cap I_\omega) \in [\varepsilon, 1-\varepsilon]\} \tag{3.8}$$

is positive. By the Poincaré Recurrence Theorem and the Lebesgue Density Theorem it is possible to pick a $(s,\omega) \in A \cap (I \times B_\varepsilon)$ that is recurrent to $I \times B_\varepsilon$ and such that $(s,\omega)$ is a density point of $A \cap I_\omega$ within $I_\omega$. We claim that there exist a sufficiently large $n$ and a multi-index $\mathbf{i} \in \mathcal{I}^n$ for which

$$m_\omega(A \cap I_{\omega,\mathbf{i}}) > (1-\varepsilon)\, m_\omega(I_{\omega,\mathbf{i}}) \tag{3.9}$$

and

$$T^n(I_{\omega,\mathbf{i}}) = I_{\omega_n} \subset I \times B_\varepsilon. \tag{3.10}$$

In fact, among the infinitely many $n$ that verify $T^n(s,\omega) \in I \times B_\varepsilon$, we can choose, by Lemma 3.2, one for which $I_{\omega,\mathbf{i}_n(s,\omega)}$ is so small that (3.9) is verified for $\mathbf{i} = \mathbf{i}_n(s,\omega)$. The equality in (3.10) is true by the Markov property of $\phi(\cdot,\omega_{n-1}) \circ \cdots \circ \phi(\cdot,\omega)$ (having used again notation (3.4)).

Since the restriction of $T^n$ to $I_{\omega,\mathbf{i}}$ is linear and $A$ is invariant, we deduce from (3.10) that $m_{\omega_n}(A \cap I_{\omega_n}) > 1-\varepsilon$, which contradicts (3.8), because $\omega_n \in B_\varepsilon$. Therefore an invariant set, mod $\mu$, can only occur in the form $I \times B$. That $B$ is measurable is a consequence of the next lemma. Q.E.D.

**Lemma 3.4** *For $i = 1,2$, let $(\Sigma_i, \mathscr{A}_i, \nu_i)$ be two probability spaces, the second of which complete. If $B_1 \in \mathscr{A}_1$, $\nu_1(B_1) > 0$, and $B_1 \times B_2 \in \mathscr{A}_1 \otimes \mathscr{A}_2$, then $B_2 \in \mathscr{A}_2$.*

PROOF. See [L1, Lemma A.1].

**Theorem 3.5** *$(\mathcal{M}, \mu, T)$ is ergodic.*

PROOF. Suppose the system is not ergodic. By Lemma 3.3, we have an invariant set $I \times B$, with $\Pi(B) \in (0,1)$. Set $B^c := \Omega \setminus B$. Without loss of generality, we assume that the trajectory of *no* point of $I \times B$ intersects $I \times B^c$ (otherwise it is easy to modify the following argument to deal with a negiglible set of exceptions).

By the ergodicity of $(\Omega, \Pi, \{\tau_z\})$ and the almost sure irreducibility of the random environment, one can find an $\omega \in B$ which is transitive in the sense of (1.5), and a $y \in \mathbb{Z}^d$, such that

$$\tau_y \, \omega \in B^c. \tag{3.11}$$

The transitivity of $\omega$ means that there exist $n \in \mathbb{Z}^+$ and $s \in I$ such that $X_n(s,\omega) = y$, whence, using (3.4),

$$\begin{aligned} T^n(s,\omega) &= (s_n, \tau_{D(s_{n-1},\omega_{n-1})} \circ \cdots \circ \tau_{D(s,\omega)} \, \omega) \\ &= (s_n, \tau_{X_n(s,\omega)} \, \omega) \\ &= (s_n, \tau_y \, \omega) \in I \times B^c, \end{aligned} \tag{3.12}$$

the inclusion descending from by (3.11). This contradicts the initial assumption. Q.E.D.

We now prove a strong stochastic property for the specific (vector-valued) observable $D$, namely the CLT for the family $\{D \circ T^k\}$. In the language of RWREs, this can be formulated as follows.

**Theorem 3.6** *The stochastic process $\{X_n\}$, defined in (2.9), satisfies the CLT with mean zero and covariance matrix $C := \{c_{\alpha\beta}\}_{\alpha,\beta=1}^d$, where*

$$c_{\alpha\beta} := \int_{\mathcal{M}} D^{(\alpha)}(s,\omega) D^{(\beta)}(s,\omega) \, \mu(ds d\omega),$$

*having denoted by $D^{(\alpha)}$ the $\alpha$-th component of the vector $D$.*

PROOF. By elementary martingale theory [HH] it suffices to prove that $\{X_n\}$ is a (multidimensional) martingale whose increments $D_n = X_{n+1} - X_n$ have covariance matrix $C$ for all $n$ (if a specific reference is needed, the first theorem of [W] implies the result).

From the considerations outlined in the beginning of Section 3 it is not hard to see that $\mathscr{F}_n$, the $\sigma$-algebra generated by $X_1, \ldots, X_n$ (equivalently, by $D_0, \ldots, D_{n-1}$) corresponds to the partition $\mathscr{D}_n := \{A_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{I}^n}$, with

$$A_{\mathbf{i}} := \bigcup_{\omega\in\Omega} I_{\omega,\mathbf{i}} \,. \tag{3.13}$$

Therefore, denoting by $\mathbb{E}_\mu$ the conditional expectation w.r.t. $\mu$, we have

$$\begin{aligned} \mathbb{E}_\mu(D_n|\mathscr{F}_n) &= \sum_{\mathbf{i}\in\mathcal{I}^n} \left[ \frac{1}{\mu(A_{\mathbf{i}})} \int_{A_{\mathbf{i}}} D_n(s,\omega)\, \mu(dsd\omega) \right] 1_{A_{\mathbf{i}}} = \\ &= \sum_{\mathbf{i}\in\mathcal{I}^n} \left[ \frac{1}{\mu(A_{\mathbf{i}})} \int_{\Omega} \int_{I_{\mathbf{i}}(\omega)} D_n(s,\omega)\, ds\, \Pi(d\omega) \right] 1_{A_{\mathbf{i}}}, \end{aligned} \tag{3.14}$$

where $1_{A_{\mathbf{i}}}$ is the indicator function of $A_{\mathbf{i}}$. For $\mathbf{i}, \omega$ fixed and $s$ ranging in $I_{\mathbf{i}}$ (we are dropping the dependence on $\omega$ from all the notation), the first $n$ positions of the walk are determined, say by the values $x_k := X_k(s,\omega)$ $(k = 0, 1, \ldots, n)$. Thus, the inner integral in (3.14) becomes

$$\begin{aligned} \int_{I_{\mathbf{i}}} D_n(s,\omega)\, ds &= \sum_{j=1}^{N} d_j\, m\big(I_{(\mathbf{i},j)}\big) \\ &= \sum_{j=1}^{N} d_j\, p_{0x_1} \cdots p_{x_{n-1},x_n}\, p_{x_n,x_n+d_j} = 0, \end{aligned} \tag{3.15}$$

by (1.2)-(1.3). Hence the r.h.s. of (3.14) is identically zero, proving the martingale property. As concerns the covariances of $D_n = D \circ T^n$, their constancy in $n$ follows from the invariance of $\mu$. Q.E.D.

# 4 Cocycles and recurrence

The upcoming definitions and results apply to a general dynamical system.

**Definition 4.1** *Let* $(\Sigma, \nu, F)$ *be a probability-preserving dynamical system, and* $f$ *a measurable function* $\Sigma \longrightarrow \mathbb{Z}^d$. *The family of functions* $\{S_n\}_{n\in\mathbb{N}}$, *defined by* $S_0(\xi) \equiv 0$ *and, for* $n \geq 1$,

$$S_n(\xi) := \sum_{k=0}^{n-1} (f \circ F^k)(\xi)$$

*is called a* commutative, $d$-dimensional, discrete cocycle *or, more precisely, the* cocycle of $f$.

**Definition 4.2** *The discrete cocycle* $\{S_n\}$ *is called* recurrent *if, for* $\nu$*-almost all* $\xi \in \Sigma$*, there exists a subsequence* $\{n_j = n_j(\xi)\}$ *such that*

$$\forall j \in \mathbb{N}, \qquad S_{n_j}(\xi) = 0.$$

A remarkable sufficient condition for cocycle recurrence was given by Schmidt [S]:

**Theorem 4.3** *Assume that* $(\Sigma, \nu, F)$ *is ergodic and denote by* $R_n$ *the distribution of* $S_n/n^{1/d}$*, relative to* $\nu$*. If there exists a positive-density sequence* $\{n_k\}_{k\in\mathbb{N}}$ *and a constant* $K > 0$ *such that*

$$R_{n_k}(\mathcal{B}(0,\rho)) \geq K\rho^d$$

*for all sufficiently small balls* $\mathcal{B}(0,\rho) \subset \mathbb{R}^d$ *(of center 0 and radius* $\rho$*), then the cocycle* $\{S_n\}$ *is recurrent.*

**Remark 4.4** Schmidt proved the above only in the case where $F$ is an automorphism (i.e., it is invertible mod $\nu$) [S]. It is easy, however, to extend his proof to the full generality claimed by Theorem 4.3. See Section A.2 of the Appendix.

In dimension 1 and 2, if $\{S_n\}$ satisfies the CLT with zero mean (even with a degenerate covariance matrix), it satisfies the main hypothesis of Theorem 4.3. Therefore, coming back to our system, since $\{X_n\}$ is the cocycle of $D$ via (2.9), and in view of Theorems 3.5 and 3.6, we obtain the following

**Theorem 4.5** *If* $d \leq 2$*, the RWRE described in Section 1 is* almost surely recurrent*. This means that, for* $\Pi$*-a.e.* $\omega$*, the random walk* $\{X_n\}$*, subject to the law* $P_\omega$*, verifies* $X_{n_j} = 0$*, with probability 1, for a subsequence* $\{n_j\}$*.*

# 5 Conclusions and generalizations

We have presented a fairly natural way—at least to a hyperbolic dynamicist—to represent a RWRE as a probability-preserving dynamical system. This is achieved by implementing the local dynamics of the particle in terms of one-dimensional perfect Markov maps and then considering a sort of "union" of all of them. (Notice, however, that the system we have introduced in Section 2 is not the only one that is called 'the point of view of the particle' in the field.)

In this representation, elementary considerations of ergodic theory can produce interesting, new and old, results in a nearly effortless way: e.g., in the martingale case, the CLT with no ellipticity assumption and, using a powerful theorem by Schmidt, recurrence in dimension $d \leq 2$.

In fact, the results contained in this note can be generalized in a number of ways, from the more to the less evident. For example:

1. There is no need for $p_{xy}$ to be zero when $y - x \notin \Lambda$, as long as it decays so fast that the function $D$ of (2.6) has square-integrable modulus. The condition $\sum_y |y|^2 \, p_{0y} \le K$, for a.a. $\omega$, suffices.

2. If we know that the random environment is ergodic for the action of a subgroup $\Gamma \subset \mathbb{Z}^d$ (i.e., $(\Omega, \Pi, \{\tau_z\}_{z \in \Gamma})$ is ergodic, which is a stronger hypothesis than the one we made in Section 1), we can relax the condition of almost sure irreducibility of $p$ and ask that (1.5) be verified only for $y \in \Gamma$. (In fact, the proof of Theorem 3.5 works equally well if $\{\tau_z\}$ is restricted to $z \in \Gamma$.) In particular, if $(\Omega, \Pi)$ is an i.i.d. environment, it suffices to require the existence of a (one-dimensional) subgroup $\Gamma$ such that (1.5) holds $\forall y \in \Gamma$.

3. More importantly, the bistochasticity condition (1.1) can be done away with.

4. Finally, one can prove not only the annealed CLT, but the *quenched* invariant principle, i.e., for a *fixed* typical environment $\omega$, the convergence of the rescaled trajectory
$$t \mapsto \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k \tag{5.1}$$
to a $d$-dimensional Brownian motion, for $t \in [0, 1]$. The annealed invariant principle follows.

These advances will be presented in [L3].

The techniques exposed in this paper can also be used separately. For instance, one can apply Theorem 4.3 to the random walks of [KO]. These are more general bistochastic RWREs than the ones discussed here, in that they do not have zero local drift as in (1.3), but zero *mean* drift:
$$\int_\Omega \sum_y y \, p_{0y}(\omega) \, \Pi(d\omega) = 0. \tag{5.2}$$
Subject to a uniform ellipticity condition, namely that, $\Pi$-almost surely,
$$p_{0y} \ge \varepsilon, \qquad \forall y \in \Lambda; \tag{5.3}$$
and to an additional, rather cumbersome but essential, hypothesis they call *condition (H)*, Komorowski and Olla [KO, Thm. 2.2] prove the annealed zero-mean CLT for $\{X_n\}$, i.e., they prove Theorem 3.6 for the dynamical system $(\mathcal{M}, \mu, T)$ adapted to their case. Since the ergodicity of that system is guaranteed by (5.3) (as in Theorem 3.5), Theorem 4.5, that is the a.s. recurrence in dimension 1 and 2, holds true in this case as well.

# A Appendix

## A.1 An example of a bistochastic environment with zero local drift

As it may not be immediately intuitive how to construct a random environment satisfying the assumptions of Section 1, we give here an explicit example, emphasizing that the $(\Omega, \Pi)$ produced below is not as general as we are able to deal with in the present paper (see also Section 5).

We start by fixing a finite $\Lambda_0 \subset \mathbb{Z}^d$, of cardinality $N_0$. Calling $\mathbb{B} = \mathbb{B}(\Lambda_0)$ the set of all bistochastic matrices indexed by the elements of $\Lambda_0$, it is known that $\mathbb{B}$ can be identified with a convex polytope of $\mathbb{R}^{(N_0-1)^2}$ (each matrix has $N_0^2$ entries and $2N_0 - 1$ independent conditions on them). The Birkhoff–Von Neumann Theorem states that any element of $\mathbb{B}$ can be expressed as a convex combination of the $N_0!$ permutation matrices of $\mathbb{B}$, which are thus the extremal points of the polytope (the permutation matrices are those that are obtained by permuting the columns of the identity matrix) [B].

Now endow $\mathbb{B}$ with any absolutely continuous probability $\pi_0$ (w.r.t. the Lebesgue measure in $\mathbb{R}^{(N_0-1)^2}$) and define $(\Omega, \Pi) := (\mathbb{B}, \pi_0)^{\mathbb{Z}^d}$, in the sense of the tensor product of measure spaces. The generic element of $\Omega$ is denoted $\omega = \{\omega^{(\zeta)}\}_{\zeta \in \mathbb{Z}^d}$, where $\omega^{(\zeta)} = \{\omega^{(\zeta)}_{xy}\}_{x,y \in \Lambda_0} \in \mathbb{B}$. To keep the notation simple, pretend that $\omega^{(\zeta)}_{xy}$ exists for all $x, y \in \mathbb{Z}^d$, equaling 0 where not otherwise defined.

Setting

$$o_{xy} = o_{xy}(\omega) := \frac{1}{N_0} \sum_{\zeta \in \mathbb{Z}^d} \omega^{(\zeta)}_{x+\zeta, y+\zeta} \tag{A.1}$$

gives rise to a bistochastic environment $o = o(\omega) := \{o_{xy}\}_{xy}$, as it can easily be verified. If we further set

$$p_{xy} = p_{xy}(\omega) := \frac{o_{xy} + o_{x,-y}}{2} \tag{A.2}$$

and $p = p(\omega) := \{p_{xy}\}_{xy}$, we obtain a bistochastic environment which is also *balanced* in the sense of Lawler [La], namely $p_{xy} = p_{x,-y}$. This is a special case of the zero-local-drift condition (1.3). Also, (1.2) holds true for $\Lambda := \Lambda_0 \cup -\Lambda_0$.

For $z \in \mathbb{Z}^d$, we define $\tau_z : \Omega \longrightarrow \Omega$ via

$$(\tau_z \omega)^{(\zeta)} := \omega^{(\zeta - z)}. \tag{A.3}$$

Since the $\omega^{(\zeta)}$ are i.i.d. random matrices, obviously $\{\tau_z\}$ leaves $\Pi$ invariant and is ergodic. Equality (1.4) is easily checked for $o$ and thus for $p$.

Finally, since $\pi_0$ has a density on $\mathbb{B} \subset \mathbb{R}^{(N_0-1)^2}$, the probability that $\omega^{(\zeta)}$ has a zero entry is null, which proves the (nonuniform) ellipticity of $o$ and $p$, implying (1.5).

## A.2 Partial proof of Schmidt's Theorem

In this section we show that Theorem 4.3, which was proved by Schmidt only for $F$ invertible [S], easily extends to the case of a general endomorphism $F$ (note that Conze has independently given a weaker result than Theorem 4.3 which does not require invertibility [C]).

In fact, if $(\Sigma, \mathscr{A}, \nu, F)$ is a probability-preserving, noninvertible, dynamical system (here $\mathscr{A}$ is the $\sigma$-algebra defined on $\Sigma$), one can consider its *natural extension* in the sense or Rohlin [R]. Shying away from the details of its construction (which can be found, e.g., in [R, CFS]), we simply recall that the natural extension of $(\Sigma, \mathscr{A}, \nu, F)$ is a probability-preserving invertible dynamical system $(\bar{\Sigma}, \bar{\mathscr{A}}, \bar{\nu}, \bar{F})$ for which there exists a measurable projection $\pi : \bar{\Sigma} \longrightarrow \Sigma$ with the following properties: First, the commutation condition

$$\pi \circ \bar{F} = F \circ \pi, \tag{A.4}$$

which explains in what sense $\bar{F}$ extends $F$. Second, setting

$$\bar{\mathscr{A}_0} := \pi^* \mathscr{A} := \left\{ \pi^{-1} A \;\middle|\; A \in \mathscr{A} \right\}, \tag{A.5}$$

one has that $(\bar{\Sigma}, \bar{\mathscr{A}_0}, \bar{\nu})$ and $(\Sigma, \mathscr{A}, \nu)$ are isomorphic measure spaces. Furthermore $\bar{\mathscr{A}_0}$ is $\bar{F}$-invariant (this means, using again the terminology of (A.5), that $\bar{F}^* \bar{\mathscr{A}_0} \subset \bar{\mathscr{A}_0}$) but in general not $\bar{F}^{-1}$-invariant.

Hence any $\mathscr{A}$-measurable function $f : \Sigma \longrightarrow \mathbb{R}$ is isomorphically associated to the $\bar{\mathscr{A}_0}$-measurable function $\bar{f} := f \circ \pi : \bar{\Sigma} \longrightarrow \mathbb{R}$. Moreover, $\forall k \in \mathbb{N}$, $f \circ F^k$ and $\bar{f} \circ \bar{F}^k$ have the same distribution (because, by (A.4), $\overline{f \circ F^k} = \bar{f} \circ \bar{F}^k$). Finally, it is known that $(\bar{\Sigma}, \bar{\mathscr{A}}, \bar{\nu}, \bar{F})$ is ergodic if and only if $(\Sigma, \mathscr{A}, \nu, F)$ is ergodic [CFS].

Coming back to Theorem 4.3 for a noninvertible $F$, one can apply Schmidt's proof to the natural extension, which is ergodic, and its cocycle $\{\bar{S}_n\}$ (obvious definition). The latter verifies the main hypothesis of the theorem because $\{S_n\}$ does, as explained above. The fact that $\{S_n\}$ is recurrent if and only if $\{\bar{S}_n\}$ is concludes the proof. Q.E.D.

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