The boundary control approach to inverse spectral theory

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Abstract. We establish connections between four approaches to inverse spectral problems: the classical Gelfand–Levitan theory, the Simon theory, the approach proposed by Remling, and the Boundary Control method. We show that the Boundary Control approach provides simple and physically motivated proofs of the central results of other theories. We demonstrate also the connections between the dynamical and spectral data and derive the local version of the classical Gelfand–Levitan equations.

In this paper we consider the Schrödinger operator

$$H = -\partial_x^2 + q\,(x) \tag{0.1}$$

on $L^2\,(\mathbb{R}_+)\,, \mathbb{R}_+ := [0,\infty)$, with a real-valued locally integrable potential q and Dirichlet boundary condition at $x = 0$. Let $d\rho(\lambda)$ be the *spectral measure* corresponding to H, and $m(z)$ be the (principal or Dirichlet) *Titchmarsh-Weyl m-function.*

1. Three approaches to inverse spectral theory

In this section we give a brief review of three different approaches to inverse problems for the operator (0.1): the Gelfand–Levitan theory, the Simon theory and the Remling approach. In the next section we describe the Boundary Control method and its connections with the other approaches.

1.1. Gelfand–Levitan theory. Determining the potential q from the spectral measure is the main result of the seminal paper by Gelfand and Levitan [**16**]. To formulate the result let us define the following functions:

$$\sigma(\lambda) = \begin{cases} \rho(\lambda) - \frac{2}{3\pi}\lambda^{\frac{3}{2}}, & \lambda \geqslant 0, \\ \rho(\lambda), & \lambda < 0 \end{cases} \tag{1.1}$$

$$F(x,t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}x \sin\sqrt{\lambda}t}{\lambda}\, d\sigma(\lambda). \tag{1.2}$$

Let $\varphi(x,\lambda)$ be a solution to the equation

$$-\varphi'' + q(x)\varphi = \lambda\varphi, \quad x > 0, \tag{1.3}$$

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with the Cauchy data

$$\varphi(0,\lambda)=0, \quad \varphi'(0,\lambda)=1. \tag{1.4}$$

The so-called transformation operator transforms the solutions of (1.3), (1.4) with zero potential to the functions $\varphi(x,\lambda)$:

$$\varphi(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x K(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,dt. \tag{1.5}$$

The kernel $K(x,t)$ satisfies the integral (Gelfand–Levitan) equation

$$F(x,t)+K(x,t)+\int_0^x K(x,s)\,F(s,t)\,ds=0\,, \quad 0\leqslant t<x, \tag{1.6}$$

and the potential can be recovered by the rule

$$q(x)=2\frac{d}{dx}K(x,x). \tag{1.7}$$

1.2. Simon approach. In [**21**] Barry Simon proposed a new approach to inverse spectral theory which has got a further development in the paper by Gesztesy and Simon [**18**] (see also an excellent survey paper [**17**]). The inverse data in this approach is the Titchmarsh–Weyl m-function which is equivalent to the knowledge of the spectral measure. It was shown in [**21**] that there exists a unique real valued function $A\in L^1_{loc}(\mathbb{R}_+)$ (the $A-$*amplitude*) such that

$$m(-k^2)=-k-\int_0^\infty A(t)e^{-2tk}\,dt\,. \tag{1.8}$$

The absolute convergence of the integral was proved for $q\in L^1(\mathbb{R}_+)$ and $q\in L^\infty(\mathbb{R}_+)$ in [**18**] for sufficiently large $\Re k$. In general situation one has an asymptotic equality

$$m(-k^2)=-k-\int_0^a A(t)e^{-2tk}\,dt+\mathcal{O}(e^{-2ak}) \tag{1.9}$$

(see [**21, 18**] for details).

So, if we know the $m-$function, we know the $A-$amplitude. Then the following local approach for solving the inverse problem was put forward in [**21**]. Locality means that the $A-$amplitude on $[0,a]$ completely determines q on the same interval (and vice versa). Based on representation (1.9) Simon proved the local version of the Borg–Marchenko uniqueness theorem: $m_1(-k^2)-m_2(-k^2)=\mathcal{O}(e^{-2ak})$ if and only if $q_1(x)=q_2(x)$ for $x\in[0,a]$.

If $A(\cdot,x)$ denotes the $A-$amplitude of the problem on $[x,\infty)$, then this family satisfies the nonlinear integro-differential equation

$$\frac{\partial A(t,x)}{\partial x}=\frac{\partial A(t,x)}{\partial t}+\int_0^t A(s,x)A(t-s,x)\,ds=0\,. \tag{1.10}$$

If one solve this equation with the initial condition $A(t,0)=A(t)$ in the domain $\{(x,t):0\le x\le a,\ 0\le t\le a-x\}$, then the potential on $[0,a]$ is determined by

$$\lim_{t\downarrow 0}A(t,x)=q(x)\,,\ 0\le x\le a. \tag{1.11}$$

The $A-$amplitude has the explicit representation through the spectral measure by the formula derived in [**18**]:

$$A(t) = -2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{-\varepsilon\lambda} \frac{\sin(2t\sqrt{\lambda})}{\sqrt{\lambda}} \, d\rho(\lambda) \quad a.e. \tag{1.12}$$

Without the Abelian regularization the integral need not be convergent (even conditionally) [**18**].

1.3. Remling approach. Remling [**19, 20**] proposed another local approach to inverse spectral problems based on the theory of de Branges spaces. He introduced the integral operator $\mathcal{K}$ acting in the space $\mathcal{F}^T := L^2(0,T)$:

$$(\mathcal{K}f)(x) = \int_0^T k(x,t)\, f(t)\, dt\,, \tag{1.13}$$

where

$$k(x,t) = \frac{1}{2}[\phi(x-t) - \phi(x+t)]\,, \quad \phi(x) = \int_0^{|x|/2} A(t)\, dt\,. \tag{1.14}$$

Remling proved that given a function $A \in L^1(0,T)$, there exists a unique $q \in L^1(0,T)$ such that A is the $A-$amplitude of this q if and only if the operator $I + \mathcal{K}$ is positive definite in $\mathcal{F}^T$. The same positivity condition was proved in [**20**] to be necessary and sufficient for solvability of the equation (1.10).

He proved the following representation of the $A-$amplitude through the regularized spectral measure $d\sigma$:

$$A(t) = -2 \int_{\mathbb{R}} \frac{\sin(2t\sqrt{\lambda})}{\sqrt{\lambda}} \, d\sigma(\lambda) \tag{1.15}$$

with the convergence in the sense of distributions.

Remling derived also two linear integral equations,

$$y(x,t) + \int_0^x k(t,s) y(x,s)\, ds = t\,, \tag{1.16}$$

$$z(x,t) + \int_0^x k(t,s) z(x,s)\, ds = \psi(t)\,, \tag{1.17}$$

where $0 \le t \le x \le T$ and $\psi(t) = -1 - \int_0^t \phi(s)\, ds$. The potential $q(x)$ on $[0,T]$ is uniquely determined by any of the functions y or z.

2. The Boundary Control method.

The Boundary Control (BC) method in inverse problems was developed about two decades ago by M. Belishev and his colleagues [**7, 14, 13, 11, 2**]. As well as methods of Simon and Remling, the BC method provides the local approach to inverse problems developing ideas of A. Blagoveshchenskii [**15**] who seems to have been the first proposed the local approach to the 1d wave equation. It is worth to notice that the papers by Simon, Gesztesy and Remling are based on the spectral approach, and locality is proved there using hard analysis, power analytical tools. In the BC method locality naturally follows from the finite speed of the front propagation in the wave equation.

The BC method uses the deep connection between inverse problems of mathematical physics, functional analysis and control theory for partial differential equations and offers an interesting and powerful alternative to previous identification techniques based on spectral or scattering methods. This approach has several advantages, namely: (i) it maintains linearity (does not introduce spurious nonlinearities); (ii) it is applicable to a wide range of linear point and/or distributed systems and reconstruction situations; (iii) it can identify coefficients occurring in highest order terms; (iv) it is, in principle, dimension-independent; and, finally, (v) it lends itself to straightforward algorithmic implementations. Being originally proposed for solving the boundary inverse problem for the multidimensional wave equation, the BC method has been successfully applied to all main types of linear equations of mathematical physics (see the review papers [**9, 10**] and references therein). In this paper we use this method in 1d situation applying it to inverse problems for the operator (0.1) and demonstrate its connections with the methods described above. We do not give the detailed proofs of our results here, they will be provided in a forthcoming paper.

2.1. The main operators of the BC method. The main ideas of the BC method can be explained on the example of the 1d wave equation

$$\left\{\begin{array}{l} u_{tt}(x,t)-u_{xx}(x,t)+q(x)u(x,t)=0, \quad x>0,\ t>0, \\ u(x,0)=u_t(x,0)=0,\ u(0,t)=f(t). \end{array}\right. \tag{2.1}$$

Here $q \in L^1_{loc}(\mathbb{R}_+)$ and f is an arbitrary $L^2_{loc}(\mathbb{R}_+)$ function referred to as a *boundary control.* The solution $u^f(x,t)$ of the problem (2.1) can be written in terms of the integral kernel $w(x,s)$ which is the unique solution to the Goursat problem:

$$\left\{\begin{array}{l} w_{tt}(x,t)-w_{xx}(x,t)+q(x)w(x,t)=0, \quad 0<x<t, \\ w(0,t)=0,\ w(x,x)=-1/2\int_0^x q(s)\,ds. \end{array}\right. \tag{2.2}$$

Using the successive approximations, one can prove the following

PROPOSITION 1. (a) *If $q\in L^1_{loc}(\mathbb{R}_+)$, then the Goursat problem* (2.2) *has a unique generalized solution $w(x,t)$ which is an absolutely continuous function and*

$$w_x(\cdot,t), w_t(\cdot,t), w_x(x,\cdot), w_t(x,\cdot) \in L_{1,\,loc}(\mathbb{R}_+). \tag{2.3}$$

The equation in (2.2) *holds almost everywhere and boundary conditions are satisfied in the classical sense.*

(b) *If $q\in C^1_{loc}(\mathbb{R}_+)$, then the solution to the Goursat problem* (2.2) *is classical, all its derivatives up to second order are continuous.*

The Goursat problem was studied in [**22**, Sec. II.4] for smooth q, but the method works for $q\in L_1(0,a)$ as well (see [**4, 5, 6, 3**]).

The next proposition can be proved by direct calculations.

PROPOSITION 2. (a) *If $q \in C^1_{loc}(\mathbb{R}_+)$ and $f\in C^2_{loc}(\mathbb{R}_+)$, $f(0)=f'(0)=0$, then the classical solution $u^f(x,t)$ to the initial-boundary value problem* (2.1) *admits the representation*

$$u^f(x,t)=\left\{\begin{array}{l} f(t-x)+\int_x^t w(x,s)f(t-s)\,ds, \quad x<t, \\ 0, \quad x\geqslant t. \end{array}\right. \tag{2.4}$$

(b) *If $q \in L^1_{loc}(\mathbb{R}_+)$ and $f \in \mathcal{F}^T$, the formula (2.4) represents a unique generalized solution to the initial-boundary value problem* (2.1) *and* $u^f \in C([0,T]; \mathcal{H}^T)$, *where*

$$\mathcal{H} = L^2_{loc}(0,\infty) \quad \textit{and} \quad \mathcal{H}^T := \{u \in \mathcal{H} : \ \textit{supp}\ u \subset [0,T]\,\}.$$

The **response operator** (the dynamical Dirichlet-to-Neumann map) R^T for the system (2.1) is defined in $\mathcal{F}^T$ by

$$(R^T f)(t) = u^f_x(0,t), \ t \in (0,T), \tag{2.5}$$

with the domain $\{f \in C^2([0,T]) : \ f(0) = f'(0) = 0\}$. According to (2.4) it has a representation

$$(R^T f)(t) = -f'(t) + \int_0^t r(s) f(t-s)\, ds, \tag{2.6}$$

where $r(t) := w_x(0,t)$ is called the **response function**.

The response operator R^T is completely determined by the response function on the interval $[0,T]$, and the dynamical inverse problem can be formulated as follows. Given $r(t),\ t \in [0,2T]$, find $q(x),\ x \in [0,T]$.

Notice that from (2.2) one can derive the formula

$$r(t) = -\frac{1}{2} q\Big(\frac{t}{2}\Big) - \frac{1}{2}\int_0^t \ q\Big(\frac{t-\zeta}{2}\Big) v(\zeta,t)\, d\zeta.\,, \tag{2.7}$$

where

$$v(\xi,\eta) = w\Big(\frac{\eta-\xi}{2}, \frac{\eta+\xi}{2}\Big).$$

To solve the dynamical inverse problem by the BC method let us introduce a couple more operators. Proposition 2 implies in particular that the **control operator** W^T,

$$W^T : \mathcal{F}^T \mapsto \mathcal{H}^T, \ W^T f = u^f(\cdot, T),$$

is bounded. The next statement claims that the operator W^T is boundedly invertible.

Proposition 3. *Let $q \in L^1_{loc}(\mathbb{R}_+)$ and $T > 0$, then for any function $z \in \mathcal{H}^T$, there exists a unique control $f \in \mathcal{F}^T$ such that*

$$u^f(x,T) = z(x). \tag{2.8}$$

Proof. According to (2.4), condition (2.8) is equivalent to the following integral Volterra equation of the second kind

$$z(x) = f(T-x) + \int_x^T \ w(x,\tau) f(T-\tau)\, d\tau \quad x \in (0,T)\,. \tag{2.9}$$

The kernel $w(x,t)$ is continuous and therefore equation (2.9) is uniquely solvable, which proves the proposition. □

The **connecting operator** $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$, plays a central role in the BC method. It connects the outer space (the space of controls) of the dynamical system (2.1) with the inner space (the space of waves) being defined by its bilinear product:

$$\big\langle C^T f, g\big\rangle_{\mathcal{F}^T} = \big\langle u^f(\cdot,T), u^g(\cdot,T)\big\rangle_{\mathcal{H}^T} \tag{2.10}$$

In other words,

$$C^T = (W^T)^* W^T\,, \tag{2.11}$$

and Propositions 2, 3 imply that this operator is positive definite, bounded and boundedly invertible on $\mathcal{F}^T$. The remarkable fact is that C^T can be explicitly expressed through R^{2T} (or through $r(t)$, $t \in [0, 2T]$).

PROPOSITION 4. *For $q \in L^1_{loc}(0,\infty)$ and $T > 0$,*

$$(C^T f)(t) = f(t) + \int_0^T [p(2T-t-s) - p(|t-s|)] f(s)\, ds\,,\ \ 0 < t < T\,, \tag{2.12}$$

where

$$p(t) := \frac{1}{2} \int_0^t r(s)\, ds\,. \tag{2.13}$$

PROOF. One can easily check that for any $f, g \in C_0^\infty(0,T)$ the function $U(s,t) := \big(u^f(\cdot,s), u^g(\cdot,t)\big)_{\mathcal{H}}$ satisfies the equation

$$U_{tt} - U_{ss} = (R^T f(s)g(t) - f(s)(R^T g)(t)\,,\quad s,t > 0\,,$$

with the boundary and initial conditions

$$U(0,t) = 0\,,\ \ U(s,0) = U_t(s,0) = 0\,.$$

Using the D'Alambert formula gives representation (2.12). □

2.2. The Gelfand–Levitan type equations. Let us consider the Cauchy problem:

$$-y'' + q(x)y = 0, \quad x > 0\,; \quad y(0) = \alpha\,,\ y'(0) = \beta\,, \tag{2.14}$$

and let f^T be a solution of the control problem

$$(W^T f^T)(x) = \begin{cases} y(x),\ 0 < x < T, \\ 0,\ x > T. \end{cases} \tag{2.15}$$

For any $g \in C_0^\infty(0,T)$ the identity

$$u^g(x,T) = \int_0^T \varkappa^T(t) u^g_{tt}(x,t)\, dt\,,\quad \varkappa^T(t) := T - t$$

is valid, and we have

$$\begin{aligned}(C^T f^T, g) = \int_0^T y(x) u^g(x,T)\, dx &= \int_0^T y(x) \int_0^T \varkappa^T(t) u^g_{tt}(x,t)\, dt\, dx \\ &= \int_0^T \varkappa^T(t)\, [y(x)u^g_x(x,T) - y_x(x)u^g(x,T)]_0^T)\, dt \\ = \int_0^T \beta\varkappa^T(t)g(t) - \alpha\varkappa^T(t)(R^T g)(t)\, dt &= (\beta\varkappa^T - \alpha(R^T)^*\varkappa^T, g).\end{aligned}$$

Here $(R^T)^*$ is the operator adjoint to R^T in $\mathcal{F}^T$:

$$((R^T)^* f)(t) = f'(t) + \int_t^T r(s-t) f(s)\, ds. \tag{2.16}$$

We have used the fact that the solution $u^g(x,t)$ is classical and $u^g(T,T) = u^g_x(T,T) = 0$ (see (2.4)).

Let us denote by y_i, f_i^T, $i = 0, 1$, the functions corresponding the cases $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = 0$. Since g is an arbitrary smooth function, the functions f_0^T and f_1^T satisfy the equations

$$(C^T f_0^T)(t) = T - t\,,\ \ (C^T f_1^T)(t) = -((R^T)^* \varkappa^T)(t)\,,\quad t \in [0,T]\,. \tag{2.17}$$

Using (2.12) these equations can be rewritten in more detail:

$$f_0^T(t) + \int_0^T c^T(t,s)\, f_0^T(s)\, ds \, = T - t\,,\ t \in [0,T]\,, \tag{2.18}$$

$$f_1^T(t) + \int_0^T c^T(t,s)\, f_1^T(s)\, ds \, = 1 - \int_t^T r(s-t)\,(T-s)\, ds\,,\ t \in [0,T]\,, \tag{2.19}$$

where $c^T(t,s) := p(2T-t-s) - p(|t-s|)$.

Using any of functions one can easily find the potential q in the following way. From equation (2.4) it follows that $u^f(t-0,t) = f(+0)$, and in particular, $y_i(T) = f_i^T(+0)$. Let us denote $f_i^T(+0)$ by $\mu_i(T)$. Then

$$q(T) = \frac{\mu_i''(T)}{\mu_i(T)}\,. \tag{2.20}$$

Equations (2.17)–(2.20) were obtained for a matrix valued q of a class C^1 in [**2**]. Using Proposition 1 we prove that they are valid also for $q \in L^1_{loc}(\mathbb{R}_+)$. The remarkable fact that a small modification of these equations holds valid in multidimensional situation [**8**].

In [**6**] we showed that the Titchmarsh–Weyl m-function (the spectral Dirichlet-to-Neumann map) and the response operator (the dynamical Dirichlet-to-Neumann map) are connected by the Laplace (or Fourier) transform and established the relation between the $A-$amplitude and the response function:

$$A(t) = -2r(2t)\,. \tag{2.21}$$

Using this relation it is easy to check that the positivity condition of Remling's operator $I + \mathcal{K}$ is equivalent to the fact that the operator C^T is positive definite. Equations (2.18), (2.19) are reduced by simple changes of variables to equations (1.16), (1.17).

The fact that the positivity of C^T give the necessary and sufficient conditions of the solvability of the inverse problem was known in the BC community for a long time. A. Blagoveshchenskii [**15**] in 1971 obtained the necessary and sufficient conditions of the solvability of the inverse problem for the 1d wave equation (with smooth density) which are equivalent to the positivity of C^T. (Certainly these conditions were in other terms — the BC method and the operator C^T were proposed fifteen years later). Belishev and Ivanov [**12**] considered the two velocity system with smooth matrix-valued potential. In a particular case when two velocities are equal, their necessary and sufficient condition is the positivity of C^T. In [**1**] necessary and sufficient condition for solvability of a nonselfadjoint inverse problem with a matrix-valued potential in terms of C^T was formulated.

We proved that given $r \in L^1(0,2T)$, there exists a unique $q \in L^1(0,T)$ such that r is the response function corresponding to the problem (2.1) with this q if and only if the operator C^T constructed by this r according to (2.12) is positive definite. The fact that r and q belong to the same functional class is confirmed by formula (2.7).

2.3. Gelfand-Levitan equations. Spectral representation of r and c^T. Using the BC approach we derive the local version of the classical Gelfand-Levitan equations (1.6). The proof is based on the fact that the kernel K of the transformation operator (1.5) satisfies a Goursat problem. On the other hand, we show that the kernel v of the operator $(W^T)^{-1}$ (which is inverse to W^T) satisfies a similar Goursat problem. The BC version of the Gelfand-Levitan equations reads as

$$v(x,t)+c^T(x,t)+\int_x^T v(x,s)c^T(s,t)\,ds=0,\quad 0<x<t<T. \tag{2.22}$$

We demonstrate that the kernel v is connected with K by the rule $v(T-x,T-t)=K(x,s)$ and c^T is similarly related to F defined in (1.2): $c^T(T-x,T-t)=F(x,t)$. Therefore, equations (2.22) can be rewritten in a classical form (1.6). On the other hand, equations (2.22) have clearly a local character since $v(x,t)$ and $c^T(x,t)$ are completely determined by $q(x)$ on the interval $[0,T]$.

To complete the review of the connections between four approaches to inverse spectral problems, we derive spectral representations of the functions r and c^T. The next proposition refines the similar statement of Remling about the $A-$amplitude.

Proposition 5. *For the response function r the representation formula*

$$r(t)=\int_{-\infty}^{\infty}\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,d\sigma(\lambda), \tag{2.23}$$

holds almost everywhere on $\mathbb{R}_+$.

The last our statement demonstrates relation of c^T to the classical object, the function F defined by (1.2).

Proposition 6. *The kernel $c^T(s,t)$ admits the following representation:*

$$c^T(s,t)=\int_{-\infty}^{\infty}\frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}\,d\sigma(\lambda),\quad s,t\in(0,T), \tag{2.24}$$

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