A uniqueness result for $\Delta u - \lambda u + V(|x|)u^p = 0$ on $\mathbb{R}^2$

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Abstract

Uniqueness of positive radial solutions decaying at infinity is proved for a class of semilinear elliptic equations on $\mathbb{R}^2$. Complementary results for the same kind of equations were obtained in the early 90's, on $\mathbb{R}^N$ with $N \geq 3$, and in finite balls of $\mathbb{R}^N$ with $N \geq 2$. The new result presented here plays a crucial role in the global bifurcation problem, previously studied by the author.

1 Introduction

The purpose of this note is to fill a gap in the literature about uniqueness of positive radial solutions of semilinear elliptic problems of the form

$$\begin{cases} \Delta u + f(|x|, u) = 0, \quad x \in B(0,R) \subset \mathbb{R}^N, \\ u(R) = 0, \end{cases} \tag{1.1}$$

where $R > 0$ can be finite or infinite. We are particularly interested in the case $R = \infty$, where the problem arises when seeking ground states in many models of mathematical physics. In this context, $u(R) = 0$ is interpreted as $\lim_{|x|\to\infty} u(x) = 0$. We shall specialize our discussion to the situation where

$$f(r, s) = -\lambda s + V(r)s^p \tag{1.2}$$

with $\lambda > 0$, $p > 1$ and $V : (0, \infty) \to (0, \infty)$ a nonincreasing function, subject to the hypotheses (H1) to (H4) below. We will also suppose that $N = 2$ throughout. Our interest in this problem was stimulated by the study of bifurcation of solutions to (1.1)-(1.2) with $R = \infty$, and issues of stability of standing waves for related nonlinear Schrödinger equations, see [4, 5, 6, 7]. In fact, Theorem 1.1 below provides

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a definitive answer to the question raised in Remark 3(d) of [5], thereby implying the existence of a global branch of solutions of (1.1)-(1.2) in dimension $N = 2$.

Existence of positive radial solutions of (1.1) with $R = \infty$ is an old story, going back to the seminal works of Strauss [12], Stuart [13] and Berestycki and Lions [1], where solutions are obtained under very general conditions. The existence of solutions in finite balls is an easier problem, already solved to a large extent at that time, see for instance the references in [1, 9]. It turns out that the uniqueness problem for (1.1) is quite hard in general, both in the case $R < \infty$ and in the case $R = \infty$, and there is an extensive literature about uniqueness and nonuniqueness results, see e.g. [2, 11, 14, 10, 8, 15, 16, 17] and the references therein. Two important strategies have proved powerful in showing uniqueness. The first one, due to Coffman [2] and later generalized by Kwong [8], is based on a shooting argument for the initial value problem

$$\begin{cases} u'' + \dfrac{1}{r}u' - \lambda u + V(r)u^p = 0, \quad r > 0, \quad u = u(r;\alpha) \geq 0, \\ u(0) = \alpha > 0, \quad \lim\limits_{r\to 0} ru'(r) = 0. \end{cases} \tag{IVP}$$

Let us emphasize here that due to the possible singularity of V at $r = 0$, the solutions of (IVP) may not be differentiable at the origin (see Remarque 1.2.8 of [7]). Under our hypotheses, local existence, uniqueness and continuity with respect to $\alpha > 0$ for solutions of (IVP) follow by the contraction mapping principle. Since there are no further singularities after $r = 0$, the solutions can then be uniquely extended to their maximal interval of positivity using standard ODE theory. Furthermore, arguments similar to the Appendix of [3] show that u and u' are differentiable with respect to α and that the function $w(r;\alpha) := \frac{\partial}{\partial\alpha}u(r;\alpha)$ satisfies

$$\begin{cases} w'' + \dfrac{1}{r}w' - \lambda w + pV(r)u^{p-1}w = 0, \quad r > 0, \\ w(0) = 1, \quad \lim\limits_{r\to 0} rw'(r) = 0. \end{cases} \tag{1.3}$$

The basic idea in [8] is to study the set of zeros and the global behaviour of w and to compare it with u. We state here the equations in the particular case we shall be dealing with in the present note. In fact, Kwong obtained sharp results for the analogous problem in any dimension $N \geq 1$, but for the autonomous nonlinearity $f(r,s) \equiv f(s) = -s + s^p$. We will essentially follow Kwong's proof, using Sturm's oscillation theory, and generalize it to the nonautonomous case (1.2).

A second method to prove uniqueness is the "separation of graphs" introduced by Peletier and Serrin in [11], where they consider an autonomous nonlinearity $f(s)$ satisfying a starshapeness condition. The method is suited for problems on the whole space and, in dimension $N = 1$, it has been extended by Toland [14] to nonautonomous nonlinearities with similar hypotheses as those we make here (see also Theorem 2.1 in [6]). However, in dimension $N \geq 2$, it seems very difficult to apply the separation of graphs to nonautonomous nonlinearities because of the first order term in the ODE.

Two important results in higher dimensions are due to Yanagida [15, 16]. In [15], very general nonlinearities of the form $f(r,s) = g(r)s + h(r)s^p$ are considered, in the whole space as well as in finite balls, but the proof only works in dimension $N \geq 3$. In [16], a class of nonlinearities comprising (1.2) is treated in dimension $N \geq 2$, but only for $R < \infty$, though some hints are given for the case $R = \infty$. Unfortunately, the uniqueness result in the whole space cannot be simply derived from that in finite balls, whence the purpose of the present work. Nevertheless, we shall strongly benefit from [16] in our proof.

Let us now introduce some notation and expose our results. We define the *solution sets* as

$$N := \{\alpha > 0 : \text{there exists } r > 0 \text{ such that } u(r;\alpha) = 0\},$$

$$G := \{\alpha > 0 : u(r;\alpha) > 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r;\alpha) = 0\},$$

$$P := \{\alpha > 0 : u(r;\alpha) > 0 \text{ for all } r > 0 \text{ and } \alpha \notin G\}.$$

Clearly, $P \cup G \cup N = (0,\infty)$ as a disjoint union. Let us then define

$$z(\alpha) := \sup\{s > 0 : u(r;\alpha) > 0 \text{ for all } r \in [0,s)\}.$$

If $\alpha \in N$, $z(\alpha)$ is the smallest zero of u. Otherwise, $z(\alpha) = \infty$.

The precise hypotheses we make on the function V are the following.

(H1) $V \in C^1(0,\infty)$.

(H2) $V > 0$ and $V' \leq 0$ on $(0,\infty)$.

(H3) $r^k V(r) \in L^\infty(0,\infty)$ for some $k \in (0,2)$.

(H4) The function $h(r) := rV'(r)/V(r)$ is nonincreasing on $(0,\infty)$.

In addition, we suppose that $\lambda > 0$ and $p > 1$.

Note that, in particular, $V(r) \to 0$ as $r \to \infty$. Examples of functions satisfying the hypotheses above are $V(r) = r^{-k}$ or $V(r) = 1/(1+r^2)^{k/2}$.

Our main result is the following.

Theorem 1.1 *There exists $\alpha_0 > 0$ such that the solution sets have the following structure:*

$$P = (0,\alpha_0), \quad G = \{\alpha_0\}, \quad N = (\alpha_0,\infty).$$

As we shall see, Theorem 1.1 is an easy consequence of Lemmas 2.7 and 2.8 in Section 2. Roughly speaking, these lemmas imply the following qualitative dependence of the solutions on $\alpha > 0$. As α grows from zero, we are first in the solution set P and the corresponding solution is bounded away from zero. Then at the value $\alpha = \alpha_0$, the solution suddenly changes its asymptotic behaviour, vanishing as $r \to 0$. Finally, after α has crossed α_0, we are in the solution set N, the solution $u(r;\alpha)$ has a finite zero $z(\alpha) > 0$, and the function $z : N \to (0,\infty)$ is decreasing.

When there is no risk of confusion, we shall merely write u and w for $u(\cdot\,;\alpha)$ and $w(\cdot\,;\alpha)$ respectively. Also, by solution we always mean nontrivial solution.

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2 Proof of Theorem 1.1

The proof requires a series of lemmas and is given at the end of this section. We assume that $\lambda > 0$, $p > 1$ and we make the hypotheses (H1) to (H4) throughout.

Remark 1 Let us mention here two useful consequences of (H1)-(H4). Setting $h_0 := \lim_{r\to 0} h(r)$ and using the identities

$$V(r) = V(1)\exp\int_1^r \frac{h(s)}{s}\,\mathrm{d}s \quad\text{and}\quad r^k V(r) = V(1)\exp\int_1^r \frac{h(s)+k}{s}\,\mathrm{d}s,$$

it is not difficult to check that $h_0 + k \geq 0$ and that $h_0 = 0$ if $\lim_{r\to 0} V(r) < \infty$.

Lemma 2.1 *Suppose that* $V(0) := \lim_{r\to 0} V(r)$ *exists and is finite. Then*

$$0 < \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)} \quad\Longrightarrow\quad \alpha \in P.$$

Proof: For given $\alpha > 0$ and $u = u(r;\alpha)$, consider the Liapounov function $E(r)$ defined by

$$E(r) = \frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1}$$

for $r \in [0,\infty)$ if $\alpha \in P \cup G$, and $r \in [0, z(\alpha)]$ if $\alpha \in N$. It follows from (IVP) that $E'(r) = -u'(r)^2/r + V'(r)u(r)^{p+1}/(p+1) \leq 0$ for $r > 0$. Hence E is nonincreasing. Now for $0 < \alpha < [(\frac{p+1}{2})\frac{\lambda}{V(0)}]^{1/(p-1)}$ we have $E(0) < 0$ and so $E < 0$. Let us first suppose that $\alpha \in N$. Then $E(z(\alpha)) = \frac{1}{2}u'(z(\alpha))^2 \geq 0$, a contradiction. Hence $u > 0$. Furthermore, $\alpha \in G$ would imply that $E(r) \to 0$ as $r \to \infty$, also a contradiction. $\square$

Lemma 2.2 *For* $\alpha \in G \cup N$ *and* $u = u(r; \alpha)$, *we have* $u'(r) < 0$ *for all* $r \in (0, z(\alpha))$ *and* $u'(z(\alpha)) < 0$ *if* $\alpha \in N$.

Proof: The result follows by easy modifications to the proof of Lemma 3 in [11]. □

Lemma 2.3 *For* $\alpha \in G \cup N$, *w has at least one zero in* $(0, z(\alpha))$.

Proof: Let us first suppose that $\alpha \in N$, $z(\alpha) < \infty$. The Lagrange identity for (IVP) and (1.3), taken between $r = 0$ and $r = z(\alpha)$, yields

$$z(\alpha)u'(z(\alpha))w(z(\alpha)) = (p-1)\int_0^{z(\alpha)} rV(r)u^p w \, \mathrm{d}r.$$

Note that the integral is convergent by (H3). Since $u'(z(\alpha)) < 0$, it is impossible to have $w > 0$ on $(0, z(\alpha))$. Hence w has at least one zero in $(0, z(\alpha))$.

Suppose now that $\alpha \in G$. By contradiction, let us assume that $w > 0$ on $(0, \infty)$. Then we have

$$rw(r)^2\left(\frac{u}{w}\right)'(r) = r\{u'(r)w(r) - u(r)w'(r)\} = (p-1)\int_0^r sV(s)u^p w \, \mathrm{d}s > 0$$

for all $r > 0$. Therefore, the function u/w is positive and increasing on $(0, \infty)$. By Lemme C.1 of [7], there exist two independent solutions ξ_0, ξ_1 of (1.3) that satisfy

$$\xi_0(r) \sim r^{-1/2}e^{-\sqrt{\lambda}r} \quad \text{and} \quad \xi_1(r) \sim r^{-1/2}e^{\sqrt{\lambda}r} \quad \text{as } r \to \infty.$$

Thus,

$$w(r) \sim r^{-1/2}[\alpha_0 e^{-\sqrt{\lambda}r} + \alpha_1 e^{\sqrt{\lambda}r}] \quad \text{as } r \to \infty$$

for some constants α_0, α_1. Furthermore, since $w > 0$ by hypothesis, we must have $\alpha_1 \geq 0$. If $\alpha_1 = 0$, then $w(r) \to 0$ exponentially as $r \to \infty$ and so w satisfies the hypotheses of Lemme 1.4.9 of [7], which implies that w must change sign, a contradiction. On the other hand, we know by Théorème 1.2.7 and Lemme C.1 of [7] that $u(r) \sim r^{-1/2}e^{-\sqrt{\lambda}r}$ as $r \to \infty$. Therefore, if $\alpha_1 > 0$, there is a constant C such that

$$\lim_{r\to\infty}\frac{u(r)}{w(r)} = \lim_{r\to\infty}\frac{Cr^{-1/2}e^{-\sqrt{\lambda}r}}{\alpha_1 r^{-1/2}e^{\sqrt{\lambda}r}} = \frac{C}{\alpha_1}\lim_{r\to\infty} e^{-2\sqrt{\lambda}r} = 0.$$

This contradiction concludes the proof. □

For $\alpha \in G \cup N$, let us define $\theta(r) := -ru'(r)/u(r)$ for $r \in [0, z(\alpha))$. We know that

$$\theta(0) = 0 \quad \text{and} \quad \lim_{r\to z(\alpha)} \theta(r) = \infty.$$

(If $\alpha \in G$, the second limit follows by Théorème 1.2.7 and Lemme C.1 of [7].) Furthermore, Lemma 4.3 and Remark 4.1(ii) of [16] imply[1] that $\theta'(r) > 0$ for all

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[1] Let us note that, in the context of (1.2), the results we borrow here from [16] only work in dimension $N = 2$.

$r \in (0, z(\alpha))$. We define $\rho := \theta^{-1}$ so that, for all $\beta > 0$, there exists a unique $r = \rho(\beta) > 0$ such that $\theta(r) = \beta$. The function ρ is continuous and increasing with

$$\rho(0) = 0 \quad \text{and} \quad \lim_{\beta \to \infty} \rho(\beta) = z(\alpha).$$

We shall now study some properties of the auxiliary function

$$v_\beta(r) := ru'(r) + \beta u(r) = -u(r)\{\theta(r) - \beta\}, \quad r \in [0, z(\alpha)).$$

First, it is clear that, for any $\beta > 0$,

$$v_\beta(r) > 0 \text{ if } r < \rho(\beta) \quad \text{and} \quad v_\beta(r) < 0 \text{ if } r > \rho(\beta).$$

Furthermore, v_β satisfies the differential equation

$$v_\beta'' + \frac{1}{r} v_\beta' - \lambda v_\beta + pV(r)u^{p-1} v_\beta = \varphi_\beta(r), \tag{2.1}$$

where

$$\varphi_\beta(r) := [\beta(p-1) - 2]V(r)u(r)^p - rV'(r)u(r)^p + 2\lambda u(r).$$

Lemma 2.4 *Let $\alpha \in G \cup N$. There exist $\overline{\beta} > 0$ and a unique function $\sigma : [0, \overline{\beta}] \to [0, \infty)$ with the following properties:*

(a) σ is continuous and decreasing, $\sigma(0) > 0$ and $\sigma(\overline{\beta}) = 0$;

(b) for all $\beta > 0$ we have

$$\varphi_\beta(r) < 0 \textit{ if } r < \sigma(\beta) \quad \textit{and} \quad \varphi_\beta(r) > 0 \textit{ if } r > \sigma(\beta).$$

Proof: Fix $\beta > 0$. For all $r \in (0, z(\alpha))$, $\varphi_\beta(r) = V(r)u(r)^p[\beta(p-1) - 2 - \xi(r)]$ where

$$\xi(r) = r\frac{V'(r)}{V(r)} - \frac{2\lambda}{u(r)^{p-1}V(r)}.$$

We are interested in the sign of $\beta(p-1) - 2 - \xi(r)$. The function ξ is negative and strictly decreasing on $(0, z(\alpha))$ with $\lim_{r \to z(\alpha)} \xi(r) = -\infty$. Thus, we need only show that the function $\Xi(r) = [2 + \xi(r)]/(p-1)$ satisfies $\Xi(0) > 0$. Indeed, on setting $\overline{\beta} = \Xi(0)$, we then have

$$\beta < \Xi(r) \iff \beta(p-1) - 2 - \xi(r) < 0 \quad \text{and} \quad \beta > \Xi(r) \iff \beta(p-1) - 2 - \xi(r) > 0$$

for all $\beta \in [0, \overline{\beta}]$. Since Ξ is continuous and decreasing, the function $\sigma := \Xi^{-1} |_{[0,\overline{\beta}]}$ has properties (a) and (b). Now $\Xi(0) > 0$ is equivalent to $\xi(0) > -2$. To check this property we consider two cases. If $V(0) = \infty$ we have $\xi(0) = h_0 \geq -k > -2$ by Remark 1. On the other hand, if $V(0) < \infty$, we have $h_0 = 0$ and $\xi(0) = -2\lambda/\alpha^{p-1}V(0)$. Hence $\xi(0) > -2$ if and only if $\alpha > [\lambda/V(0)]^{1/(p-1)}$, which is true by Lemma 2.1. The proof is complete. □

The following lemma is now an obvious consequence of the properties of functions ρ and σ.

Lemma 2.5 *Let $\alpha \in G \cup N$. There exists a unique $\beta_0 > 0$ such that $\rho(\beta_0) = \sigma(\beta_0)$.*

Setting $v = v_{\beta_0}$ and $\rho_0 = \rho(\beta_0)$, we will next apply Sturm's oscillation theory to the equations (1.3) and (2.1).

Lemma 2.6 *For $\alpha \in G \cup N$, w has a unique zero $r_0 \in (0, z(\alpha))$. Furthermore, $w(z(\alpha)) < 0$ if $\alpha \in N$ and $\lim\limits_{r \to \infty} w(r) = -\infty$ if $\alpha \in G$.*

Proof: We have seen that

$$v'' + \frac{1}{r}v' + [pV(r)u(r)^{p-1} - \lambda]v < 0 \quad \text{and} \quad v > 0 \quad \text{on } (0, \rho_0)$$

whereas

$$v'' + \frac{1}{r}v' + [pV(r)u(r)^{p-1} - \lambda]v > 0 \quad \text{and} \quad v < 0 \quad \text{on } (\rho_0, z(\alpha)).$$

Moreover, $v(0) = \beta_0\alpha > 0$ and $\lim_{r\to 0} rv'(r) = 0$. Let us further remark that, if $\alpha \in N$, $v(z(\alpha)) = z(\alpha)u'(z(\alpha)) < 0$ by Lemma 2.2 so that v has a unique zero ρ_0 in $[0, z(\alpha)]$. Let $\tau \in (0, z(\alpha))$ be the first zero of w (which exists by Lemma 2.3). Since w satisfies

$$w'' + \frac{1}{r}w' + [pV(r)u(r)^{p-1} - \lambda]w = 0 \quad \text{for } r \in (0, z(\alpha)) \tag{2.2}$$

with the initial data $w(0) = 1$ and $\lim_{r\to 0} rw'(r) = 0$, it follows by Sturm's comparison theorem (see e.g. Lemma 1 in [8]) that $\rho_0 \in (0, \tau)$ (v oscillates faster that w). Consequently,

$$v'' + \frac{1}{r}v' + [pV(r)u(r)^{p-1} - \lambda]v > 0 \quad \text{and} \quad v < 0 \quad \text{on } (\tau, z(\alpha)).$$

Since $w(\tau) = 0$ and v has no zero larger that ρ_0, Sturm's theorem then implies that w has no further zero in $(\tau, z(\alpha))$ and we can set $r_0 = \tau$. If $z(\alpha) < \infty$, this actually holds on $(\tau, z(\alpha)]$ and we have $w(z(\alpha)) < 0$, as expected.

To handle the case $z(\alpha) = \infty$, we shall apply Lemma 6 of [8]. We first remark that the disconjugacy interval (d, ∞) of (2.2) is such that $d < \rho_0 < r_0$. Indeed, if we had $\rho_0 \le d$, we could find a solution $\widetilde{w}$ of (2.2), linearly independent of w, such that $\widetilde{w}(\widetilde{r}) = 0$ for some $\widetilde{r} > d \ge \rho_0$. But then, since $w + \widetilde{w}$ is a solution of (2.2) having two zeros in (ρ_0, ∞), v should have another zero in this interval, a contradiction. Therefore, $w(r_0) = 0$ with $r_0 \in (d, \infty)$ and Lemma 6 of [8] implies that $\lim_{r\to\infty} w(r) = -\infty$, which finishes the proof. $\square$

Lemma 2.7 *Let $\alpha \in G$. There exists $\epsilon > 0$ such that $(\alpha, \alpha + \epsilon) \subset N$.*

Proof: Since w is unbounded by Lemma 2.6, we have $r_0 \in (d, \infty)$ by Lemma 6 of [8], where (d, ∞) is the disconjugacy interval of (2.2). Now let $d < r_1 < r_0 < r_2$. Since $w(r_1) > 0$ and $w(r_2) < 0$, there is $\epsilon > 0$ such that, for all $\widetilde{\alpha} \in (\alpha, \alpha + \epsilon)$,

$$\widetilde{u}(r_1) > u(r_1) \quad \text{and} \quad \widetilde{u}(r_2) < u(r_2)$$

where $\widetilde{u} = u(r; \widetilde{\alpha})$. Hence, for a given $\widetilde{\alpha} \in (\alpha, \alpha + \epsilon)$, the graph of $\widetilde{u}$ intersects the graph of u at some point $r_3 \in (r_1, r_2)$. We will prove that there exists $\widetilde{r} \in (r_3, \infty)$ such that $\widetilde{u}(\widetilde{r}) = 0$, so that $\widetilde{\alpha} \in N$. By contradiction, let us suppose that $\widetilde{u}(r) > 0$ for all $r > r_3$.

We first show that $\widetilde{u}(r) < u(r)$ for all $r > r_3$. By contradiction, assume that $\widetilde{u}(r_4) = u(r_4)$ for some $r_4 > r_3$ and $u - \widetilde{u} > 0$ on (r_3, r_4). The function $z := u - \widetilde{u}$ satisfies

$$z'' + \frac{1}{r} z' + \left[V(r) \frac{u^p - \widetilde{u}^p}{u - \widetilde{u}} - \lambda \right] z = 0 \quad \text{on } (r_3, r_4). \tag{2.3}$$

We shall compare this equation with (2.2). Since $z(r_3) = z(r_4) = 0$ and $\frac{u^p - \widetilde{u}^p}{u - \widetilde{u}} < pu^{p-1}$ on (r_3, r_4), we can apply Sturm's theorem as follows. Let y be any solution of (2.2), linearly independent of w. Then y must vanish somewhere in (r_3, r_4). Thus, since $\{w, y\}$ is a basis of solutions of (2.2), it is impossible to find a positive solution of (2.2) on (d, ∞), a contradiction. Hence $z(r) > 0$ for all $r > r_3$.

It follows now that (2.2) is a Sturm majorant of (2.3) on (r_3, ∞). Let $\widetilde{w}$ be a solution of (2.2) such that $\widetilde{w}(r_3) = 0$. Since $r_3 \in (d, \infty)$, Lemma 6 of [8] implies that $\widetilde{w}$ is unbounded and we can assume that $\widetilde{w}(r) \to +\infty$ as $r \to \infty$. Also, $\widetilde{w} > 0$ on (r_3, ∞) since no solution can vanish more than once on the disconjugacy interval. Then the strong version of Sturm's theorem (Lemma 1 in [8]) implies that

$$\frac{\widetilde{w}'(r)}{\widetilde{w}(r)} \leq \frac{z'(r)}{z(r)} \quad \text{for all } r > r_3.$$

By integration, this yields

$$\ln \widetilde{w}(r) \leq \ln \frac{\widetilde{w}(r_4)}{z(r_4)} + \ln z(r) \quad \text{for } r_3 < r_4 < r.$$

Hence $z(r) \equiv u(r) - \widetilde{u}(r) \to +\infty$ as $r \to \infty$, which is impossible since $0 < \widetilde{u}(r) < u(r)$ on (r_3, ∞) and $u(r) \to 0$ as $r \to \infty$. Therefore, $\widetilde{u}$ must vanish at some point $\widetilde{r} \in (r_3, \infty)$ and the proof is complete. $\square$

Lemma 2.8 *Let* $\alpha^* \in N$. *Then* $[\alpha^*, \infty) \subset N$ *and* $z : [\alpha^*, \infty) \to (0, \infty)$ *is monotone decreasing.*

Proof: We first remark that N is an open subset of $(0, \infty)$. Indeed, for $\widehat{\alpha} \in N$, there is an $\widehat{r} > 0$ such that $u(\widehat{r}; \widehat{\alpha}) < 0$. By continuous dependence on the initial data, we then have $u(\widehat{r}; \alpha) < 0$ for all α sufficiently close to $\widehat{\alpha}$. Thus, N is open. On the other hand, since the graph of a solution cannot be tangent to the r-axis, $z : N \to (0, \infty)$ is continuous.

By Lemma 2.6, we know that $w(z(\alpha^*)) < 0$. So for $\varepsilon > 0$ small enough, we have

$$(\alpha^*, \alpha^* + \varepsilon) \subset N \quad \text{and} \quad u(z(\alpha^*); \alpha) < 0 \quad \text{for all } \alpha \in (\alpha^*, \alpha^* + \varepsilon).$$

For such an α, the intermediate value theorem then implies that there exists an $r \in (0, z(\alpha^*))$ so that $u(r; \alpha) = 0$. Hence $z(\alpha) \le r < z(\alpha^*)$ for all $\alpha \in (\alpha^*, \alpha^* + \varepsilon)$ and it easily follows that z is decreasing in a right neighbourhood of α^*. Let us now set

$$\overline{\alpha} := \sup\{\alpha > \alpha^* : [\alpha^*, \alpha) \subset N \text{ and } z : [\alpha^*, \alpha) \to (0, \infty) \text{ is decreasing}\}.$$

By contradiction, we assume that $\overline{\alpha} < \infty$. Then there exists $z(\overline{\alpha}) := \lim_{\alpha \to \overline{\alpha}} z(\alpha) \in [0, \infty)$. By continuity, we have $u(z(\overline{\alpha}); \overline{\alpha}) = 0$ and so $\overline{\alpha} \in N$. But then, $[\overline{\alpha}, \overline{\alpha} + \varepsilon) \subset N$ for $\varepsilon > 0$ small enough, contradicting the definition of $\overline{\alpha}$. The proof is complete. ☐

Proof of Theorem 1.1: First, by Lemmas 2.7 and 2.8, G contains at most one point. By Théorème 1.1.8 of [7] we know that G is not empty and so $G = \{\alpha_0\}$ for some $\alpha_0 > 0$. Now, by Lemma 2.7 and 2.8, we have $(\alpha_0, \infty) \subset N$. On the other hand, there is no $\alpha_1 < \alpha_0$ such that $\alpha_1 \in N$ since otherwise we would have $[\alpha_1, \infty) \subset N$, contradicting $\alpha_0 \in G$. Hence $N = (\alpha_0, \infty)$ and we must have $P = (0, \alpha_0)$, finishing the proof. ☐

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