Abrikosov Lattice Solutions of the Ginzburg-Landau Equations

T. Tzaneteas*† and I. M. Sigal ‡§

Dept. of Mathematics, Univ. of Toronto, Toronto, Canada, M5S 2E4

October 31, 2009

Abstract

Building on the earlier work of Odeh, Barany, Golubitsky, Turksi and Lasher we give a proof of existence of Abrikosov vortex lattices in the Ginzburg-Landau model of superconductivity.
Keywords: magnetic vortices, superconductivity, Ginzburg-Landau equations, Abrikosov vortex lattices, bifurcations.

1 Introduction

1.1 The Ginzburg-Landau Model. The Ginzburg-Landau model of superconductivity describes a superconductor contained in $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, in terms of a complex order parameter $\psi : \Omega \to \mathbb{C}$, and a magnetic potential $A : \Omega \to \mathbb{R}^n$[1]. The key physical quantities for the model are

- the density of superconducting pairs of electrons, $n_s := |\psi|^2$;
- the magnetic field, $B := \operatorname{curl} A$;
- and the current density, $J := \operatorname{Im}(\bar{\psi}\nabla_A \psi)$.

In the case $n = 2$, $\operatorname{curl} A := \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$ is a scalar-valued function. The covariant derivative ∇_A is defined to be $\nabla - iA$. The Ginzburg-Landau theory specifies that a state (ψ, A), in the absence of an external magnetic field, has energy

$$\mathcal{E}_\Omega(\psi, A) := \int_\Omega |\nabla_A \psi|^2 + |\operatorname{curl} A|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2, \tag{1}$$

where κ is a positive constant that depends on the material properties of the superconductor.

It follows from the Sobolev inequalities that for bounded open sets Ω, $\mathcal{E}_\Omega$ is well-defined and C^∞ as a functional on the Sobolev space H^1. The critical points of this functional must satisfy the well-known Ginzburg-Landau equations inside Ω:

$$\Delta_A \psi = \kappa^2(|\psi|^2 - 1)\psi, \tag{2a}$$

$$\operatorname{curl}^* \operatorname{curl} A = \operatorname{Im}(\bar{\psi}\nabla_A \psi). \tag{2b}$$

Here $\Delta_A = -\nabla_A^* \nabla_A$, ∇_A^* and curl^* are the adjoints of ∇_A and curl. Explicitly, $\nabla_A^* F = -\operatorname{div} F + iA \cdot F$, and $\operatorname{curl}^* F = \operatorname{curl} F$ for $n = 3$ and $\operatorname{curl}^* f = (\frac{\partial f}{\partial x_2}, -\frac{\partial f}{\partial x_1})$ for $n = 2$.

There are two immediate solutions to the Ginzburg-Landau equations that are homogeneous in ψ. These are the perfect superconductor solution where $\psi_S \equiv 1$ and $A_S \equiv 0$, and the *normal* (or non-superconducting) solution where $\psi_N = 0$ and A_N is such that $\operatorname{curl} A_N =: b$ is constant.

---

*This paper is part of the first authors PhD thesis.

†Supported in part by Ontario graduate fellowship and by NSERC under Grant NA 7901

‡Supported by NSERC Grant NA7601

§Corresponding author; E-mail address: im.sigal@utoronto.ca

[1] The Ginzburg-Landau theory is reviewed in every book on superconductivity. For reviews of rigorous results see the papers [10, 11] and the books [23, 15, 17, 22]

It is well-known that there exists a critical value κ_c (in the units used here, $\kappa_c = 1/\sqrt{2}$), that separates superconductors into two classes with different properties: Type I superconductors, which have $\kappa < \kappa_c$ and exhibit first-order phase transitions from the non-superconducting state to the superconducting state, and Type II superconductors, which have $\kappa > \kappa_c$ and exhibit second-order phase transitions and the formation of vortex lattices. Existence of the vortex lattice solutions is the subject of the present paper.

1.2 Results. In 1957, Abrikosov [1] discovered solutions of (2) whose physical characteristics n_s, B, and J are periodic with respect to a two-dimensional lattice, while independent of the third dimension, and which have a single flux per lattice cell[2]. (In what follows we call such solutions, with n_s and B non-constant, *lattice solutions*, or, if a lattice $\mathcal{L}$ is fixed, $\mathcal{L}$-*lattice solutions*. In physics literature they are called variously mixed states, Abrikosov mixed states, Abrikosov vortex states.) Due to an error of calculation he concluded that the lattice which gives the minimum average energy per lattice cell is the square lattice. Abrikosov's error was corrected by Kleiner, Roth, and Autler [18], who showed that it is in fact the triangular lattice which minimizes the energy.

Since then these Abrikosov lattice solutions have been studied in numerous experimental and theoretical works. Of more mathematical studies, we mention the articles of Eilenberger [14] and Lasher [19].

The rigorous investigation of Abrikosov solutions began soon after their discovery. Odeh [21] proved the existence of non-trivial minimizers and obtained a result concerning the bifurcation of solutions at the critical field strength. Barany, Golubitsky, and Tursky [8] investigated this bifurcation for certain lattices using equivariant bifurcation theory, and Takáč [24] has adapted these results to study the zeros of the bifurcating solutions.

Except for a variational result of [21] (see also [13]), work done by both physicists and mathematicians has followed the general strategy of [1].

In this paper we combine and extend the previous technique to give a self-contained proof of the existence of Abrikosov lattice solutions. To formulate our results we mention that lattices $\mathcal{L} \subset \mathbb{R}^2$ are characterized by the area $|\Omega_\mathcal{L}|$ of the lattice cell $\Omega_\mathcal{L}$ and the shape τ, given by the ratio of basis vectors identified as complex numbers (for details see Section 3). We will prove the following results, whose precise formulation will be given below (Theorem 4).

Theorem 1. *Let $\mathcal{L}$ be a lattice with $\left||\Omega_\mathcal{L}| - \frac{2\pi}{\kappa^2}\right| \ll 1$.*

(I) If $|\Omega_\mathcal{L}| > \frac{2\pi}{\kappa^2}$, there exists an $\mathcal{L}$-lattice solution. If $|\Omega_\mathcal{L}| \leq \frac{2\pi}{\kappa^2}$, then there is no $\mathcal{L}$-lattice solution in a neighbourhood of the branch of normal solutions.

(II) The above solution is close to the branch of normal solutions and is unique, up to symmetry, in a neighbourhood of this branch.

(III) The solutions above are real analytic in $|\Omega_\mathcal{L}|$ in a neighbourhood of $\frac{2\pi}{\kappa^2}$.

(IV) The lattice shape for which the average energy per lattice cell is minimized approaches the triangular lattice as $|\Omega_\mathcal{L}| \to \frac{2\pi}{\kappa^2}$.

Remark 2.

(a) [21, 12] showed that for all $|\Omega_\mathcal{L}| > \frac{2\pi}{\kappa^2}$ there exists a global minimizer of $\mathcal{E}_{\Omega_\mathcal{L}}$.

(b) [21, 8] proved results related to our solutions in (I).

(c) [19] proved partial results on (IV).

Among related results, a relation of the Ginzburg-Landau minimization problem, for a fixed, finite domain and for increasing Ginzburg-Landau parameter κ and external magnetic field, to the Abrikosov lattice variational problem was obtained in [3, 5]. [12] (see also [13]) have found boundaries between superconducting, normal and mixed phases.

All the rigorous results above deal with Abrikosov lattices with one quantum of magnetic flux per lattice cell. partial results for higher magnetic fluxes were proven in [9, 4]. This problem will be addressed in our subsequent paper.

Acknowledgements
The second author is grateful to Yuri Ovchinnikov for many fruitful discussions. A part of this work was done during I.M.S.'s stay at the IAS, Princeton.

[2]Such solutions correspond cylindrical samples. In 2003, Abrikosov received the Nobel Prize for this discovery

2 Properties of the Ginzburg-Landau Equations

2.1 Symmetries. The Ginzburg-Landau equations exhibit a number of symmetries, that is, transformations which map solutions to solutions. The most important of these symmetries is the gauge symmetry, defined for any sufficiently regular function $\eta : \Omega \to \mathbb{R}$, which maps $(\psi, A) \mapsto (T_\eta\psi, T_\eta A)$, where

$$T_\eta\psi = e^{i\eta}\psi, \qquad T_\eta A = A + \nabla\eta. \tag{3}$$

There are also the translation symmetry, defined for each $t \in \mathbb{R}^2$, which maps $(\psi, A) \mapsto (T_t\psi, T_t A)$, where

$$T_t\psi(x) := \psi(x + t), \qquad T_t A(x) := A(x + t), \tag{4}$$

and rotation and reflection symmetry, defined for each $R \in O(2)$ (the set of orthogonal 2×2 matrices), which maps $(\psi, A) \mapsto (T_R\psi, T_R A)$, where

$$T_R\psi(x) := \psi(Rx), \qquad T_R A(x) := R^{-1}A(Rx). \tag{5}$$

2.2 Flux Quantization. One can show that under certain boundary conditions (e.g., 'gauge-periodic', see below, or if $\Omega = \mathbb{R}^2$ and $\mathcal{E}_\Omega < \infty$) the magnetic flux through Ω is quantized.

3 Lattice States

Our focus in this paper is on states (ψ, A) defined on all of $\mathbb{R}^2$, but whose physical properties, the density of superconducting pairs of electrons, $n_s := |\psi|^2$, the magnetic field, $B := \operatorname{curl} A$, and the current density, $J := \operatorname{Im}(\bar{\psi}\nabla_A\psi)$, are doubly-periodic with respect to some lattice $\mathcal{L}$. We call such states $\mathcal{L}-$*lattice states*.

One can show that a state $(\psi, A) \in H^1_{loc}(\mathbb{R}^2; \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ is a $\mathcal{L}$-lattice state if and only if translation by an element of the lattice results in a gauge transformation of the state, that is, for each $t \in \mathcal{L}$, there exists a function $g_t \in H^2_{loc}(\mathbb{R}^2; \mathbb{R})$ such that

$$\psi(x + t) = e^{ig_t(x)}\psi(x) \text{ and } A(x + t) = A(x) + \nabla g_t(x)$$

almost everywhere.

It is clear that the gauge, translation, and rotation symmetries of the Ginzburg-Landau equations map lattice states to lattice states. In the case of the gauge and translation symmetries, the lattice with respect to which the solution is periodic does not change, whereas with the rotation symmetry, the lattice is rotated as well. It is a simple calculation to verify that the magnetic flux per cell of solutions is also preserved under the action of these symmetries.

Note that (ψ, A) is defined by its restriction to a single cell and can be reconstructed from this restriction by lattice translations.

3.1 Flux quantization. The important property of lattice states is that the magnetic flux through a lattice cell is quantized:

$$\Phi(A) := \int_\Omega \operatorname{curl} A = 2\pi n \tag{6}$$

for some integer n. Here Ω is any fundamental cell of the lattice. Indeed, if $|\psi| > 0$ on the boundary of the cell, we can write $\psi = |\psi|e^{i\theta}$ and $0 \leq \theta < 2\pi$. The periodicity of n_s and J ensure the periodicity of $\nabla\theta - A$ and therefore by Green's theorem, $\int_\Omega \operatorname{curl} A = \oint_{\partial\Omega} A = \oint_{\partial\Omega} \nabla\theta$ and this function is equal to $2\pi n$ since ψ is single-valued.

We let b be the average magnetic flux per lattice cell, $b := \frac{1}{|\Omega|}\Phi(A)$. Equation (6) then imposes a condition on the area of a cell, namely,

$$|\Omega| = \frac{2\pi n}{b}.$$

Due to the physical interpretation of b as being related to the applied magnetic field, from now on we use $b = \frac{2\pi n}{|\Omega|}$ as a parameter of our problem. We note that due to the reflection symmetry of the problem we can assume that $b \geq 0$.

3.2 Lattice Shape. In order to define the shape of a lattice, we identify $x \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$, and view $\mathcal{L}$ as a subset of $\mathbb{C}$. It is a well-known fact (see [6]) any lattice $\mathcal{L} \subseteq \mathbb{C}$ can be given a basis r, r' such that the ratio $\tau = \frac{r'}{r}$ satisfies the inequalities:

(i) $|\tau| \geq 1$.

(ii) $\operatorname{Im}\tau > 0$.

(iii) $-\frac{1}{2} < \operatorname{Re}\tau \leq \frac{1}{2}$, and $\operatorname{Re}\tau \geq 0$ if $|\tau| = 1$.

Although the basis is not unique, the value of τ is, and we will use that as a measure of the shape of the lattice.

Using the rotation symmetry we can assume that if $\mathcal{L}$ has has as a basis $\{re_1, r\tau\}$, where r is a positive real number and $e_1 = (1, 0)$.

3.3 Fixing the Gauge. The gauge symmetry allows one to fix solutions to be of a desired form. We will use the following preposition, first used by [21] and proved in [24]. We provide an alternate proof in Appendix C.

Proposition 3. *Let (ψ, A) be an $\mathcal{L}$-lattice state, and let b be the average magnetic flux per cell. Then there is a $\mathcal{L}$-lattice state $(\phi, A_0 + a)$ that is gauge-equivalent to a translation of (ψ, A), where $A_0(x) = \frac{b}{2}x^{\perp}$ (where $x^{\perp} = (-x_2, x_1)$), and ϕ and a satisfy the following conditions.*

(i) a is doubly periodic with respect to $\mathcal{L}$: $a(x + t) = a(x)$ for all $t \in \mathcal{L}$.

(ii) a has mean zero: $\int_{\Omega} a = 0$.

(iii) a is divergence-free: $\operatorname{div} a = 0$.

(iv) $\phi(x + t) = e^{\frac{ib}{2}t\wedge x}\phi(x)$, where $t \wedge x = t_1x_2 - t_2x_1$, for $t = re_1, r\tau$.

3.3 Lattice Energy. Lattice states clearly have infinite total energy, so we will instead consider the average energy per cell, defined by

$$E(\psi, A) := \frac{1}{|\Omega|}\mathcal{E}_{\Omega}(\psi, A). \tag{7}$$

Here, Ω is a primitive cell of the lattice with respect to which (ψ, A) is a lattice state and $|\Omega|$ is its Lebesgue measure. We seek minimizers of this functional under the condition that the average magnetic flux per lattice cell is fixed: $\frac{1}{|\Omega|}\Phi(A) = b$.

In terms of the minimization problem, we see that the perfect superconductor is a solution only when $\Phi(A)$ is fixed to be $\Phi(A) = 0$. On the other hand, there is a normal solution, ($\psi_N = 0,\ A_N,\ \operatorname{curl} A_N = \text{constant}$), for any condition on $\Phi(A)$.

We define the energy of the lattice with the flux n per cell as

$$\mathcal{E}_n(\mathcal{L}) := \inf E(\psi, A), \tag{8}$$

where the infimum is taken over all smooth $\mathcal{L}$-lattice states satisfying (i) through (iv) of Proposition 3.

3.4 Result. Precise Formulation. The following theorem gives the precise formulation of Theorem 1 from the introduction.

Theorem 4. *Let $n = 1$.*

(I) For every b sufficiently close to but less than the critical value $b_c = \kappa^2$, there exists an $\mathcal{L}-$lattice solution of the Ginzburg-Landau equations with one quantum of flux per cell and with average magnetic flux per cell equal to b.

(II) This solution is unique, up to the symmetries, in a neighbourhood of the normal solution.

(III) The family of these solutions is real analytic in b in a neighbourhood of b_c.

(IV) If $\kappa^2 > 1/2$, then the global minimizer $\mathcal{L}_b$ of the average energy per cell, $\mathcal{E}_1(\mathcal{L})$, approaches the $\mathcal{L}_{triangular}$ as $b \to b_c$ in the sense that the shape τ_b approaches $\tau_{triangular} = e^{i\pi/3}$ in $\mathbb{C}$.

The rest of this papers is devoted to the proof of this theorem.

4 Rescaling

In this section we rescale (ψ, A) to eliminate the dependence of the size of the lattice on b. Our space will then depend only on the number of quanta of flux and the shape of the lattice.

Suppose, then, that we have a $\mathcal{L}$-lattice state (ψ, A), where $\mathcal{L}$ has shape τ. Now let b be the average magnetic flux per cell of the state and n the quanta of flux per cell. From the quantization of the flux, we know that

$$b = \frac{2\pi n}{|\Omega|} = \frac{2\pi n}{r^2 \operatorname{Im}\tau},$$

We set $\sigma := \left(\frac{n}{b}\right)^{\frac{1}{2}}$. The last two relations give $\sigma = \left(\frac{\operatorname{Im}\tau}{2\pi}\right)^{\frac{1}{2}} r$. We now define the rescaling $(\hat{\psi}, \hat{A})$ to be

$$(\hat{\psi}(x), \hat{A}(x)) := (\sigma\psi(\sigma x), \sigma A(\sigma x)).$$

Let $\mathcal{L}^\tau$ be the lattice spanned by r^τ and $r^\tau\tau$, with Ω^τ being a primitive cell of that lattice. Here

$$r^\tau := \left(\frac{2\pi}{\operatorname{Im}\tau}\right)^{\frac{1}{2}}. \tag{9}$$

We note that $|\Omega^\tau| = 2\pi n$. We summarize the effects of the rescaling above:

(i) $(\hat{\psi}, \hat{A})$ is a $\mathcal{L}^\tau$-lattice state.

(ii) $E(\psi, A) = \hat{\mathcal{E}}_\lambda(\hat{\psi}, \hat{A})$, where $\lambda = \frac{\kappa^2 n}{b}$ and

$$\hat{\mathcal{E}}_\lambda(\psi, A) = \frac{\kappa^4}{2\pi\lambda^2} \int_{\Omega^\tau} \left(|\nabla_A\psi|^2 + |\operatorname{curl} A|^2 + \frac{\kappa^2}{2}(|\psi|^2 - \frac{\lambda}{\kappa^2})^2 \right) dx. \tag{10}$$

(iii) ψ and A solve the Ginzburg-Landau equations if and only if $\hat{\psi}$ and $\hat{A}$ solve

$$(-\Delta_A - \lambda)\psi = -\kappa^2|\psi|^2\psi, \tag{11a}$$

$$\operatorname{curl}^* \operatorname{curl} A = \operatorname{Im}\{\bar{\psi}\nabla_A\psi\} \tag{11b}$$

for $\lambda = \frac{\kappa^2 n}{b}$. The latter equations are valid on Ω^τ with the boundary conditions given in the next statement.

(iv) If (ψ, A) is of the form described in Proposition 3, then

$$\hat{A} = A_0^n + a, \text{ where } A_0^n(x) := \frac{n}{2}x^\perp,$$

where $x^\perp = (-x_2, x_1)$, and $\hat{\psi}$ and a satisfy

(a) a is double periodic with respect to $\mathcal{L}^\tau$,

(b) $\int_{\Omega^\tau} a = 0$,

(c) $\operatorname{div} a = 0$,

(d) $\hat{\psi}(x + t) = e^{\frac{in}{2}t\wedge x}\hat{\psi}(x)$ for $t = r^\tau$, $r^\tau\tau$.

In what follows we drop the hat from $\hat{\psi}$, $\hat{A}$, and $\hat{\mathcal{E}}_\lambda$.

We are now state our problem in terms of the fields ψ and a. We define the Hilbert space $\mathscr{L}_n(\tau)$ to be the closure under the L^2-norm of the space of all smooth ψ on Ω^τ satisfying the quasiperiodic boundary condition (d) in part (iv) above. $\mathscr{H}_n(\tau)$ is then the space of all $\psi \in \mathscr{L}_n(\tau)$ whose (weak) partial derivatives up to order 2 are square-integrable.

Similarly, we define the Hilbert space $\vec{\mathscr{L}}(\tau)$ to be the closure of the space of all smooth a on Ω^τ that satisfy periodic boundary conditions, have mean zero, and are divergence free, and $\vec{\mathscr{H}}(\tau)$ is then the subspace of $\vec{\mathscr{L}}(\tau)$ consisting of those elements whose partial derivatives up to order 2 are square-integrable.

Our problem then is, for each $n = 1, 2, \ldots$, find $(\psi, a) \in \mathscr{H}_n(\tau) \times \vec{\mathscr{H}}(\tau)$ so that $(\psi, A_0^n + a)$ solves the rescaled Ginzburg-Landau equations (11), and among these find the one that minimizes the average energy $\mathcal{E}_\lambda$.

5 Reduction to Finite-dimensional Problem

In this section we reduce the problem of solving Eqns (11) to a finite dimensional problem. We address the latter in the next section. Substituting $A = A_0^n + a$, we rewrite (11) as

$$(L^n - \lambda)\psi + 2ia \cdot \nabla_{A_0^n}\psi + |a|^2\psi + \kappa^2|\psi|^2\psi = 0, \tag{12a}$$

$$(M + |\psi|^2)a - \operatorname{Im}\{\bar{\psi}\nabla_{A_0^n}\psi\} = 0, \tag{12b}$$

where

$$L^n := -\Delta_{A_0^n} \text{ and } M := \operatorname{curl}^* \operatorname{curl}. \tag{13}$$

The operators L^n and M are elementary and well studied. Their properties that will be used below are summarized in the following theorems, whose proofs may be found in Appendix B.

Theorem 5. *L^n is a self-adjoint operator on $\mathscr{H}_n(\tau)$ with spectrum $\sigma(L^n) = \{(2k+1)n : k = 0, 1, 2, \dots\}$ and $\dim_{\mathbb{C}} \operatorname{null}(L^n - n) = n$.*

Theorem 6. *M is a strictly positive operator on $\vec{\mathscr{H}}(\tau)$ with discrete spectrum.*

We first solve the second equation (12b) for a in terms of ψ, using the fact that M is a strictly positive operator, and therefore $M + |\psi|^2$ is invertible. We have $a = a(\psi)$, where

$$a(\psi) = (M + |\psi|^2)^{-1} \operatorname{Im}(\bar{\psi}\nabla_{A_0^n}\psi). \tag{14}$$

We collect the elementary properties of the map a in the following preposition, where we identify with a real Banach space using $\psi \leftrightarrow \overrightarrow{\psi} := (\operatorname{Re}\psi, \operatorname{Im}\psi)$.

Proposition 7. *The unique solution, $a(\psi)$, of* (12b) *maps $\mathscr{H}_n(\tau)$ to $\vec{\mathscr{H}}(\tau)$ and has the following properties:*

(a) $a(\cdot)$ is analytic as a map between real Banach spaces.

(b) $a(0) = 0$.

(c) For any $\alpha \in \mathbb{R}$, $a(e^{i\alpha}\psi) = a(\psi)$.

Proof. The only statement that does not follow immediately from the definition of a is (a). It is clear that $\operatorname{Im}(\bar{\psi}\nabla_{A_0^n}\psi)$ is real-analytic as it is a polynomial in ψ and $\nabla\psi$, and their complex conjugates. We also note that $(M - z)^{-1}$ is complex-analytic in z on the resolvent set of M, and therefore, $(M + |\psi|^2)^{-1}$ is analytic. (a) now follows. ☐

Now we substitute the expression (14) for a into (12a) to get a single equation $F(\lambda, \psi) = 0$, where the map $F : \mathbb{R} \times \mathscr{H}_n(\tau) \to \mathscr{L}_n(\tau)$ is defined as

$$F(\lambda, \psi) = (L^n - \lambda)\psi + 2ia(\psi) \cdot \nabla_{A_0^n}\psi + |a(\psi)|^2\psi + \kappa^2|\psi|^2\psi. \tag{15}$$

The following proposition lists some properties of F.

Proposition 8.

(a) F is analytic as a map between real Banach spaces,

(b) for all λ, $F(\lambda, 0) = 0$,

(c) for all λ, $D_\psi F(\lambda, 0) = L^n - \lambda$,

(d) for all $\alpha \in \mathbb{R}$, $F(\lambda, e^{i\alpha}\psi) = e^{i\alpha}F(\lambda, \psi)$.

(e) for all ψ, $\langle\psi, F(\lambda, \psi)\rangle \in \mathbb{R}$.

Proof. The first property follows from the definition of F and the corresponding analyticity of $a(\psi)$. (b) through (d) are straightforward calculations. For (e), we calculate that

$$\langle\psi, F(\lambda,\psi)\rangle = \langle\psi,(L^n-\lambda)\psi\rangle + 2i\int_{\Omega^\tau}\bar\psi\alpha(\psi)\cdot\nabla\psi + 2\int_{\Omega^\tau}(\alpha(\psi)\cdot A_0)|\psi|^2 + \int_{\Omega^\tau}|\alpha(\psi)|^2|\psi|^2 + \kappa^2\int_{\Omega^\tau}|\psi|^4.$$

The final three terms are clearly real and so is the first because $L^n-\lambda$ is self-adjoint. For the second term we calculate the complex conjugate and see that

$$\overline{2i\int_{\Omega^\tau}\bar\psi\alpha(\psi)\cdot\nabla\psi} = -2i\int_{\Omega^\tau}\psi\alpha(\psi)\cdot\nabla\bar\psi = 2i\int_{\Omega^\tau}(\nabla\psi\cdot\alpha(\psi))\bar\psi,$$

where we have integrated by parts and used the fact that the boundary terms vanish due to the periodicity of the integrand and that $\operatorname{div} a(\psi)=0$. Thus this term is also real and (e) is established. □

Now we reduce the equation $F(\lambda,\psi)=0$ to an equation on the finite-dimensional subspace $\mathrm{null}(L^n-n)$. To this end we use the standard method of Lyapunov-Schmidt reduction. Let $X:=\mathscr{H}_n(\tau)$ and $Y:=\mathscr{L}_N(\tau)$ and let $K=\mathrm{null}(L^n-n)$. We let P be the Riesz projection onto K, that is,

$$P := -\frac{1}{2\pi i}\oint_\gamma (L^n-z)^{-1}\,dz, \tag{16}$$

where $\gamma\subseteq\mathbb{C}$ is a contour around 0 that contains no other points of the spectrum of L. This is possible since 0 is an isolated eigenvalue of L. P is a bounded, orthogonal projection, and if we let $Z:=\mathrm{null}\,P$, then $Y=K\oplus Z$. We also let $Q:=I-P$, and so Q is a projection onto Z.

The equation $F(\lambda,\psi)=0$ is therefore equivalent to the pair of equations

$$PF(\lambda, P\psi+Q\psi)=0, \tag{17}$$
$$QF(\lambda, P\psi+Q\psi)=0. \tag{18}$$

We will now solve (18) for $w=Q\psi$ in terms of λ and $v=P\psi$. To do this, we introduce the map $G:\mathbb{R}\times K\times Z\to Z$ to be $G(\lambda,v,w):=QF(\lambda,v+w)$. Applying the Implicit Function Theorem to G, we obtain a real-analytic function $w:\mathbb{R}\times K\to Z$, defined on a neighbourhood of $(n,0)$, such that $w=w(\lambda,v)$ is a unique solution to $G(\lambda,v,w)=0$, for (λ,v) in that neighbourhood. We substitute this function into (17) and see that the latter equation in a neighbourhood of $(n,0)$ is equivalent to the equations

$$\psi = v + w(\lambda,v) \tag{19}$$

and

$$\gamma(\lambda,v) := PF(\lambda, v+w(\lambda,v)) = 0 \tag{20}$$

(the *bifurcation equation*). Note that $\gamma:\mathbb{R}\times K\to\mathbb{C}$. We have shown that in a neighbourhood of $(n,0)$ in $\mathbb{R}\times X$, (λ,ψ) solves $F(\lambda,\psi)=0$ if and only if (λ,v), with $v=P\psi$, solves (20).

Finally we note that γ inherits the symmetry of the original equation:

Lemma 9. *For every $\alpha\in\mathbb{R}$, $\gamma(\lambda,e^{i\alpha}v)=e^{i\alpha}\gamma(\lambda,v)$.*

Proof. We first check that $w(\lambda,e^{i\alpha}v)=e^{i\alpha}w(\lambda,v)$. We note that by definition of w, $G(\lambda,e^{i\alpha}v,w(\lambda,e^{i\alpha}v))=0$, but by the symmetry of F, we also have $G(\lambda,e^{i\alpha}v,e^{i\alpha}w(\lambda,v))=e^{i\alpha}G(\lambda,v,w(\lambda,v))=0$. The uniqueness of w then implies that $w(\lambda,e^{i\alpha}v)=e^{i\alpha}w(\lambda,v)$. We can now verify that

$$\gamma(\lambda,e^{i\alpha}v)=PF(\lambda,e^{i\alpha}v+w(\lambda,e^{i\alpha}v))=e^{i\alpha}PF(\lambda,v+w(\lambda,v))\rangle = e^{i\alpha}\gamma(\lambda,v).$$ □

Solving the bifurcation equation (20) is a subtle problem unless $n=1$. The latter case is tackled in the next section.

6 Bifurcation Theorem. $n = 1$

In this section we look at the case $n = 1$, and look at solutions near the trivial solution. For convenience we drop the index $n = 1$ from the notation. We will see that as $b = \frac{\kappa^2}{\lambda}$ decreases past the critical value $b = \kappa^2$, a branch of non-trivial solutions bifurcates from the trivial solution. More precisely, we have the following result.

Theorem 10. *For every τ there exists a branch, (λ_s, ψ_s, A_s), $s \in \mathbb{C}$ with $|s|^2 < \epsilon$ for some $\epsilon > 0$, of nontrivial solutions of the rescaled Ginzburg-Landau equations (11), unique (apart from the trivial solution $(1, 0, A_0)$) in a sufficiently small neighbourhood of $(1, 0, A_0)$ in $\mathbb{R} \times \mathscr{H}(\tau) \times \vec{\mathscr{H}}(\tau)$, and s.t.*

$$\begin{cases} \lambda_s = 1 + g_\lambda(|s|^2), \\ \psi_s = s\psi_0 + s g_\psi(|s|^2), \\ A_s = A_0 + g_A(|s|^2), \end{cases}$$

where $(L-1)\psi_0 = 0$, g_ψ is orthogonal to $\mathrm{null}(L-1)$, *$g_\lambda : [0, \epsilon) \to \mathbb{R}$, $g_\psi : [0, \epsilon) \to \mathscr{H}(\tau)$, and $g_A : [0, \epsilon) \to \vec{\mathscr{H}}(\tau)$ are real-analytic functions such that $g_\lambda(0) = 0$, $g_\psi(0) = 0$, $g_A(0) = 0$ and $g_\lambda'(0) > 0$. Moreover,*

$$g_\lambda'(0) = \left(\kappa^2 - \frac{1}{2}\right) \frac{\int_{\Omega^\tau} |\psi_0|^4}{\int_{\Omega^\tau} |\psi_0|^2} + \frac{1}{4\pi} \int_{\Omega^\tau} |\psi_0|^2. \tag{21}$$

Proof. The proof of this theorem is a slight modification of a standard result from the bifurcation theory. It can be found in Appendix A, Theorem 15, whose hypotheses are satisfied by F as shown above (see also [21, 8]). The latter theorem gives us a neighbourhood of $(1, 0)$ in $\mathbb{R} \times \mathscr{H}(\tau)$ such that the only non-trivial solutions are given by

$$\begin{cases} \lambda_s = 1 + g_\lambda(|s|^2), \\ \psi_s = s\psi_0 + s g_\psi(|s|^2). \end{cases}$$

Recall $a(\psi)$ defined in (14). We now define $\tilde{g}_A(s) = a(\psi_s)$, which is real-analytic and satisfies $\tilde{g}_A(-t) = a(-\psi_t) = \tilde{g}_A(t)$, and therefore is really a function of t^2, $g_A(t^2)$. Hence $A_s = A_0 + g_A(|s|^2)$.

Finally, to prove (21) we multiply the equation $F(\lambda, \psi) = 0$ scalarly by ψ_0 and use that L is self-adjoint and $(L-1)\psi_0 = 0$ to obtain

$$\langle \psi_0, (\lambda - 1)\psi \rangle = 2i \langle \psi_0, a(\psi) \cdot \nabla_{A_0} \psi \rangle + \langle \psi_0, |a(\psi)|^2 \psi \rangle + \kappa^2 \langle \psi_0, |\psi|^2 \psi \rangle.$$

Let $a_1 := g_A'(0)$. Substituting here the expansions obtained in the first part of the theorem, we find

$$g_\lambda'(0) \|\psi_0\|^2 = 2i \int_{\Omega^\tau} \bar{\psi}_0 a_1 \cdot \nabla_{A_0} \psi_0 + \kappa^2 \int_{\Omega^\tau} |\psi_0|^4. \tag{22}$$

In order to simplify this expression we first note that by differentiating (12b) w.r. to $|s|^2$ at $s = 0$, we obtain

$$\mathrm{curl}^* \,\mathrm{curl}\, a_1 = \mathrm{Im}(\bar{\psi}_0 \nabla_{A_0} \psi_0).$$

Now for the first term on the r.h.s. of (22), taking the imaginary part of (22) we see that $\mathrm{Re}(\int_{\Omega^\tau} \bar{\psi}_0 a_1 \cdot \nabla_{A_0} \psi_0) = 0$ and therefore

$$\begin{aligned} 2i \int_{\Omega^\tau} \bar{\psi}_0 a_1 \cdot \nabla_{A_0} \psi_0 \, dx &= -2 \int_{\Omega^\tau} a_1 \cdot \mathrm{Im}(\bar{\psi}_0 \nabla_{A_0} \psi_0) \, dx \\ &= -2 \int_{\Omega^\tau} a_1 \cdot \mathrm{curl}^* \,\mathrm{curl}\, a_1 \, dx \\ &= -2 \int_{\Omega^\tau} (\mathrm{curl}\, a_1)^2 \, dx. \end{aligned}$$

Here in the second step we integrated by parts. Next we show that

$$-\frac{1}{2} \mathrm{curl}^* |\psi_0|^2 = \mathrm{Im}(\bar{\psi}_0 \nabla_{A_0} \psi_0). \tag{23}$$

Using the notations of Appendix B (with $n = 1$), we have that $L_- \psi_0 = 0$:

$$\partial_{x_1} \psi_0 + i \partial_{x_2} \psi_0 + \frac{1}{2} x_1 \psi_0 + \frac{i}{2} x_2 \psi_0 = 0$$

Multiplying this relation by $\bar\psi_0$ and subtracting and adding the complex conjugate of the result, we obtain the two relations

$$\begin{cases}\bar\psi_0\partial_{x_1}\psi_0 - \psi_0\partial_{x_1}\bar\psi_0 = -i\bar\psi_0\partial_{x_2}\psi_0 - i\psi_0\partial_{x_2}\bar\psi_0 - ix_2|\psi_0|^2,\\ \bar\psi_0\partial_{x_2}\psi_0 - \psi_0\partial_{x_2}\bar\psi_0 = i\bar\psi_0\partial_{x_1}\psi_0 + i\psi_0\partial_{x_1}\bar\psi_0 + ix_1|\psi_0|^2.\end{cases}$$

This means that

$$\begin{aligned}\mathrm{Im}(\bar\psi_0\nabla_{A_0}\psi_0) &= \begin{pmatrix} -\frac{i}{2}(\bar\psi_0\partial_{x_1}\psi_0 - \psi_0\partial_{x_1}\bar\psi_0) + \frac{n}{2}x_2|\psi_0|^2 \\ -\frac{i}{2}(\bar\psi_0\partial_{x_2}\psi_0 - \psi_0\partial_{x_2}\bar\psi_0) - \frac{n}{2}x_1|\psi_0|^2\end{pmatrix}\\ &= \begin{pmatrix} -\frac{1}{2}(\bar\psi_0\partial_{x_2}\psi_0 + \psi_0\partial_{x_2}\bar\psi_0) \\ \frac{1}{2}(\bar\psi_0\partial_{x_1}\psi_0 + \psi_0\partial_{x_1}\bar\psi_0)\end{pmatrix}\\ &= \begin{pmatrix} -\frac{1}{2}\partial_{x_2}|\psi_0|^2 \\ \frac{1}{2}\partial_{x_1}|\psi_0|^2\end{pmatrix},\end{aligned}$$

which gives (23).

(23) implies that $\mathrm{curl}\, a_1 = -\frac{1}{2}|\psi_0|^2 + C$ for some constant C. This C can be determined using the fact that, since A has mean zero, a_1 does as well, and this gives $C = \frac{1}{4\pi}\int_{\Omega^\tau}|\psi_0|^2$, which establishes

$$\mathrm{curl}\, a_1 = -\frac{1}{2}|\psi_0|^2 + \frac{1}{4\pi}\int_{\Omega^\tau}|\psi_0|^2. \tag{24}$$

Using this equation we finish the calculation above:

$$2i\int_{\Omega^\tau}\bar\psi_0 a_1\cdot\nabla_{A_0}\psi_0\,dx = -\frac{1}{2}\int_{\Omega^\tau}|\psi_0|^4\,dx + \frac{1}{4\pi}\left(\int_{\Omega^\tau}|\psi_0|^2\,dx\right)^2. \tag{25}$$

Substituting this expression into (22) and rearranging terms we arrive at (21). And that completes the proof of Theorem 10. □

Theorem 10 implies (I) - (III) of Theorem 4. □

Finally, we mention

Lemma 11. *Recall that* $\mathrm{Im}\,\tau > 0$. *Let* (λ_s, ψ_s, A_s) *be the solution branch constructed above and let* $m_\tau = (\sqrt{\mathrm{Im}\,\tau})^{-1}\begin{pmatrix}1 & \mathrm{Re}\,\tau\\ 0 & \mathrm{Im}\,\tau\end{pmatrix}$. *Then* $(\lambda_s, \tilde\psi_s, \tilde A_s)$, *where the functions* $(\tilde\psi_s, \tilde A_s)$ *are defined on a* τ*-independent square lattice and are given by*

$$\begin{cases}\tilde\psi_s(x) = \psi_s(m_\tau x),\\ \tilde A_s(x) = M_\tau^t A_s(m_\tau x),\end{cases} \tag{26}$$

depend $\mathbb{R}$*-analytically on* τ .

We sketch the proof of this lemma. The transformation above maps functions on a lattice of the shape τ into functions on a τ-independent square lattice, but leads to a slightly more complicated expression for the Ginzburg-Landau equations. Namely, let $U_\tau\psi(x) := \psi(m_\tau x)$ and $V_\tau a(x) := m_\tau^t a(m_\tau x)$. Applying U_τ and V_τ to the equations (12a) and (12b), we conclude that $(\tilde\psi_s, \tilde A_s)$ satisfy the equations

$$(L_\tau^n - \lambda)\psi + 2i(m_\tau^t)^{-1}a\cdot(m_\tau^t)^{-1}\nabla_{A_0^n}\psi + |(m_\tau^t)^{-1}a|^2\psi + \kappa^2|\psi|^2\psi = 0, \tag{27a}$$

$$(M_\tau + |\psi|^2)a + \tilde F_\tau^a(\psi) = 0, \tag{27b}$$

where

$$L_\tau^n := -U_\tau\Delta_{A_0^n}U_\tau^{-1} \text{ and } M_\tau := V_\tau\,\mathrm{curl}^*\,\mathrm{curl}\,V_\tau^{-1}. \tag{28}$$

Here we used that $V_\tau A_0^n = A_0^n$ and $U_\tau\nabla\psi = (m_\tau^t)^{-1}U_\tau\psi$. (The latter relation is a straightforward computation and the former one follows from the facts that for any matrix m, $(mx)^\perp = (\det m)(m^t)^{-1}x^\perp$, and that in our case, $\det m_\tau = 1$.) Note that the gauge in the periodicity condition will still depend on $\mathrm{Im}\,\tau$. These complications, however, are inessential and the same techniques as above can be applied in this case. The important point here is to observe that the function ψ_0, constructed in Appendix B, the function $w(\lambda, s\psi_0)$, where $w(\lambda, v)$ is the solution of (18), and the bifurcation equation (20) depend on τ real-analytically. We leave the details of the proof to the interested reader.

7 Abrikosov Function

In this section, we continue with the case $n = 1$. We prove the Abrikosov relation between the energy per cell and the Abrikosov function, $\beta(\tau)$, which is defined by

$$\beta(\tau) := \frac{\int_{\Omega^\tau} |\psi_0^\tau|^4}{\left(\int_{\Omega^\tau} |\psi_0^\tau|^2\right)^2}, \tag{29}$$

where ψ_0^τ is a non-zero element in the nullspace of $L^n - 1$ acting on $\mathscr{H}_n(\tau)$. Since the nullspace is a one-dimensional complex subspace, β is well-defined.

Recall that $b = \frac{\kappa^2}{\lambda}$. Since the function $g_\lambda(|s|^2)$ given in Theorem 10 obeys $g_\lambda(0) = 0$ and $g_\lambda'(0) \neq 0$, the function $b_s = \kappa^2(1 + g_\lambda(|s|^2))^{-1} =: \kappa^2 + g_b(|s|^2)$ can be inverted to obtain $|s| = s(b)$. Absorbing $\hat{s} = \frac{s}{|s|}$ into ψ_0^τ, we can define the family $(\psi_{s(b)}^\tau, A_{s(b)}^\tau, b_{s(b)}^\tau)$ of $\mathcal{L}^\tau$-periodic solutions of the Ginzburg-Landau equations parameterized by average magnetic flux b and their energy

$$E_b(\tau) := \mathcal{E}_{\kappa^2/b}(\psi_{s(b)}^\tau, A_{s(b)}^\tau).$$

Clearly, $\psi_{s(b)}^\tau, A_{s(b)}^\tau, b_{s(b)}^\tau$ are analytic in b. We note the relation between the new perturbation parameter $\mu := \kappa^2 - b$ and the bifurcation parameter $|s|^2$:

$$\mu = \frac{g_\lambda(|s|^2)}{\lambda}\kappa^2 = g_\lambda'(0)\kappa^2|s|^2 + O(|s|^4). \tag{30}$$

The relation between the Abrikosov function and the energy of the Abrikosov lattice solutions is as follows.

Theorem 12. *In the case $\kappa > \frac{1}{\sqrt{2}}$, the minimizers, τ_b, of $\tau \mapsto E_b(\tau)$ are related to the minimizer, τ_*, of $\beta(\tau)$, as $\tau_b - \tau_* = O(\mu^{1/2})$, In particular, $\tau_b \to \tau_*$ as $b \to \kappa^2$.*

Proof. We first show that the theorem is a consequence of the following proposition, which is proved below.

Proposition 13. *We have*

$$E_b(\tau) = \frac{\kappa^2}{2} + \kappa^4 - 2\kappa^2\mu + \left(\frac{\kappa^4}{4\pi} - \frac{1}{1 + 4\pi(\kappa^2 - \frac{1}{2})\beta(\tau)}\right)\mu^2 + O(\mu^3). \tag{31}$$

To prove the theorem we note that $E_b(\tau)$ is of the form $E_b(\tau) = e_0 + e_1\mu + e_2(\tau)\mu^2 + O(\mu^3)$. The first two terms are constant in τ, so we consider $\tilde{E}_b(\tau) = e_2(\tau) + O(\mu)$. τ_b is also the minimizer of $\tau \mapsto \tilde{E}_b(\tau)$ and τ_*, of $e_2(\tau)$. We have the expansions $\tilde{E}_b(\tau_*) - \tilde{E}_b(\tau_b) = \frac{1}{2}\tilde{E}_b''(\tau_b)(\tau_* - \tau_b)^2 + O((\tau_* - \tau_b)^3)$ and $\tilde{E}_b(\tau_*) - \tilde{E}_b(\tau_b) = -\frac{1}{2}e_2''(\tau_b)(\tau_* - \tau_b)^2 + O((\tau_* - \tau_b)^3) + O(\mu)$, which imply the desired result. That concludes the proof of the theorem. □

Prof of Proposition 13. Recall that $\mu := \kappa^2 - b$. Using the real-analyticity of the function g_b, g_ψ^τ, and g_A^τ, we can express $\lambda(\mu) := \kappa^2/b$, $\psi^\tau(\mu) := \psi_{s(b)}^\tau$ and $A^\tau(\mu) := A_{s(b)}^\tau$ as

$$\lambda(\mu) = 1 + \frac{1}{\kappa^2}\mu + O(\mu^2) \tag{32}$$

$$\psi^\tau(\mu) = \mu^{1/2}\psi_0^\tau + \mu^{3/2}\psi_1^\tau + O(\mu^{5/2}) \tag{33}$$

$$A^\tau(\mu) = A_0 + \mu a_1^\tau + O(\mu^2). \tag{34}$$

We will first show that

$$E_b(\tau) = \frac{\kappa^2}{2} + \kappa^4 - 2\kappa^2\mu + \frac{\kappa^4}{4\pi}\left(1 - \frac{1}{\kappa^2}\int_{\Omega^\tau} |\psi_0^\tau|^2\right)\mu^2 + O(\mu^3). \tag{35}$$

Multiplying (11a) scalarly by ψ and integrating by parts gives

$$\int_{\Omega^\tau} |\nabla_A \psi^\tau|^2 = \kappa^2 \int_{\Omega^\tau} \left(\lambda|\psi^\tau|^2 - \kappa^2|\psi^\tau|^4\right).$$

Substituting this into the expression for the energy we find

$$E_b(\tau) = \frac{\kappa^4}{2\pi\lambda^2}\int_{\Omega^\tau}\left(\frac{\lambda^2}{2\kappa^2} - \frac{\kappa^2}{2}|\psi^\tau|^4 + |\operatorname{curl} A^\tau|^2\right). \tag{36}$$

Using the expansion above gives

$$E_b(\tau) = \frac{\kappa^2}{2} + \kappa^4 - 2\kappa^2\mu + \frac{\kappa^4}{2\pi}\left(1 - \frac{\kappa^2}{2}\int_{\Omega^\tau} |\psi_0^\tau|^4 + \int_{\Omega^\tau} |\operatorname{curl} a_1^\tau|^2\right)\mu^2 + O(\mu^3), \tag{37}$$

where we have used the fact that $\operatorname{curl} A_0 = 1$. As we proved above in (24), we can show that

$$\operatorname{curl} a_1^\tau = -\frac{1}{2}|\psi_0^\tau|^2 + \frac{1}{4\pi}\int_{\Omega^\tau} |\psi_0^\tau|^2.$$

Substituting this expression into (37), we obtain

$$E_b(\tau) = \frac{\kappa^2}{2} + \kappa^4 - 2\kappa^2\mu + \frac{\kappa^4}{4\pi}\left(1 - \left(\kappa^2 - \frac{1}{2}\right)\int_{\Omega^\tau} |\psi_0^\tau|^4 - \frac{1}{4\pi}\left(\int_{\Omega^\tau} |\psi_0^\tau|^2\right)^2\right)\mu^2 + O(\mu^3). \tag{38}$$

Now, if we differentiate (12a) twice w.r. to $\mu^{1/2}$ at $\mu = 0$ to obtain

$$(L-1)\psi_1^\tau = \frac{1}{\kappa^2}\psi_0^\tau - \kappa^2|\psi_0^\tau|^2\psi_0^\tau - 2ia_1^\tau \cdot \nabla_{A_0}\psi_0^\tau.$$

Now using the fact that $L-1$ is self-adjoint and that $(L-1)\psi_0^\tau = 0$, we find

$$\begin{aligned} 0 &= \int_{\Omega^\tau} \bar{\psi}_0^\tau (L-1)\psi_1^\tau \, dx \\ &= \int_{\Omega^\tau} \bar{\psi}_0^\tau \left(\frac{1}{\kappa^2}\psi_0^\tau - \kappa^2|\psi_0^\tau|^2\psi_0^\tau - 2ia_1^\tau \cdot \nabla_{A_0}\psi_0^\tau\right) dx \\ &= \frac{1}{\kappa^2}\int_{\Omega^\tau} |\psi_0^\tau|^2 \, dx - \kappa^2 \int_{\Omega^\tau} |\psi_0^\tau|^4 \, dx - \int_{\Omega^\tau} \bar{\psi}_0^\tau (2ia_1^\tau \cdot \nabla_{A_0}\psi_0^\tau) \, dx. \end{aligned}$$

An analogous calculation to the one in the proof of Theorem 10 then gives

$$0 = \frac{1}{\kappa^2}\int_{\Omega^\tau} |\psi_0^\tau|^2 \, dx - \kappa^2 \int_{\Omega^\tau} |\psi_0^\tau|^4 \, dx + \frac{1}{2}\int_{\Omega^\tau} |\psi_0^\tau|^4 \, dx - \frac{1}{4\pi}\left(\int_{\Omega^\tau} |\psi_0^\tau|^2 \, dx\right)^2,$$

This relation gives (35) from (38), but also by dividing by $\left(\int |\psi_0^\tau|^2\right)^2$ and rearranging we then obtain (31). □

The following result was discovered numerically in the physics literature and proven in [2] using earlier result of [20]:

Theorem 14. *The function $\beta(\tau)$ has exactly two critical points, $\tau = e^{i\pi/3}$ and $\tau = e^{i\pi/2}$. The first is minimum, whereas the second is a maximum.*

Theorems 10, 12, 14, after rescaling to the original variables, imply Theorem 4, which, as was mentioned above, a precise restatement of Theorem 1 of Introduction.

A Bifurcation with Symmetry

In this appendix we present a variant of a standard result in Bifurcation Theory.

Theorem 15. *Let X and Y be complex Hilbert spaces, with X a dense subset of Y, and consider a map $F : \mathbb{R}\times X \to Y$ that is analytic as a map between real Banach spaces. Suppose that for some $\lambda_0 \in \mathbb{R}$, the following conditions are satisfied:*

1. *$F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$,*
2. *$D_\psi F(\lambda_0, 0)$ is self-adjoint and has an isolated eigenvalue at 0 of (geometric) multiplicity 1,*
3. *For non-zero $v \in \operatorname{null} D_\psi F(\lambda_0, 0)$, $\langle v, D_{\lambda,\psi} F(\lambda_0, 0)v\rangle \neq 0$,*
4. *For all $\alpha \in \mathbb{R}$, $F(\lambda, e^{i\alpha}\psi) = e^{i\alpha}F(\lambda, \psi)$.*
5. *For all $\psi \in X$, $\langle \psi, F(\lambda, \psi)\rangle \in \mathbb{R}$.*

Then $(\lambda_0, 0)$ is a bifurcation point of the equation $F(\lambda,\psi) = 0$. In fact, there is a family of non-trivial solutions, (λ,ψ), unique in a neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R} \times X$, and this family has the form

$$\begin{cases} \lambda = \phi_\lambda(|s|^2), \\ \psi = sv + s\phi_\psi(|s|^2), \end{cases}$$

for $s \in \mathbb{C}$ with $|s| < \epsilon$, for some $\epsilon > 0$. Here $v \in \operatorname{null} D_\psi F(\lambda_0, 0)$, and $\phi_\lambda : [0,\epsilon) \to \mathbb{R}$ and $\phi_\psi : [0,\epsilon) \to X$ are unique real-analytic functions, such that $\phi_\lambda(0) = \lambda_0$, $\phi_\psi(0) = 0$.

Proof. The analysis of Section 6 reduces the problem to the one of solving the bifurcation equation (20). Since the projection P, defined there, is rank one and self-adjoint, we have

$$P\psi = \frac{1}{\|v\|^2}\langle v, \psi\rangle v, \text{ with } v \in \operatorname{null} D_\psi F(\lambda_0, 0). \tag{39}$$

We can therefore view the function γ in the bifurcation equation (20) as a map $\gamma : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$, where

$$\gamma(\lambda, s) = \langle v, F(\lambda, sv_0 + w(\lambda, sv))\rangle.$$

We now look for non-trivial solutions of this equation, by using the Implicit Function Theorem to solve for λ in terms of s. Note that if $\gamma(\lambda, t) = 0$, then $\gamma(\lambda, e^{i\alpha}t) = 0$ for all α, and conversely, if $\gamma(\lambda, s) = 0$, then $\gamma(\lambda, |s|) = 0$. So we need only to find solutions of $\gamma(\lambda, t) = 0$ for $t \in \mathbb{R}$. We now show that $\gamma(\lambda, t) \in \mathbb{R}$. Since the projection Q is self-adjoint, and since $Qw(\lambda, v) = w(\lambda, v)$ we have

$$\langle w(\lambda, tv), F(\lambda, tv + w(\lambda, tv)\rangle = \langle w(\lambda, tv), QF(\lambda, tv + w(\lambda, tv)\rangle = 0.$$

Therefore, for $t \neq 0$,

$$\langle v, F(\lambda, tv + \Phi(\lambda, tv))\rangle = t^{-1}\langle tv + w(\lambda, tv), F(\lambda, tv + w(\lambda, tv))\rangle,$$

and this is real by condition (5) of the theorem. Thus we can restrict γ to a function $\gamma_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

By a standard application of the Implicit Function Theorem to $t^{-1}\gamma_0(\lambda, t) = 0$, in which (1)-(3) are used (see for example [7]), there is $\epsilon > 0$ and a real-analytic function $\widetilde{\phi}_\lambda : (-\epsilon, \epsilon) \to \mathbb{R}$ such that $\widetilde{\phi}_\lambda(0) = \lambda_0$ and if $\gamma_0(\lambda, t) = 0$ with $|t| < \epsilon$, then either $t = 0$ or $\lambda = \phi_\lambda(t)$. Recalling that $\gamma(\lambda, e^{i\alpha}t) = e^{i\alpha}\gamma(\lambda, t)$, we have shown that if $\gamma(\lambda, s) = 0$ and $|s| < \epsilon$, then either $s = 0$ or $\lambda = \phi_\lambda(|s|)$.

We also note that by the symmetry, $\widetilde{\phi}_\lambda(-t) = \widetilde{\phi}_\lambda(|t|) = \widetilde{\phi}_\lambda(t)$, so $\widetilde{\phi}_\lambda$ is an even real-analytic function, and therefore must in fact be a function solely of $|t|^2$. We therefore set $\phi_\lambda(t) = \widetilde{\phi}_\lambda(\sqrt{t})$, and so ϕ_λ is real-analytic.

We now define $\phi_\psi : (-\epsilon, \epsilon) \to \mathbb{R}$ to be

$$\phi_\psi(t) = \begin{cases} t^{-1}w(\phi_\lambda(t), tv) & t \neq 0, \\ 0 & t = 0, \end{cases} \tag{40}$$

ϕ_ψ is also real-analytic and satisfies $s\phi_\psi(|s|^2) = w(\phi_\lambda(|s|^2), sv)$ for any $s \in \mathbb{C}$ with $|s|^2 < \epsilon$.

Now we know that there is a neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R} \times \operatorname{null} D_\psi F(\lambda_0, 0)$ such that in that neighbourhood $F(\lambda, \psi) = 0$ if and only if $\gamma(\lambda, s) = 0$ where $P\psi = sv$. By taking a smaller neighbourhood if necessary, we have proven that $F(\lambda, \psi) = 0$ in that neighbourhood if and only if either $s = 0$ or $\lambda = \phi_\lambda(|s|^2)$. If $s = 0$, we have $\psi = sv + s\phi_\psi(|s|^2) = 0$ which gives the trivial solution. In the other case, $\psi = sv + s\phi_\psi(|s|^2)$ and that completes the proof of the theorem. □

B The Operators L and M

In this appendix we prove Theorems 6 and 5. The proofs below are standard.

Proof of Theorem 6. The fact that M is positive follows immediately from its definition. We note that its being strictly positive is the result of restricting its domain to elements having mean zero. □

Proof of Theorem 5. First, we note that L^n is clearly a positive self-adjoint operator. To see that it has discrete spectrum, we first note that the inclusion $H^2 \hookrightarrow L^2$ is compact for bounded domains in $\mathbb{R}^2$ with Lipschitz boundary (which certainly includes lattice cells). Then for any z in the resolvent set of L^n, $(L^n - z)^{-1} : L^2 \to H^2$ is bounded and therefore $(L^n - z)^{-1} : L^2 \to L^2$ is compact.

In fact we find the spectrum of L^n explicitly. We introduce the harmonic oscillator creation and annihilation operators

$$L^n_{\pm} = \partial_{x_1} \mp i\partial_{x_2} \mp \frac{n}{2}x_1 + \frac{in}{2}x_2. \tag{41}$$

One can verify that these operators satisfy the following.

1. $[L^n_+, L^n_-] = 2n$.
2. $L^n - n = -L^n_+ L^n_-$.
3. $L^n - n = -L^n_- L^n_+$.

As for the harmonic oscillator (see for example [16]), this gives the explicit information about $\sigma(L)$ as stated in the theorem.

For the dimension of the null space of L, we need the following lemma.

Lemma 16. $\operatorname{null}(L^n - n) = \operatorname{null} L^n_-$*.*

Proof. If $L^n_-\psi = 0$, we immediately have $(L^n - n)\psi = -L^n_+ L^n_- \psi = 0$. For the reverse inclusion we use the fact that $\|L^n_-\psi\|^2 = \langle L^n_-\psi, L^n_-\psi\rangle = \langle \psi, -L^n_+ L^n_-\psi\rangle = \langle \psi, (L^n - n)\psi\rangle$. ☐

We can now prove the following.

Proposition 17. $\operatorname{null} L^n - n$ *is given by*

$$\operatorname{null}(L^n - n) = \{\, e^{\frac{in}{2}x_2(x_1+ix_2)} \sum_{k=-\infty}^{\infty} c_k e^{ki\sqrt{2\pi \operatorname{Im}\tau}(x_1+ix_2)} \mid c_{k+n} = e^{in\pi\tau} e^{2ki\pi\tau} c_k \}. \tag{42}$$

and therefore, in particular, $\dim_{\mathbb{C}} \operatorname{null} L^n_- = n$*.*

Proof. A simple calculation gives the following operator equation

$$e^{\frac{n}{4}|x|^2} L^n_- e^{-\frac{n}{4}|x|^2} = \partial_{x_1} + i\partial_{x_2}.$$

This immediately proves that $\psi \in \operatorname{null} L^n_-$ if and only if $\xi = e^{\frac{n}{4}|x|^2}\psi$ satisfies $\partial_{x_1}\xi + i\partial_{x_2}\xi = 0$. We now identify $x \in \mathbb{R}^2$ with $z = x^1 + ix^2 \in \mathbb{C}$ and see that this means that ξ is analytic. We therefore define the entire function Θ to be

$$\Theta(z) = e^{-\frac{n(r^\tau)^2}{4\pi^2}z^2} \xi\left(\frac{r^\tau z}{\pi}\right).$$

The quasiperiodicity of ψ transfers to Θ as follows.

$$\Theta(z + \pi) = \Theta(z),$$

$$\Theta(z + \pi\tau) = e^{-2inz} e^{-in\pi\tau z} \Theta(z).$$

To complete the proof, we now need to show that the space of the analytic functions which satsify these relations form a vector space of dimension n. It is easy to verify that the first relation ensures that Θ have a absolutely convergent Fourier expansion of the form

$$\Theta(z) = \sum_{k=-\infty}^{\infty} c_k e^{2kiz}.$$

The second relation, on the other hand, leads to relation for the coefficients of the expansion. Namely, we have

$$c_{k+n} = e^{in\pi\tau} e^{2ki\pi\tau} c_k$$

And that means such functions are determined solely by the values of $c_0, \ldots, c_{n-1}$ and therefore form an n-dimensional vector space. ☐

This completes the proof of Theorem 5. ☐

C Fixing the Gauge

We provide here an alternate proof of Proposition 3, largely based on ideas in [14]. We begin by defining the function $B : \mathbb{R} \to \mathbb{R}$ to be

$$B(\zeta) = \frac{1}{r}\int_0^r \operatorname{curl} A(\xi, \zeta)\, d\xi.$$

It is clear that $b = \frac{1}{r\tau_2}\int_0^{r\tau_2} B(\zeta)\, d\zeta$. A calculation shows that $B(\zeta + r\tau_2) = B(\zeta)$.

We now define $P = (P_1, P_2) : \mathbb{R}^2 \to \mathbb{R}^2$ to be

$$P_1(x) = bx_2 - \int_0^{x_2} B(\zeta)\, d\zeta,$$
$$P_2(x) = \int_{\frac{\tau_1}{\tau_2}x_2}^{x_1} \operatorname{curl} A(\xi, x_2)\, d\xi + \frac{\tau \wedge x}{\tau_2} B(x_2).$$

A calculation shows that P is doubly-periodic with respect to $\mathcal{L}$.

We now define $\eta' : \mathbb{R}^2 \to \mathbb{R}$ to be

$$\eta'(x) = \frac{b}{2}x_1 x_2 - \int_0^{x_1} A_1(\xi, 0)\, d\xi - \int_0^{x_2} A_2(x_1, \zeta) - P_2(x_1, \zeta)\, d\zeta.$$

η' satisfies

$$\nabla\eta = -A + A_0 + P.$$

Now let η'' be a doubly-periodic solution of the equation $\Delta\eta'' = -\operatorname{div} P$. Also let $C = (C_1, C_2)$ be given by

$$C = -\frac{1}{|\Omega|}\int_\Omega P + \nabla\eta\, dx,$$

where Ω is any fundamental cell, and set $\eta''' = C_1 x_1 + C_2 x_2$.

We claim that $\eta = \eta' + \eta'' + \eta'''$ is such that $A + \nabla\eta$ satisfies (i) - (iii) of the proposition. We first note that $A + \nabla\eta = A - A + A_0 + P + \nabla\eta'' + C$. By the above, $A' = P + \nabla\eta'' + C$ is periodic. We also calculate that $\operatorname{div} A' = \operatorname{div} P + \Delta\eta'' = 0$. Finally $\int A' = \int P + \nabla\eta - C = 0$.

All that remains is to prove (iv). This will follow from a gauge transformation and translation of the state. We note that

$$A_0(x+t) + A'(x+t) = A_0(x) + A'(x) + \frac{b}{2}\begin{pmatrix} -t_2 \\ t_1 \end{pmatrix}.$$

This means that $A_0(x+t) + A'(x+t) = A_0(x) + A'(x) + \nabla g_t(x)$, where $g_t(x) = \frac{b}{2}t \wedge x + C_t$ for some constant C_t. To establish (iv), we need to have it so that $C_t = 0$ for $t = r, r\tau$. First let l be such that $r \wedge l = -\frac{C_r}{b}$ and $r\tau \wedge l = -\frac{C_{r\tau}}{b}$. This l exists as it is the solution to the matrix equation

$$\begin{pmatrix} 0 & r \\ -r\tau_2 & r\tau_1 \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} -\frac{C_r}{b} \\ -\frac{C_{r\tau}}{b} \end{pmatrix},$$

and the determinant of the matrix is just $r^2\tau_2$, which is non-zero because $(r, 0)$ and $r\tau$ form a basis of the lattice. Let $\zeta(x) = \frac{b}{2}l \wedge x$. A straight forward calculation then shows that $e^{i\zeta(x)}\psi(x+l)$ satisfies (iv) and that $A(x+l) + \nabla\zeta(x)$ still satisfies (i) through (iii). This proves the proposition.

References

[1] A. A. Abrikosov, *On the magnetic properties of superconductors of the second group*, J. Explt. Theoret. Phys. (USSR) **32** (1957), 1147–1182.

[2] A. Aftalion, X. Blanc, and F. Nier, *Lowest Landau level functional and Bargmann spaces for Bose-Einstein condensates*, J. Funct. Anal. 241 (2006), 661–702.

[3] A. Aftalion and S. Serfaty, *Lowest Landau level approach in superconductivity for the Abrikosov lattice close to H_{c_2}*, Selecta Math. (N.S.) 13 (2007), 183–202.

[4] Y. Almog, *On the bifurcation and stability of periodic solutions of the Ginzburg-Landau equations in the plane*, SIAM J. Appl. Math. 61 (2000), 149–171.

[5] Y. Almog, *Abrikosov lattices in finite domains*, Commun. Math. Phys. 262 (2006), 677-702.

[6] L. V. Alfors, *Complex analysis*, McGraw-Hill, New York, 1979.

[7] A. Ambrosetti and G. Prodi, *A primer of nonlinear analysis*, Cambridge University Press, Cambridge, 1993.

[8] E. Barany, M. Golubitsky, and J. Turksi, *Bifurcations with local gauge symmetries in the Ginzburg-Landau equations*, Phys. D 56 (1992), 36–56.

[9] S. J. Chapman, *Nucleation of superconductivity in decreasing fields*, European J. Appl. Math. 5 (1994), 449–468.

[10] S. J. Chapman, S. D. Howison, and J. R. Ockedon, *Macroscopic models of superconductivity*, SIAM Rev. 34 (1992), 529–560.

[11] Q. Du, M. D. Gunzburger, and J. S. Peterson, *Analysis and approximation of the Ginzburg-landau model of superconductivity*, SIAM Rev. 34 (1992), 54–81.

[12] M. Dutour, *Phase diagram for Abrikosov lattice*, J. Math. Phys. 42 (2001), 4915–4926.

[13] M. Dutour, *Bifurcation vers ltat dAbrikosov et diagramme des phases*, Thesis Orsay, http://www.arxiv.org/abs/math-ph/9912011.

[14] G. Eilenberger, *Zu Abrikosovs Theorie der periodischen Lösungen der GL-Gleichungen für Supraleiter 2. Art*, Z. Physik 180 (1964), 32–42.

[15] S. Fournais, B. Helffer, *Spectral Methods in Surface Superconductivity.* Prog. Nonlin. Diff. Eqns. **77** (2010).

[16] S. J. Gustafson and I. M. Sigal, *Mathematical concepts of quantum mechanics*, Springer, 2006.

[17] A. Jaffe and C. Taubes, *Vortices and monopoloes: structure of static gauge theories*, Progress in Physics 2. Birkhäuser, Boston, Basel, Stuttgart, 1980.

[18] W.H. Kleiner, L. M. Roth, and S. H. Autler, *Bulk solution of Ginzburg-Landau equations for type II superconductors: upper critical field region*, Phys. Rev. 133 (1964), A1226–A1227.

[19] G. Lasher, *Series solution of the Ginzburg-Landau equations for the Abrikosov mixed state*, Phys. Rev. 140 (1965), A523–A528.

[20] S. Nonnenmacher and A. Voros, *Chaotic eigenfunctions in phase space*, J. Statist. Phys. 92 (1998), 431–518.

[21] F. Odeh, *Existence and bifurcation theorems for the Ginzburg-Landau equations*, J. Math. Phys. 8 (1967), 2351–2356.

[22] J. Rubinstein, *Six lectures on superconductivity*, Boundaries, interfaces, and transitions (Banff, AB, 1995), 163–184, CRM Proc. Lecture Notes, 13, Amer. Math. Soc., Providence, RI, 1998.

[23] E. Sandier and S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations and their Applications, Vol 70, Birkhäuser, 2007.

[24] P. Takáč, *Bifurcations and vortex formation in the Ginzburg-Landau equations*, ZAMM Z. Angew. Math. Mech. 81 (2001), 523–539.

[25] D. R. Tilley and J. Tilley, *Superfluidity and superconductivity.* 3rd edition. Institute of Physics Publishing, Bristol and Philadelphia, 1990.

[26] M. Tinkham, *Introduction to superconductivity*, McGraw-Hill Book Co., New York, 1996.