Padé summability of the cubic oscillator

Vincenzo Grecchi[1]

Marco Maioli[2]

André Martinez[3]

Abstract.
We prove the Padé (Stieltjes) summability of the perturbation series of the energy levels of the cubic anharmonic oscillator, $H_1(\beta) = p^2 + x^2 + i\sqrt{\beta}x^3$, as suggested by the numerical studies of Bender and Weniger. At the same time, we give a simple and independent proof of the positivity of the eigenvalues of the $\mathcal{PT}$-symmetric operator $H_1(\beta)$ for real β (Bessis-Zinn Justin conjecture). All the $n \in \mathrm{N}$ zeros of an eigenfunction, real at $\beta = 0$, become complex with negative imaginary part, for complex, non-negative $\beta \neq 0$.

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1 Introduction

The anharmonic oscillators are interesting non-solvable models of quantum physics, as the cubic one, for their simplicity. New interest comes from the theory of the $\mathcal{PT}$-symmetric operators. In particular, the interest is directed to the summability of the perturbation series, also in connection to similar problems in quantum field theory.
Many years ago [1], the Padé summability (PS) of the perturbation series of the energy levels of the quartic anharmonic oscillator with Hamiltonian $K_{4,1}(\beta) = p^2 + x^2 + \beta x^4$ was proved.

[1]Dipartimento di Matematica, Università di Bologna, I-40127-Bologna, Italy - E.mail: grecchi@dm.unibo.it

[2]Dipartimento di Matematica, Università di Modena, I-41100-Modena, Italy - E.mail: marco.maioli@unimore.it

[3]Dipartimento di Matematica, Università di Bologna, I-40127-Bologna, Italy - E.mail: martinez@dm.unibo.it

Some years later [2], the Borel summability of the perturbation series of each eigenvalue $E_{n,\alpha}(\beta)$, $n \in \mathrm{N}$, of the cubic anharmonic oscillator,

$$H_\alpha(\beta) = p^2 + \alpha x^2 + i\sqrt{\beta}x^3, \tag{1}$$

was also proved for a fixed $\alpha > 0$. This result was later extended [3], giving the distributional Borel summability [4] of the perturbation series, in the case of negative β.
The conjecture of Bessis-Zinn Justin (BZJ), was proved by Dorey et al. [11] at $\alpha = 0$. Shin [5] extended the proof to $\alpha \in \mathrm{R}$, and proved the positivity of the eigenvalues $\{E_{n,\alpha}(\beta)\}_n$ for $\alpha \geq 0$, $\beta > 0$. Strangely enough, Bessis didn't suggest, as far as we know, that the reality of the eigenvalues was a consequence of his loved PS. Some years later, Bender and Weniger have given numerical evidence of PS [8].
The BZJ conjecture was later extended by Bender and Boettcher (BB) [6], to the family of $\mathcal{PT}$-symmetric (PTS) Hamiltonians,

$$H_{N,\alpha}(1) = p^2 + \alpha x^2 - (ix)^N,$$

$\alpha \geq 0$, $N \geq 2$, with analytic eigenfunctions $\phi(z)$, where $z = x + iy$, vanishing at infinity on the two Stokes angular sectors of the complex plane,

$$S^N_{\pm 1} = \{|\arg(iz) \pm \frac{2\pi}{(N+2)}| < \frac{\pi}{(N+2)}\}. \tag{2}$$

The last conjecture was proved, as part of a more general result, by Vladimir Bouslaev and one of us [9] (see also [10]), in the relevant case of $N = 4$.
Shin has proved the BB conjecture, in the general case, for $\alpha \leq 0$ [5].
From now on, we restrict the discussion to the cubic oscillator. The family of operators $H_1(\beta)$ is an analytic family of type A on the cut plane $C_c = \{\beta \in \mathrm{C}; \beta \neq 0, |\arg(\beta) = \theta| < \pi\}$, and we have the spectral equivalence [12],

$$H_1(\beta) \sim \alpha^{-1/2} H_\alpha(1), \tag{3}$$

where $\alpha = \exp(-i2\theta/5)$. For β at the boundary of the cut, for instance, $\beta = b\exp(-i\pi) = -b - i0^+$, $b > 0$, the mechanical problem defined by the Hamiltonian $H_\alpha(\beta)$, for $\alpha = a > 0$, is uncomplete in both classical and quantum cases and can be defined by the physical hypothesis of the disapperance of the particle when it reaches infinity. In the quantum case, this means that one must define the Hamiltonian by the Gamow-Siegert condition at $-\infty$ [2]. The eigenvalues have the meaning of resonances and the eigenfunctions have the meaning of metastable states for the dynamical problem. Thus we expect, and we prove, a negative imaginary part of the

eigenvalues, related, in the usual way, to the lifetime of the metastable state. We consider the eigenfunction $\psi_{n,a,\beta}(z)$, for a fixed $a > 0$, where $n \in \mathrm{N}$ is the number of its nodes, and β is on the complex cut plane C_c. The n nodes, numerically studied in [7] for positive β, are stable at $\beta = 0$ and are the only zeros on the lower half complex plane $\mathrm{C}_- = \{z \in \mathrm{C}; \Im(x) = y < 0\}$. On the other side, there are no zeros on the strip $0 \leq \Im(z) \leq y_+ = 2a\Re\sqrt{\beta}/3b$.
We use the Loeffel-Martin method and the complex semiclassical Sibuya picture to prove the confinement of the nodes. We use also the hypothesis of the boundedness of each eigenvalue for bounded parameters. This hypothesis is verified by the Bohr-Sommerfeld quantization rule and the invariance of the number of nodes.
The crossings of eigenvalues and the branch point singularities are forbidden by the unique characterization of the eigenfunctions by the number of their nodes, and the simplicity of the spectrum. Let us remember that we have the extended $\mathcal{PT}$ symmetry (see [2]) of the complex Hamiltonians,

$$H_1(\beta) = PH_1^*(\bar{\beta})P.$$

The isolation and analyticity of each eigenvalue on the cut plane C_c and the unique sum of the perturbation series, imply the extended $\mathcal{PT}$ symmetry of the eigenfunctions, $\psi_{n,1,\beta}(x) = \bar{\psi}_{n,1,\bar{\beta}}(-x)$, and eigenvalues $E_{n,1}(\beta) = \bar{E}_{n,1}(\bar{\beta})$. The identity, obtained by complex scaling for $\beta \neq 0$, $|\arg(\beta)| < \pi$,

$$\{E_{n,1}(\beta) = \alpha^{-1/2} E_{n,\alpha}(1)\}_{n\in\mathrm{N}}, \tag{4}$$

where $\alpha = \beta^{-2/5}$, allows the global analytic continuation on the Riemann surface of $\beta^{1/5}$, of the set of the eigenvalues. In particular, we prove the power law behavior of the eigenvalues at $\beta = \infty$ by the scaling law (4) and the analyticity of $\{E_{n,\alpha}(1)\}_n$ at $\alpha = 0$.
We prove the PS of the perturbation series of each eigenvalue to the eigenvalue itself. In order to be more precise, let us fix $n \in \mathrm{N} = \{0, 1, 2, ...\}$, and set the simplified notations for the once subtracted eigenvalue, $f(\beta) = (E_{n,1}(\beta) - E_{n,1}(0))/\beta$, for any β on the cut complex plane C_c. Thus, for $\beta \in \mathrm{C}_c$, we have the Stieltjes representation for $f(\beta)$, and the asymptotics for small $b = |\beta|$ given by the formal perturbation series [2],

$$f(\beta) = \int_0^\infty \frac{1}{(1+\beta\lambda)}\rho(\lambda)d\lambda \sim \Sigma(\beta) = \sum_{k=0}^{\infty} c_{k+1}\beta^k,$$

where $\rho(\lambda)$ is non-negative, and the $\{c_j\}_{j\in\mathrm{N}}$ are the perturbation coefficients of $E_{n,1}(\beta)$.

Thus, we prove, in a new way, the positivity of the eigenvalues, for positive β,

$$E_{n,1}(\beta) = E_{n,1}(0) + \beta f(\beta) = E_{n,1}(0) + \beta \int_0^\infty \frac{1}{(1+\beta\lambda)} \rho(\lambda) d\lambda \geq E_{n,1}(0) > 0.$$

The PS of the perturbation series to the eigenvalue is defined by the limit,

$$f(\beta) = \lim_{k\to\infty} R_k^k(\beta),$$

where the $R_k^k(\beta) = P_k(\beta)/Q_k(\beta)$, are the diagonal Padé approximants, $P_k(\beta)$, $Q_k(\beta)$ are polynomials of order k, with $Q_k(0) = 1$, completely defined by the asymptotics for $|\beta|$ small,

$$|R_k^k(\beta) - \Sigma^{2k+1}(\beta)| = O(|\beta|^{2k+1}),$$

where $\Sigma^k(\beta) = \sum_{j=0}^{k-1} c_{j+1}\beta^j$.
The semiclassical behavior, for large positive λ, of the discontinuity,

$$\ln(\rho(\lambda)) = -C^{-1}\lambda(1 + O(\ln(\lambda)/\lambda)),$$

where $C = 15/8$, agrees with the asymptotics of the perturbation coefficients for large j, as computed in [8], for $n = 0$,

$$c_j = (-1)^{j+1} 4\sqrt{15} C^j (2\pi)^{-3/2} \Gamma(j + 1/2)(1 + O(1/j)).$$

For numerical aspects, as the interesting similarity of this perturbation series with the one of the quartic anharmonic oscillator, see reference [8].

In Section 2 we discuss the operators for β at the boundaries of the complex cut plane. In Section 3 we consider the stability, analyticity and asymptotics of the eigenvalues and the nodes of the eigenfunctions for small $|\beta|$. In Section 4 we confine the nodes on the lower complex half plane. In Section 5 we prove the stability of the nodes for small parameter. In Section 6 we prove the stability of the nodes for large parameter. In Section 7 we prove the boundedness of an eigenvalue for bounded parameters. In Section 8 we prove the power law behavior in the parameter at infinity. In Section 9 we prove the Padé summability of the perturbation series.

2 The imaginary part of the eigenvalues on the cut

Let $\beta = b\exp(i\theta)$, $b > 0$, the family of operators $H_1(\beta)$ is an analytic family of type A on the cut plane $C_c = \{\beta \in \mathrm{C}; \beta \neq 0, |\arg(\beta)| < \pi\}$, and we have the spectral equivalence (see [2])

$$H_1(\beta) = PH_1^*(\bar{\beta})P \sim \alpha^{-1/2}H_\alpha(b), \tag{5}$$

where $\alpha = \exp(-i2\theta/5)$ and $P\psi(x) = \psi(-x)$. The identity of the sets of eigenvalues,

$$\{E_{n,1}(\beta)\}_{n\in \mathrm{N}} = \alpha^{-1/2}\{E_{n,\alpha}(b)\}_{n\in \mathrm{N}}, \tag{6}$$

defines the global analytic continuation from C_c to all the Riemann surface of $\beta^{1/5}$, of the set of the eigenvalues, $\{E_{n,1}(\beta))\}_{n\in \mathrm{N}}$.
In particular, here we are interested in the eigenvalues $\{E_{n,1}(\beta)\}_{n\in \mathrm{N}}$ for β on the closed cut plane $\bar{C}_c = \{\beta \in \mathrm{C}; \beta \neq 0, |\arg(\beta)| \leq \pi\}$.
The operators $H_1(\beta)$ on the borders of the cut, $\arg(\beta) = \theta = \pm\pi$, are uncomplete, and are defined by the choice of the fundamental behavior at $\pm\infty$ respectively [2]. This choice is fixed by the spectral equivalence (3).
In the case of $\theta = -\pi$, $i\sqrt{\beta}$ positive, the Hamiltonian is real, but both the classical problem and the quantum one, are uncomplete. In the quantum case, all the solutions are L^2 at $-\infty$. The choice of the Gamow-Siegert behavior (corresponding to negative current density) at $-\infty$, gives eigenvalues with the meaning of resonances and eigenstates with the meaning of metastable states.
On the other case, for $\theta = \pi$, $i\sqrt{\beta}$ negative, with the choice of the anti-Gamow-Siegert behavior at $+\infty$, we have the correct definition of the operator and we verify the spectral equivalence, $H_1(\beta) \sim \sqrt{\alpha}H_\alpha(b)$, $\alpha = \exp(-i2\pi/5)$.
Actually, we have the following behavior of the eigenfunction $\psi_{n,1,\beta}$ for x large:

$$\psi_{n,1,\beta}(x) \sim \frac{K}{x^{3/4}}\exp(-\sqrt{i}\sqrt[4]{\beta}x^{5/2}) = \frac{K}{x^{3/4}}\exp(-i\sqrt[4]{b}x^{5/2}). \tag{7}$$

The same behavior is defined by the scaled eigenfunction, satisfying the L^2 condition,

$$\psi_{n,\alpha}(b)(x) \sim \frac{K_1}{x^{3/4}}\exp(-\sqrt{i}\sqrt[4]{b}x^{5/2}) \to 0, \tag{8}$$

as $x \to \infty$.
Now, let us consider the translated Hamiltonian

$$T_{-\epsilon}H(1,\beta)T_{-\epsilon}^{-1} = H_{-\epsilon,1}(\beta),$$

where the complex translation, is defined by $T_{-\epsilon}\psi(x) = \psi_{-\epsilon}(x) = \psi(x - i\epsilon)$, for $\epsilon > 0$. Because of the translation analyticity of the Hamiltonian, we have the spectral equivalence,

$$H_{-\epsilon,1}(\beta) \sim H_1(\beta)$$

for $\beta \in \dot{C}_c$. This equivalence can be extended to the case of $\theta = \pi$, if the operator $H_1(\beta)$ is defined, as above, by the correct condition at $+\infty$. Let us consider an eigenfunction $\psi_{-\epsilon,n,1,\beta}(x) = \psi_{n,1,\beta}(x - i\epsilon)$ of $H_{-\epsilon,1}(\beta)$, for $\beta = b\exp(i\pi) \neq 0$. We have the L^2 behavior,

$$\psi_{n,1,\beta}(x - i\epsilon) \sim \frac{K_2}{x^{3/4}} \exp(-i\sqrt[4]{b}x^{5/2}(1 - 5i\epsilon/x)) \to 0, \tag{9}$$

as $x \to +\infty$, if $\psi_{n,1,\beta}$ has the anti-Gamow-Siegert behavior.
Let us consider the numerical range of the translated operator, for $\beta = b\exp(i\pi)$,

$$H_{-\epsilon,1}(\beta) = p^2 + V_{1,\beta}(x - i\epsilon),$$

for $\epsilon > 0$, where $V_{1,\beta}(x - i\epsilon) = (x - i\epsilon)^2 - \sqrt{b}(x - i\epsilon)^3$ and

$$\Im V(x - i\epsilon) = -\epsilon(2x - \sqrt{b}(3x^2 - \epsilon^2)) \geq -\frac{\epsilon}{3\sqrt{b}}(1 + 3b\epsilon^2).$$

Thus, the intersection of the numerical ranges of the operators $\{H_{-\epsilon,1}(\beta)\}_\epsilon$, for all $\epsilon > 0$, is contained on $\overline{\mathrm{C}}_+ = \{z \in \mathrm{C} : \Im z \geq 0\}$. Thus, we have "anti-resonances" $E_n(1, b\exp(i\pi)) = E_n(1, -b + i0^+)$, with $\Im E_n(1, -b - i0^+) \geq 0$, as the usual anti-resonances [16].
In a similar way, we prove that $\Im E_{n,1}(-b - i0^+) \leq 0$. Thus, we have:
Lemma 1. *An eigenvalue $E(\beta) = E_{n,1}(\beta)$, $n \in \mathrm{N}$, $\beta = b\exp(i\pi) = -b+i0^+$, $b > 0$, of $H_1(\beta) \sim \alpha^{-1/2}H_\alpha(b)$, $\alpha = \exp(-4i\pi/5)$, has a non-negative imaginary part: $\Im E(b\exp(i\pi)) \geq 0$. On the other side, for $\beta = b\exp(-i\pi) = -b - i0^+$, $b > 0$, we have $\Im E(b\exp(-i\pi)) \leq 0$.*

3 Analyticity, symmetry and stability of the nodes for small parameter

Let us consider the analytic family of type A of compact resolvent operators,

$$H_\alpha(\beta), \tag{10}$$

on the domain $D = D(p^2) \cap D(x^3)$ for fixed $\alpha \in \mathrm{C}$, β on the cut plane

$$\mathrm{C}_c = \{\beta \in \mathrm{C}; b = |\beta| > 0, |\arg(\beta) = \theta| < \pi, \}$$

([2], Theorem 2.9).
We fix, for example, $\alpha = 1$.

We remember that $H_1(\beta)$, for β on the borders of C_c, for instance at $\arg(\beta) = -\pi$ (the other case, $\arg(\beta) = -\pi$, is similar), is defined by the Gamow-Siegert condition at $-\infty$.
The eigenvalue $E_{n,1}(\beta)$, for a fixed $n = 0, 1, ...$, of $H_1(\beta)$ is an eigenvalue also of the operator $\alpha^{1/2} H_\alpha(b)$, $E_{n,1}(\beta) = \alpha^{1/2} E_{n,\alpha}(b)$, (the index n is related to the number of nodes of the eigenfunction) where

$$\alpha = (b/\beta)^{2/5} = \exp(-2i\theta/5).$$

In particular $E_{n,1}(b \exp(\pm\pi)) = \sqrt{\alpha} E_{n,\alpha}(b)$, where $\alpha = \exp(\mp 2i\pi/5)$.
Moreover, the eigenvalue $E_{n,1}(\beta)$ of $H_1(\beta)(p, x)$, for $\beta = b \exp(\pm i\pi)$, is also an eigenvalue of the translated operator,

$$H_1(\beta)(p, x \pm i\epsilon) = p^2 + (x \pm i\epsilon)^2 + i\sqrt{\beta}(x \pm i\epsilon)^3,$$

for $\epsilon > 0$.
For β on the completed cut plane, $\overline{\mathrm{C}}_c = \{\beta \in \mathrm{C}; \beta \neq 0, |\arg(b)| \leq \pi, \}$ we have the spectral equivalence for scaling:

$$H(\alpha, b) \sim (\alpha)^{-1/2} H(1, \beta), \tag{11}$$

where $\alpha = (b/\beta)^{2/5} = \exp(-2i\theta/5)$ with $|\arg(\alpha)| < \pi/2$.
In place of the limit of $H_1(\beta)$, as $\beta \to 0$, we consider the norm resolvent limit $H_\alpha(b) \to H_\alpha(0)$, for $\alpha = (b/\beta)^{2/5}$ fixed, as $b \to 0$. Let us notice that $H(\alpha, 0)$, for $\alpha \neq 0$, is defined on the domain $D = D(p^2) \cap D(x^2)$ (see Theorem 2.13 on reference [2], and its extension on reference [3]).
We have the result of strong asymptotism of the eigenvalues:

Theorem 1.
For $n = 0, 1, ...$, let $E_{n,1}(\beta)$ be an eigenvalue, and let c_k, $k \in \mathrm{N}$, be its perturbation coefficients,

$$f(\beta) = \frac{(E_{n,1}(\beta) - E_{n,1}(0))}{\beta}.$$

Then, there exists $b_n > 0$ such that $f(\beta)$ is analytic on the bounded sector,

$$\Omega_n = \{\beta \in \mathrm{C}; 0 < |\beta| < b_n, |\arg(\beta)| \leq \pi\},$$

and there exist numbers $A, C > 0$, such that

$$|f(\beta) - \Sigma^N(\beta)| < AC^N N! |\beta|^N,$$

where $\Sigma^N(\beta) = \sum_{k=0}^{N-1} c_{k+1}(\beta)^k$, *uniformly for* $N-1 \in \mathrm{N}$ *and* $\beta \in \Omega_n$.

Proof See reference [2], Theorem 3.2.,(where β is our $i\sqrt{\beta}$), extended in reference [3] (for $k=1$).

Remark 1: the stability of the nodes.

Together with the stability of the eigenvalues, we have the stability of the eigenfunctions. In particular, we are interested in the stability of their zeros, or nodes.

We have the limit of the eigenvalue $E_n(1,\beta) \to E_n(1,0)$ and the strong limit of the eigenvector $\psi_{n,1,\beta} \to \psi_{n,1,0}$ as $b = |\beta| \to 0^+$, for $\beta \in \Omega_n$, $\arg(\beta)$ fixed, $|\arg(\beta)| \leq \pi$. Thus we have the limit $\psi_{n,1,\beta}(z) \to \psi_{n,1,0}(z)$ as $b = |\beta| \to 0^+$, for $\beta \in \Omega_n$, $\arg(\beta)$ fixed, $|\arg(\beta)| \leq \pi$, uniformly for z on a compact of the complex plane.

Since the perturbed eigenfunctions are entire, as the unperturbed ones, we have the stability of the n zeros of $\psi_{n,1,0}(z)$ for $b = |\beta|$ small.

For any fixed regular closed curve $\gamma = \partial\Gamma$ on the complex plane, oriented in the positive sense, around the segment of extremes (x_-, x_+), where $x_\pm = \pm\sqrt{E_n(1,0)} = \pm\sqrt{(2n+1)}$, we have the constant number of zeros (nodes) in Ω_n:

$$n = \frac{1}{2i\pi}\oint_\gamma \frac{\psi'_{n,1,\beta}(z)}{\psi_{n,1,\beta}(z)}dz = \frac{1}{2i\pi}\oint_\gamma \frac{\psi'_{n,1,0}(z)}{\psi_{n,1,0}(z)}dz,$$

for $\beta \in \Omega'_n$, where

$$\Omega'_n = \{\beta \in \mathrm{C}; 0 < |\beta| \leq b'_n, |\arg(\beta)| \leq \pi\},$$

and $0 < b'_n \leq b_n$.

Let us set $\psi_\beta = \psi_{n,1,\beta}$ and $\psi_0 = \psi_{n,1,0}$ and apply the theorem of Rouché [17]. Since the zeros of $\psi_0(z)$ are not on γ, there exists $M > 0$, such that $|\psi_0| \geq M > 0$ uniformly on γ. Moreover, $|\psi_\beta(z) - \psi_0(z)| \to 0$ uniformly for z on the compact γ, because of the analyticity. Thus, we have $|\psi_0(z)| > |\psi_\beta(z) - \psi_0(z)|$ for z on γ and for $\beta \in \Omega'_n$, so that the Rouché theorem applies.

We shall see (Theorem 2) that, for $\beta \in \mathrm{C}_c \bigcap \Omega'_n$, where C_c is the complex plane cut on the negative real axis, the n zeros (nodes) are confined on $\Gamma \bigcap \mathrm{C}_-$, where $\mathrm{C}_- = \{z \in \mathrm{C}; \Im(z) < 0\}$. The n nodes are the only zeros of $\psi(z)$ on C_-.

4 Absence of zeros of the eigenfunctions on a strip containing the real axis

We consider the operator

$$H(a,\beta) = p^2 + ax^2 + i\sqrt{\beta}x^3,$$

and the eigenvalue $E_{n,a}(\beta)$, with eigenfunction $\psi_{n,a,\beta}$, where $n = 0, 1, ...,$ $a \geq 0$, $|\beta| = b > 0$, $\theta = \arg(\beta)$, $|\theta| \leq \pi$.
Let us fix $|\theta| < \pi$, $a > 0$.
We call $z = x+iy$ the x variable extended to the complex plane. We consider the eigenvalues $E = E_{n,a}(\beta)$, and eigenfunctions $\psi_E(z) = \psi_{n,a,\beta}(z)$, where the label n is related to the number of zeros stable at $\beta = 0$.

On the strip

$$A(a,\beta) = \{z \in \mathrm{C}; 0 \leq \Im(z) \leq y_+ = \frac{2a\Re\sqrt{\beta}}{3b} = \frac{2a}{3\sqrt{b}}\cos(\frac{\theta}{2})\},$$

there are no zeros of the eigenfunction $\psi_{n,a,\beta}(z)$.

Theorem 2. *On the strip*

$$A(a,\beta) = \{z \in \mathrm{C}; 0 \leq \Im(z) \leq y_+ = \frac{2a\Re\sqrt{\beta}}{3b} = \frac{2a}{3\sqrt{b}}\cos(\frac{\theta}{2})\},$$

there are no zeros of any eigenfunction $\psi_E(z)$ of $H(a,\beta)$, with eigenvalue E, where $a > 0$, and $b = |\beta| > 0$, $\theta = \arg(\beta)$, $|\theta| \leq \pi$.
Proof.
Let us, at first, set $|\theta| < \pi$, and consider the translated operator $H_{a,\beta,y} = p^2 + V_y$, where

$$V_y = V_y(x) = a(x+iy)^2 + i\sqrt{\beta}(x+iy)^3 =$$

$$= ax^2 - ay^2 - 3\sqrt{\beta}yx^2 + \sqrt{\beta}y^3 + 2aiyx - 3i\sqrt{\beta}y^2x + i\sqrt{\beta}x^3.$$

Let $\psi_y(x) = \psi_E(x+iy)$. We have:

$$\psi_E(x+iy) = \psi_y(x) \neq 0$$

for every $x \in \mathrm{R}$, for $0 \leq y \leq y_+ = 2a\Re(\beta)/3b$.
For $0 \leq y \leq y_+$,

$$-\Im(\psi_y(r)\frac{d\overline{\psi_y(r)}}{dr}) = \int_r^\infty \Im(V_y(x) - E)|\psi_y(x)|^2 dx > 0, \tag{12}$$

or

$$-\Im(\psi_y(r)\frac{d\overline{\psi_y(r)}}{dr}) = -\int_{-\infty}^{r} \Im(V_y(x)-E)|\psi_y(x)|^2 dx > 0, \tag{13}$$

for any $r \in \mathrm{R}$.
The proof is based on the monotonicity of

$$f(x) = \Im(V_y(x)-E) = R(x^3-3y^2x)+2axy-3Iyx^2+c,$$

where $R = \Re(\sqrt{\beta})$, $I = \Im(\sqrt{\beta})$ and c is a constant, that is, the non-negatitivity of $f'(x)$,

$$f'(x) = \Im(V_y(x)-E)' = 3Rx^2-6Iyx-3Ry^2+2ay = Ax^2+Bx+C \geq 0,$$

where: $A = 3R$, $B = -6Iy$, $C = -3Ry^2+2ay$. We impose the non-positivity of the discriminant:

$$(B^2-4AC)/4 = 12y[3by-2Ra] \leq 0,$$

proved for

$$0 \leq y \leq y_+ = \frac{2Ra}{3b} = \frac{2a}{3\sqrt{b}}\cos(\frac{\theta}{2}).$$

We have absence of zeros for $0 \leq \Im z \leq y_+$.
In the case of $|\arg(\beta)| = \pi$, we have the limits of the equations (12)(13), and for any $r \in \mathrm{R}$,

$$\int_{r}^{\infty} \Im(V(x)-E)|\psi_\epsilon(x)|^2 dx = -\int_{r}^{\infty} \Im(E)|\psi_E(x)|^2 dx > 0, \text{ or}$$

$$\int_{-\infty}^{r} \Im(E)|\psi_E(x)|^2 dx > 0. \tag{14}$$

This means that the imaginary part of the eigenvalue is different from zero. Thus, the imaginary part of the eigenvalue is different from zero, and we extend Lemma 1:

Lemma 2.
An eigenvalue $E(\beta) = E_{n,1}(\beta)$, $n \in \mathrm{N}$, $\beta = b\exp(i\pi) = -b+i0^+$, $b>0$, *of* $H_1(\beta) \sim \alpha^{-1/2}H_\alpha(b)$, $\alpha = \exp(-4i\pi/5)$, *has a positive imaginary part:* $\Im E(b\exp(i\pi)) > 0$. *On the other side, for* $\beta = b\exp(-i\pi) = -b-i0^+$, $b>0$, *we have (14),* $\Im E(b\exp(-i\pi)) < 0$.

5 The semiclassical confinement of the nodes for small parameter

We prove the absence, for $|\beta| > 0$ small, $|\arg(\beta)| \le \pi$, of any zero of the eigenfunction $\psi_\beta(z) = \psi_{n,1,\beta}(z)$ of $H_1(\beta)$, with eigenvalue $E_\beta = E_{n,1}(\beta)$, for a fixed $n = 0, 1, ...$, on the lower half plane, $z \in \mathrm{C}_-$, for large $|z|$.
Let us consider the semiclassical quantity,

$$p_\beta(z) = \sqrt{V_\beta(z) - E_\beta}, \tag{15}$$

where $V_\beta(z) = z^2 + i\sqrt{\beta}z^3$ and $E_\beta = E_{n,1}(\beta)$ is the eigenvalue of the Hamiltonian with eigenfunction $\psi_\beta(z)$. There are three zeros of $p_\beta(x)$. Two zeros $z_\pm(\beta)$, converge $z_\pm(\beta) \to z_\pm = \pm\sqrt{E_0}$ as $\beta \to 0$, in the sector $|arg(\beta)| < \pi$. The third one, $z_0(\beta)$ diverges, $z_0(\beta) \sim i/\sqrt{\beta}$, as $\beta \to 0$, in the sector $|arg(\beta)| < \pi$.
Let $n = 0, 1, ..$ fixed, $z \in \mathrm{C}_-$, $|z| >> |z_\pm|$, and $\beta \in \Omega'_n$, we define:

$$f(\beta, z) = f(n, 1, \beta, z) = \frac{|\psi'_\beta(z)|}{|p_\beta(z)\psi_\beta(z)|}, \tag{16}$$

where $p_\beta(z)$ is defined above (15). We have,

$$f(\beta, z) \to 1, \tag{17}$$

for $|z| \to \infty$, uniformly for z on the sector $|\arg(iz)| \le \pi/2 - \epsilon$, for any ϵ $0 < \epsilon < \pi/2$, $0 \le |\beta| \le b'_n$, for fixed $\arg\beta$, $|\arg(\beta)| < \pi$. This means that no node of $\psi_\beta(z)$ goes to (or comes from) infinity on the sector $|\arg(iz)| \le \pi/2$, for this set of parameters.

Theorem 3.
Let $\psi_\beta(z) = \psi_{n,1,\beta}(z)$ be an eigenfunction with n nodes and eigenvalue $E = E_n$.
No one of its nodes goes to (or comes from) infinity on the sector $|\arg(iz)| < \pi/2$, for $|\beta| \le b'_n$, $|\arg(\beta)| < \pi$.

Remark 2.
We can extend the limit (17) uniformly for $0 \le b = |\beta| \le b'_n$ fixed, and $|\arg\beta| \le \pi$. Thus we extend the barrier for the zeros at infinity on the sector $|\arg(iz)| \le \pi/2$, to the full $\beta-$sector

$$\overline{\Omega}'_n = \{\beta \in \mathrm{C}; 0 \le |\beta| \le b'_n, |\arg(\beta)| \le \pi\},$$

6 The semiclassical confinement of the nodes for large parameter

Let the Hamiltonian be $H_a = p^2 + V_a = H_a(\beta) = p^2 + V_a(\beta)$, and $E = E_{n,a}(\beta)$ for fixed $n = 0, 1, ..,$ be an eigenvalue with eigenfunction $\psi_E(z) = \psi_{n,a,\beta}(z)$ for fixed β, $|\beta| \geq b'_n$, $|\arg(\beta)| \leq \pi$, and $0 \leq a \leq 1$. We have $E \neq 0$, $|\arg(E)| \leq \pi/2$, because of the numerical range.
Now, we make the hypothesis of boundendness of an eigenvalue for the parameters restricted on a compact. This allows to prove the stability of the nodes of its eigenfunction. We will prove later that the eigenvalue is bounded if the number of nodes is stable.
Hypothesis I.
The eigenvalue $E = E_{n,a}(\beta)$ for fixed $n = 0, 1, ..,$ β, $|\beta| = b'_n > 0$, where b'_n is given in Theorem 1, is uniformly bounded for $|\arg(\beta)| \leq \pi$, and $0 \leq a \leq 1$.

Let us recall [13] the 5 Stokes angular sectors of the complex x plane, for $\beta \neq 0$,

$$S_k = S_k(\arg(\beta)) = \{z \in \mathrm{C}; |\arg(iz) + \frac{1}{5}\arg(\beta) - \frac{2k\pi}{5}| < \frac{\pi}{5}\},$$

$-2 \leq k \leq 2$.
The eigenfunction $\psi_E(z)$ is an entire function and,

$$(\psi_E(z), \psi'_E(z)) \to 0$$

as $|z| \to \infty$, for $\arg(z)$ in each one of the two Stokes angular sectors $S_{\pm 1}$.
On the other side, $\psi_E(z)$ is purely divergent in the other three sectors $S_0, S_{\pm 2}$, and has no zeros [13] in the full angular sector of the complex plane

$$S = S(\arg(\beta)) = S_{-2} \bigcup \bar{S}_{-1} \bigcup S_0 \bigcup \bar{S}_1 \bigcup S_2 = \{x \in \mathrm{C}; |\arg(iz) + \frac{1}{5}\arg(\beta)| < \pi\}$$

for large $|z|$.
We have the following result.

Theorem 3'.
Let β be fixed, with $|\beta| = b'_n > 0$, $|\arg(\beta)| \leq \pi$, and $0 \leq a \leq 1$, $\psi_a(z) = \psi_{n,a,\beta}(z)$ be an eigenfunction, with eigenvalue $E = E_a = E_{n,a}(\beta)$, where the index $n = 0, 1, ..,$ is the number of its nodes at $a = 1$.
Moreover, we assume the Hypothesis I.
Then, no one of its nodes goes to (or comes from) infinity on the sector

$$S_- = \{z \in \mathrm{C}; |\arg(iz)| \leq \pi/2\}.$$

Remark 3.
Considering also Theorem 2, we have the invariance of the number of nodes.
Proof of the Theorem 3'.
Let us consider the function

$$f_a(z) = \frac{|\psi'_a(z)|}{|p_a(z)\psi_a(z)|}, \tag{18}$$

where $p_a(z) = \sqrt{V_a(z) - E_{n,a}} \sim \sqrt[4]{\beta}(iz)^{3/2}$ for large $|z|$ with a, E fixed.
Since $\psi_a(z)$ is the analytic solution of the Schrödinger equation, with energy $E_{n,a}$, the zeros of $\psi_a(z)$ are simple and $f_a(z)$ has a pole where $\psi_a(z)$ has a zero. We have,

$$f_a(z) \to 1$$

as $|z| \to \infty$, uniformly for $|\arg(iz)| \leq \pi/2$, $a \in [0,1]$, E on a compact set. This means that no zero of $\psi_a(z)$ goes to (or comes from) infinity on the sector $|\arg(iz)| \leq \pi/2$, for this set of parameters.

7 Boundedness of the eigenvalues

We prove the boundedness of the eigenvalues $E_{n,a}(\beta)$ for bounded parameters (n, a, β). In particular, $n = 0, 1, ...$ is fixed, $a \in [0, a_n]$, where $a_n = (1/b'_n)^{2/5} > 0$, $\beta \in$ C, $|\beta| = 1$, $|\arg(\beta) = \theta| \leq \pi$.
It is better to use the following scaling:

$$E_{n,\alpha}(1) = \exp(-i\theta/5)E_{n,a}(\beta),$$

where $\alpha = a\exp(-i2\theta/5)$.
Thus, we should prove the boundedness of $E_{n,\alpha}(1)$ for $|\alpha| \in [0, a_n]$, and $|\arg(\alpha)| \leq 2\pi/5$ for $\alpha \neq 0$.
For our non self-adjoint operators, we use an argument slightly different from the one of the reference [1]. We directly use the semiclassical quantization and the stability of the nodes.
Theorem 4.
For any $n = 0, 1, ..,$ $\alpha_0 = 0$, or $\alpha_0 \neq 0$ with $|\arg(\alpha_0)| \leq 2\pi/5$, $E_{n,\alpha}(1)$ is bounded and continuous at $\alpha = \alpha_0$.
Proof.
Let us consider the three parameter operators,

$$H(\hbar, \alpha, \beta) = -\hbar^2 p^2 + \alpha x^2 + i\sqrt{\beta}x^3,$$

and the eigenvalues $E_n(\hbar,\alpha,\beta)$ for $n=0,1,....$ We have the spectral equivalence for real scaling:

$$H(1,\alpha,1) \sim H(\lambda^{-2},\lambda^2\alpha,\lambda^3),$$

so that

$$E_n(1,\alpha,1) = E_n(\lambda^{-2},\lambda^2\alpha,\lambda^3),$$

for $n=0,1,..\ \lambda\neq 0$.
Because of the analyticity of the family of operators $H_\alpha(1) = H(1,\alpha,1)$, boundedness implies continuity.
We prove the boundedness by absurd.
Let us fix $n=0,1,..,$ and α_0, $0\leq a_0 = |\alpha_0| \leq a_n$, $|\arg(\alpha_0)|\leq 2\pi/5$ for $a_0<0$, and suppose $|E_n(1,\alpha,1)|\to\infty$ as $\alpha\to\alpha_0$. For α near α_0, we scale the Hamiltonian and use the identity:

$$\lambda^{6/5}E_n(1,\alpha,1) = E_n(\hbar,\alpha',1) := \epsilon = \exp(i\theta),$$

where $\epsilon=\epsilon(\alpha)$, $\lambda=\hbar=|E_n(1,\alpha,1)|^{-5/6}>0$, $\alpha'=\lambda^{2/5}\alpha$ and $|\theta|\leq\pi/2$. We set $\epsilon_0=\epsilon(\alpha_0)=E_n(\hbar,0,1)\neq 0$.
Thus, we study the semiclassical eigenvalue problem $H(\hbar,\alpha',1)\psi_n=\epsilon\psi_n$, by the Bohr-Sommerfeld quantization rule:

$$J(\alpha) = i\int_\gamma \sqrt{V(z)-\epsilon}\,dz = \pi(2n+1)\hbar + O((\hbar)^2), \tag{19}$$

where the phase of $\sqrt{V(z)-\epsilon}$ vanishes as $|z|\to\infty$, with $\arg(z)=-\pi/6$.
Thus, the n nodes are confined on the fixed compact domain Ω bounded by the regular curve $\partial\Omega=\gamma_1\cup\gamma_2$, where γ_1 is the arc of circle $\gamma_1=\{|z|=R; |\arg(iz)|\leq(\pi/2)+\theta'\}$, (see Theorems 3-4) and γ_2 the segment of extrems $(-R\exp(i\theta'), R\exp(i\theta'))$, where $\theta'=\arg(\alpha)/4$ (see Theorem 2). The limit $\alpha\to\alpha_0$ implies $\epsilon\to\epsilon_0$, $\alpha',\hbar\to 0$, and, for the classical action,

$$J(\alpha)\to J(\alpha_0) = i\int_\gamma\sqrt{iz^3-\epsilon_0}\,dz = \epsilon_0^{5/6} i\int_\gamma\sqrt{iy^3-1}\,dy =$$

$$= \epsilon_0^{5/6}\int_\gamma\sqrt{1-iy^3}\,dy = \epsilon_0^{5/6}2\Re(2\exp(-i\pi/6)\int_0^1\sqrt{1-x^3}\,dx) =$$

$$= \epsilon_0^{5/6}4\sin(\frac{\pi}{3})\int_0^1\sqrt{1-x^3}\,dx = \epsilon_0^{5/6}2\sqrt{\pi}\,\sin(\frac{\pi}{3})\frac{\Gamma(1+(1/3))}{\Gamma((1/3)+(3/2))}\neq 0, \tag{20}$$

where $y=z\epsilon_0^{-2/3}$, and where the phase of $\sqrt{iz^3-\epsilon_0}$ vanishes as $|z|\to\infty$, for $\arg(z)=-\pi/6$, and where γ, in this semiclassical approximation, has been

distorted to a regular path encircling the origin and both the turning points $z_\pm$.
As a result, for the left hand of equation (19) we have,

$$J(\alpha) \to 0, \tag{21}$$

as $\alpha \to \alpha_0$, $\hbar \to 0$, in contradiction with the limit of the left hand of equation (19), as written in equation (20). The proof is similar for $\alpha_0 = 0$.
Let us notice that the same analysis gives the correct semiclassical behavior of the eigenvalues [6], [5], for large n. From the equations (20) and (19), we have,

$$\epsilon_0^{5/6} 2\sqrt{\pi}\,\sin(\frac{\pi}{3})\frac{\Gamma(1+(1/3))}{\Gamma((1/3)+(3/2))} \sim \pi(2n+1)\hbar,$$

where,

$$\epsilon_0 = E_n(\hbar, 0, 1) \to (\frac{\Gamma[(3/2)+(1/3)]\sqrt{\pi}\,A}{\sin(\pi/3)\Gamma[1+(1/3)]})^{6/5},$$

as $n \to \infty$, $n\hbar \to A > 0$ [6].

8 The power law behavior at infinity

We prove here the algebraic behavior of the eigenvalues for large parameter. We use the scaling formula:

$$\sqrt{\alpha}E_{n,1}(\beta) = E_{n,\alpha}(1)$$

for $n \in \mathrm{N}$, where $\alpha = \beta^{-2/5}$. Let us recall that Theorem 4, in the special case of $\alpha_0 = 0$, implies continuity and boundedness of each eigenvalue $E_{n,\alpha}(1)$ in the limit $\alpha \to 0$.
The analyticity, of type A, of the family of operators $H_\alpha(1)$ (see [2] Theorem 2.10), with the control of the nodes, and the simplicity of the spectrum, imply the stability at $\alpha = 0$ and the $\alpha-$analyticity in a neighborhood of the origin of each eigenvalue $E_{n,\alpha}(1)$.
Therefore, if $\alpha = \beta^{-2/5}$, $\sqrt{\alpha}E_{n,1}(\beta) = E_{n,\alpha}(1) \to E_{n,0}(1)$ for $\beta \to \infty$. Thus, for $|\beta| = b$ large, $E_{n,\alpha}(\beta)$ grows as $b^{1/5}$, and has an algebraic singularity there:

$$E_{n,1}(\beta) = \beta^{1/5}E_{n,\beta^{-2/5}}(1) \sim \beta^{1/5}E_{n,0}(1).$$

Let us notice that we have $\arg(E_{n,1}(b\exp(\pm i\pi))) \to \pm\pi/5$, and

$$\pm b^{-1/5}\Im(E_{n,1}(b\exp(\pm i\pi))) \to E_{n,0}(1)\sin(\pi/5) > 0,$$

as $b \to \infty$.

9 Global analyticity, symmetry, and Padé summability on the cut plane

Let $E(\beta) = E_{n,1}(\beta)$, $n = 0, 1, 2, ...,$

$$f(\beta) = \frac{E(\beta) - E(0)}{\beta},$$

$f(\beta)$ is bounded holomorphic on the completed cut complex plane $\bar{C}_c = \{\beta \in \mathrm{C}; \beta \neq 0, |\arg(\beta) = \theta| \leq \pi\}$, (see Theorems 1-2-3'-4). Moreover, we have the symmetry of the eigenvalues: $E_{n,1}(\beta) = \bar{E}_{n,1}(\bar{\beta})$, so that we have, $f(\beta) = \bar{f}(\bar{\beta})$. For the Cauchy theorem, we have,

$$f(\beta) = \frac{1}{2i\pi} \oint_\gamma \frac{f(z)}{z - \beta} = \frac{1}{2i\pi} \oint_\gamma \frac{1}{1 - (\beta/z)} \frac{f(z)}{z} dz = \int_0^\infty \frac{1}{(1 + \beta x)} \rho(x) dx,$$

where γ is any curve turning around β in the positive way, we have the dispersion relation of a Stieltjes function, where

$$\rho(1/b) = -b(f(-b + i0^+) - f(-b - i0^+))/2i\pi =$$

$$= -b\Im f(-b + i0^+)/\pi = \Im E_n(-b + i0^+)/\pi \geq 0,$$

for Lemma 1. We have the asymptotism to the formal power series:

$$f(\beta) \sim \Sigma(\beta) = \sum_{j=0}^{\infty} a_j (-\beta)^j$$

for $|\beta|$ small, where the

$$a_j = |c_{j+1}| = \int_0^\infty x^j \rho(x) dx, \tag{22}$$

are the moments of the measure $\rho(x)dx$. Thus, the problem of the moments

$$a_j = |c_{j+1}| = \int_0^\infty x^j d\mu(x), \tag{23}$$

has the solution $d\mu(x) = \rho(x)dx$. Because of the bound on the perturbation coefficients $|c_j| < AC^j j!$ (see Theorem 1 and references [2], [3]), the Carleman theorem condition (see [20] page 330) is satisfied,

$$\sum_n (1/a_n)^{1/2n} = \infty,$$

and the unicity of the solution $d\mu(x) = \rho(x)dx$.
Let us recall the definition of the diagonal Padé approximants $R_n^n(\beta)$ of

the formal power series $\Sigma(\beta) = \sum_{j=0}^{\infty} a_j(-\beta)^j$, with partial sums $\Sigma^N(\beta) = \sum_{j=0}^{N-1} a_j(-\beta)^j$, $\beta \in C$. The diagonal Padé approximants, $R_n^n(\beta)$, $n \geq 0$, are the rational fractions,

$$R_n^n(\beta) = \frac{P_n(\beta)}{Q_n(\beta)},$$

where $P_j(\beta), Q_j(\beta)$, are polynomials of degree j, with the condition $Q_j(0) = 1$, defined by the asymptotic condition, $|R_n^n(\beta) - \Sigma^{2n+1}(\beta)| = O(\beta^{2n+1})$, for $|\beta| \to 0$.
As a general result, the Padé approximants $R_n^n(\beta)$ on Stieltjes asymptotic expansions, don't have poles or zeros on the complex cut plane, and there converge

$$R_n^n(\beta) \to f_\mu(\beta) = \int_0^\infty \frac{1}{(1+\beta x)} d\mu(x),$$

where $d\mu$ is a measure solution of the moment problem (23). In this case, necessarily we have $d\mu(x) = \rho(x)dx$ and $f_\mu(\beta) = f(\beta)$.
Thus, we have the result:

Theorem 5.
The function

$$f(\beta) = \frac{E(\beta) - E(0)}{\beta},$$

is a Stieltjes function,

$$f(\beta) = \int_0^\infty \frac{1}{(1+\beta x)} \rho(x) dx, \tag{24}$$

for β on the cut complex plane, where

$$\rho(1/b) = \Im(E_n(-b + i0^+))/\pi > 0, \tag{25}$$

and,

$$\ln(\rho(x)) = -C^{-1}x(1 + O(\ln(x)/x)), \tag{26}$$

where $C^{-1} = 8/15 = 2B(2, 3/2) = 2\int_0^1 x\sqrt{1-x}dx$, [16] for large positive x. The diagonal Padé approximants of the perturbation series, converge to f,

$$R_n^n(\beta) \to f(\beta),$$

as $n \to \infty$, uniformly for β on compacts of the cut complex plane.

Proof.

The inequality (25) is proved by the $\mathcal{PT}$ symmetry of the eigenfunctions and eigenvalues, $E_n(-b+i0^+) - E_n(-b-i0^+) = 2i\Im(E_n(-b+i0^+))$ and by Lemma 2. We only have to discuss the asymptotic behavior of the discontinuity function.
For the semiclassical behavior of the discontinuity (26), we consider the semiclassical scaling, where $b > 0$ plays the role of semiclassical parameter, with the anti-Gamow-Siegert condition at $+\infty$:

$$H(b,1,\exp(i\pi)) \sim bH(1,1,b\exp(i\pi)).$$

In the case of the semiclassical operator $H(b,1,\exp(i\pi))$, we have a "double well problem", with the barrier width $C^{-1} = 8/15 = 2\int_0^1 x\sqrt{1-x}dx$, and $\hbar = b$. This value of the barrier, implies the behavior of $\rho(x)$, as $x \to \infty$, as given in (26), and the behavior of the perturbation coefficients $c_j = (-1)^j a_{j-1}$, $a_j = \int_0^\infty x^j d\rho(x)$, $a_j \sim DC^j j!$, as $j \to \infty$, for some $D > 0$, compatible with the behavior:

$$c_j = (-1)^{j-1} 4\sqrt{15} C^j (2\pi)^{-3/2} \Gamma(j+1/2)(1+O(1/j)),$$

for large j, obtained numerically [8] in the case of n=0.

Remark 4.
We have proved, in a new way, that the eigenvalue $E(\beta) = E_{n,1}(\beta)$, $n = 0,1,2,...$, is real and positive for positive β,

$$E(\beta) = E(0) + \beta f(\beta) = E(0) + \beta \int_0^\infty \frac{1}{(1+\beta x)} \rho(x) dx \geq E(0),$$

and $E_{n,1}(\beta) \sim \beta^{1/5} E_{n,0}(1)$ for large positive β.

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