# Results on Convergence in Norm of Exponential Product Formulas and Pointwise of the Corresponding Integral Kernels

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**Abstract.** For the last one and a half decades it has been known that the exponential product formula holds also *in norm* in nontrivial cases. In this note, we review the results on its convergence in norm as well as pointwise of the integral kernels in the case for Schrödinger operators, with error bounds. Optimality of the error bounds is elaborated.

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## 1. Introduction

The Trotter product formula, Trotter–Kato product formula or exponential product formula is usually a product formula which in strong operator topology approximates the group/semigroup with generator being a sum of two operators. It is often a useful tool to study Schrödinger evolution groups/semigroups in quantum mechanics and to study Gibbs semigroups in statistical mechanics.

To think of a typical case, let $A$ and $B$ be selfadjoint operators in a Hilbert space $\mathcal{H}$ with domains $D[A]$ and $D[B]$ and $H := A + B$ their operator sum with domain $D[H] = D[A] \cap D[B]$. Assume that $H$ is selfadjoint or essentially selfadjoint on $D[H]$ and denote its closure by the same $H$. Then Trotter [44] proved the unitary product formula

$$[e^{-itB/2n}e^{-itA/n}e^{-itB/2n}]^n - e^{-itH} \to 0, \quad \text{strongly},$$
$$[e^{-itA/n}e^{-itB/n}]^n - e^{-itH} \to 0, \quad \text{strongly}, \quad n \to \infty,$$

and also, when $A$ and $B$ are nonnegative, the selfadjoint product formula

$$[e^{-tB/2n}e^{-tA/n}e^{-tB/2n}]^n - e^{-tH} \to 0, \quad \text{strongly},$$
$$[e^{-tA/n}e^{-tB/n}]^n - e^{-tH} \to 0, \quad \text{strongly}. \quad n \to \infty,$$

The convergence is *locally uniform*, i.e. uniform on compact $t$-intervals, respectively in the real line $\mathbf{R}$ and in the closed half line $[0,\infty)$. Kato [29] dicovered the latter selfadjoint product formula to hold also for the form sum $H := A\dot{+}B$ with form domain $D[H^{1/2}] = D[A^{1/2}] \cap D[B^{1/2}]$, which we assume for simplicity is dense in $\mathcal{H}$. However, it remains to be an open problem whether the unitary product formula for the form sum holds.

However, since around 1993 we have begun to know that selfadjoint product formulas converge even in (*operator*) *norm*, though in some special cases, by the following two first results. Rogava [37] proved, when $B$ is $A$-bounded and $H = A + B$ is selfadjoint, among others, the abstract product formula that

$$\|[e^{-tA/n}e^{-tB/n}]^n - e^{-tH}\| = O(n^{-1/2}\log n), \quad n \to \infty,$$

locally uniformly in $[0,\infty)$. Helffer [13] proved, when $H := -\Delta + V(x)$ is a Schrödinger operator in $L^2(\mathbf{R}^d)$ with nonnnegative potential $V(x)$ satisfying $|\partial_x^\alpha V(x)| \le C_\alpha$ ($|\alpha| \ge 2$) so that $H$ is selfadjoint on the domain $D[-\Delta]\cap D[V]$, the symmetric product formula that

$$\|[e^{-tV/2n}e^{-t(-\Delta)/n}e^{-tV/2n}]^n - e^{-tH}\| = O(n^{-1}), \quad n \to \infty,$$

locally uniformly in $[0,\infty)$. Many works were done to extend these results before 2000, e.g. in [5, 20, 22, 32, 33, 35] for the abstract product formula, [9, 10, 17, 18, 19, 41] for the Schrödinger operators, and after that, e.g. in [23, 27, 16], [3, 4, 6, 7] for the abstract product formula. In most of them, use was made of operator-theoretic methods, though of a probabilistic method in [17, 18, 19, 41].

In this note, we want to describe more recent results on convergence in norm for exponential product formulas and also pointwise of the corresponding integral kernels, mainly based on our works since around 2000, [23, 27, 24, 25, 26]. As for the error bounds, although it is easy to see by the Baker–Campbell–Hausdorff formula (e.g. [45], [40]) that with both operators $A$ and $B$ being bounded, the *nonsymmetric* product formula has an optimal error bound $O(n^{-1})$ while the *symmetric* one does $O(n^{-2})$, it was shown in [27] that even the *symmetric* product formula has an optimal error bound $O(n^{-1})$ in general, if both $A$ and $B$ are unbounded. However, in [25] (cf. [26]), a better upper sharp error bound $O(n^{-2})$ has been obtained for the *symmetric* product formula with the Schrödinger operator $-\Delta + V(x)$ having nonnegative potentials $V(x)$ growing polynomially at infinity, in spite that both $-\Delta$ and $V$ are unbounded operators. In this note we mention, with a sketch of proof, a latest complementary result [2] which settles the sharp optimal error bound is in fact $O(n^{-2})$ with the symmetric product formula for the harmonic oscillator, by estimating the error not only from above but also from below, in norm as well as pointwise.

Theorems are described in Section 2. Optimality of error bounds is discussed separately in Section 3. The idea of proof is briefly mentioned in Section 4. In Section 5 we give concluding remarks, and also refer to a connection of the exponential product formula with the Feynman path integral.

It should be also noted that in almost the same context with the notion of norm ideals (e.g. [12], [38]) we are able to deal with the trace norm convergence as in [46, 30, 31, 34, 14, 21, 42]. For an extensive literature on this we refer to [47].

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## 2. Theorems

We begin with our result which extends ultimately Rogava and Helffer's.

**Theorem 2.1.** *(Ichinose-Tamura-Tamura-Zagrebnov 2001*[23, 27]*) Let $A$ and $B$ be nonnegative selfadjoint operators, and assume $H = A+B$ is selfadjoint on $D[H] = D[A] \cap D[B]$. Then as $n \to \infty$,*

$$\|[e^{-tB/2n}e^{-tA/n}e^{-tB/2n}]^n - e^{-tH}\| = O(n^{-1}), \tag{2.1}$$

$$\|[e^{-tA/n}e^{-tB/n}]^n - e^{-tH}\| = O(n^{-1}). \tag{2.2}$$

*The convergence is locally uniform in the closed half line $[0, \infty)$, while on the whole half line $[0, \infty)$, if $H$ is strictly positive, i.e. $H \geq \eta I$ for some $\eta > 0$. The error bound $O(n^{-1})$ in (*2.1*) and (*2.1*) is optimal.*

We can go beyond this result. First, focussing on the Schrödinger operator $-\Delta + V(x)$, we ask whether norm convergence implies pointwise convergence of integral kernels. The answer is yes, though strong convergence does not. This problem is discussed for Schrödinger operators with potentials of polynomial growth (Theorem 2.2), with positive Coulomb potential (Theorem 2.3), and also for the Dirichlet Laplacian (Theorem 2.4). Pointwise convergence of integral kernels for Schrödinger semigroups is important, because it gives a time-sliced approximation to the imaginary-time Feynman path integral.

Next, we ask, for the unitary exponential/Trotter product formula, whether there are nontrivial cases where it converges in norm, though it does not in general hold (see [15]). The answer is yes. In fact, it holds for the Dirac operator and relativistic Schrödinger operator (Theorem 2.5).

Let $H = H_0 + V := -\Delta + V(x)$ with $V(x)$ a real-valued function. By $K^{(n)}(t, x, y)$ we denote the integral kernel of $[e^{-tH_0/2n}e^{-tV/n}e^{-tH_0/2n}]^n$, and by $e^{-tH}(x, y)$ that of $e^{-tH}$.

**Theorem 2.2.** *(Ichinose-Tamura 2004* [25]*) (positive potential of polynomial growth) Assume that* $V(x)$ *is in* $C^\infty(\mathbf{R}^d)$*, bounded below and satisfies* $|\partial_x^\alpha V(x)| \le C_\alpha \langle x\rangle^{m-\delta|\alpha|}$ *with some* $0<\delta\le 1$ *(*$\langle x\rangle = (1+x^2)^{1/2}$*).*
*(i) (In norm)*

$$\|[e^{-tH_0/2n}e^{-tV/n}e^{-tH_0/2n}]^n - e^{-tH}\|_{L^2} = O(n^{-2}), \tag{2.3}$$

*locally uniformly in the open half line* $(0,\infty)$*.*

*(ii) (Integral kernel)*

$$\begin{aligned}&[K^{(n)}(t,x,y) - e^{-tH}(x,y)] = O(n^{-2}),\\ &\text{in } C^\infty(\mathbf{R}^d\times\mathbf{R}^d)\text{-topology, locally uniformly in } (0,\infty),\end{aligned} \tag{2.4}$$

*i.e. together with all* $x$*,* $y$*-derivatives.*

This theorem improves the result of Takanobu [41], who used a probabilistic method with the Feynman–Kac formula (see Sect. 5) to show uniform pointwise convergence of the integral kernels, roughly speaking, with error bound $O(n^{-\rho/2})$, if $V(x)$ satisfies $V(x)\ge C(1+|x|^2)^{\rho/2}$ and $|\partial_x^\alpha V(x)| \le C_\alpha(1+|x|^2)^{(\rho-\delta|\alpha|)_+/2}$ for some constants $C$, $C_\alpha \ge 0$ and $\rho\ge 0$, $0<\delta\le 1$. The claim of Theorem 2.2 is a little bit sharpened in Theorems 3.1 and 3.2, in the next section, in the case of the harmonic oscillator.

**Theorem 2.3.** *(Ichinose-Tamura 2006* [26]*) (positive Coulomb potential) Let* $H := -\Delta + V(x)$ *with* $V(x)\ge 0$*. Assume that* $V(-\Delta+1)^{-\alpha}$*:* $L^2(\mathbf{R}^d)\to L^2(\mathbf{R}^d)$ *is bounded for some* $0<\alpha<1$*, and that* $V\in C^\infty$ *near a neighbourhood* $U$ *of both* $p$ *and* $q$ *(after* $p, q \in \mathbf{R}^d$ *taken). Then*

$$\begin{aligned}&[K(t/n)^n(x,y) - e^{-tH}(x,y)] = O(n^{-1}),\\ &\text{in } C^\infty(U)\text{-topology, locally uniformly in } (0,\infty).\end{aligned} \tag{2.5}$$

The condition is satisfied if $V$ is in $L^2(\mathbf{R}^3)+L^\infty(\mathbf{R}^3)$, in particular, if $V$ is the positive Coulomb potential $1/|x|$. We don't know what happens at the singularities of $V(x)$.

**Theorem 2.4.** *(Ichinose-Tamura 2006* [26]*) (Dirichlet Laplacian) Let* $\Omega\subset\mathbf{R}^d$ *be a bounded domain with smooth boundary and* $\chi_\Omega$ *the indicator function of* $\Omega$*. Let* $H_0 := -\Delta$ *in* $L^2(\mathbf{R}^d)$*, and* $H := -\Delta_D$ *the Dirichlet Laplacian in* $\Omega$ *with domain* $D[H] = H^2(\Omega)\cap H_0^1(\Omega)$*. Then for* $0<\sigma<\frac{1}{6}$*,*

$$\begin{aligned}&\left(\chi_\Omega\, e^{-tH_0/n}\,\chi_\Omega\right)^n(x,y) - e^{-tH}(x,y) = O(n^{-\sigma}),\\ &\text{locally uniformly in } (t,x,y)\in(0,\infty)\times\Omega\times\Omega.\end{aligned} \tag{2.6}$$

We don't know what happens when $x$ or $y$ approaches the boundary of $\Omega$.

**Corollary.**

$$\|[\chi_\Omega\, e^{-tH_0/n}\,\chi_\Omega]^n f - e^{-tH}f\|_{L^2} \to 0, \quad f\in L^2(\Omega).$$

Consequently, Theorem 2.4 is a *stronger* statement than this corollary, though the latter is also obtained by Kato [29] as an abstract result: If $A$ is a nonnegative selfadjoint operator and $P$ an orthogonal projection in a Hilbert space $\mathcal{H}$, then $(Pe^{-tA/n}P)^n \to e^{-tA_P}$, *strongly*, as $n \to \infty$, where $A_P := (A^{1/2}P)^*(A^{1/2}P)$. In passing, however, it is an open question whether it holds that $(Pe^{-itA/n}P)^n \to e^{-itA_P}P$, strongly (*Zeno product formula*). A partial answer was given in [11].

All Theorems 2.2–2.4 hold with order of products exchanged, e.g. in Theorem 2.2, $[e^{-tV/2n}e^{-tH_0/n}e^{-tV/2n}]^n$ instead of $[e^{-tH_0/2n}e^{-tV/n}e^{-tH_0/2n}]^n$.

**Theorem 2.5.** *(Ichinose-Tamura 2004* [24]*)(Unitary Trotter in norm) Let $A$ and $B$ be selfadjoint, and assume $H := A + B$ to be essentially selfadjoint in a Hilbert space $\mathcal{H}$. Assume that there exists a dense subspace $\mathcal{D}$ of $\mathcal{H}$ with $\mathcal{D} \subset D[A] \cap D[B]$ such that $e^{-itA}$, $e^{-itB} : \mathcal{D} \to \mathcal{D}$. Further assume that the commutators $[A, B]$, $[A, [A, B]]$ and $[B, [A, B]]$ are bounded on $\mathcal{H}$. Then*

$$\|\big(e^{-itB/2n}e^{-itA/n}e^{-itB/2n}\big)^n - e^{-itH}\| = O(n^{-2}), \quad n \to \infty, \tag{2.7}$$

*locally uniformly in the real line* $\mathbf{R}$.

As important applications we have ones to the Dirac operator $H = H_0 + V = (i\alpha \cdot \nabla + m\beta) + V(x)$ in $L^2(\mathbf{R}^3)^4$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta$ are the 4 Dirac matrices, with $\partial^\gamma V(x)(|\gamma| = 2)$ being bounded, as well as to the relativistic Schrödinger operator $H = H_0 + V = \sqrt{-\Delta + m^2} + V(x)$ on $L^2(\mathbf{R}^d)$ with $\partial_x^\gamma V(x)$ being bounded for $1 \leq |\gamma| \leq 4$ ($0 \leq |\gamma| \leq 4$, if $m = 0$). In these cases, $H$ are essentially selfadjoint, and satisfy the conditions in the theorem. So it holds that

$$\|[e^{-itV/2n}e^{-itH_0/n}e^{-itV/2n}]^n - e^{-itH}\|_{L^2} = O(n^{-2}), \quad n \to \infty, \tag{2.8}$$

locally uniformly in $\mathbf{R}$.

However, this theorem does not apply to Schrödinger operators except for the *Stark Hamiltonian* $(-\Delta + V(x)) + a \cdot x$ in $L^2(\mathbf{R}^d)$, where $a$ is a constant real vector in $\mathbf{R}^d$.

Finally it should be noted that we have shown in Theorems 2.2–2.4 that the convergence is unifrom only on compact $t$-intervals which are away from 0, though in Theorems 2.1 and 2.5, on ones which are allowed to be not away from 0.

## 3. Optimalty of Error Bounds

In this section we discuss optimality of error bounds. The error bound $O(1/n)$ in Theorem 2.1 is optimal, because if both $A$ and $B$ are bounded operators, by the Baker–Campbell–Hausdorff formula we know

$$[e^{-tA/n}e^{-tB/n}]^n - e^{-tH} = R'_n \cdot n^{-1},$$
$$[e^{-tB/2n}e^{-tA/n}e^{-tB/2n}]^n - e^{-tH} = R_n \cdot n^{-2},$$

for some $R'_n$ and $R_n$ being uniformly bounded operators which in general are not the zero operator. From this, optimality in the former non-symmetric case is evident. But even in the symmetric case it is optimal. Indeed, there exist unbounded nonnegative selfadjoint operators $A$, $B$ such that $H = A + B$ is selfadjoint and

$$\|[e^{-tB/2n}e^{-tA/n}e^{-tB/2n}]^n - e^{-tH}\| \geq c(t)n^{-1}$$

for some continuous function $c(t)$ with $c(t) > 0,\ t > 0$ and $c(0) = 0$ ([27]).

However, further in some special symmetric case in Theorem 2.2 where $-\Delta$, $V$ are taken as $A$, $B$, we have seen the symmetric product formula hold with a *sharp* error bound $O(n^{-2})$. We can make more precise this result with the 1-dimensional harmonic oscillator $H := H_0 + V := \frac{1}{2}(-\partial_x^2 + x^2)$ in $L^2(\mathbf{R})$.

**Theorem 3.1.** *(Azuma-Ichinose 2007* [2]*) There exists bounded continuous functions $C(t) \geq 0$ and $c(t) \geq 0$ in $t \geq 0$, which are positive except $t = 0$ with $C(0) = c(0) = 0$, independent of $n$, such that for $n = 1, 2, \ldots,$*

$$c(t)n^{-2} \leq \|[e^{-\frac{t}{2n}V}e^{-\frac{t}{n}H_0}e^{-\frac{t}{2n}V}]^n - e^{-tH}\| \leq C(t)n^{-2}, \qquad t \geq 0\,. \tag{3.1}$$

This theorem mentions an error bound from below, extending the harmonic oscillator case of Theorem 2.2 which treats only the right-half inequality with $C(t) = C$ being a positive constant depending on each compact $t$-interval in the *open* half line $(0, \infty)$.

It is anticipated that the same is true for the Schrödinger operator $H = -\Delta + V(x)$ with growing potentials like $V(x) = |x|^{2m}$ treated in Theorem 2.2.

Theorem 3.1 is obtained as a corollary from the following theorem of its integral kernel version. Here one calculates explicitly the integral kernel $K^{(n)}(t, x, y)$ of $[e^{-tV/2n}e^{-tH_0/n}e^{-tV/2n}]^n$ to estimate its difference from the integral kernel $e^{-tH}(x, y)$ of $e^{-tH}$.

**Theorem 3.2.** *(Azuma-Ichinose 2007* [2]*) There exists a bounded operator $R(t)$ and uniformly bounded operators $\{Q^{(n)}(t)\}_{n=1}^\infty$ with integral kernels $R(t, x, y)$ and $Q^{(n)}(t, x, y)$ being uniformy bounded continuous functions in $(0, \infty) \times \mathbf{R} \times \mathbf{R}$ such that*

$$K^{(n)}(t, x, y) - e^{-tH}(x, y) = \big[R(t, x, y) + Q^{(n)}(t, x, y)n^{-1}\big]n^{-2}; \tag{3.2}$$

*they satisfy*

$$\sup_{x,y} |R(t, x, y)|,\ \sup_n \sup_{x,y} |Q^{(n)}(t, x, y)| \to 0,\ t \to 0\ ; \quad \sup_{x,y} |R(t, x, y)| \to 0,\ t \to \infty.$$

*$R(t, x, y)$ is explicitly given by*

$$\begin{aligned} R(t, x, y) \;&=\; e^{-tH}(x, y)\frac{t^2}{12}\Big[t\Big(\frac{1}{4}\frac{e^t + e^{-t}}{e^t - e^{-t}} + \frac{(e^t + e^{-t})xy - (x^2 + y^2)}{(e^t - e^{-t})^2}\Big) \\ &\qquad + \frac{1}{16}\Big(1 + \frac{4xy - (e^t + e^{-t})(x^2 + y^2)}{e^t - e^{-t}}\Big)\Big]. \end{aligned} \tag{3.3}$$

*If $t > 0$, $R(t, x, y)$ can become positive and negative.*

**Lemma 3.3.** $K^{(n)}(t,x,y) =$

$$\frac{1}{\sqrt{\pi}}\left(\frac{\sqrt{1+\frac{t^2}{4n^2}}}{\left(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n-\left(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n}\right)^{1/2}$$

$$\times\exp\left[\frac{2\sqrt{1+\frac{t^2}{4n^2}}}{\left(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n-\left(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n}xy\right]$$

$$\times\exp\Bigg\{\Bigg[-\frac{t}{4n}-\frac{n}{2t}\Bigg(1-$$

$$\frac{\left(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^{n-1}-\left(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^{n-1}}{\left(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n-\left(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n}\Bigg)\Bigg](x^2+y^2)\Bigg\}. \tag{3.4}$$

*Proof.* Calculate the Gaussian integral

$$\begin{aligned}&K^{(n)}(t,x,y)\\ \equiv\ &\left(\frac{n}{2\pi t}\right)^{\frac{n}{2}}\overbrace{\int_{\mathbf{R}}\cdots\int_{\mathbf{R}}}^{n-1}\prod_{j=1}^{n}\left[e^{-\frac{t}{4n}x_j^2}e^{-\frac{(x_j-x_{j-1})^2}{2t/n}}e^{-\frac{t}{4n}x_{j-1}^2}\right]dx_1\cdots dx_{n-1},\end{aligned}$$

where $x=x_n,\ y=x_0$. We shall encounter with continued fraction to lead to the final expression (3.4) of the lemma.

To show Theorem 3.2, we simply calculate the difference $K^{(n)}(t,x,y)-e^{-tH}(x,y)$, though it is not so simple.

Here we mention what the operator with $R(t,x,y)$ as its integral kernel is. By the Baker–Campbell–Hausdorff formula (e.g. [45], [40]), if $A$ and $B$ are bounded operators, we have

$$\begin{aligned}&[e^{-tB/2n}e^{-tA/n}e^{-tB/2n}]^n-e^{-t(A+B)}\\ =\ &\exp\big(-t(A+B)-n^{-2}\frac{t^2}{24}[2A+B,[A,B]]-O_p(n^{-3})\big)\\ =\ &e^{-t(A+B)}-n^{-2}\frac{t^2}{24}\int_0^t e^{-(t-s)(A+B)}[2A+B,[A,B]]e^{-s(A+B)}ds+O_p(n^{-3}),\end{aligned}$$

where $O_p(n^{-3})$ is an operator with norm of $O(n^{-3})$. In our case where $A=-\frac{1}{2}\partial_x^2,\ B=\frac{1}{2}x^2$, we can show $R(t,x,y)$ is just the integral kernel of the operator

$$-\frac{t^2}{24}\int_0^t e^{-(t-s)H}[2H_0+V,[H_0,V]]e^{-sH}ds,$$

which *does* make sense, though $H_0$ and $V$ are unbounded operators. We have $[2H_0 + V, [H_0, V]] = -4H_0 + 2V = -4H + 6V$.

## 4. Idea of Proof

Put $K(\tau) = e^{-\tau B/2}e^{-\tau A}e^{-\tau B/2}$. Note that $0 \le K(\tau) \le 1$. Then we need to estimate the difference between $K(t/n)^n$ and $e^{-tH}$. The general technique of proof is: (i) to establish an appropriate version of Chernoff's theorem ([8]):

$$\big[(1+\tau^{-1}(1-K(\tau))^{-1} - (1+H)^{-1}\big] \to 0, \ \ \tau \downarrow 0$$
$$\Longrightarrow \quad [K(t/n)^n - e^{-tH}] \to 0, \ \ n \to \infty$$

and/or (ii) to do telescoping:

$$e^{-tH} - K(t/n)^n = \sum_{k=1}^{n} e^{-(k-1)tH/n}(e^{-tH/n} - K(t/n))K(t/n)^{n-k}$$

to estimate each summand on the right. The former method (i) seems to be more efficient than the latter (ii).

In fact, to prove Theorem 1, we use the former method, establishing the following norm version of Chernoff's theorem with error bounds. The case without error bounds was noted by Neidhardt–Zagrebnov [33].

**Lemma 4.1.** *(Ichinose-Tamura* [23]*) I. Let* $\{F(t)\}_{t\ge 0}$ *be a family of selfadjoint operators with* $0 \le F(t) \le 1$*, and* $H \ge 0$ *a selfadjoint operator in a Hilbert space* $\mathcal{H}$*. Define* $S_t := t^{-1}(1 - F(t))$*. Then*
(a) For $0 < \alpha \le 1$, $\|(1+S_t)^{-1} - (1+H)^{-1}\| = O(t^{\alpha})$, $\quad t \downarrow 0$
*implies*
(b) For every fixed $\delta > 0$,
$\|F(t/n)^n - e^{-tH}\| = \delta^{-2}t^{-1+\alpha}e^{\delta t}O(n^{-\alpha})$, $\ n \to \infty$, $\ t > 0$.

*Therefore for* $\alpha = 1$ *this convergence is uniform on each compact interval* $[0, L]$ *in the closed half line* $[0, \infty)$*.*

*II. Moreover, in case* $H \ge \eta I$ *for some constant* $\eta > 0$*, if for every* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that* $F(t) \le 1 - \delta(\varepsilon)$ *for all* $t \ge \varepsilon$*, then*

$$\|F(t/n)^n - e^{-tH}\| = (1+2/\eta)^2 t^{-1+\alpha}O(n^{-\alpha}), \quad n \to \infty, \ \ t > 0.$$

*Therefore for* $\alpha = 1$ *this convergence is uniform on the whole closed half line* $[0, \infty)$*.*

Condition II is satisfied, e.g. for $F(\tau) = e^{-\tau B/2}e^{-\tau A}e^{-\tau B/2}$. For the proof, we refer to [23].

For the proof of Theorems 2.2–2.5 we employ the latter method (ii), and further, for Theorems 2.2–2.4, make a crucial use of Agmon's kernel theorem:

**Lemma 4.2.** *(Agmon's kernel theorem* [1]*)*

*Let $T : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ be a bounded operator with ranges of $T$ and its adjoint $T^*$ satisfying $R[T],\ R[T^*] \subset H^m(\mathbf{R}^d),\ m > d$. If $T$ is an integral operator with integral kernel $T(x,y)$ being a bounded continuous function in $\mathbf{R}^d \times \mathbf{R}^d$ such that*

$$(Tf)(x) = \int T(x,y)f(y)\,dy, \qquad f \in L^2,$$

*then*

$$|T(x,y)| \leq C(\|T\|_m + \|T^*\|_m)^{\frac{d}{m}} \|T\|^{1-\frac{d}{m}},$$

*where $\|T\|_m := \|T\|_{\mathcal{L}(L^2 \to H^m)}$ is the operator norm of $T$ as a bounded operator of $L^2(\mathbf{R}^d)$ into the Sobolev space $H^m(\mathbf{R}^d)$.*

Indeed, we estimate the $\mathcal{L}(L^2 \to H^m)$-operator norm of the difference $T = [e^{-tV/2n}e^{t\Delta/n}e^{-tV/2n}]^n - e^{t(-\Delta+V)}$.

## 5. Concluding Remarks

We have so far considered the case where the *operator sum* $H = A + B$ of two nonnegative selfadjoint operators $A$ and $B$ is selfadjoint. However, otherwise, the exponential product formula in norm does not in general hold for the *form sum* $H = A + B$ of two selfadjoint operators $A \geq 0,\ B \geq 0$, even if it is essentially selfadjoint on $D[A] \cap D[B]$ (see [43]). Nevertheless, there is some case where it holds:

**Theorem 5.1.** *(Ichinose-Neidhardt-Zagrebnov 2004* [16]*) Let $H = A \dot{+} B$ be the form sum of $A$ and $B$. If $D[H^\alpha] \subseteq D[A^\alpha] \cap D[B^\alpha]$ for some $\frac{1}{2} < \alpha < 1$, and $D[A^{\frac{1}{2}}] \subseteq D[B^{\frac{1}{2}}]$, then*

$$\|[e^{-tB/2n}e^{-tA/n}e^{-tB/2n}]^n - e^{-tH}\| = O(n^{-(2\alpha-1)}), \tag{5.1}$$

$$\|[e^{-tA/n}e^{-tB/n}]^n - e^{-tH}\| = O(n^{-(2\alpha-1)}), \tag{5.2}$$

*locally uniformly in* $[0,\infty)$.

This error bound in (5.1)/(5.2) is also optimal. For this we refer to [43]. The condition for the domains of $A$ and $B$ is not symmetric. It is an open question whether one may improve it so as to become symmetric with respect to $A$ and $B$.

Finally, as we should like to mention, there is a very nice Feynman path integral formula which represents the Schrödinger semigroup, called the *Feynman–Kac formula*

$$\begin{aligned} &(e^{-tH}f)(x) = (e^{-t(-\Delta+V)}f)(x) \\ = &\int_{B\in C([0,\infty)\to\mathbf{R}^d),B(0)=x} \exp[-\int_0^t V(B(s))ds]f(B(t))d\mu(B), \end{aligned}$$

where $\mu(\cdot)$ is the Wiener measure on the path space $C([0,\infty) \to \mathbf{R}^d)$ (e.g. ([39]). We may use this formula to get whatever results, in fact, a lot of them. This is a big advantage! But disadvantage is that it is only restricted to the Schrödinger operator or Laplacian. For instance, if we think of the semigroup for the relativistic Schrödinger operator $H = \sqrt{-\Delta + m^2} + V(x)$, we have to establish another Feynman–Kac formula (cf. [28]).

Indeed, the Feynman–Kac formula is one of the realizations of Feynman path integral as a *true integral* on a path space. However, as Nelson [36] noted, the exponential/Trotter product formula also can give a meaning to the Feynman path integral as a *time-sliced approximation* by finite-dimensional integrals (cf. [15]). What it has advantage at is that we may apply it to the sum $H = A + B$ of any two selfadjoint operators $A, B$ bounded from below.

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