Translation Invariant States on Twisted Algebras on a Lattice

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Abstract

We construct an algebra with twisted commutation relations and equip it with the shift. For appropriate irregularity of the non-local commutation relations we prove that the tracial state is the only translation-invariant state.

Keywords: Generalized Price-Powers-shift, generalized Jordan-Wigner- transformation, invariant states

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1 Introduction

Whereas the dynamics of quantum systems with finitely many degrees of freedom is fully controllable via the spectrum of the unitary operator that implements it, this is not longer true with infinitely many degrees of freedom. While these systems can always be described by operators acting on suitable Hilbert spaces, it may however happen, typically in the thermodynamic limit of interacting quantum systems, that not all of these operators represent observable quantities. In such cases, what one does is to focus not upon the dynamics, rather on its equilibrium (time-invariant) states.

A great step forward in our understanding of thermodynamics, in particular of its second law, would follow by showing that for interacting systems the class of invariant states is drastically reduced in comparison to those of quasifree evolutions, where e.g. every state in equilibrium with respect to any (space-)translation invariant dynamics is also (space-) translation invariant. For interacting systems, one hopes that invariant states might be characterized by only a few thermodynamic quantities.

Unfortunately, scarcely any control on the dynamics of realistic physical systems is available; however, some steps forward can be made following a toy model approach and constructing algebras and discrete dynamics on these algebras such that only one invariant state exists. A particular instance of invariant state is the so-called tracial, or totally mixed, state ω whose two-point correlation functions are such that $\omega(AB) = \omega(BA)$ for all A, B in the algebra of operators.

Examples of quantum dynamical systems where the tracial state is the only invariant state are the Price-Powers shifts [1, 2],[3], [4] or the irrational rotation algebras [5],[4]. In both cases, what forbids the existence of invariant states different from the tracial one is that operators in the course of time anticommute infinitely often and sufficiently irregularly with one another. Such a lack of asymptotic commutativity (abelianess) is indeed expected in real interacting quantum systems.

In the following, we give another example of such quantum dynamical systems inspired very much by the Price-Powers shifts. In this latter case, the algebra is created by self-adjoint operators $e_k = e_k^*$, $e_k^2 = 1$, $k \in \mathbb{N}$, that commute or anticommute, $e_k e_p = (-1)^{g(|p-k|)} e_p e_k$, as prescribed by a so-called *bitstream*, namely a two-valued function on the integers, $g : \mathbb{N} \mapsto \{0, 1\}$, $g(0) = 0$. In the following, we will consider Weyl-operators in place of the e_k and organize them in such a way that shifted Weyl-operators remain in Weyl-like relation with one another in a so-called *complementary* manner [6] so that any two of them create a full matrix algebra $M_{d\times d}$.

We shall show that, under the hypothesis of sufficiently irregular complementary Weyl-like relations, the tracial state is the only translationally invariant state as for the Powers-Price shifts. The paper is organized as follows:

- in chapter 2. we define the algebra and the complementary relations, together with some representations either as an AF-algebra (in some cases a UHF algebra) or as a quantum-spin chain by means of generalized Jordan-Wigner transformations;
- in chapter 3. we show that, under the assumption of sufficiently random commutation relations, only the tracial state can be translation invariant.

2 The Algebra and its Automorphism

2.1 Definitions

We start with infinitely many finite-dimensional algebras $\mathcal{A}_m$, $m \in \mathbb{N}$, all isomorphic to $d \times d$ matrix algebras, created by the operators $W^{(m)}_{\vec{k}}$, $\vec{k} \in \mathbb{Z}^2_d := \{(k_1, k_2),\ k_i = 0, 1, \ldots, d-1\}$, that satisfy the commutation relations of a discrete Weyl group

$$W^{(m)}_{\vec{k}_1} W^{(m)}_{\vec{k}_2} = e^{\frac{2\pi i \sigma(\vec{k}_1, \vec{k}_2)}{d}} W^{(m)}_{\vec{k}_2} W^{(m)}_{\vec{k}_1} \ , \tag{1}$$

with symplectic form $\sigma(\vec{k}_1, \vec{k}_2) := k_{11}k_{22} - k_{12}k_{21}$.

Note that $W^{(m)}_{\vec{0}} = \mathbb{1}$, and $W^{(m)}_{-\vec{k}} = (W^{(m)}_{\vec{k}})^{-1}$.

The relations between Weyl operators with different upper indices are *twisted* by means of a sequence of 2×2 matrices A_n, $n \in \mathbb{Z}$, with entries in $\{0, 1, \ldots, d-1\}$; explicitly,

$$\begin{aligned} W^{(p)}_{\vec{k}_p} W^{(q)}_{\vec{k}_q} &= \mathrm{e}^{2\pi i\, u_{\vec{k}_p\vec{k}_q}(q-p)}\, W^{(q)}_{\vec{k}_q} W^{(p)}_{\vec{k}_p} \ , & (2)\\ u_{\vec{k}_p\vec{k}_q}(q-p) &:= \frac{1}{d}\, \sigma(\vec{k}_p, A_{q-p}\vec{k}_q) \ . & (3) \end{aligned}$$

Setting $A_0 = \mathbb{1}$, the single-site relations (1) are a particular instance of (2).

The finite products define elements of an infinite discrete group. We denote them as

$$W_I := W^{(1)}_{\vec{k}_1} W^{(2)}_{\vec{k}_2} \cdots W^{(\ell)}_{\vec{k}_\ell} \ , \tag{4}$$

where I denotes a sequence of vectors $\{\vec{k}_m\}_{m\in\mathbb{Z}}$ with only finitely many components, $(\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_\ell)$, possibly different from the vector $\vec{0}$. We define a star operation as $(W^{(m)}_{\vec{k}})^* = (W^{(m)}_{\vec{k}})^{-1}$, and $(UV)^* = V^*U^*$ as usual. So the Weyl operators and their products are unitary elements of the C^* algebra generated by the finite products W_I, which are assumed to be linearly independent. We shall denote this algebra by $\mathcal{A}$.

We shall further equip $\mathcal{A}$ with an automorphism $\alpha : \mathcal{A} \mapsto \mathcal{A}$ such that $\alpha^n \left(W^{(0)}_{\vec{k}}\right) = W^{(n)}_{\vec{k}}$. Then, generic algebraic relations read

$$\begin{aligned} W_I \alpha^n(W_J) &= \Big(\prod_{a=1}^{n_I} W^{(a)}_{\vec{k}_a}\Big) \Big(\prod_{b=1}^{n_J} W^{(b+n)}_{\vec{k}_b}\Big) = \mathrm{e}^{2\pi i\, u_n(I;J)}\, \alpha^n(W_J)\, W_I \ , & (5)\\ u_n(I;J) &:= \frac{1}{d} \sum_{a=1}^{n_I} \sum_{b=1}^{n_J} \sigma(\vec{k}_a\, ,\, A_{b+n-a}\vec{k}_b) = \sum_{a=1}^{n_I} \sum_{b=1}^{n_J} u_{\vec{k}_a\vec{k}_b}(b+n-a) \ . & (6) \end{aligned}$$

Remarks 1

1. Every sequence of 2×2 matrices $\{A_n\}_{n\in\mathbb{N}}$ generates its own algebra $\mathcal{A}$; however, for special sequences $\{A_n\}_{n\in\mathbb{N}}$ (as will be shown in the next section), the corresponding algebras built by $W^{(m)}_{\vec{k}}, m = 1,,, l$ will be isomorphic $\forall l$ to the tensor product over l local sizes and therefore the total algebra can be considered as the same $\mathcal{A}$ equipped with the usual shift and a different automorphism α with finite speed.

2. We will refer to the Weyl operators $W^{(m)}_{\vec{k}}$ in (1) as to the letters of the algebra $\mathcal{A}$ and to the products W_I as in (4) as to the words of $\mathcal{A}$. The norm of every word equals 1 and $(W_I)^d = \mathbb{1}$, as the eigenvalues of all W_I are the pure phases $e^{\frac{2\pi i\ell}{d}}$, $0 \leq \ell \leq d-1$. Linear combinations of words will be referred to as sentences, as in the case of the Price-Powers-shift. A state $\omega : \mathcal{A} \mapsto \mathcal{C}$ over the algebra amounts to a positive, normalized functional over $\mathcal{A}$: it is thus fixed by giving its values on all words.

We proceed in studying the algebras, showing the existence of a non-trivial representation. This is not a trivial problem, see [7]. Consider the set $\mathcal{I}$ of multiindices I and define the composition law of addition: $\mathcal{I} \times \mathcal{I} \mapsto \mathcal{I}$: if $I = \{\vec{k}^I_m\}_{m\in\mathbb{Z}}$ and $J = \{\vec{k}^J_m\}_{m\in\mathbb{Z}}$, then

$$(I, J) \mapsto I + J = \{\vec{k}^I_m + \vec{k}^J_m\}_{m\in\mathbb{Z}} \ ,$$

where the sum of vectors $\vec{k}^I_m + \vec{k}^J_m$ is understood modulo d. Then, the family of operators $\{\widetilde{W}_I\}_{I\in\mathcal{I}}$ satisfying the multiplication law $\widetilde{W}_I\,\widetilde{W}_J = \widetilde{W}_{I+J}$, form an Abelian group G on which the shift defines an automorphism with an associated shift-invariant measure δ such that $\delta(\widetilde{W}_I) = 0$ unless $I = I_0 := \{\vec{0}\}_{m\in\mathbb{Z}}$. Therefore, one can consider the Hilbert space $\ell_2(G)$ spanned by the orthonormal elements $\widetilde{W}_I$ and represent the Weyl operators W_I introduced before by

$$\Pi(W_I)\widetilde{W}_J = \underbrace{\mathrm{e}^{i\pi\, u_0(I,J)}}_{\omega(I;J)}\, \widetilde{W}_{I+J} \ . \tag{7}$$

$\omega(I; J)$ is a cocycle, namely

$$\omega(I_1, I_2 + I_3)\,\omega(I_2, I_3) = \omega(I_1 + I_2, I_3)\,\omega(I_1, I_3) \ ;$$

In this way $\mathcal{A}$ is represented as a sub-algebra $\Pi(\mathcal{A}) \subseteq \mathcal{B}(\ell_2(G))$ (it is known as the regular representation) of the bounded operators on $\ell_2(G)$ and thus all considerations in [8] are therefore applicable to $\mathcal{A}$. For instance, $\mathcal{A}$ has a trivial center if to any word W_I there exists another word W_J such that W_I and W_J do not commute. In this case the trace on the algebra

$$tr(W_I) = 0 \quad \forall I \neq \emptyset, \quad tr(1) = 1 \tag{8}$$

is unique and it is implemented in the regular representation by

$$tr(W_I) = \langle \widetilde{W}_{I_0} | \Pi(W_I) | \widetilde{W}_{I_0} \rangle \ .$$

Evidently. the trace is invariant under the shift automorphism $\alpha : \mathcal{A} \mapsto \mathcal{A}$

Like in [8], the main interest is in finding conditions on the cocycle (7) such that no other invariant states exist other than the tracial state. In [8] the main tool was the high degree of anticommutativity, a generalization of the fact, that translation invariant states over Fermi systems have to be even. Though this criterion is sufficient only if $d = 2$, non-commutativity as embodied in (2) will nevertheless turn out to be just as powerful in restricting the class of invariant states. We will indeed give other arguments to enlarge the class of automorphisms that allow only the tracial state as invariant state. Though not optimal, the result indicates that delocalization by the dynamics as we describe it in (2.2)is an effect which is worth studying in more detail and in more realistic thermodynamic systems.

2.2 Spin-Chain Representation

We have already given a representation of the algebra over $\mathcal{B}(l_2(G))$. However, in order to bring it in closer contact with physical models we seek connections with spin chains and their automorphisms.

This demands different representations. To demonstrate the differences we make a short detour, considering finite algebras defined on a finite ring of N lattice points (= upper indices) with a cyclic shift, instead of an infinite set of points with a non-recurrent shift. We get different dimensions of the Hilbert spaces. In the above mentioned representation there are $(d^2)^N$ basis vectors. For a spin-chain one would expect only d^N as necessary. But this, as will turn out, is possible for a restricted set of defining sequences only. In Section 2.3 we give then a representation with a double-spin-chain, possible for any defining sequence, employing again $(d^2)^N$ basis vectors for the finite algebra on a ring.

More precisely, we shall try to represent the Weyl operators $W_{\vec{k}}^{(m)}$, $m = 0, \ldots, N$, as elements of the full matrix algebra $\bigotimes_{n=0}^{N}(M_{d\times d})_n$. We proceed step by step: let us define the Weyl operator at site $0 \leq j \leq N$ to be

$$W_{\vec{k}}^{(j)} = W_{A_{0,j}\vec{k}} \otimes W_{A_{1,j}\vec{k}} \otimes W_{A_{j,j}\vec{k}} \otimes 1_{j+1} \ldots \otimes 1_N \ . \tag{9}$$

The unknowns in the construction are the 2×2 matrices with integer entries from $\{0, 1, \ldots, d-1\}$ that we have to adjust in order to fulfil the commutation relations (2). Therefore, from (1), one gets the condition

$$\sum_{\ell=0}^{j} \sigma(A_{\ell,j}\vec{k}, A_{\ell,j}\vec{m}) = \sigma(\vec{k}, \vec{m}) \ , \tag{10}$$

forall $\vec{k}$, $\vec{m}$ and $0 \leq j \leq N$ which is equivalent to $\sum_{\ell=0}^{j} Det(A_{\ell,n}) = 1$. If d is prime, then all Weyl operators are unitarily isomorphic so that the algebra created by them is $M_{d\times d}$; therefore, in the rest of this section, d will be assumed to be a prime number.

We have to control whether this ansatz can really be satisfied and how far the matrices $A_{\ell,k}$ are determined by the matrices A_n in (3). It turns out that $A_{0,j} = A_j$ while $A_{00} = 1$. The other matrices $A_{\ell,k}$ have to be calculated recursively from (2) and (3). More precisely to evaluate

$$\sigma(A_{0,1}\vec{k}, A_{0,2}\vec{l}) + \sigma(A_{1,1}\vec{k}, A_{1,2}\vec{l}) = \sigma(\vec{k}, A_{0,1}\vec{l}) \tag{11}$$

we define the linear map $A \mapsto \widehat{A}$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to \widehat{A} := \begin{pmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{pmatrix}$$

With this map, it follows that $A_{1,2} = \hat{A}_{1,1}^{-1}(A_{0,1} - \hat{A}_{0,1}A_{0,2})$. This fixes $A_{1,2}$ if we take into account that the freedom in $A_{1,1}$ is reduced to an isomorphism inside of the local algebra. However, we can only be sure that there exists a solution $A_{1,2}$ if $A_{1,1}$ is invertible which surely holds if $Det(A_{1,1}) \neq 0$. Similarly

$$A_{n-k,n} = \hat{A}^{-1}_{n-k,n-k}(A_{0,n-k} - \sum_{l=0}^{n-k-1} \hat{A}_{l,n-k}A_{l,n}) \ .$$

The equations are uniquely solvable (up to trivial local isomorphisms) under the constraint

$$Det(A_{0,j}) + \ldots + Det(A_{j-1,j}) \neq 1 \quad \forall j \ . \tag{12}$$

This implies in addition that $Det(A_{j,j}) \neq 0$; as a consequence the algebra generated by the Weyl operators (9) is isomorphic to the full matrix algebra.

Only the sequence of matrices A_n is to our disposal, whereas the matrices $A_{\ell,k}$ are linearly depending on them; however, there remains enough freedom to find sequences that meet the condition (12). A special example corresponds to choosing $A_{0,j} = \delta_{1,j}\mathbb{1}$; this in turn corresponds to the usual shift on the lattice algebra. More generically, one may choose $A_{0,j}$ to belong to a left ideal with determinant 0 for all $j \neq 0$; it then follows that, for all $k > 0$, $A_{n-k,n}$ also belongs to this ideal and therefore $Det(A_{n,n}) = 1 \quad \forall n$.

As a consequence, provided $Det(A_{n,n}) \neq 0 \quad \forall n$ we can consider the algebra to be the same algebra i.e. the spin chain $M_{d\times d}^{\otimes\infty}$, but equipped with different automorphisms corresponding to the different sequences $A_{0,n}$. Notice that in the spin-chain representation, the algebra is fairly simple, while the automorphism (which we will again denote by α and which corresponds to the shift in the regular representation) is complicated.

Some discussions on the spin chain representations and their importance for physics follows in the Conclusion.

2.3 The Generalized Jordan-Wigner Transformation

We have already mentioned the representation of every algebra $\mathcal{A}$ as a C^* algebra $\Pi(\mathcal{A}) \subseteq \mathcal{B}(\ell_2(G))$. We can give another representation that can be considered as a generalization of the Jordan-Wigner transformations, that relates the spin lattice with the Fermions on a lattice. We represent $\mathcal{A}$ as a subalgebra of the doubled spin chain $\bigotimes_{m=-\infty}^{\infty}(M_{d\times d}\otimes M_{d\times d})_m = \bigotimes_{n=-\infty}^{\infty}(M_{d\times d})_n$, where $W_{\vec{k}}^{(0)}$ with $\vec{k} = (k_1, k_2)$ is identified with the infinite tensor product

$$\left(\bigotimes_{n=1}^{+\infty}(W_{0,b_n})_{-2n}\otimes(W_{0,a_n})_{-2n+1}\right)\otimes(W_{k_1,0})_0\otimes(W_{k_2,k_1})_1\bigotimes_{n=2}^{+\infty}(1)_n \ , \tag{13}$$

where the Weyl operator $W_{k_1,0}$ is at site $n=0$, while W_{k_2,k_1} is at site $n=1$, whereas the operators W_{0,b_n} are located at sites $-2n$ and W_{0,a_n} at sites $-2n+1$. Moreover, the components a_n and b_n are determined by $\vec{k}$ and the commutation relations via

$$A_n \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} b_n \\ a_n \end{pmatrix} \ .$$

Furthermore, the action of the shift automorphism α is now represented as a 2-step translation along the lattice.

Notice that, since the contributions from the infinite tails commute with each other by construction, finite tensor products of the form $W_{k_1,k_2}^{(0)}\cdots W_{l_1,l_2}^{(N)}$ may be effectively represented as elements of the matrix algebra $\bigotimes_{n=0}^{2N}(M_{d\times d})_n$. It thus follows that the commutant consists of operators of the form

$$\bigotimes_{n=2j-1}^{-\infty}(1)_n\otimes(W_{\ell_1,\ell_2})_{2j}\otimes(W_{\ell_2,0})_{2j+1}\otimes\left(\bigotimes_{k=j+1}(W_{0,b_{k-j}(\ell)})_{2k}\otimes(W_{0,a_{k-j}(\ell)})_{2k+1}\right),$$

plus operators belonging to the center. (The center becomes trivial if (12) holds. But this condition is not needed here.)

We can consider the operators to act on the vector $|\Omega >= \otimes_{-\infty<k<\infty}|0>$ where at each lattice point $W_{0,k}|0>=|0>$. Representing $\widetilde{W}^0_{k_1,k_2}$ by

$$\left(\bigotimes_{n=1}^{+\infty} 1 \otimes 1\right) \otimes (W_{k_1,0})_0 \otimes (W_{k_2,})_1 \bigotimes_{n=2}^{+\infty}(1)_n$$

and letting it acting on $|\Omega >= |\widetilde{W}_{I_0}\rangle$ we reproduce $l^2(G)$ and therefore the regular representation.

Example As a concrete illustration of the algebraic construction of above, let us consider the Price-Powers shift. This corresponds to $d=2$, and to a generating sequence of matrices which are either $A_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ corresponding to $g(n)=0$ or $A_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ corresponding to $g(n)=1$. Then, we set $e_0 = W^{(0)}_{1,0}$ and observe that operators at odd places have the form $W_{0,k}$ and therefore all commute. Consequently, we can remove them from the tensor product (13) and represent

$$e_m = \bigotimes_{n=1}^{+\infty}(W_{0,b_n})_{k-n} \otimes (W_{1,0})_k \bigotimes_{n=k+1}^{+\infty} (1)_n \ .$$

2.4 Preliminary Remarks on Invariant States

We now turn to the problem of finding the invariant states under the 2-step shift. If we are only interested in the local effects of such automorphism, we can take the periodic shift in $\bigotimes_{n=0}^{2N}(M_{d\times d})$, which is unitarily implemented.

The algebra created by finitely many Weyl operators is imbedded in $\bigotimes_{n=0}^{2N}(M_{d\times d})$, so that we can conclude that the automorphism $\alpha : \mathcal{A} \mapsto \mathcal{A}$ is unitarily implemented. Therefore, we can construct states on the local algebra defined by density operators that commute with the unitary that implements the periodic shift. However, in general, these density operators will not have a limit when $N \to \infty$.

Another possibility to construct α-invariant states is to start from vectors in the infinite tensor product $\bigotimes_{n=-\infty}^{\infty} |\psi_n\rangle$ that are invariant under the shift. The simple choice where all $|\psi_n\rangle$ are identical to an eigenvector $|\phi\rangle$ of $W_{(0,1)}$ gives, independently of the sequence $\{A_n\}_{n\in\mathbb{N}}$, already the tracial state; indeed, since for every W_I at some position $\langle\phi|W_{k,l}|\phi\rangle = 0$. Therefore, the representation is isomorphic to the regular representation.

If we choose some other vector and assume that $A_n \neq 1$ for infinitely many n, again we obtain the tracial state; in fact, either $|\langle\psi|W_{0,b_n}|\psi\rangle\,\langle\psi|W_{0,a_n}|\psi\rangle| < 1$, infinitely often so that

$$\prod_{n=-\infty}^{+\infty} |\langle\psi|W_{0,b_n}|\psi\rangle\,\langle\psi|W_{0,a_n}|\psi\rangle| = 0 \ .$$

If for some $N > 0$ $A_n = 0$ for all $n > N$, we can choose a vector ϕ over $\bigotimes_{n=1}^{2N} M^{(n)}_{d\times d}$ that is appropriately entangled over the lattice points to guarantee that $\langle\phi|W_{k_1,0)}|\phi\rangle \neq 0$. By averaging this vector over the period N, one gets an expectation value still $\neq 0$. However, in general, one expects that it decreases with N. Therefore, if for every N we can find $n > N$ such that $A_n \neq 0$ in order to obtain another invariant state in the limit $N \to \infty$, it is necessary to have correlations between infinitely many lattice points; this can hardly be satisfied because of monogamy of entanglement. Though we are unable to exclude that other invariant states might be constructed, our considerations already indicate that the tracial state will turn out to be the only invariant state under appropriate conditions on the defining sequence of matrices A_n.

3 Invariant states

Given $(\mathcal{A}, \alpha)$, let ω be an invariant state such that $\omega \circ \alpha = \omega$ and consider the corresponding GNS representation π_ω of $\mathcal{A}$ as a C^* algebra of bounded operators on the GNS Hilbert space $\mathcal{H}_\omega$ with cyclic vector $|\Omega\rangle$. Namely, $\omega(A) = \langle\Omega|\pi_\omega(A)|\Omega\rangle$; further, the shift automorphism α is implemented by a unitary U_ω such that $U_\omega|\Omega\rangle = |\Omega\rangle$. The following simple lemmas hold.

Lemma 1. *Let P_0^ω denote the projection onto the U_ω-invariant subspace of $\mathcal{H}_\omega$. If*

$$\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = 0 \tag{14}$$

is true for all W_I, then ω is tracial that is $\omega(XY) = \omega(YX)$ for all $X, Y \in \mathcal{A}$.

Proof. Since $P_0^\omega \geq |\Omega\rangle\langle\Omega|$, the assumption implies

$$|\omega(W_I)|^2 = \langle\Omega|\pi_\omega(W_I^*)|\Omega\rangle\langle\Omega|\pi_\omega(W_I)|\Omega\rangle \leq \langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = 0\ ,$$

whence, by Cauchy-Schwartz, $\omega(W_I\, W_J) \neq 0$ only if $W_I W_J = 1$. Thus, $\omega(W_I\, W_J) = \omega(W_J\, W_I)$ for all $W_{I,J}$, whence $\omega(XY) = \omega(YX)$ for $\mathcal{A}$ is generated by linear combinations of W_Is. ☐

Lemma 2. *If ω is α-invariant, then, setting $u_n(I) := u_n(I; I)$ in (6),*

$$\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi\, i\, u_n(I)}\langle\Omega|\pi_\omega(W_I)U_\omega^{-n}\pi_\omega(W_I^*)|\Omega\rangle\ . \tag{15}$$

Proof. The mean ergodic theorem of von Neumann ([9]) and (6) imply

$$\begin{aligned}\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\omega\Big(W_I^*\alpha^n\Big(W_I\Big)\Big)\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi\, i\, u_n(I)}\,\omega\Big(W_I\alpha^{-n}\Big(W_I^*\Big)\Big)\ .\end{aligned}$$

☐

Remarks 1

1. If $W_I = 1$, then $\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = 1$, $u_n(I) = 0$ for all $n \in \mathbb{N}$ and (15) is trivially satisfied.

2. Using the spectral decomposition $U_\omega = \int_{Sp(U_\omega)} \mathrm{d}P_\omega^\lambda\, \mathrm{e}^{2\pi\, i\lambda}$, (15) reads

$$\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = \lim_{N\to\infty}\int_{Sp(U_\omega)} \mathrm{d}\mu_I(\lambda)\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi\, i(u_n(I)-n\lambda)}\ , \tag{16}$$

where $\mathrm{d}\mu_I(\lambda) := \mathrm{d}(\langle\Omega|\pi_\omega(W_I)P_\omega^\lambda\pi_\omega(W_I^*)|\Omega\rangle$.

We now concentrate on the sequences $u(I) := \{u_n(I)\}_{n\in\mathbb{N}}$ and $u_n^\lambda(I) := \{u_n(I) - \lambda\, n\}_{n\in\mathbb{N}}$ and study their spectrum [9]. In order to properly introduce this notion, consider a sequence $v = \{v_n\}_{n\in\mathbb{N}}$ taking its values $v_n \in \mathbb{C}$ in a compact subset of the complex numbers. For all $k \in \mathbb{N}$ the partial sums

$$S_N(k) := \frac{1}{N}\sum_{n=0}^{N-1} v_n^* v_{n+k} \tag{17}$$

are bounded, thus, the sequence $S(k) := \{S_N(k)\}_{N\in\mathbb{N}}$ has accumulation points and, by a Cantor-like diagonalization argument (for details see the Appendix), there exists at least one subsequence $\{N_j\}_{j\in\mathbb{N}}$ such that the limit

$$s_k(v) := \lim_{j\to\infty}\frac{1}{N_j}\sum_{n=0}^{N_j-1} v_n^* v_{n+k}$$

exists for all $k \in \mathbb{N}$. By setting $s_{-k} := s_k^*$, one obtains a positive-definite sequence (details are again in the Appendix), that is a sequence $s(v) = \{s_k(v)\}_{k\in\mathbb{Z}}$ such that

$$\sum_{i,j} z_i^* s_{i-j}(v) z_j \geq 0$$

for all sequences $\{z_i\}_{i\in\mathbb{Z}}$ such that $\sum_{i\in\mathbb{Z}} |z_i|^2 < \infty$. Then, by Bochner's theorem,

$$s_k(v) = \int_0^1 \mathrm{d}\mu_v(x)\, \mathrm{e}^{2\pi\, ikx}\ ,$$

where $\mathrm{d}\mu_v(x)$ is a positive (*correlation*) measure on $[0, 1)$ such that

$$\int_0^1 \mathrm{d}\mu_v(x) = \lim_{j\to\infty}\frac{1}{N_j}\sum_{n=0}^{N_j-1} |v_n|^2\ .$$

If the correlation measure of a sequence v is the Lebesgue measure, then $s_k(v) = 0$ whenever $k \neq 0$ and the sequence v is said to be *uniformly distributed*. Therefore, it makes sense to introduce the

Definition 1 (Spectrum of a sequence)**.** *[9, 10] Given a sequence $v := \{v_n\}_{n\in\mathbb{N}}$ with values in a compact subspace of $\mathbb{C}$, its Fourier-Bohr spectrum is given by*

$$\mathrm{Sp}(v) := \left\{\mu \in [0, 1)\ :\ \limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=0}^{N-1} v_n\, \mathrm{e}^{-2\pi\, in\mu}\right| \neq 0\right\}\ .$$

In other words, the spectrum of a sequence v is the subset of values $\mu \in [0, 2\pi)$ such that the sequences $v(\mu) = \{v_n \exp(-2\pi i n\mu)\}_{n\in\mathbb{N}}$ are not uniformly distributed. Equivalently, $\mu \notin Sp(v)$ if and only if

$$\lim_{j\to\infty} \frac{1}{N_j} \sum_{n=0}^{N_j-1} v_n \,\mathrm{e}^{-2\pi\, i\mu\, n} = 0 \tag{18}$$

for all converging subsequences of partial sums.

By means of the spectral properties of sequences, we can derive sufficient conditions that force the invariant state ω to be tracial.

Lemma 3. *Let $v(I) := \{\mathrm{e}^{2\pi i\, u_n(I)}\}_{n\geq 0}$; then, the dynamical system $(\mathcal{A}, \alpha)$ has the tracial state as its only invariant state if for each I, $Sp(v(I))$ is either $\emptyset$ or $\{0\}$ with*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} v_n(I) = \sum_{j=0}^{d-1} p_j(I) e^{\frac{2\pi i}{d} j} \Big(=: \nu(v(I))\Big) \qquad (*)$$

for some $p_j(I) \geq 0$, $j \in D$, $p_0 \neq 1$, $\sum_{j=0}^{d-1} p_j(I) = 1$.

Proof. If $Sp(v(I)) = \emptyset$, then, using (18) and dominated convergence, the right hand side of (16) vanishes and the result follows from Lemmas 1 and 2.

If for some I $Sp(v(I)) = \{0\}$ and relation (*) holds for such I, then, using again (18) and dominated convergence,

$$\langle\Omega|\pi_\omega(W_I^*)P_0^\omega \pi_\omega(W_I)|\Omega\rangle = \nu(v(I))\langle\Omega|\pi_\omega(W_I)P_0^\omega \pi_\omega(W_I^*)|\Omega\rangle \ . \tag{19}$$

Since $|\nu(v(I))| < 1$, by exchanging W_I and W_I^*, one gets

$$\langle\Omega|\pi_\omega(W_I^*)P_0^\omega \pi_\omega(W_I)|\Omega\rangle < \langle\Omega|\pi_\omega(W_I)P_0^\omega \pi_\omega(W_I^*)|\Omega\rangle < \langle\Omega|\pi_\omega(W_I^*)P_0^\omega \pi_\omega(W_I)|\Omega\rangle \ .$$

Thus, $\langle\Omega|\pi_\omega(W_I^*)P_0^\omega \pi_\omega(W_I)|\Omega\rangle = 0$ and Lemmas 1 and 2 apply. □

For some given I, for instance a singleton $I = \{1\}$, there surely exist sequences of matrices $\{A_n\}_{n\in\mathbb{N}}$, with entries from $\{0, 1, \ldots, d-1\}$, such that $Sp(v(I)) = \emptyset$, or $Sp(v(I)) = \{0\}$.

However, in order to use the previous Lemma, we have to make sure that there exist sequences $\{A_n\}_{n\in\mathbb{N}}$ such that conditions 1 or 2 in the previous Lemma are fulfilled for all I.

In the following, we shall consider the 4 entries $a_{ij}(n)$ of the matrices A_n as random processes with values from $\{0, 1, \ldots\, d-1\}$. Then, we shall focus upon the space $\mathcal{X}$ of sequences $\vec{x} = \{\vec{x}_n\}_{n\in\mathbb{N}}$, where $\vec{x}_n = (a_{11}(n), a_{12}(n), a_{21}(n), a_{22}(n))$ are 4-valued vectors with the entries of the matrices A_n as components. If we want in addition to meet the requirements in (2.2), e.g. that $Det(A_{n,n}) = 1 \quad \forall n$, we can restrict to 2 entries $(a_{11}(n), a_{12}(n))$ but still keep enough randomness.

We equip $\mathcal{X}$ with the shift-automorphism $(\sigma(\vec{x}))_n = \vec{x}_{n+1}$ and with a σ-invariant measure μ (defined on the σ-algebra of cylinders). Concretely, if f is a measurable function on $\mathcal{X}$, then its mean value with respect to μ is given by

$$\mu(f) = \int_{\mathcal{X}} \mathrm{d}\mu(\vec{x})\, f(\vec{x}) \ ;$$

furthermore, $\mu(f \circ \sigma) = \mu(f)$.

We shall assume that the dynamical system $(\mathcal{X}, \mu, \sigma)$ is mixing (hence ergodic), that is, if f and g are two essentially bounded functions on $\mathcal{X}$ with respect to μ, then

$$\lim_{k \to +\infty} \mu(f \cdot (g \circ \sigma^k)) = \mu(f)\, \mu(g) \ , \tag{20}$$

We shall call a sequence $\vec{x} \in \mathcal{X}$ typical if for all measurable functions it is self-averaging and mixing in the following sense:

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ \sigma^n(\vec{x}) \quad = \quad \mu(f) \tag{21}$$

$$\lim_{k \to +\infty} \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ \sigma^n(\vec{x})\, g \circ \sigma^{n+k}(\vec{x}) \quad = \quad \mu(f)\, \mu(g) \ . \tag{22}$$

As a concrete example, consider $(\mathcal{X}, \mu, \sigma)$ as the product of 4 identically distributed independent Bernoulli processes.

Theorem 1. *Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of matrices which provides a typical sequence $\vec{x} \in \mathcal{X}$ with respect to a shift-invariant measure μ as explained above. Let $u(I) = \{u_n(I)\}_{n \in \mathbb{N}}$ be the sequences defined in Lemma 2; then, for all I, the spectrum of $u(I)$ is $\emptyset$ or $\{0\}$.*

Proof: Observe that the quantities $\mathrm{e}^{-2\pi\, i\, u_n(I)}$ can be regarded as measurable functions on $\mathcal{X}$; more precisely, let P_j project out of the sequence $\vec{x}$ the j-th component ($P_j \vec{x} = \vec{x}_j$). Then, consider the expression of $u_n(I)$ given by (6); it turns out that one can write

$$\begin{aligned} \mathrm{e}^{2\pi\, i\, u_n(I)} &= \prod_{a=1}^{n_I} \prod_{b=1}^{n_I} G_{\vec{k}_a \vec{k}_b} \circ \sigma^{b-a} \circ P_n(\vec{x}) \quad \text{where} \\ G_{\vec{k}_a \vec{k}_b} \circ \sigma^{b-a} \circ P_n(\vec{x}) &= \mathrm{e}^{\frac{2\pi\, i}{d}\, \sigma(\vec{k}_a, A_{b-a+n} \vec{k}_b)} \ . \end{aligned}$$

Therefore, the function on the left hand side of the first equality belongs to the class of function on $\mathcal{X}$ that have been used to define typical sequences (see conditions (21) and (22)).

Now, we consider the following quantity

$$\begin{aligned} (*) := \lim_{K \to \infty} \lim_{N \to \infty} \frac{1}{NK} \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} \mathrm{e}^{-2\pi\, i\, (u_n(I) + \lambda\, n)}\, \mathrm{e}^{2\pi\, i (u_{n+k}(I) + \lambda(n+k))} = \\ = \lim_{K \to \infty} \lim_{N \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathrm{e}^{2\pi\, i\, \lambda\, k}\, \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi\, i (u_{n+k}(I) - u_n(I))} \ . \end{aligned}$$

Notice that the second average is just the function $S_N(k)$ in (17); then, using the conditions defining typical sequences, one gets

$$\lim_{k \to +\infty} \lim_{N \to +\infty} S_N(k) = \left| \mu \left(\prod_{a=1}^{n_I} \prod_{b=1}^{n_I} G_{\vec{k}_a \vec{k}_b} \circ \sigma^{b-a} \right) \right|^2 = (**) \ .$$

Therefore, one concludes that in the limits $N \to +\infty$ and $K \to +\infty$, the expression $(*)$ converges to 0 if $\lambda \neq 0$, or to $(**)$ if $\lambda = 0$. Thus, the spectrum of $u_n(I)$ is either empty or consists of $\lambda = 0$, because for a typical sequence the expression $(*)$ equals

$$\lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi\, i\,(u_n(I)+\lambda\, n)} \right|^2 .$$

□

Finally, by applying Lemma 3, we obtain

Theorem 2. *Given a sequence of 2×2 matrices $\{A_n\}_{n\in\mathbb{N}}$ with integer entries such that it is typical in the sense of conditions (21) and (22), then only the tracial state on the twisted Weyl algebra $\mathcal{A}$ is α-invariant.*

4 Conclusions

We have constructed quantum algebras and automorphisms on them such that they permit only one invariant state, namely the tracial state. The main tool in the proof was the application of results on classical random sequences. We should notice however, that our proof can also be applied to the Price-Powers shift, but there it does not cover all possibilities for which the result holds. Indeed, also bitstreams not fulfilling conditions (21) and (22) may have the tracial state as the only shift-invariant state. Indeed, these two conditions are only sufficient, but not necessary.

Also, the failure in section 2.3 to construct invariant states does not refer to the special construction of the automorphism, rather it is long-range non-commutativity which seems to be important. If we can represent the operators as in (9) we can embody this non-commutativity in

$$[[\cdots[\alpha^n(A)\,,\,B_0],B_1],\cdots\,B_n] \neq 0 \tag{23}$$

for an appropriate sequence of B_k where the B_k is localized at the lattice point k. Therefore the operator $\alpha^n(A)$ is not only spread as it happens for quasifree evolution but got delocalized also in a multiplicative sense. Nevertheless it has still finite velocity in the sense that, at every step, an operator in the local algebra $\bigotimes_{n=0}^{N}(M_{d\times d})_n$ is mapped into an operator located in the algebra $\bigotimes_{n=0}^{N+1}(M_{d\times d})_n$.

We expect that the occurrence of non-trivial multicommutators that do not vanish should be typical of interacting quantum systems. Of course, in general, the dynamics is such that one deals with continuous automorphism groups and with multicommutators by far more complicated. However, the preceding analysis indicates that in the present abstract model multicommutators are responsible for the nonexistence of invariant states. This gives a hint that also in more general situations multicommutators should play an important role in the search for invariant states.

5 Appendix

The Cantor-like argument needed, following equation (17)

Define S_N as a function on the integers such that

$$S_N(k) := \frac{1}{N} \sum_{n=0}^{N-1} v_n^* v_{n+k}$$

as in (17), where $|S_N(k)| \leq K$ by assumption. Now $S_N(1)$ need not converge, but one can extract a subset $\{N(j,1),\ j = 1,2,3\dots\} \in \mathbb{N}$ such that $S_{N(j,1)}(1)$ converges, as $j \to \infty$. Then we proceed inductively, define for each k a smaller subset $\{N(j,k),\ j = 1,2,3\dots\} \subset \{N(j,k-1)\}$, such that $S_{N(j,k)}(k)$ converges as $j \to \infty$, with k fixed.

Now, and here is the Cantor diagonalization trick, one considers the set $\{N_j := N(j,j)\}$. The sequences $S_{N(j,j)}(k)$ converge, as $j \to \infty$, for each k.

Positive-definiteness

Consider a set $\{z_i\}$, $|i| \leq M$, extend the set $\{v_n\}$, defining $v_n := 0$ for $n < 0$, and transform

$$\sum_{i,j} z_i^* s_{i-j} z_j = \lim_{N\to\infty} \frac{1}{N} \sum_{i,j} \sum_{m=i}^{N+1+i} v_{m-i}^* z_i^* v_{m-j} z_j \,.$$

Considering the bounds $|v_n| < V$, $|z_i| < Z$, one gets

$$\sum_{i,j} z_i^* s_{i-j} z_j = \lim_{N\to\infty} \frac{1}{N} \left[\sum_{m\in\mathbb{Z}} \left(\sum_i v_{m-i} z_i \right)^* \left(\sum_j v_{m-j} z_j \right) + \mathrm{O}\left(M^2 \cdot V^2 \cdot Z^2\right) \right].$$

The error-term vanishes in the limit $N \to \infty$ (to be taken over the subset of N where limits of the S_N exist). Then one may consider approaching ℓ^2 sequences of z_i by finite sequences.

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