DEGENERATE KAM THEORY FOR PARTIAL DIFFERENTIAL EQUATIONS

BAMBUSI D., BERTI M., MAGISTRELLI E.

Abstract. This paper deals with *degenerate* KAM theory for lower dimensional elliptic tori of infinite dimensional Hamiltonian systems, depending on *one* parameter only. We assume that the linear frequencies are analytic functions of the parameter, satisfy a weak non–degeneracy condition of Rüssmann type and an asymptotic behavior. An application to nonlinear wave equations is given.

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Introduction

KAM theory ensures the persistence of invariant tori of nearly integrable Hamiltonian systems, filled by quasi–periodic solutions with frequencies satisfying strong non–resonance conditions of diophantine type. In order to verify such severe non–resonance properties, KAM theory always requires some non–degeneracy condition concerning the dependence of the frequencies on the parameters of the system (actions, potentials, masses, ...). The Kolmogorov non–degeneracy condition in [Kol54] is the simplest one and states that the frequency–to–action map is a diffeomorphism.

In concrete systems the Kolmogorov condition could be not verified (or it could be very difficult to check it). For example, it is never satisfied in the spatial solar system, see e.g. Herman-Féjoz [Féj04]. This problem strongly motivated the search of weaker non–degeneracy conditions.

For finite dimensional Hamiltonian systems, degenerate KAM theory has been widely developed since Arnold [Arn63] and Pjartly [Pja69]. We quote also other important works by Bruno [Bru92], Cheng-Sun [CS94] and Xu-You-Qiu [XYQ97]. Then new contributions were given by Rüssmann [Rüs90]-[Rüs01] not only for Lagrangian (i.e. maximal dimensional) tori but also for lower dimensional elliptic/hyperbolic tori. For recent developments we refer to Sevryuk [Sev07].

Concerning infinite dimensional Hamiltonian systems, KAM theory was first extended by Kuksin [Kuk93] and Wayne [Way90] for parameter dependent nonlinear wave and Schrödinger equations. The aim is to continue finite dimensional elliptic tori under the influence of an infinite dimensional perturbation. The integrable unperturbed linear partial differential equation (in short, PDE), for example the Klein-Gordon equation $u_{tt} - u_{xx} + \xi u = 0$, is completely degenerate as it amounts to infinitely many *isochronous* harmonic oscillators where there is no frequency–amplitude modulation. Replacing the scalar parameter ξ by some potential, this amounts to introducing infinitely many external parameters into the system, which may be adjusted and thus substitute the Kolmogorov non–degeneracy condition. As a result, one finds a Cantor set of potentials for which the PDE possesses small amplitude quasi–periodic solutions. However, this Cantor set surely excludes any open interval of constant potentials.

A different approach was taken in Kuksin–Pöschel [KP96], [Pös96a]. After one symplectic change of variables, the nonlinear PDE is approximated by the integrable forth order (partial) Birkhoff normal form. Then one introduces the unperturbed actions as parameters. Kuksin and Pöschel proved that, for cubic wave and Schrödinger equations, the frequency–to–action map of the Birkhoff normal form is non–degenerate and then KAM theory applies.

The present paper deals with degenerate KAM theory for lower dimensional elliptic tori of PDEs, in particular when the frequencies of the linearized system depend on *one* parameter only.

We extend to partial differential equations the results in Rüssmann [Rüs01] developed in the context of finite dimensional systems, see Section 1 for the precise statements of the main theorems, and we give an application to the nonlinear wave equation, see Section 3.

The main assumption in [Rüs01] is that the frequencies are analytic functions of the parameters and satisfy a weak non–degeneracy condition. For maximal dimensional tori this property is equivalent to the fact that the range of the frequency map is not contained in any hyperplane.

Rüssmann's proof goes into some steps. First, he uses properties of the zero set of analytic functions to show that the *qualitative* weak non–degeneracy assumption implies a *quantitative* non–degeneracy property. Second, he shows that, notwithstanding the fact that the frequencies change during the KAM iteration process, the set of non–resonant frequencies met at each step has large measure. Third, he proves that the same is true for the final frequencies on the limiting perturbed torus constructed through the iteration. For the last two steps Rüssmann introduces the concept of "chain of frequencies".

For infinite dimensional systems, the main difficulty in extending the approach of Rüssmann is met at step 1, where one has to bound the maximal order of the zeros of *infinitely* many analytic functions, a fact which is generically impossible. Here we exploit the asymptotic growth of the frequencies to reduce the effective number of functions to a finite one. This idea allows to deduce a quantitative non–resonant property of the kind of the second order Melnikov non–resonance conditions, typical of infinite dimensional KAM theory, see Proposition 3.

Concerning the other steps, we avoid the Rüssmann construction of chains, making use of the recent formulation of the KAM theorem in Berti-Biasco [BB10]. An advantage of this formulation is an explicit characterization of the Cantor set of parameters which satisfy the Melnikov non–resonance conditions at all the steps of the KAM iteration, in terms of the *final* frequencies only. This approach *completely* separates the question of the existence of admissible non–resonant frequencies from the iterative construction of the invariant tori. This procedure considerably simplifies the measure estimates (also for finite dimensional systems), as it allows to perform them only at the final step and not at each step of the iteration, see Section 2.

We apply these abstract results to nonlinear wave (NLW) equations with Dirichlet boundary conditions

$$u_{tt} - u_{xx} + V(x)u + \xi u + f(x,u) = 0$$

requiring *only* $f(x,u) = O(u^2)$. Using the mass $\xi \in \mathbb{R}$ as a parameter we prove in Theorem 2 the persistence of Cantor families of small amplitude elliptic invariant tori of NLW. This result generalizes the one in [Pös96b], valid for $f(x,u) = u^3+$ higher order terms, to arbitrary analytic nonlinearities. Actually, in [Pös96b] the fourth order Birkhoff normal form of NLW is non–degenerate and the action–to–frequency map is a diffeomorphism. For general nonlinearities this property could be hard to verify, if ever true. The use of degenerate KAM theory allows to avoid this computation and then it is more versatile.

Finally we recall that a KAM theorem for degenerate PDEs was already proved by Xu–You–Qiu [XYQ96] which extended to the infinite dimensional case the method introduced in [XYQ97]. The main difference is that such authors assume a quantitative (weak) non–degeneracy assumption whose verification is usually very hard. On the contrary our non–degeneracy assumption (which follows Rüssmann) is quite easy to be verified. In particular, since it is based on properties of analytic functions it is enough to verify it for one value of the parameter, a task usually not very difficult.

The paper is organized as follows: in Section 1 we present the main results. In Section 2 we prove the measure estimates. In Section 3 we consider the application to the nonlinear wave equation. Finally in section 4 we deduce the quantitative non–resonance condition (2.1) from the qualitative non–resonance condition (ND) and the analyticity and asymptotic behavior of the linear frequencies, see assumption (A).

Notations. For $l \in \mathbb{Z}^\infty$ define the norms

$$|l| := \sum_j |l_j|, \quad |l|_\delta := \sum_j j^\delta |l_j|, \quad \langle l \rangle_d := \max \left\{ 1, \left| \sum_j j^d l_j \right| \right\}.$$

Given $a, b \in \mathbb{R}^M$, $M \leq +\infty$, denote the scalar product $\langle a, b \rangle := \sum_{j=1}^M a_j b_j$.

We define the set

$$\mathcal{Z}_N := \left\{ (k,l) \in \mathbb{Z}^N \times \mathbb{Z}^\infty \setminus (0,0) \colon |l| \leq 2 \right\} \tag{0.1}$$

and we split $\mathcal{L} := \{ l \in \mathbb{Z}^\infty \colon |l| \leq 2 \}$ as the union of the following four disjoint sets

$$\begin{aligned} &\mathcal{L}_0 := \{ l = 0 \}, \quad \mathcal{L}_1 := \{ l = e_j \}, \\ &\mathcal{L}_{2+} := \{ l = e_i + e_j \text{ for } i \neq j \}, \quad \mathcal{L}_{2-} := \{ l = e_i - e_j \text{ for } i \neq j \}, \end{aligned} \tag{0.2}$$

where $e_i := (0, \ldots 0, \underbrace{1}_{i-th}, 0, \ldots)$ and $i, j \geq N+1$.

Given a map $\Omega \colon \mathcal{I} \ni \xi \mapsto \Omega(\xi) \in \mathbb{R}^\infty$ we define the norm $|\Omega|_{-\delta} := \sup_{\xi \in \mathcal{I}} \sup_j |\Omega_j| j^{-\delta}$ and the C^μ-norm, $\mu \in \mathbb{N}$, as

$$|\Omega|^{C^\mu}_{-\delta} := \sum_{\nu=0}^{\mu} \left| \frac{d^\nu}{d\xi^\nu} \Omega(\xi) \right|_{-\delta}.$$

The $| \ |^{C^\mu}$ norm of a map $\omega : \mathcal{I} \to \mathbb{R}^N$, $N < \infty$, is defined analogously.

1. Statement of the main results

Fix an integer $N \geq 1$ and consider the phase space

$$\mathcal{P}^{a,p} := \mathbb{T}^N \times \mathbb{R}^N \times \ell^{a,p} \times \ell^{a,p} \quad \ni (x, y, z, \overline{z})$$

for some $a > 0$, $p > 1/2$, where $\mathbb{T}^N$ is the usual N-torus and $\ell^{a,p}$ is the Hilbert space of complex valued sequences $z = (z_1, z_2, \ldots)$ such that

$$\|z\|^2_{a,p} := \sum_{j \geq 1} |z_j|^2 j^{2p} e^{2aj} < +\infty \,,$$

endowed with the symplectic structure $\sum_{j=1}^N dx_j \wedge dy_j + \mathrm{i} \sum_{j \geq N+1} dz_j \wedge d\overline{z}_j$.

Consider a family of Hamiltonians

$$H := Z + P \tag{1.1}$$

depending on *one* real parameter ξ varying in a compact set $\mathcal{I} \subset \mathbb{R}$, where Z is the normal form

$$Z := \sum_{j=1}^{N} \omega_j(\xi) y_j + \sum_{j \geq N+1} \Omega_j(\xi) z_j \overline{z}_j \,, \tag{1.2}$$

with frequencies $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$, $\Omega = (\Omega_{N+1}, \Omega_{N+2}, \ldots) \in \mathbb{R}^\infty$, real analytic in ξ, and P is a small perturbation, also real analytic in ξ.

The equations of motion of the unperturbed system Z are

$$\dot{x} = \omega(\xi), \quad \dot{y} = 0, \quad \dot{z} = \mathrm{i}\Omega(\xi) z, \quad \dot{\overline{z}} = -\mathrm{i}\Omega(\xi)\overline{z} \,.$$

For each $\xi \in \mathcal{I}$ the torus $\mathcal{T}_0^N = \mathbb{T}^N \times \{0\} \times \{0\} \times \{0\}$ is an invariant N–dimensional torus for Z with frequencies $\omega(\xi)$ and with an elliptic fixed point in its normal space, described by the $z\overline{z}$-coordinates, with frequencies $\Omega(\xi)$. The aim is to prove the persistence of a large family of such N–dimensional elliptic invariant tori in the complete Hamiltonian system, provided the perturbation P is sufficiently small.

To this end we shall use the abstract KAM theorem in [BB10]. An advantage of its formulation is an explicit characterization of the Cantor set of parameters which satisfy the Melnikov non–resonance conditions at all the steps of the KAM iteration, in terms of the *final* frequencies only, see (1.7). This approach *completely* separates the question of the existence of admissible non–resonant frequency vectors from the iterative construction of N–dimensional invariant tori.

We now state a simplified version of the KAM theorem in [BB10] sufficient for the applications of this paper.

1.1. **KAM theorem.** We assume:

(A) *Analyticity and Asymptotic condition*: There exist $d \geq 1$, $\delta < d-1$, $0 < \eta < 1$ fixed, and functions $\nu_j : \mathcal{I} \to \mathbb{R}$ such that

$$\Omega_j(\xi) = j^d + \nu_j(\xi) j^\delta, \qquad j \geq N+1 \,,$$

where each $\nu_j(\xi)$ extends to an analytic function on the complex neighborhood of $\mathcal{I}$

$$\mathcal{I}_\eta := \bigcup_{\xi \in \mathcal{I}} \{\xi' \in \mathbb{C} \colon |\xi - \xi'| < \eta\} \subseteq \mathbb{C}.$$

Also the function $\omega \colon \mathcal{I} \to \mathbb{R}^N$ has an analytic extension on $\mathcal{I}_\eta$. Moreover there exists $\Gamma \geq 1$ such that

$$\sup_{\mathcal{I}_\eta} \sup_j |\nu_j(\xi)| \leq \Gamma \,, \quad \sup_{\mathcal{I}_\eta} |\omega(\xi)| \leq \Gamma \,.$$

Consider the complexification of $\mathcal{P}^{a,p}$ and define a complex neighborhood $\mathcal{D}_{a,p}(s,r)$ of the torus $\mathcal{T}_0^N$ by

$$\mathcal{D}_{a,p}(s,r) := \Big\{ |\mathrm{Im}\, x| < s, |y| < r^2, \|z\|_{a,p} + \|\bar{z}\|_{a,p} < r \Big\} \tag{1.3}$$

for some $s, r > 0$, where $|\cdot|$ denotes the max–norm for complex vectors.

For $W = (X, Y, U, V) \in \mathbb{C}^N \times \mathbb{C}^N \times \ell^{a,p}(\mathbb{C}) \times \ell^{a,p}(\mathbb{C})$, define the weighted phase space norm

$$|\!|\!| W |\!|\!|_{p,r} := |X| + r^{-2}|Y| + r^{-1}\|U\|_{a,p} + r^{-1}\|V\|_{a,p} \,.$$

Finally set

$$\mathcal{E} := \mathcal{I}_\eta \times D_{a,p}(s,r)\,.$$

(R) *Regularity condition*: There exist $s > 0, r > 0$ such that, for each $\xi \in \mathcal{I}$, the Hamiltonian vector field $X_P := (\partial_y P, -\partial_x P, \mathrm{i}\partial_{\bar{z}} P, -\mathrm{i}\partial_z P)$ is a real analytic map

$$X_P \colon \mathcal{D}_{a,p}(s,r) \longrightarrow \mathcal{P}^{a,\bar{p}}, \qquad \begin{cases} \bar{p} \geq p & \text{for } d > 1 \\ \bar{p} > p & \text{for } d = 1 \end{cases}$$

with $p - \bar{p} \leq \delta < d - 1$, real analytic in $\xi \in \mathcal{I}_\eta$ and

$$|\!|\!| X_P |\!|\!|_{r,\bar{p},\mathcal{E}} := \sup_{\mathcal{E}} |\!|\!| X_P |\!|\!|_{\bar{p},r} < +\infty\,.$$

KAM Theorem. [BB10] *Consider the Hamiltonian system $H = Z + P$ on the phase space $\mathcal{P}^{a,p}$. Assume that the frequency map of the normal form Z is analytic and satisfies condition (A). Let $9r^2 < \gamma < 1$. Suppose the perturbation P satisfies (R) and*

$$\sum_{2i+j_1+j_2=4} \sup_{\mathcal{E}} |\partial_y^i \partial_z^{j_1} \partial_{\bar{z}}^{j_2} P| \leq \frac{\sqrt{\gamma}}{3r}\,. \tag{1.4}$$

Then there is $\epsilon_* > 0$ such that, if the KAM–condition

$$\varepsilon := \gamma^{-1} |\!|\!| X_P |\!|\!|_{r,\bar{p},\mathcal{E}} \leq \epsilon_* \tag{1.5}$$

holds, then

1. *there exist C^∞-maps $\omega^* \colon \mathcal{I} \to \mathbb{R}^N$, $\Omega^* \colon \mathcal{I} \to \ell_\infty^{-d}$, satisfying, for any $\mu \in \mathbb{N}$,*

$$|\omega^* - \omega|^{C^\mu} \leq M(\mu)\varepsilon\gamma^{1-\mu}, \quad |\Omega^* - \Omega|_{-\delta}^{C^\mu} \leq M(\mu)\varepsilon\gamma^{1-\mu} \tag{1.6}$$

   for some constant $M(\mu) > 0$,
2. *there exists a smooth family of real analytic torus embeddings*

$$\Phi : \mathbb{T}^N \times \mathcal{I}^* \to \mathcal{P}^{a,\bar{p}}$$

where $\mathcal{I}^*$ is the Cantor set

$$\mathcal{I}^* := \Big\{ \xi \in \mathcal{I} \,:\, |\langle k, \omega^*(\xi)\rangle + \langle l, \Omega^*(\xi)\rangle| \geq \frac{2\gamma \langle l \rangle_d}{1+|k|^\tau}\,,\ \forall (k,l) \in \mathcal{Z}_N \Big\}\,, \tag{1.7}$$

such that, for each $\xi \in \mathcal{I}^*$, the map Φ restricted to $\mathbb{T}^N \times \{\xi\}$ is an embedding of a rotational torus with frequencies $\omega^*(\xi)$ for the Hamiltonian system H, close to the trivial embedding $\mathbb{T}^N \times \mathcal{I} \to \mathcal{T}_0^N$.

Remark. The KAM Theorem 5.1 in [BB10] provides also explicit estimates on the map Φ and a normal form in an open neighborhood of the perturbed torus.

Remark. The above KAM theorem follows by Theorem 5.1 in [BB10] and remark 5.1, valid for Hamiltonian analytic also in ξ. Actually (1.4), (1.5) and $9r^2 < \gamma < 1$ imply the assumptions (5.5) and (H3) of Theorem 5.1 of [BB10]. Estimate (1.6) is (5.15) in [BB10].

Remark. The main difference between the above KAM theorem and those in Kuksin [Kuk93] and Pöschel [Pös96a], concerns, for the assumptions, the analytic dependence of H in the parameters ξ, which is only Lipschitz in [Kuk93], [Pös96a]. For the results, the main difference is the explicit characterization of the Cantor set $\mathcal{I}^*$. Note that we do *not* only claim that the frequencies of the preserved torus satisfy the second order Melnikov non-resonance conditions, fact already proved in [Pös96a]. The above KAM Theorem states that also the *converse* is true: *if* the parameter ξ belongs to $\mathcal{I}^*$, then the KAM torus with frequencies $\omega^*(\xi)$ is preserved.

The main result of the next section proves that $\mathcal{I}^*$ is non–empty, under some weak non–degeneracy assumptions.

1.2. **The measure estimates.**

We first give the following definition.

Definition 1. A function $f = (f_1, \dots, f_M) \colon \mathcal{I} \to \mathbb{R}^M$ is said to be *non–degenerate* if for any vector $(c_1, \dots, c_M) \in \mathbb{R}^M \setminus \{0\}$ the function $c_1 f_1 + \ldots + c_M f_M$ is not identically zero on $\mathcal{I}$.

We assume:

(ND) *Non–degeneracy condition*: The frequency map (ω, Ω) satisfies
- *i)* $(\omega, 1) \colon \mathcal{I} \to \mathbb{R}^N \times \mathbb{R}$ is non–degenerate
- *ii)* for any $l \in \mathbb{Z}^\infty$ with $0 < |l| \leq 2$ the map $(\omega, \langle l, \Omega\rangle) \colon \mathcal{I} \to \mathbb{R}^N \times \mathbb{R}$ is non–degenerate.

Remark. Condition *i)* implies that $\omega \colon \mathcal{I} \to \mathbb{R}^N$ is non–degenerate. Actually *i)* means that, for any $(c_1, \dots, c_N) \in \mathbb{R}^N \setminus \{0\}$, the function $c_1\omega_1 + \ldots + c_N\omega_N$ is not identically constant on $\mathcal{I}$.

Remark. The non–degeneracy of the first derivative of the frequency map (ω', Ω'), namely

i') $\omega' \colon \mathcal{I} \to \mathbb{R}^N$ is non–degenerate

ii') for any $l \in \mathbb{Z}^\infty$ with $0 < |l| \le 2$ the map $(\omega', \langle l, \Omega' \rangle) \colon \mathcal{I} \to \mathbb{R}^N \times \mathbb{R}$ is non–degenerate,

implies (ND).

Theorem 1. (Measure estimate) *Assume that the frequency map* (ω, Ω) *fulfills assumptions (A) and (ND). Take*

$$M(\mu_0)\varepsilon\gamma^{1-\mu_0} \le \beta/4\,, \quad M(\mu_0+1)\varepsilon\gamma^{-\mu_0} \le 1\,, \tag{1.8}$$

where $\mu_0 \in \mathbb{N}$, $\beta > 0$ *are defined in* (2.1) *and* $M(\mu_0)$ *in* (1.6). *Then there exist constants* τ, $\gamma_* > 0$, $\mu_* \ge \mu_0$, *depending on* d, N, μ_0, β, η *such that*

$$|\mathcal{I} \setminus \mathcal{I}^*| \le (1 + |\mathcal{I}|)\Big(\frac{\gamma}{\gamma_*}\Big)^{\frac{1}{\mu_*}}$$

for all $0 < \gamma \le \gamma_*$.

In [Rüs01] the constant β is called the "amount of non–degeneracy" and μ_0 the "index of non–degeneracy".

1.3. **Application: wave equation.** The previous results apply to the nonlinear wave equation with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - u_{xx} + V(x)u + \xi u + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.9}$$

where $V(x) \ge 0$ is an analytic, 2π-periodic, even potential $V(-x) = V(x)$, the mass ξ is a real parameter on an interval $\mathcal{I} := [0, \xi_*]$, the nonlinearity $f(x,u)$ is real analytic, odd in the two variables, i.e. for all $(x,u) \in \mathbb{R}^2$,

$$f(-x,-u) = -f(x,u)\,,$$

and

$$f(x,0) = (\partial_u f)(x,0) = 0\,. \tag{1.10}$$

For every choice of the indices $\mathcal{J} := \{j_1 < j_2 < \ldots < j_N\}$ the linearized equation $u_{tt} - u_{xx} + V(x)u + \xi u = 0$ possesses the quasi–periodic solutions

$$u(t,x) = \sum_{h=1}^{N} A_h \cos(\lambda_{j_h} t + \theta_h)\, \phi_{j_h}(x)$$

where $A_h, \theta_h \in \mathbb{R}$, and ϕ_j, resp. $\lambda_j^2(\xi)$, denote the simple Dirichlet eigenvectors, resp. eigenvalues, of $-\partial_{xx} + V(x) + \xi$. For $V(x) \ge 0$ (that we can assume with no loss of generality), all the Dirichlet eigenvalues of $-\partial_{xx} + V(x)$ are strictly positive.

Theorem 2. *Under the above assumptions, for every choice of indexes* $\mathcal{J} := \{j_1 < j_2 < \ldots < j_N\}$*, there exists* $r_* > 0$ *such that, for any* $A = (A_1, \ldots, A_N) \in \mathbb{R}^N$ *with* $|A| =: r \le r_*$*, there is a Cantor set* $\mathcal{I}^* \subset \mathcal{I}$ *with asymptotically full measure as* $r \to 0$*, such that, for all the masses* $\xi \in \mathcal{I}^*$*, the nonlinear wave equation* (1.9) *has a quasi–periodic solution of the form*

$$u(t,x) = \sum_{h=1}^{N} A_h \cos(\widetilde{\lambda}_h t + \theta_h)\, \phi_{j_h}(x) + o(r),$$

where $o(r)$ *is small in some analytic norm and* $\widetilde{\lambda}_h - \lambda_{j_h} \to 0$ *as* $r \to 0$*.*

2. Proof of Theorem 1

The first step is to use the analyticity of the linear frequencies to transform the non–degeneracy assumption (ND) into a quantitative non–resonance property, extending Rüssmann's Lemma 18.2 in [Rüs01] to infinite dimensions.

Proposition 3. *Let* $(\omega, \Omega) : \mathcal{I} \mapsto \mathbb{R}^N \times \mathbb{R}^\infty$ *satisfy assumptions (A) and (ND) on* $\mathcal{I}$*. Then there exist* $\mu_0 \in \mathbb{N}$ *and* $\beta > 0$ *such that*

$$\max_{0 \le \mu \le \mu_0} \left| \frac{d^\mu}{d\xi^\mu} (\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle) \right| \ge \beta(|k| + 1) \tag{2.1}$$

for all $\xi \in \mathcal{I}$*,* $(k, l) \in \mathcal{Z}_N$*.*

Technically, this is the most difficult part of the paper and its proof is developed in Section 4.

As a Corollary of Proposition 3 and by (1.6), also the final frequencies (ω^*, Ω^*) satisfy a non–resonance property similar to (2.1).

Lemma 1. *Assume* $M(\mu_0)\varepsilon\gamma^{1-\mu_0} \le \beta/4$*, where* μ_0 *and* β *are defined in Proposition 3 and* $M(\mu_0)$ *is the constant in* (1.6)*. Then*

$$\max_{0 \le \mu \le \mu_0} \left| \frac{d^\mu}{d\xi^\mu} \langle k, \omega^*(\xi) \rangle + \langle l, \Omega^*(\xi) \rangle \right| \ge \frac{\beta}{2}(|k| + 1) \tag{2.2}$$

for all $\xi \in \mathcal{I}$ *and* $(k, l) \in \mathcal{Z}_N$*.*

Proof. By (2.1) and (1.6) we get, for all $0 \le \mu \le \mu_0$,

$$\begin{aligned} \left| \frac{d^\mu}{d\xi^\mu} \langle k, \omega^*(\xi) \rangle + \langle l, \Omega^*(\xi) \rangle \right| &\ge \left| \frac{d^\mu}{d\xi^\mu} \langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle \right| \\ &- \left| \frac{d^\mu}{d\xi^\mu} \langle k, \omega^*(\xi) - \omega(\xi) \rangle + \langle l, \Omega^*(\xi) - \Omega(\xi) \rangle \right| \\ &\ge \beta(|k| + 1) - 2(|k| + 1) M(\mu_0) \varepsilon \gamma^{1-\mu} \\ &\ge (\beta/2)(|k| + 1) \end{aligned}$$

since $M(\mu_0)\varepsilon\gamma^{1-\mu_0} \le \beta/4$. □

We now proceed with the proof of Theorem 1. By (1.7) we have

$$\mathcal{I} \setminus \mathcal{I}^* \subset \bigcup_{(k,l)\in\mathcal{Z}_N} \mathcal{R}_{kl}(\gamma) \tag{2.3}$$

with resonant regions

$$\mathcal{R}_{kl}(\gamma) := \left\{ \xi \in \mathcal{I} \colon \frac{|\langle k, \omega^*(\xi)\rangle + \langle l, \Omega^*(\xi)\rangle|}{1+|k|} < \frac{2\gamma}{1+|k|^{\tau+1}} \langle l \rangle_d \right\}.$$

In the following we assume $0 < \gamma \leq 1/8$.

Lemma 2. *There is $L_* > 1$ such that*

$$\langle l \rangle_d \geq \max\{L_*, 8\Gamma|k|\} \quad \Longrightarrow \quad \mathcal{R}_{kl}(\gamma) = \emptyset\,.$$

Proof. The asymptotic assumption (A) and (1.6) imply that

$$\frac{\langle l, \Omega^* \rangle}{\langle l \rangle_d} \to 1 \quad \text{as} \quad \langle l \rangle_d \to +\infty\,, \quad \text{uniformly in } \xi \in \mathcal{I}\,.$$

So $|\langle l, \Omega^*\rangle| \geq \langle l \rangle_d/2$ for $\langle l \rangle_d \geq L_* > 1$. If $|k| \leq (1/8\Gamma)\langle l \rangle_d$ then $\mathcal{R}_{kl}(\gamma)$ is empty, because, for all $\xi \in \mathcal{I}$,

$$|\langle k, \omega^*(\xi)\rangle + \langle l, \Omega^*(\xi)\rangle| \geq \frac{\langle l \rangle_d}{2} - 2\Gamma|k| \geq 2\gamma\langle l \rangle_d \geq 2\gamma \langle l \rangle_d \frac{1+|k|}{1+|k|^{\tau+1}}$$

provided $0 < \gamma \leq 1/8$. $\square$

As a consequence, in the following we restrict the union in (2.3) to $\langle l \rangle_d < \max\{L_*, 8\Gamma|k|\}$.

Lemma 3. *There exists $B := B(\mu_0, \beta, \omega, \Omega, \eta) > 0$ such that, for any $(k,l) \in \mathcal{Z}_N$ satisfying $\langle l \rangle_d < \max\{L_*, 8\Gamma|k|\}$ and for all γ with*

$$0 < \gamma < \frac{\beta}{8(\mu_0+1)\max\{L_*, 8\Gamma\}}\,, \tag{2.4}$$

then

$$|\mathcal{R}_{kl}(\gamma)| \leq B(1+|\mathcal{I}|)\alpha^{\frac{1}{\mu_0}} \qquad \text{where} \qquad \alpha := \frac{2\gamma}{1+|k|^{\tau+1}}\langle l \rangle_d\,. \tag{2.5}$$

Proof. We use Theorem 17.1 in [Rüs01]. The C^∞–function

$$g^*_{kl}(\xi) := \frac{\langle k, \omega^*(\xi)\rangle + \langle l, \Omega^*(\xi)\rangle}{1+|k|}$$

satisfies, by (2.2),

$$\min_{\xi\in\mathcal{I}} \max_{0\leq\mu\leq\mu_0} \left| \frac{d^\mu}{d\xi^\mu} g^*_{kl}(\xi) \right| \geq \frac{\beta}{2}\,.$$

Moreover $\langle l \rangle_d < \max\{L_*, 8\Gamma|k|\}$ and (2.4) imply

$$\alpha < \max\{2L_*, 16\Gamma\}\gamma < \frac{\beta}{4(\mu_0+1)}\,.$$

Then the assumptions of Theorem 17.1 in [Rüs01] are satisfied and so

$$|\mathcal{R}_{kl}(\gamma)| \le B(\mu,\beta,\eta)(1+|\mathcal{I}|)\alpha^{\frac{1}{\mu_0}}|g^*_{kl}|^{\mu_0+1}_\eta$$

where

$$|g^*_{kl}|^{\mu_0+1}_\eta := \sup_{\xi\in\mathcal{I}_\eta\cap\mathbb{R}} \max_{0\le\nu\le\mu_0+1}\left|\frac{d^\nu}{d\xi^\nu}g^*_{kl}(\xi)\right|.$$

By (1.8), (1.6) and $\langle l\rangle_d \le \max\{L_*, 8\Gamma|k|\}$, we have that the norm $|g^*_{kl}|^{\mu_0+1}_\eta$ is controlled by a constant depending on ω,Ω and this implies (2.5). $\square$

Now the measure estimate proof continues as in [Pös96a].

Lemma 4. *Assume $d>1$, and*

$$\tau > \mu_0\left(N+\frac{2}{d-1}\right). \tag{2.6}$$

Then there is $\gamma_* := \gamma_*(N,\mu_0,\omega,\Omega,\beta,\eta,d) > 0$, such that, for any $\gamma\in(0,\gamma_*)$,

$$\left|\bigcup_{(k,l)\in\mathcal{Z}_N}\mathcal{R}_{kl}(\gamma)\right| \le (1+|\mathcal{I}|)\left(\frac{\gamma}{\gamma_*}\right)^{\frac{1}{\mu_0}}.$$

Proof. By Lemma 2 we have

$$\left|\bigcup_{(k,l)\in\mathcal{Z}_N}\mathcal{R}_{kl}(\gamma)\right| \le \sum_{\substack{0\le|k|\le\frac{L_*}{8\Gamma}\\ \langle l\rangle_d<L_*}}|\mathcal{R}_{kl}(\gamma)| + \sum_{\substack{|k|>\frac{L_*}{8\Gamma}\\ \langle l\rangle_d<8\Gamma|k|}}|\mathcal{R}_{kl}(\gamma)|\,. \tag{2.7}$$

We first estimate the second sum. By Lemma 3 and

$$\mathrm{card}\,\{l\colon \langle l\rangle_d < 8\Gamma|k|\} \le (8\Gamma|k|)^{\frac{2}{d-1}}$$

we get

$$\begin{aligned}\sum_{\substack{|k|>\frac{L_*}{8\Gamma}\\ \langle l\rangle_d<8\Gamma|k|}}|\mathcal{R}_{kl}(\gamma)| &\le \sum_{\substack{|k|>\frac{L_*}{8\Gamma}\\ \langle l\rangle_d<8\Gamma|k|}} B(1+|\mathcal{I}|)\left(\frac{2\gamma}{|k|^{\tau+1}}\langle l\rangle_d\right)^{\frac{1}{\mu_0}}\\ &\le C_1(1+|\mathcal{I}|)\gamma^{\frac{1}{\mu_0}}\sum_{k\in\mathbb{Z}^N\setminus\{0\}}(8\Gamma|k|)^{\frac{2}{d-1}}|k|^{-\frac{\tau}{\mu_0}}\\ &\le C_2(1+|\mathcal{I}|)\gamma^{\frac{1}{\mu_0}}\end{aligned}$$

by (2.6), for some constant $C_1, C_2 > 0$ depending on $N,\mu_0,\omega,\Omega,\beta,\eta,d$. Similarly the first sum in (2.7) is estimates by

$$\sum_{\substack{0\le|k|\le\frac{L_*}{8\Gamma}\\ \langle l\rangle_d<L_*}}|\mathcal{R}_{kl}(\gamma)| \le C_3(1+|\mathcal{I}|)\gamma^{\frac{1}{\mu_0}}$$

with $C_3>0$, and so the thesis follows for some $\gamma_*>0$ small enough. $\square$

Lemma 5. *Assume $d = 1$ and*

$$\tau > \mu_0(N+1)\Big(1 - \frac{\mu_0}{\delta}\Big). \tag{2.8}$$

Then there are positive constants γ_* and μ_* depending on $N, \mu_0, \omega, \Omega, \beta, \eta, \delta$ such that

$$\Big| \bigcup_{(k,l)\in\mathcal{Z}_N} \mathcal{R}_{kl}(\gamma) \Big| \le (1 + |\mathcal{I}|)\Big(\frac{\gamma}{\gamma_*}\Big)^{-\frac{\delta}{\mu_0(\mu_0-\delta)}}.$$

Proof. For $(k,l) \in \mathcal{Z}_N^+ := \mathcal{Z}_N \cap (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_{2+})$, where these sets are defined in (0.2), we estimate, as in the case $d > 1$,

$$\Big| \bigcup_{(k,l)\in\mathcal{Z}_N^+} \mathcal{R}_{kl}(\gamma) \Big| \le C_4(1 + |\mathcal{I}|)\gamma^{\frac{1}{\mu_0}} \tag{2.9}$$

for some $C_4 > 0$.

Let now $(k,l) \in \mathcal{Z}_N^- := \mathbb{Z}^N \times \mathcal{L}_{2-}$ and assume, without loss of generality, $i > j$, then $\langle l\rangle_d = i - j$. By the asymptotic behavior of Ω^* (see assumption (A) and (1.6)) and remembering that $\delta < 0$, there is a constant $a > 0$ such that

$$\Big| \frac{\Omega_i^* - \Omega_j^*}{i - j} - 1 \Big| \le \frac{a}{j^{-\delta}}, \qquad \text{for all } i > j\,. \tag{2.10}$$

Hence $\langle l, \Omega^*\rangle = \Omega_i^* - \Omega_j^* = i - j + r_{ij}$, with $|r_{ij}| \le \frac{a}{j^{-\delta}} m$ and $m := i - j$. Then we have $|\langle k, \omega^*\rangle + \langle l, \Omega^*\rangle| \ge |\langle k, \omega^*\rangle + m| - |r_{ij}|$, provided $|\langle k, \omega^*\rangle + m| \ge \left|\frac{a}{j^{-\delta}} m\right|$, from which follows that, for fixed k, l,

$$\mathcal{R}_{kl} \cap \mathcal{S}^+ \subseteq \mathcal{Q}_{kj}^m := \Big\{ \xi \in \mathcal{I} \colon \frac{|\langle k, \omega^*(\xi)\rangle + m|}{1 + |k|} < \frac{2\gamma}{1 + |k|^{\tau+1}} m + \frac{am}{(1 + |k|)j^{-\delta}} \Big\}$$

where we have set for simplicity $\mathcal{R}_{kl} := \mathcal{R}_{kl}(\gamma)$, and

$$\mathcal{S}^+ := \Big\{ \xi \in \mathcal{I} \colon \frac{|\langle k, \omega^*(\xi)\rangle + m|}{1 + |k|} \ge \frac{am}{(1 + |k|)j^{-\delta}} \Big\}.$$

Calling $\mathcal{S}^-$ the complementary set of $\mathcal{S}^+$, we have

$$\mathcal{R}_{kl} = \big(\mathcal{R}_{kl} \cap \mathcal{S}^-\big) \cup \big(\mathcal{R}_{kl} \cap \mathcal{S}^+\big) \subseteq \mathcal{Q}_{kj}^m$$

so we need to estimate $\mathcal{Q}_{kj}^m$. Notice first that $\mathcal{Q}_{kj}^m \subset \mathcal{Q}_{kj_0}^m$ if $j > j_0$, for some j_0 to be fixed later. For γ small enough the result in Lemma 2 applies also the set $\mathcal{Q}_{kj_0}^m$ and so we get

$$\Big| \bigcup_{(k,l)\in\mathcal{Z}_N^-} \mathcal{R}_{kl} \Big| \le \sum_{\substack{|k| \le \frac{L_*}{8\Gamma} \\ m < L*}} \Big(|\mathcal{Q}_{kj_0}^m| + \sum_{j<j_0} |\mathcal{R}_{kl}| \Big) + \sum_{\substack{|k| > \frac{L_*}{8\Gamma} \\ m < 8\Gamma|k|}} \Big(|\mathcal{Q}_{kj_0}^m| + \sum_{j<j_0} |\mathcal{R}_{kl}| \Big)$$

We start with the sum over $m < 8\Gamma|k|$, that we denote with (S_2). Using Lemma 3 we get

$$\begin{aligned}(S_2) &\leq C_5(1+|\mathcal{I}|)\left(\left(\frac{a}{|k|j_0^{-\delta}}\right)^{\frac{1}{\mu_0}} + \left(\frac{2\gamma}{|k|^{\tau+1}}\right)^{\frac{1}{\mu_0}} j_0\right)\sum_{m<8\Gamma|k|} m^{\frac{1}{\mu_0}} \\ &\leq C_6(1+|\mathcal{I}|)\gamma^{\frac{-\delta}{\mu_0(\mu_0-\delta)}}|k|^{1+\frac{\delta\tau}{\mu_0(\mu_0-\delta)}}\end{aligned}$$

having choosen j_0 as

$$j_0 := \left(\frac{a}{2}|k|^\tau\gamma^{-1}\right)^{\frac{1}{\mu_0-\delta}}.$$

Summing in $|k| \geq L_*/(8\Gamma)$ and using (2.8) yields

$$\sum_{\substack{|k|\geq L_*/(8\Gamma)\\ m<8\Gamma|k|}}\left(\left|\mathcal{Q}^m_{kj_0}\right| + \sum_{j<j_0}|\mathcal{R}_{kl}|\right) \leq C_7(1+|\mathcal{I}|)\gamma^{\frac{-\delta}{\mu_0(\mu_0-\delta)}},$$

with $C_7 > 0$. The estimate of the first sum follows in a similar way. Hence we have obtained the thesis for $\gamma_* > 0$ small enough. □

3. Proof of Theorem 2

We write (1.9) as an infinite dimensional Hamiltonian system introducing coordinates $q,\ p \in \ell^{a,p}$ by

$$u = \sum_{j\geq 1}\frac{q_j}{\sqrt{\lambda_j}}\phi_j\,, \quad v := u_t = \sum_{j\geq 1} p_j\sqrt{\lambda_j}\phi_j\,, \quad \lambda_j(\xi) := \sqrt{\mu_j+\xi}\,,$$

where μ_j and ϕ_j, are respectively the simple Dirichlet eigenvalues and eigenvectors of $-\partial_{xx} + V(x)$, normalized and orthogonal in $L^2(0,\pi)$. Note that $\mu_j > 0$ for all $j \geq 1$ because $V(x) \geq 0$. The Hamiltonian of (1.9) is

$$\begin{aligned}H_{NLW} &= \int_0^\pi\left(\frac{v^2}{2} + \frac{1}{2}(u_x^2 + V(x)u^2 + \xi u^2) + F(x,u)\right)dx \\ &= \frac{1}{2}\sum_{j\geq 1}\lambda_j(q_j^2+p_j^2) + G(q) \qquad (3.1)\end{aligned}$$

where $\partial_u F(x,u) = f(x,u)$ and

$$G(q) := \int_0^\pi F\Big(x, \sum_{j\geq 1} q_j\lambda_j^{-1/2}\phi_j\Big)\,dx\,. \qquad (3.2)$$

Note that since f satisfies only (1.10) then $G(q)$ could contain cubic terms.

Now we reorder the indices in such a way that $\mathcal{J} := \{j_1 < \ldots < j_N\}$ corresponds to the first N modes. More precisely we define a reordering $k \to j_k$ from $\mathbb{N} \to \mathbb{N}$ which is bijective and increasing both from $\{1,\ldots,N\}$ onto $\mathcal{J}$ and from $\{N+1, N+2, \ldots\}$ onto $\mathbb{N}\setminus\mathcal{J}$.

Introduce complex coordinates

$$z_k := \frac{1}{\sqrt{2}}(p_{j_k} + \mathrm{i}q_{j_k}), \quad \bar{z}_k := \frac{1}{\sqrt{2}}(p_{j_k} - \mathrm{i}q_{j_k})$$

and action-angle coordinates on the first N-modes

$$z_k := \sqrt{I_k + y_k}e^{\mathrm{i}x_k}, \quad 1 \le k \le N,$$

with

$$I_k \in \left(\frac{r^{2\theta}}{2}, r^{2\theta}\right], \quad \theta \in (0,1). \tag{3.3}$$

Then the Hamiltonian (3.1) assumes the form (1.1)-(1.2) with frequencies

$$\omega(\xi) := (\lambda_{j_1}(\xi), \dots, \lambda_{j_N}(\xi)), \quad \Omega(\xi) := (\lambda_{j_{N+1}}(\xi), \lambda_{j_{N+2}}(\xi), \dots).$$

The asymptotic assumption (A) holds with $d = 1$, $\delta = -1$ and $\eta = \mu_1/2$. Also the regularity assumption (R) holds with $\bar{p} = p + 1$, see Lemma 3.1 of [CY00].

By conditions (3.2), (1.10) and (3.3) the perturbation satisfies

$$\varepsilon := \gamma^{-1} \lvert X_P \rvert_{r,\bar{p},\mathcal{E}} = O(\gamma^{-1}r^{3\theta-2}), \qquad \sum_{2i+j_1+j_2=4} \sup_{\mathcal{I}\times\mathcal{D}(s,r)} \left|\partial_y^i \partial_z^{j_1} \partial_{\bar{z}}^{j_2} P\right| = O(1).$$

Fixed

$$\theta \in (2/3, 1), \quad \gamma := r^{\sigma}, \quad 0 < \sigma < (3\theta - 2)/\mu_0,$$

then, for $r > 0$ small enough, the KAM conditions (1.4)-(1.5) are verified as well as the smallness condition (1.8). It remains to verify assumption (ND).

Lemma 6. *The non–degeneracy condition (ND) holds.*

Proof. It is sufficient to prove that, for any $(c_0, c_1, \dots, c_N, c_h, c_k) \in \mathbb{R}^{N+3} \setminus \{0\}$ with $k > h > N$, the function $c_0 + c_1\lambda_{j_1} + \dots + c_N\lambda_{j_N} + c_h\lambda_{j_h} + c_k\lambda_{j_k}$ is not identically zero on $\mathcal{I} = [0, \xi_*]$. For simplicity of notation we denote $\lambda_l := \lambda_{j_l}$.

Suppose, by contradiction, that there exists $(c_0, c_1, \dots, c_N, c_h, c_k) \neq 0$ such that $c_0 + c_1\lambda_1 + \dots + c_N\lambda_N + c_h\lambda_h + c_k\lambda_k \equiv 0$. Then, taking the first $N+2$ derivatives, we get the system

$$\begin{cases} c_0 + c_1\lambda_1 + \dots + c_N\lambda_N + c_h\lambda_h + c_k\lambda_k = 0 \\ c_1\frac{d}{d\xi}\lambda_1 + \dots + c_N\frac{d}{d\xi}\lambda_N + c_h\frac{d}{d\xi}\lambda_h + c_k\frac{d}{d\xi}\lambda_k = 0 \\ \vdots \\ c_1\frac{d^{N+2}}{d\xi^{N+2}}\lambda_1 + \dots + c_N\frac{d^{N+2}}{d\xi^{N+2}}\lambda_N + c_h\frac{d^{N+2}}{d\xi^{N+2}}\lambda_h + c_k\frac{d^{N+2}}{d\xi^{N+2}}\lambda_k = 0. \end{cases}$$

Since this system admits a non–zero solution, the determinant of the associated matrix is zero. On the other hand this determinant is c_0 times the determinant of the $(N+2)\times(N+2)$ minor

$$D=\begin{bmatrix} \frac{d}{d\xi}\lambda_1(\xi) & \dots & \frac{d}{d\xi}\lambda_N(\xi) & \frac{d}{d\xi}\lambda_h(\xi) & \frac{d}{d\xi}\lambda_k(\xi) \\ \frac{d^2}{d\xi^2}\lambda_1(\xi) & \dots & \frac{d^2}{d\xi^2}\lambda_N(\xi) & \frac{d^2}{d\xi^2}\lambda_h(\xi) & \frac{d^2}{d\xi^2}\lambda_k(\xi) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{d^{N+2}}{d\xi^{N+2}}\lambda_1(\xi) & \dots & \frac{d^{N+2}}{d\xi^{N+2}}\lambda_N(\xi) & \frac{d^{N+2}}{d\xi^{N+2}}\lambda_h(\xi) & \frac{d^{N+2}}{d\xi^{N+2}}\lambda_k(\xi) \end{bmatrix}$$

which is different from 0, as we prove below. This implies $c_0=0$. Moreover the unique solution $(c_1,\dots c_N,c_h,c_k)$ of the system associated to D is zero. This is a contradiction.

In order to prove that the determinant of D is different from zero, we first observe that, by induction, for any $r\geq 1$,

$$\frac{d^r}{d\xi^r}\lambda_i(\xi)=\frac{(2r-3)!!}{2^r}\frac{(-1)^{r+1}}{(\mu^i+\xi)^{r-\frac{1}{2}}},$$

where, for n odd, $n!!:=n(n-2)(n-4)\dots 1$ and $(-1)!!:=1$. Setting $x_i=(\mu_i+\xi)^{-1}$ and using the linearity of the determinant, we obtain

$$\det D=\prod_{r=1}^{N+2}(-1)^{r+1}\frac{(2r-3)!!}{2^r}\left(\prod_{i=1}^{N}(\mu_i+\xi)^{-\frac{1}{2}}\right)(\mu_h+\xi)^{-\frac{1}{2}}(\mu_k+\xi)^{-\frac{1}{2}}$$
$$\cdot\det\begin{bmatrix} 1 & \dots & 1 & 1 & 1 \\ x_1 & \dots & x_N & x_h & x_k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{N+1} & \dots & x_N^{N+1} & x_h^{N+1} & x_k^{N+1} \end{bmatrix}$$

The last is a Vandermonde determinant which is not zero since all the x_i are all different from each other. For a similar quantitative estimate we refer to [Bam99]. □

In conclusion the KAM Theorem and Theorem 1 apply proving Theorem 2.

4. Quantitative non–resonance property: Proof of Proposition 3

Split the set $\mathcal{L}$ as in (0.2) and discuss the four cases separately.

Case $l\in\mathcal{L}_0$**.** *There exist* $\mu_0\in\mathbb{N}$, $\beta>0$ *such that*

$$\max_{0\leq\mu\leq\mu_0}\left|\frac{d^\mu}{d\xi^\mu}\langle k,\omega(\xi)\rangle\right|\geq\beta(1+|k|)$$

for all $\xi\in\mathcal{I}$, $k\in\mathbb{Z}^N\setminus\{0\}$.

Proceed by contradiction and assume that for all $\mu_0\in\mathbb{N}$ and for all $\beta>0$ there exist $\xi_{\mu_0,\beta}\in\mathcal{I}$, $k_{\mu_0,\beta}\in\mathbb{Z}^N\setminus\{0\}$ such that

$$\max_{0\leq\mu\leq\mu_0}\left|\frac{d^\mu}{d\xi^\mu}\langle\frac{k_{\mu_0,\beta}}{1+|k_{\mu_0,\beta}|},\omega(\xi_{\mu_0,\beta})\rangle\right|<\beta.$$

In particular, for all $\lambda := \mu_0 \in \mathbb{N}$, $\beta := 1/(\lambda+1)$, there exist $\xi_\lambda \in \mathcal{I}$, $k_\lambda \in \mathbb{Z}^N \setminus \{0\}$ such that

$$\max_{0\leq\mu\leq\lambda}\left|\frac{d^\mu}{d\xi^\mu}\langle\frac{k_\lambda}{1+|k_\lambda|},\omega(\xi_\lambda)\rangle\right| < \frac{1}{\lambda+1}\,,$$

namely, for all $\mu \geq 0$, for any $\lambda \geq \mu$, we have

$$\left|\frac{d^\mu}{d\xi^\mu}\langle\frac{k_\lambda}{1+|k_\lambda|},\omega(\xi_\lambda)\rangle\right| < \frac{1}{\lambda+1}. \tag{4.1}$$

By compactness there exist converging subsequences $\xi_{\lambda_h} \to \overline{\xi} \in \mathcal{I}$ and $\frac{k_{\lambda_h}}{1+|k_{\lambda_h}|} \to \overline{c} \in \mathbb{R}^N$ with $1/2 \leq |\overline{c}| \leq 1$ if $\lambda_h \to \infty$ as $h \to \infty$. Passing to the limit in (4.1), for any $\mu \geq 0$, we get

$$\frac{d^\mu}{d\xi^\mu}\langle\overline{c},\omega(\overline{\xi})\rangle = \lim_{h\to\infty}\frac{d^\mu}{d\xi^\mu}\langle\frac{k_{\lambda_h}}{1+|k_{\lambda_h}|},\omega(\xi_{\lambda_h})\rangle = 0\,,$$

namely the analytic function $\langle\overline{c},\omega(\xi)\rangle$ vanishes with all its derivatives at $\overline{\xi}$. Then $\langle\overline{c},\omega(\xi)\rangle \equiv 0$ on $\mathcal{I}$. This contradicts the assumption of non–degeneracy of ω.

Case $l \in \mathcal{L}_1$. *There exist $\mu_0 \in \mathbb{N}$, $\beta > 0$ such that*

$$\max_{0\leq\mu\leq\mu_0}\left|\frac{d^\mu}{d\xi^\mu}(\langle k,\omega(\xi)\rangle + \Omega_j(\xi))\right| \geq \beta(1+|k|)$$

for all $\xi \in \mathcal{I}$, $k \in \mathbb{Z}^N$, $j \geq N+1$.

Arguing by contradiction as above, we assume that for all $\lambda \in \mathbb{N}$ there exist $\xi_\lambda \in \mathcal{I}$, $k_\lambda \in \mathbb{Z}^N$, $j_\lambda \geq N+1$ such that

$$\max_{0\leq\mu\leq\lambda}\left|\frac{d^\mu}{d\xi^\mu}(\langle k_\lambda,\omega(\xi_\lambda)\rangle + \Omega_{j_\lambda}(\xi_\lambda))\right| < \frac{1}{\lambda+1}(1+|k_\lambda|). \tag{4.2}$$

The asymptotic assumption (A) implies

$$j^d \geq \Theta_1|k| + \Theta_2 \quad \Longrightarrow \quad \left|\frac{\langle k,\omega(\xi)\rangle + \Omega_j(\xi)}{1+|k|}\right| \geq \frac{1}{2}\,,\ \forall\xi\in\mathcal{I}\,,$$

with $\Theta_1 := 2\Gamma+1$, $\Theta_2 := \max\{1,(2\Gamma)^d\}$. Then, (4.2) implies that

$$j_\lambda^d < \Theta_1|k_\lambda| + \Theta_2\,, \quad \forall\lambda\geq 1\,. \tag{4.3}$$

By compactness $\xi_{\lambda_h} \to \overline{\xi}$ as $h \to \infty$. The indexes $k_\lambda \in \mathbb{Z}^N$, $j_\lambda \geq N+1$ belong to non–compact spaces and they could converge or not. Hence we have to separate the various cases.

Case k_λ bounded. By (4.3) also the sequence j_λ is bounded. So we extract constant subsequences $k_{\lambda_h} \equiv \overline{k}$, $j_{\lambda_h} \equiv \overline{\jmath}$. Passing to the limit in (4.2), we get, for any $\mu \geq 0$,

$$\frac{d^\mu}{d\xi^\mu}\left(\langle\frac{\overline{k}}{1+|\overline{k}|},\omega(\overline{\xi})\rangle + \frac{\Omega_{\overline{\jmath}}(\overline{\xi})}{1+|\overline{k}|}\right) = 0\,.$$

By the analyticity of ω, Ω, the function $\langle \overline{k}, \omega\rangle(\xi) + \Omega_{\overline{\jmath}}(\xi)$ is identically zero on $\mathcal{I}$. This contradicts the non–degeneracy of (ω, Ω_j).

Case k_λ unbounded. The quantity $\frac{k_\lambda}{1+|k_\lambda|}$ converges, up to subsequence, to $\overline{c} \in \mathbb{R}^N$, with $1/2 \leq |\overline{c}| \leq 1$.

If $\{j_\lambda\}$ is bounded, there is a subsequence $\{j_{\lambda_h}\}$ that is constantly equal to $\overline{\jmath}$. Passing to the limit in (4.2), we get, for any $\mu \geq 0$,

$$\frac{d^\mu}{d\xi^\mu}\langle \overline{c}, \omega(\overline{\xi})\rangle = \lim_{h\to\infty} \frac{d^\mu}{d\xi^\mu}\left(\langle \frac{k_{\lambda_h}}{1+|k_{\lambda_h}|}, \omega(\xi_{\lambda_h})\rangle + \frac{j^d_{\lambda_h} + \nu_{j_{\lambda_h}}(\xi_{\lambda_h}) j^\delta_{\lambda_h}}{1+|k_{\lambda_h}|}\right) = 0.$$

By the analyticity of ω we come to a contradiction with the non–degeneracy assumption on ω.

If $\{j_\lambda\}$ is unbounded there is a divergent subsequence $j_{\lambda_h} \to \infty$. Then we consider the first derivative of the function $\langle k, \omega(\xi)\rangle + \Omega_j(\xi)$, namely, recalling assumption (A) on Ω, the function $\langle k, \omega'(\xi)\rangle + \nu'_j(\xi) j^\delta$. The analyticity assumption (A) and Cauchy estimates imply that

$$\left|\frac{d^\mu}{d\xi^\mu}\nu_j(\xi)\right| \leq \frac{\Gamma}{\eta^\mu}, \quad \forall \xi \in \mathcal{I}, \ \mu \geq 0 . \tag{4.4}$$

Then, using also (4.3), there is a constant $\widetilde{\Theta}_1 > 0$ such that, for any $\mu \geq 0$,

$$\frac{d^\mu}{d\xi^\mu}\nu'_{j_{\lambda_h}} \frac{j^\delta_{\lambda_h}}{1+|k_\lambda|} \leq \widetilde{\Theta}_1 \frac{j^\delta_{\lambda_h}}{j^d_{\lambda_h}} \to 0 \quad \text{ as } \ h \to \infty$$

since $\delta < d-1$. Then, passing to the limit in (4.2) yields, for any $\mu \geq 0$,

$$\frac{d^\mu}{d\xi^\mu}\langle \overline{c}, \omega'(\overline{\xi})\rangle = 0 .$$

Hence $\langle \overline{c}, \omega'(\xi)\rangle$ and all its derivatives vanish at $\overline{\xi}$. By analyticity, $\langle \overline{c}, \omega'(\xi)\rangle$ is identically zero on $\mathcal{I}$ and then the function $\langle \overline{c}, \omega(\xi)\rangle$ is identically equal to some constant. This contradicts the non–degeneracy assumption on $(\omega, 1)$.

Case $l \in \mathcal{L}_{2+}$**.** *There exist $\mu_0 \in \mathbb{N}$, $\beta > 0$ such that*

$$\max_{0\leq\mu\leq\mu_0}\left|\frac{d^\mu}{d\xi^\mu}(\langle k, \omega(\xi)\rangle + \Omega_i(\xi) + \Omega_j(\xi))\right| \geq \beta(1+|k|)$$

for all $\xi \in \mathcal{I}$, $k \in \mathbb{Z}^N$, $i, j \geq N+1$.

This follows by arguments similar to the case $l \in \mathcal{L}_1$.

Case $l \in \mathcal{L}_{2-}$**.** *There exist $\mu_0 \in \mathbb{N}$, $\beta > 0$ such that*

$$\max_{0\leq\mu\leq\mu_0}\left|\frac{d^\mu}{d\xi^\mu}(\langle k, \omega(\xi)\rangle + \Omega_i(\xi) - \Omega_j(\xi))\right| \geq \beta(1+|k|)$$

for all $\xi \in \mathcal{I}$, $k \in \mathbb{Z}^N$, $i, j \geq N+1$, $i \neq j$.

Proceed by contradiction as above and assume that for all $\lambda \in \mathbb{N}$ there exist $\xi_\lambda \in \mathcal{I}$, $k_\lambda \in \mathbb{Z}^N$, $i_\lambda, j_\lambda \geq N+1$ such that

$$\max_{0 \leq \mu \leq \lambda} \left| \frac{d^\mu}{d\xi^\mu} \left(\langle \frac{k_\lambda}{1+|k_\lambda|}, \omega(\xi_\lambda) \rangle + \frac{\Omega_{i_\lambda}(\xi_\lambda)}{1+|k_\lambda|} - \frac{\Omega_{j_\lambda}(\xi_\lambda)}{1+|k_\lambda|} \right) \right| < \frac{1}{\lambda+1}.$$

In particular we have that for all $\lambda \geq \mu$

$$\left| \frac{d^\mu}{d\xi^\mu} \left(\langle \frac{k_\lambda}{1+|k_\lambda|}, \omega(\xi_\lambda) \rangle + \frac{\Omega_{i_\lambda}(\xi_\lambda)}{1+|k_\lambda|} - \frac{\Omega_{j_\lambda}(\xi_\lambda)}{1+|k_\lambda|} \right) \right| < \frac{1}{\lambda+1}. \tag{4.5}$$

The asymptotic behavior (A) of Ω implies

$$\begin{aligned} |\Omega_i(\xi) - \Omega_j(\xi)| &\geq |i^d - j^d| - |\nu_i(\xi) i^\delta| - |\nu_j(\xi) j^\delta| \\ &\geq \frac{|i-j|}{2} \Big(i^{d-1} + j^{d-1} \Big) - \Gamma \Big(i^\delta + j^\delta \Big) \\ &\geq \frac{1}{2} \Big(i^{d-1} + j^{d-1} \Big) - \Gamma \Big(i^\delta + j^\delta \Big) . \end{aligned} \tag{4.6}$$

Then, remembering that $\delta < d-1$, we have that

$$\min\{i,j\}^{d-1} \geq \Theta_3 |k| + \Theta_4 \implies |\langle k, \omega(\xi) \rangle + \Omega_i(\xi) - \Omega_j(\xi)| \geq \frac{1}{2}(1+|k|)$$

$\forall \xi \in \mathcal{I}$, with $\Theta_3 := 1 + 2\Gamma$ and $\Theta_4 := \max\{1, 4\Gamma^{(d-1)/(d-1-\delta)}\}$. Then (4.5) with $\mu = 0$ implies that

$$\min\{i_\lambda, j_\lambda\}^{d-1} < \Theta_3 |k_\lambda| + \Theta_4 \, , \ \forall \lambda \geq 1 \, . \tag{4.7}$$

By compactness, $\xi_{\lambda_h} \to \overline{\xi} \in \mathcal{I}$ as $h \to \infty$. The indexes $k_\lambda, i_\lambda, j_\lambda$ can be bounded or not, and we study the various cases separately.

Case k_λ bounded. If k_λ is bounded then $k_\lambda = \overline{k}$ for infinitely many λ. Then (4.7) implies that also the sequence $\min\{i_\lambda, j_\lambda\}$ is bounded. Assuming $j_\lambda < i_\lambda$, there exists a constant subsequence $j_{\lambda_h} \equiv \overline{\jmath}$.

If also i_λ is bounded, we extract a constant subsequence $i_{\lambda_h} \equiv \overline{\imath}$. Then, passing to the limit in (4.5), we obtain, for all $\mu \geq 0$,

$$\frac{d^\mu}{d\xi^\mu} \left(\langle \frac{\overline{k}}{1+|\overline{k}|}, \omega(\overline{\xi}) \rangle + \frac{\Omega_{\overline{\imath}}(\overline{\xi})}{1+|\overline{k}|} - \frac{\Omega_{\overline{\jmath}}(\overline{\xi})}{1+|\overline{k}|} \right) = 0 \, .$$

By analyticity, the function $\langle \overline{k}, \omega(\xi) \rangle + \Omega_{\overline{\imath}}(\xi) - \Omega_{\overline{\jmath}}(\xi)$ is identically zero on $\mathcal{I}$, contradicting the non–degeneracy assumption on $(\omega, \langle l, \Omega \rangle)$ with $l = e_{\overline{\imath}} - e_{\overline{\jmath}}$.

If i_λ is unbounded, we extract a divergent subsequence $\{i_{\lambda_h}\}$. Since k_λ, j_λ are bounded we deduce, by the asymptotic assumption (A), that, definitively for λ large,

$$\frac{1}{1+|k_\lambda|} \Big(\langle k_\lambda, \omega(\xi_\lambda) \rangle + \Omega_{i_\lambda}(\xi_\lambda) - \Omega_{j_\lambda}(\xi_\lambda) \Big) \geq \frac{i_\lambda^d}{2(1+|k_\lambda|)} \, ,$$

which tends to infinity for $\lambda \to +\infty$. This contradicts (4.5) with $\mu = 0$.

Case k_λ unbounded. If k_λ is unbounded, we extract a divergent subsequence such that $|k_{\lambda_h}| \to \infty$ as $h \to \infty$ and $\frac{k_{\lambda_h}}{1+|k_{\lambda_h}|} \to \bar c \in \mathbb{R}^N$ with $1/2 \le |\bar c| \le 1$.

Subcase $\max\{i_\lambda, j_\lambda\}$ *bounded.* For all $\mu \ge 0$, passing to the limit in (4.5), we have

$$\frac{d^\mu}{d\xi^\mu}\langle \bar c, \omega(\bar\xi)\rangle = 0\,.$$

This contradicts the non–degeneracy of ω.

Subcase $\max\{i_\lambda, j_\lambda\}$ *unbounded,* $\min\{i_\lambda, j_\lambda\}$ *bounded.* Assume, without loss of generality, $i_\lambda > j_\lambda$. In this case

$$\sup_{\xi\in\mathcal{I}}\sup_{\lambda} |\Omega_{j_\lambda}(\xi)| =: M < +\infty\,.$$

We extract a divergent subsequence i_{λ_h} and claim that, definitively,

$$i^d_{\lambda_h} < 2\Big(1 + (1+\Gamma)|k_{\lambda_h}| + M\Big)\,. \tag{4.8}$$

Otherwise, definitively for λ large,

$$\frac{1}{1+|k_{\lambda_h}|}\Big(\langle k_{\lambda_h}, \omega(\xi_{\lambda_h})\rangle + \Omega_{i_{\lambda_h}}(\xi_{\lambda_h}) - \Omega_{j_{\lambda_h}}(\xi_{\lambda_h})\Big) \ge 1\,,$$

which contradicts (4.5) for $\mu = 0$.

By (4.4), (4.8), and since j_{λ_h} are bounded, there is $\widetilde\Theta_2 > 0$ such that, for any $\mu \ge 0$,

$$\frac{j^\delta_{\lambda_h}}{1+|k_{\lambda_h}|}\frac{d^\mu}{d\xi^\mu}\nu'_{j_{\lambda_h}}(\xi_{\lambda_h}) \le \frac{\widetilde\Theta_2}{1+|k_{\lambda_h}|}\,, \qquad \frac{i^\delta_{\lambda_h}}{1+|k_{\lambda_h}|}\frac{d^\mu}{d\xi^\mu}\nu'_{j_{\lambda_h}} \le \widetilde\Theta_2\frac{i^\delta_{\lambda_h}}{i^d_{\lambda_h}}$$

and both tend to zero if $h \to \infty$. Hence, passing to the limit in (4.5) (start with the first derivative), we obtain, for any $\mu \ge 0$

$$\frac{d^\mu}{d\xi^\mu}\langle \bar c, \omega'(\bar\xi)\rangle = \lim_{h\to\infty}\frac{d^\mu}{d\xi^\mu}\langle \frac{k_{\lambda_h}}{1+|k_{\lambda_h}|}, \omega'(\xi_{\lambda_h})\rangle\,. \tag{4.9}$$

By analyticity, the function $\langle \bar c, \omega'(\xi)\rangle$ is identically zero on $\mathcal{I}$ and consequently the function $\langle \bar c, \omega\rangle(\xi)$ is identically equal to some constant. This contradicts the non–degeneracy assumption on the function $(\omega, 1)$.

Subcase $\min\{i_\lambda, j_\lambda\}$ *unbounded.* Relation (4.6) implies

$$|\Omega_{i_\lambda} - \Omega_{j_\lambda}| \ge \frac{1}{4}\Big(i^{d-1}_\lambda + j^{d-1}_\lambda\Big)$$

if $i^{d-1}_\lambda + j^{d-1}_\lambda \ge 4\Gamma\big(i^\delta_\lambda + j^\delta_\lambda\big)$, that is always verified definitively since $\delta < d-1$.

We claim that

$$i^{d-1}_\lambda + j^{d-1}_\lambda < 4(\Gamma+1)|k_\lambda| + 4\,.$$

Otherwise, definitively for λ large,

$$\frac{|\langle k_\lambda, \omega(\xi_\lambda)\rangle + \Omega_{i_\lambda}(\xi_\lambda) - \Omega_{j_\lambda}(\xi_\lambda)|}{1+|k_\lambda|} \geq 1$$

which contradicts (4.5) for $\mu = 0$.

We extract diverging subsequences $i_{\lambda_h}, j_{\lambda_h}$ such that

$$i_{\lambda_h}^{d-1} \leq 4(\Gamma+1)|k_{\lambda_h}| + 4 \qquad \text{and} \qquad j_{\lambda_h}^{d-1} \leq 4(\Gamma+1)|k_{\lambda_h}| + 4\,.$$

Then, using also (4.4), there is $\widetilde{\Theta}_3 > 0$ such that, for any $\mu \geq 0$,

$$\frac{i_{\lambda_h}^{\delta}}{1+|k_{\lambda_h}|}\frac{d^\mu}{d\xi^\mu}\nu'_{i_{\lambda_h}} \leq \widetilde{\Theta}_3 \frac{i_{\lambda_h}^{\delta}}{i_{\lambda_h}^{d-1}} \longrightarrow 0$$
$$\frac{j_{\lambda_h}^{\delta}}{1+|k_{\lambda_h}|}\frac{d^\mu}{d\xi^\mu}\nu'_{j_{\lambda_h}} \leq \widetilde{\Theta}_3 \frac{j_{\lambda_h}^{\delta}}{j_{\lambda_h}^{d-1}} \longrightarrow 0$$

for $h \to \infty$.

We deduce as in (4.9) that all the derivatives of $\langle \overline{c}, \omega'(\overline{\xi})\rangle$ vanish and by analyticity this contradicts the non–degeneracy assumption on $(\omega, 1)$.

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Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Milano (Italy)

E-mail address: dario.bambusi@unimi.it

Dipartimento di Matematica e applicazioni "R. Caccioppoli", Università degli Studi di Napoli Federico II, Napoli (Italy), Supported by the European Research Council under FP7 "New Connections between dynamical systems and Hamiltonian PDEs with small divisors phenomena"

E-mail address: elisa.magistrelli@unina.it

E-mail address: m.berti@unina.it