An alternative to the Schwarzschild solution

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Abstract

We propose an alternative to the famous Schwarzschild solution to Einstein field equations which arises from a different physical hypothesis. In this solution, which works for the external geometry of a spherical static star and circumnavigates the Birkhoff Theorem, Black-Hole Singularities are removed at the classical level, i.e. without arguments of quantum gravity.

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1 Introduction

The concept of Black-Hole (BH) has been considered very fascinating by scientists even before the introduction of general relativity (see [1] for an historical review). However, an unsolved problem concerning such objects is the presence of a space-time singularity in their core. Such a problem was present starting by the firsts historical papers concerning BHs [2, 3, 4]. It is a common opinion that this problem could be solved when a correct quantum gravity theory will be, finally, obtained, see [5] for recent developments.

On the other hand, fundamental issues which dominate the question about the existence or non-existence of BH horizons and singularities and some ways to avoid the development of BH singularities at a classical level, which does not require the need for a quantum gravity theory, have been discussed by various authors in the literature, see references from [6] to [19].

In general relativity the Einstein field equations connect the curvature tensor of space-time on the left hand side to the energy-momentum tensor in space-time on the right hand side [1]. A key point is that the *strong principle of equivalence* (SPOE) implies that special relativity has to hold locally for all of the laws of physics in all of space-time as seen by time-like observers [20]). Then, the frame of reference of co-moving observers within a gravitationally collapsing object have to always be able to be connected to the frame of reference of stationary observers by special relativistic transformations with physical velocities which are less than the speed of light. Recently, various arguments and computations have been introduced to support the idea that physically acceptable solutions to the Einstein field equations will only be the ones preserving the SPOE as *a law of nature* in the universe [6, 7, 8]. As consequence, the compact objects which emerge from the process of gravitational collapse could not have event horizons (EHs). In fact, the existence of EHs would prevent co-moving observers within a gravitationally collapsing object from being able to be connected to the frame of reference of stationary observers by special relativistic transformations with physical velocities which are less than the speed of light. Hence, a result of the SPOE is that objects with non-zero mass having EHs could be physically prohibited [6, 7, 8]. The preservation of the SPOE in the field equations would put an overall constraint on the nature of the non-gravitational physical elements which go into the energy-momentum tensor on the right hand side of the field equations. This constraint does not determine in a unique way the specific form of the non-gravitational dynamics of the energy-momentum tensor [7, 8, 9]. Various different theories have been constructed (e.g. *eternally collapsing objects* (ECO), *magneto-spheric eternally collapsing objects* (MECO), *non-linear electrodynamics* (NLED) *objects*), which preserve the SPOE and hence can generate highly redshifted compact objects without EHs and Singularities [7, 8], [9]-[14]. Since each of these different SPOE preserving theories have unique observational predictions associated with the interaction of their non-gravitational components with the environment of their highly redshifted compact objects without EHs and Singularities, the specific one chosen by Nature can only be determined by astrophysical observations which test these predictions.

Let us emphasize another fundamental point [15]-[19]. The basis of general relativity are 10 non-linear coupled equations. It is now well known that while such equations may yield various solutions, most of them could have no bearing with physical reality. Let us consider the Reissner-Nordström solution, originally derived as vacuum space-time exterior to a spherically symmetric charged fluid [1]. It is not only a correct solution, but obviously, a solution with a strong physical meaning. This solution is usually interpreted as a charged BH solution which forces the charged fluid with radius R_0 to be contracted to a geometrical point, $R_0 \to 0$ [1]. From a purely mathematical point of view it is granted that such a process could be possible because mathematics allows it [1]. But, from a physical point of view, let us ask "*do physics allow a charged fluid to be contracted into a geometrical non-dimensional point?*" New ideas of mathematical physicists are going beyond this purely mathematical "*take for granted*"[15]-[19]. The authors of [19] claim that a charged sphere may collapse to a quasi BH rather than to an exact BH; i.e., no trapped surfaces, no EH, while R_0 will eventually hover over the running mathematical EH.

On the other hand, the famous Schwarzschild solution is obtained for a neutral mass point and the BH mass came from the integration constant $\alpha_0 = 2GM_0/c^2$ [1]. From both of physical and mathematical points of view, integrations constants do not need always to be finite or arbitrary. Even though as algebra symbols, they may appear to assume arbitrary value. Considering the generalization given by the Birkhoff theorem, the same solution also represents the vacuum space-time exterior to a static neutral spherical fluid. In this case, the corresponding general integration constant $\alpha = 2GM/c^2$ indeed gives gravitational mass of the fluid having a radius R_0, i.e. $M = \int_0^{R_0} 4\pi\rho R^2 dR$.

Let us ask: *how does the value of M would change if we would allow $R_0 \to 0$? Is there any guarantee that M must remain finite as $R_0 \to 0$?* If this would be seen as an adiabatic gravitational collapse without emission of radiation, one would say, yes, one must have $M = M_0$ (no loss of mass-energy). But in [16] it has been shown that an adiabatic collapse could not exist in the physical reality. In that case, all those so-called adiabatic collapses could be only respectable looking mathematical exercises with respect a different, non-singular physical reality. In [18] it has been proposed that, like a consequence, when the fluid sphere would be assumed to have contracted to a point,

$$M \to M_0 = Lim_{R_0 \to 0} \int_0^{R_0} 4\pi\rho R^2 dR = 0. \tag{1}$$

In other words, the mass of a Schwarzschild BH that arises from a real, non adiabatic collapse could be zero.

In this paper, we propose an alternative to the Schwarzschild solution to Einstein field equations which starts from a different physical hypothesis. In this solution, which works for the external geometry of a spherical static source (a star) and circumnavigates the Birkhoff Theorem, Black-Hole Singularities are removed at the classical level. Arguments of quantum gravity are not taken into account.

Note that we are *not* claiming that our solution is correct and the Schwarzschild solution is wrong. Actually, we will see that the two solutions are incompatible as they arises from *different physical hypotheses*. Only the Nature can chose which is the correct one among the physical hypotheses and, consequently, which is the correct solution for the external space-time geometry of a spherical static star.

2 The solution

Following [21] and working with $G = 1$, $c = 1$ and $\hbar = 1$ (natural units), the more general line-element which respects central symmetry is

$$ds^2 = h(r,t)dr^2 + k(r,t)(\sin^2\theta d\varphi^2 + d\theta^2) + l(r,t)dt^2 + a(r,t)drdt, \tag{2}$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ $0 \leq \varphi \leq 2\pi$.

We search a line-element solution in which the metric is spatially symmetric with respect to the origin of the coordinate system, i.e. that we find again the same solution when spatial coordinates are subjected to a orthogonal transformations and rotations and it is asymptotically flat at infinity [1]. In order to obtain the standard Schwarzschild solution to Einstein field equations in vacuum one uses transformations of the type [21]

$$r = f_1(r', t'), \qquad t = f_2(r', t') \ , \tag{3}$$

where f_1 and f_1 are arbitrary functions of the new coordinates r' and t'. At this point, if one wants the standard Schwarzschild solution, r and t have to be chosen in a way that $a(r,t) = 0$ and $k(r,t) = -r^2$ [21]. In particular, the second condition implies that the standard Schwarzschild radius is determined in a way which guarantees that the length of the circumference centred in the origin of the coordinate system is $2\pi r$ [21].

In our approach, we will suppose again that $a(r,t) = 0$, but, differently from the standard analysis, we will assume that the length of the circumference centred in the origin of the coordinate system *is not* $2\pi r$. We release *a different physical assumption*, i.e. that arches of circumference *are not Euclidean*, but they are deformed by the presence of the mass of the central body M. Note that this different physical hypothesis permits to circumnavigate the Birkhoff Theorem [3] which leads to the Schwarzschild solution [1, 3]. In fact, the demonstration of the Birkhoff Theorem *starts* from a line element *in which* $k(r,t) = -r^2$ *has been chosen*, see the discussion in paragraph 32.2 of [1] and, in particular, look at Eq. (32.2) of such a paragraph.

Then, we proceed assuming $k = -mr^2$, where m is a generic function to be determined in order to obtain that the length of circumferences centred in the origin of the coordinate system are not Euclidean. In other words, m represents *a measure of the non-Euclidean behaviour* of circumferences centred in the origin of the coordinate system.

The line element (2) becomes

$$ds^2 = hdr^2 - mr^2(\sin^2\theta d\varphi^2 + d\theta^2) + ldt^2. \tag{4}$$

One puts

$$\begin{aligned} X &\equiv \tfrac{1}{3}r^3 \\ Y &\equiv -\cos\theta \\ Z &\equiv \varphi. \end{aligned} \tag{5}$$

In the X, Y, Z coordinates the line-element (4) reads

$$ds^2 = ldt^2 + \frac{h}{r^4}dX^2 - mr^2[\frac{dY^2}{1-Y^2} + dZ^2(1-Y^2)]. \tag{6}$$

Let us consider three functions

$$A \equiv -\tfrac{h}{r^4}$$
$$B \equiv mr^2 \tag{7}$$
$$C \equiv l$$

which satisfy the conditions

$$X \to \infty \; implies \; A \to \tfrac{1}{r^4} = \tfrac{1}{(3X)^{\frac{4}{3}}}, \;\; B \to r^2 = 3X^{\frac{2}{3}}, \;\; C \to 1$$
$$normalization \; condition \; AB^2C = 1. \tag{8}$$

The line-element (6) becomes

$$ds^2 = Cdt^2 - AdX^2 - B\frac{dY^2}{1-Y^2} - BdZ^2(1-Y^2). \tag{9}$$

From the metric (9) one gets the Christoffell coefficients like (only the non zero elements will be written down)

$$\begin{array}{ll}
\Gamma^t_{tX} = -\frac{1}{2C}\frac{\partial C}{\partial X} & \Gamma^X_{XX} = -\frac{1}{2A}\frac{\partial A}{\partial X} \\
\Gamma^X_{YY} = \frac{1}{2A}\frac{\partial B}{\partial X}\frac{1}{1-Y^2} & \Gamma^X_{ZZ} = \frac{1}{2A}\frac{\partial B}{\partial X}(1-Y^2) \\
\Gamma^X_{tt} = -\frac{1}{2A}\frac{\partial C}{\partial X} & \Gamma^Y_{YX} = -\frac{1}{2B}\frac{\partial B}{\partial X} \\
\Gamma^Y_{YY} = \frac{-Y}{1-Y^2} & \Gamma^Y_{ZZ} = -Y(1-Y^2) \\
\Gamma^Z_{ZX} = -\frac{1}{2B}\frac{\partial B}{\partial X} & \Gamma^Z_{ZX} = \frac{Y}{1-Y^2}.
\end{array} \tag{10}$$

By using the equation for the components of the Ricci tensor, the components of Einstein field equation in vacuum are [21]

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{dx^l} - \frac{\partial \Gamma^l_{il}}{dx^k} - \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km} = 0. \tag{11}$$

By inserting Eqs. (10) in Eqs. (11) one gets only three independent relations

$$\frac{\partial}{\partial X}(\frac{1}{A}\frac{\partial B}{\partial X}) - 2 - \frac{1}{AB}(\frac{\partial B}{\partial X})^2 = 0 \tag{12}$$

$$\frac{\partial}{\partial X}(\frac{1}{A}\frac{\partial A}{\partial X}) - \frac{1}{2}(\frac{1}{A}\frac{\partial A}{\partial X})^2 - (\frac{1}{B}\frac{\partial B}{\partial X})^2 - \frac{1}{2}(\frac{1}{C}\frac{\partial C}{\partial X})^2 = 0 \tag{13}$$

$$\frac{\partial}{\partial X}(\frac{1}{A}\frac{\partial C}{\partial X}) - \frac{1}{AC}(\frac{\partial C}{\partial X})^2 = 0. \tag{14}$$

From the second of Eqs. (8) (normalization condition) one gets also

$$\frac{1}{A}\frac{\partial A}{\partial X}+\frac{2}{B}\frac{\partial B}{\partial X}+\frac{1}{C}\frac{\partial C}{\partial X}=0. \tag{15}$$

Eq. (14) can be rewritten like

$$\frac{\partial}{\partial X}(\frac{1}{C}\frac{\partial C}{\partial X})=\frac{1}{AC}\frac{\partial A}{\partial X}\frac{\partial C}{\partial X}, \tag{16}$$

which can be integrated, giving

$$\frac{1}{C}\frac{\partial C}{\partial X}=aA, \tag{17}$$

where a is an integration constant.

By adding Eq. (13) to Eq. (16) one gets

$$\frac{\partial}{\partial X}(\frac{1}{A}\frac{\partial A}{\partial X}+\frac{1}{C}\frac{\partial C}{\partial X})=(\frac{1}{B}\frac{\partial B}{\partial X})^2+\frac{1}{2}(\frac{1}{A}\frac{\partial A}{\partial X}+\frac{1}{C}\frac{\partial C}{\partial X})^2 \tag{18}$$

Considering Eq. (15) we obtain

$$2\frac{\partial}{\partial X}(\frac{1}{B}\frac{\partial B}{\partial X})=-3(\frac{1}{B}\frac{\partial B}{\partial X})^2, \tag{19}$$

which can be integrated, giving

$$\frac{1}{B}\frac{\partial B}{\partial X}=\frac{2}{3X+b}, \tag{20}$$

where b is an integration constant. A second integration gives

$$B=d(3X+b)^{\frac{2}{3}}. \tag{21}$$

where d is an integration constant. But the first of Eqs. (8) implies $d=1$, thus

$$B=(3X+b)^{\frac{2}{3}}. \tag{22}$$

By using Eqs. (17) and (15) we obtain

$$\frac{\partial C}{\partial X}=aAC=\frac{a}{B^2}=a(3X+b)^{-\frac{4}{3}}.$$

By integrating and considering the first of Eqs. (8) one gets

$$C=1-a(3X+b)^{-\frac{1}{3}} \tag{23}$$

Then, from Eq. (15) one obtains

$$A=\frac{(3X+b)^{-\frac{4}{3}}}{1-a(3X+b)^{-\frac{1}{3}}}. \tag{24}$$

By putting Eqs. (24) and (22) in Eq. (12) one immediately sees that this last equation is automatically satisfied.

We note that the function A results singular for values $a(3X+b)^{-\frac{1}{3}}=1$. However, this is a mathematical singularity due to the particular coordinates t, X, Y, Z defined by the transformation (5). In fact, by assuming that such a singularity is located at $X=0$ we get

$$b = a^3, \tag{25}$$

i.e. we find a relation between the two integration constants b and a.
At the end we obtain

$$\begin{gathered}
A = (r^3+a^3)^{-\frac{4}{3}}[1-a(r^3+a^3)^{-\frac{1}{3}}]^{-1} \\
B = (r^3+a^3)^{\frac{2}{3}} \\
C = 1-a(r^3+a^3)^{-\frac{1}{3}}.
\end{gathered} \tag{26}$$

By inserting the functions (26) in Eq. (9) and using Eqs. (7) and (5) to return to the standard polar coordinates the line-element solution reads

$$ds^2 = (1-\frac{a}{(r^3+a^3)^{\frac{1}{3}}})dt^2 - (r^3+a^3)^{\frac{2}{3}}(\sin^2\theta d\varphi^2 + d\theta^2) + \frac{d[(r^3+a^3)^{\frac{2}{3}}]}{1-\frac{a}{(r^3+a^3)^{\frac{1}{3}}}}. \tag{27}$$

Hence, we understand that the assumption to locate the mathematical singularity of the function A at $X=0$ coincides with the physical condition that the length of the circumference centred in the origin of the coordinate system is $2\pi(r^3+a^3)^{\frac{1}{3}}$, which is different from the Euclidean value $2\pi r$. This is the fundamental physical difference between the proposed solution and the standard Schwarzschild solution. The value of the generic function m which permits that the length of circumferences centred in the origin of the coordinate system are not Euclidean is

$$m = \frac{(r^3+a^3)^{\frac{2}{3}}}{r^2}. \tag{28}$$

On the other hand, in order to determinate the value of the constant a, by following [21] one can use the weak field approximation which implies $g_{00} \cong 1 + 2\varphi$ at large distances, where $g_{00} = (1-\frac{a}{(r^3+a^3)^{\frac{1}{3}}})$ in Eq. (27) and $\varphi \equiv \frac{-M}{r}$ is the Newtonian potential. Thus, for $a \ll r$, we immediately obtain: $a = 2M = r_g$, i.e. a results exactly the gravitational radius [1, 21].

Then, we can rewrite the solution (27) in an ultimate way like

$$ds^2 = (1-\frac{r_g}{(r^3+r_g^3)^{\frac{1}{3}}})dt^2 - (r^3+r_g^3)^{\frac{2}{3}}(\sin^2\theta d\varphi^2 + d\theta^2) + \frac{d[(r^3+r_g^3)^{\frac{2}{3}}]}{1-\frac{r_g}{(r^3+r_g^3)^{\frac{1}{3}}}}. \tag{29}$$

Some comments are needed. By defining

$$\widehat{r} \equiv (r^3 + r_g^3)^{\frac{1}{3}}, \tag{30}$$

eq. (27) becomes

$$ds^2 = (1 - \frac{r_g}{\widehat{r}})dt^2 - \widehat{r}^2(\sin^2\theta d\varphi^2 + d\theta^2) + \frac{d\widehat{r}^2}{1 - \frac{r_g}{\widehat{r}}}. \tag{31}$$

Eq. (31) looks *formally equal* to standard Schwarzschild solution, but it is *physically different.* In fact, from Eq. (30) it is *always* $\widehat{r} \geq r_g$ in Eq. (31) as it is $r \geq 0$ in Eq. (29). In this way, *there are not physical singularities* in Eq. (31). In fact, $r = 0$ in Eq. (29) implies $\widehat{r} = r_g$ in Eq. (31) which corresponds to the mathematical singularity at $X = 0$. This singularity is *not physical* but is due to the particular coordinates t, X, Y, Z defined by the transformation (5).

We emphasize that the solution Eq. (31) and the Schwarzschild solution *are not* comparable as they arises from *different physical hypotheses.* As it is carefully explained in [21], the standard Schwarzschild solution arises from the hypothesis that the coordinates r and t of the two functions (3) are chosen in order to guarantee that the length of the circumference centred in the origin of the coordinate system is $2\pi r$. Indeed, in the proposed solution, r and t are chosen in order to guarantee that the length of the circumference centred in the origin of the coordinate system is *not* $2\pi r$, i.e. the circumferences centred in the origin of the coordinate system are *not* Euclidean. In particular, choosing to put the mathematical singularity of the function A at $X = 0$ is equivalent to the physical condition that the length of the circumference centred in the origin of the coordinate system is $2\pi(r^3 + r_g^3)^{\frac{1}{3}}$.

Thus, again we emphasize that we are *not* claiming that the solution proposed in this paper is correct and the Schwarzschild solution is wrong. Only the Nature can chose which is the correct one among the physical hypotheses and, consequently which is the correct solution for the external space-time geometry of a spherical static star. If the Nature decides that the length of the circumference centred in the origin of the coordinate system is $2\pi r$ the correct solution is the Schwarzschild one. If the Nature decides that the length of the circumference centred in the origin of the coordinate system is $2\pi(r^3 + r_g^3)^{\frac{1}{3}}$ the correct solution is the one of Eq. (29).

Notice that the space-time defined by the line element solution (29) is devoid of trapped surfaces. Thus, the Singularity Theorem in [22] *cannot be applied.*

We also emphasize that a large distances, i.e. where $r_g \ll r$, the proposed solution coincides with the standard Schwarzschild solution, thus, both of the weak field approximation and the analysis of astrophysical situations remain the same.

3 Conclusion remarks

In this paper an alternative to the famous Schwarzschild solution to Einstein field equations has been analysed. The solution arises from a different physical hypothesis, which assumes that that arches of circumference centred in the origin of the coordinate system *are not Euclidean.* For the external geometry of a spherical static star the Birkhoff Theorem is circumnavigated and the Black-Hole Singularities are removed at the classical level, i.e. without arguments of quantum gravity.

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