On the solvability conditions for the diffusion equation with convection terms

Vitali Vougalter[1], Vitaly Volpert[2]
[1] University of Toronto, Department of Mathematics, Toronto, ON, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

[2] Institute Camille Jordan, UMR 5208 CNRS, University Lyon 1, 69622 Villeurbanne, France
e-mail: volpert@math.univ-lyon1.fr

Abstract. Linear second order elliptic equation describing heat or mass diffusion and convection on a given velocity field is considered in $\mathbb{R}^3$. The corresponding operator L may not satisfy the Fredholm property. In this case, solvability conditions for the equation $Lu = f$ are not known. In this work, we derive solvability conditions in $H^2(\mathbb{R}^3)$ for the non self-adjoint problem by relating it to a self-adjoint Schrödinger type operator, for which solvability conditions are obtained in our previous work [13].

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1 Introduction

Reaction-diffusion equations with with convective terms have been studied extensively in recent years, in particular with applications to nonlinear propagation phenomena (see e.g. [3], [2], [4], [10], [11]). Analysis of such equations is often based on the solvability conditions of linear problems. Classical results for elliptic equations affirm that they are solvable if and only if the right-hand side is orthogonal to the solutions of the homogeneous adjoint problem. This is so-called Fredholm alternative. It appears that it may not be applicable for reaction-diffusion problems in unbounded domains. In this case, solvability conditions are not established. In this work we study reaction-diffusion equations in the case where the corresponding operator does not satisfy the Fredholm property and obtain for them solvability conditions. We consider the equation in $\mathbb{R}^3$:

$$\Delta u + v.\nabla u + c(x)u = f(x), \tag{1.1}$$

where u is the temperature or concentration distribution, the first term in the left-hand side of this equation describes its diffusion, the second term convection, and the last term its production. The conditions on the coefficient $c(x)$ will be specified below. The fluid velocity

$v = (v_1, v_2, v_3)$ is a vector-function, and dot denotes the scalar product of two vectors in three dimensions. Assume that $v = -\nabla p$, where p is the pressure of the fluid. This is Darcy's law describing fluid motion in a porous medium. If the density of the fluid is constant, then, in the presence of the sources or sinks, we have $div\ v = F$, where F is some given function. Together with Darcy's law we obtain the equation for the pressure $\Delta p = -F$. Other equations for the pressure can also be considered (see Appendix).

We study in this work solvability conditions for equation (1.1). Let us recall that the classical Fredholm solvability condition for the operator equation $Lu = f$ affirms that this equation is solvable if and only if $(\phi_i, f) = 0$ for a finite number of functionals ϕ_i from the space E^* dual to the space E which contains the image of the operator. This solvability condition is applicable if the operator L satisfies the Fredholm property, that is its image is closed, the dimension of the kernel is finite, the codimension of the image (or the number of solvability conditions) is also finite.

Elliptic problems in unbounded domains satisfy the Fredholm property if and only if the corresponding limiting operators are invertible [12]. Let us assume that $c(x) = c_0 + c_1(x)$ where c_0 is a constant and the function $c_1(x)$ converges to zero at infinity. Then the Fredholm property is satisfied if $c_0 < 0$ and it is not valid if $c_0 \geq 0$. In the latter case the image of the operator L corresponding to the left-hand side of equation (1.1) is not closed, and the solvability conditions are not known. In this work we will establish solvability conditions for the non-Fredholm operator L in the case where c_0 is non-negative. To the best of our knowledge it is the first result on the solvability conditions of such equations in $\mathbb{R}^n$ with $n > 1$. In the case $n = 1$ the situation is different and non-Fredholm operators can be studied by introduction of weighted spaces [12] or reducing them to some integro-differential equations [2]. These methods are not applicable for $n > 1$. We will use here our previous results on the solvability conditions for non-Fredholm equations of the Schrödinger type [13] where we used the spectral theory of self-adjoint operators.

In order to formulate the solvability conditions we need to introduce the homogeneous formally adjoint equation

$$\Delta w - div(vw) + c(x)w = 0. \tag{1.2}$$

We will show that equation (1.1) is solvable if and only if its right-hand side f is orthogonal in L^2 to all bounded solutions of the homogeneous adjoint equation (1.2). The exact formulation is given below in Theorem 3. This condition looks similar to the Fredholm type solvability conditions. However, this similarity is only formal since the operator does not satisfy the Fredholm property.

2 Assumptions and main result

We assume that the right-hand side of equation (1.1) satisfies the following conditions.

Assumption 1. *The function* $f(x) \in L^2(\mathbb{R}^3)$ *and* $|x|f(x) \in L^1(\mathbb{R}^3)$.

This assumption is analogous to Assumption 1.1 of [13]. Related to problem (1.1) there is the homogeneous equation

$$\Delta u + v.\nabla u + c(x)u = 0 \tag{2.3}$$

and the corresponding operator $L := \Delta + v.\nabla + c(x)$. We show that problem (1.1) with the non self-adjoint operator L involving first derivatives can be related to the nonhomogeneous problem for the standard Schrödinger operator, namely

$$H_a z = g, \tag{2.4}$$

where $H_a := -\Delta + W_a(x),\ a \geq 0$ and the potential function

$$W_a(x) := \frac{(\nabla p)^2}{4} - \frac{\Delta p}{2} - c(x)$$

tends to the constant $-a$ at infinity. We will use the notation $W_0(x) = W_a(x) + a$. It is a shallow and short-range potential (Assumption 2 below), analogously to Assumption 1.1 of [13]. Thus the essential spectrum of H_a on $L^2(\mathbb{R}^3)$ coincides with the semi-axis $[-a, \infty)$ (see e.g. [5]) and the Fredholm alternative theorem fails to work for problem (2.4). The orthogonality relations which enable us to solve such a problem in $L^2(\mathbb{R}^3)$ were established in Theorem 1 of [13] (see also [14]) and are generalized to $H^2(\mathbb{R}^3)$ in Lemma 4 below.

Assumption 2. *The potential function $W_0(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the bound*

$$|W_0(x)| \leq \frac{C}{1 + |x|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ and $x \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}} \frac{9}{8} (4\pi)^{\frac{-2}{3}} \|W_0\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}} \|W_0\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \quad \text{and} \quad \sqrt{c_{HLS}} \|W_0\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here and further down C stands for a finite positive constant and c_{HLS} given on p.98 of [7] denotes the constant in the Hardy- Littlewood-Sobolev inequality

$$\Big| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x) f_1(y)}{|x-y|^2} dx dy \Big| \leq c_{HLS} \|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

For a function $f_1(x) \in L^p(\mathbb{R}^3),\ 1 \leq p \leq \infty$ we denote its norm as $\|f_1\|_{L^p(\mathbb{R}^3)}$. S^3_r will stand for the sphere of radius r in $\mathbb{R}^3$ centered at the origin. We distinguish the two situations dependent upon the behavior at infinity of the potential function involved in (2.4): *Case I* of $a > 0$ and *Case II* when $a = 0$.

By means of Lemma 2.3 of [13], under our Assumption 2 on the potential function, the operator H_a is self-adjoint and unitarily equivalent to $-\Delta - a$ on $L^2(\mathbb{R}^3)$ via the wave operators (see [6], [9])

$$\Omega^{\pm} := s - \lim_{t \to \mp\infty} e^{it(-\Delta + W_0)} e^{it\Delta}$$

with the limit understood in the strong L^2 sense (see e.g. [8] p.34, [1] p.90). Therefore, H_a on $L^2(\mathbb{R}^3)$ has only the essential spectrum $\sigma_{ess}(H_a) = [-a, \ \infty)$. Via the spectral theorem its functions of the continuous spectrum satisfying

$$[H_a + a]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3, \tag{2.5}$$

in the integral formulation the Lippmann-Schwinger equation for the perturbed plane waves (see e.g. [8] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}([W_0]\varphi_k)(y)dy \tag{2.6}$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3 \tag{2.7}$$

form the complete system in $L^2(\mathbb{R}^3)$. The function $\varphi_0(x)$ will stand for the solution of equations (2.5) and (2.6) with the value of the wave vector $k = 0$. With a slight abuse of notations $\int_{\mathbb{R}^3} f_1(x)\bar{f}_2(x)dx$ is being denoted by $(f_1(x), f_2(x))_{L^2(\mathbb{R}^3)}$ even if the functions involved in the inner product do not belong to $L^2(\mathbb{R}^3)$, like for instance the functions of the continuous spectrum $\varphi_k(x)$ which normalization is given in (2.7). We introduce the exponentially weighted functions

$$w_k(x) := e^{-\frac{p(x)}{2}}\varphi_k(x), \quad k \in \mathbb{R}^3 \tag{2.8}$$

with the properties of $p(x)$ established in the Appendix. Our main result is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Moreover, let assumptions of Lemma A1 hold if the fluid pressure is described by model (4.1) and assumptions of Lemma A2 hold if the fluid pressure solves equation (4.2). Then equation (1.1) admits the unique solution $u(x) \in H^2(\mathbb{R}^3)$ if and only if*

$$(f(x), w(x))_{L^2(\mathbb{R}^3)} = 0, \tag{2.9}$$

where $w(x) \in L^\infty(\mathbb{R}^3)$ is an arbitrary bounded solution of problem (1.2).

3 Proof of Theorem 3

The following statement shows that the solvability conditions established in Theorem 1 of [13] hold not only in $L^2(\mathbb{R}^3)$ but in $H^2(\mathbb{R}^3)$ as well.

Lemma 4. *Let Assumption 1 be satisfied for $g(x)$ and Assumption 2 hold. Then*

I) In Case I problem (2.4) admits the unique solution $z \in H^2(\mathbb{R}^3)$ if and only if

$$(g(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0 \ for \ k \in S^3_{\sqrt{a}} \ a.e. \tag{3.1}$$

II) In Case II problem (2.4) possesses the unique solution $z \in H^2(\mathbb{R}^3)$ if and only if

$$(g(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0. \tag{3.2}$$

Proof. We prove the statement of the lemma in Case I and when $a = 0$ the argument will be analogous. Let orthogonality relations (3.1) hold. Then by means of Theorem 1 of [13] equation (2.4) admits the unique solution $z(x) \in L^2(\mathbb{R}^3)$. This enables us to estimate

$$\|W_0 z\|_{L^2(\mathbb{R}^3)} \leq \|W_0\|_{L^\infty(\mathbb{R}^3)} \|z\|_{L^2(\mathbb{R}^3)} < \infty \tag{3.3}$$

via Assumption 2. Since $g(x)$ satisfies Assumption 1, it is square integrable and equation (2.4) yields $\Delta z \in L^2(\mathbb{R}^3)$. From problem (2.4) we easily deduce

$$\|\nabla z\|^2_{L^2(\mathbb{R}^3)} + (W_0 z, z)_{L^2(\mathbb{R}^3)} - a\|z\|^2_{L^2(\mathbb{R}^3)} = (g, z)_{L^2(\mathbb{R}^3)}.$$

By means of the Schwarz inequality, the right-hand side of the identity above is finite and the absolute value of the second term in its left-hand side can be estimated from above by the finite quantity $\|W_0\|_{L^\infty(\mathbb{R}^3)} \|z\|^2_{L^2(\mathbb{R}^3)}$ via the Schwarz inequality and bound (3.3). Thus $\nabla z \in L^2(\mathbb{R}^3)$ and $z(x) \in H^2(\mathbb{R}^3)$. On the other hand, if orthogonality relations (3.1) are not satisfied, problem (2.4) does not have a square integrable and therefore $H^2(\mathbb{R}^3)$ solution (see the statement and the proof of Theorem 1 of [13]).

□

Proof of Theorem 3. We prove the theorem for $a > 0$ and in Case II the proof will be similar, using $w_0(x)$ given by (2.8) for $k = 0$. A straightforward computation using definition (2.8) and (2.5) shows that functions $w_k(x),\ k \in S^3_{\sqrt{a}}$ a.e. are solutions to (1.2).

Since functions of the continuous spectrum $\varphi_k(x) \in L^\infty(\mathbb{R}^3),\ k \in \mathbb{R}^3$ by means of Corollary 2.2 of [13] under our Assumption 2 and the exponential factor in (2.8) is bounded via Lemma A1 or Lemma A2 dependent upon the model (4.1) or (4.2) being chosen to describe the liquid pressure, we have $w_k \in L^\infty(\mathbb{R}^3),\ k \in \mathbb{R}^3$.

Let problem (1.1) admit the unique solution $u(x) \in H^2(\mathbb{R}^3)$ and $w(x) \in L^\infty(\mathbb{R}^3)$ be an arbitrary bounded solution of equation (1.2). Since $w(x)$ may not be decaying at infinity (see e.g. (2.8)), we introduce the sequence of cut off functions $\{\xi_n\}_{n=1}^\infty$ which are infinitely smooth, dependent upon the radial variable such that $\xi_n \equiv 1$ for $|x| \leq r_n,\quad \xi_n \equiv 0$ for $|x| \geq R_n$ and monotonically decreasing inside the spherical layer $r_n \leq |x| \leq R_n$. The sequences of radii $r_n,\ R_n \to \infty$ as $n \to \infty$ and chosen in such a way that R_n increases at a higher rate which enables us to achieve $\|\nabla \xi_n\|_{L^2(\mathbb{R}^3)},\ \|\Delta \xi_n\|_{L^2(\mathbb{R}^3)} \to 0$ as $n \to \infty$. From (1.1) we easily obtain

$$(\Delta u + v.\nabla u + c(x)u, w\xi_n)_{L^2(\mathbb{R}^3)} = (f, w\xi_n)_{L^2(\mathbb{R}^3)}, \quad n \in \mathbb{N}. \tag{3.4}$$

Assumption 1 on $f(x)$ along with Fact 1 of the Appendix of [13] yield $f(x) \in L^1(\mathbb{R}^3)$. This implies the estimate on the right side of (3.4)

$$|(f,w)_{L^2(\mathbb{R}^3)} - (f, w\xi_n)_{L^2(\mathbb{R}^3)}| \le C \int_{|x|>r_n} |f(x)|dx \to 0, \quad n \to \infty.$$

Note that $|(f,w)_{L^2(\mathbb{R}^3)}| \le C \int_{\mathbb{R}^3} |f(x)|dx < \infty$ and well defined. Integrating by parts in the left side of (3.4) using the fact that $w(x)$ satisfies (1.2) yields

$$2(u, \nabla w.\nabla \xi_n)_{L^2(\mathbb{R}^3)} + (u, w\Delta\xi_n)_{L^2(\mathbb{R}^3)} - (u, wv.\nabla\xi_n)_{L^2(\mathbb{R}^3)}, \quad n \in \mathbb{N}. \tag{3.5}$$

The second term in (3.5) can be estimated above in the absolute value by means of the Schwarz inequality as

$$C\|u\|_{L^2(\mathbb{R}^3)}\|\Delta\xi_n\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty.$$

Similarly for the third term of (3.5) using either Lemma A1 or Lemma A2 of the Appendix depending upon the choice of the model for the liquid pressure there is an upper bound in the absolute value as

$$C\|u\|_{L^2(\mathbb{R}^3)}\|\nabla\xi_n\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty.$$

Integrating by parts the half of the first term of (3.5) can be easily written as

$$-(u, w\Delta\xi_n)_{L^2(\mathbb{R}^3)} - (\nabla u.\nabla\xi_n, w)_{L^2(\mathbb{R}^3)}, \quad n \in \mathbb{N}.$$

The Schwarz inequality yields

$$|(\nabla u.\nabla\xi_n, w)_{L^2(\mathbb{R}^3)}| \le C\|\nabla u\|_{L^2(\mathbb{R}^3)}\|\nabla\xi_n\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty.$$

Hence by letting $n \to \infty$ we arrive at the orthogonality condition (2.9). In order to relate problems (1.1) and (2.4) we introduce the change of variables

$$u(x) := z(x)e^{\frac{p(x)}{2}}, \quad g(x) := -f(x)e^{-\frac{p(x)}{2}} \tag{3.6}$$

with the properties of the fluid pressure established in Lemma A1 of the Appendix when considering equation (4.1) and in Lemma A2 when dealing with model (4.2) under given assumptions. Then the straightforward computation yields

$$\nabla u = e^{\frac{p}{2}}\nabla z + \frac{z}{2}e^{\frac{p}{2}}\nabla p, \quad \Delta u = e^{\frac{p}{2}}\Delta z + e^{\frac{p}{2}}\nabla z.\nabla p + e^{\frac{p}{2}}\frac{z}{2}\Delta p + e^{\frac{p}{2}}\frac{z}{4}(\nabla p)^2. \tag{3.7}$$

On the other hand,

$$z(x) = u(x)e^{-\frac{p(x)}{2}}, \tag{3.8}$$

such that

$$\nabla z = e^{-\frac{p}{2}}\nabla u - e^{-\frac{p}{2}}\frac{u}{2}\nabla p, \quad \Delta z = e^{-\frac{p}{2}}\Delta u - e^{-\frac{p}{2}}\nabla p.\nabla u - e^{-\frac{p}{2}}\frac{u}{2}\Delta p + e^{-\frac{p}{2}}\frac{u}{4}(\nabla p)^2. \tag{3.9}$$

Hence suppose $u(x) \in H^2(\mathbb{R}^3)$ is a solution to problem (1.1). Then via relations (3.6) and (3.7) we easily obtain that $z(x)$ satisfies (2.4). By means of identities (3.6), (3.8) and (3.9) using the results of either Lemma A1 or Lemma A2 dependent upon which model (4.1) or (4.2) to describe the liquid pressure to use we arrive at

$$|z| \le C|u| \in L^2(\mathbb{R}^3), \quad |\nabla z| \le C|\nabla u| + C|u| \in L^2(\mathbb{R}^3),$$

$$|\Delta z| \le C|\Delta u| + C|\nabla u| + C|u| \in L^2(\mathbb{R}^3), \quad |g| \le C|f|$$

such that $z \in H^2(\mathbb{R}^3)$ and $g(x)$ satisfies Assumption 1 formulated originally for $f(x)$. Note that such a solution of problem (2.4) is unique since otherwise the difference of two solutions will solve the equation with vanishing right-hand side. The operator H_a is unitarily equivalent to $-\Delta - a$ on $L^2(\mathbb{R}^3)$ and therefore, does not have any nontrivial square integrable zero modes.

On the other hand, let (2.9) hold. This implies

$$(f(x), w_k(x))_{L^2(\mathbb{R}^3)} = 0 \tag{3.10}$$

with $w_k(x)$ given by (2.8) for $k \in S^3_{\sqrt{a}}$ a.e. such that $w_k(x) \in L^\infty(\mathbb{R}^3)$ and satisfy (1.2).

Then by means of (3.1) and Lemma 4 equation (2.4) admits the unique solution $z(x) \in H^2(\mathbb{R}^3)$. Using identities (3.8) and (3.9) we obtain that $u(x)$ is a solution to problem (1.1). Formulas (3.6) and (3.7) along with the results of either Lemma A1 or Lemma A2 depending upon the model for the liquid pressure being used yield

$$|u| \le C|z| \in L^2(\mathbb{R}^3), \quad |\nabla u| \le C|\nabla z| + C|z| \in L^2(\mathbb{R}^3),$$

$$|\Delta u| \le C|\Delta z| + C|\nabla z| + C|z| \in L^2(\mathbb{R}^3).$$

Thus $u(x) \in H^2(\mathbb{R}^3)$. To show the uniqueness of such a solution we suppose that there exist $u_1(x), u_2(x) \in H^2(\mathbb{R}^3)$ satisfying equation (1.1). Then via the change of variables (3.8) we arrive at $z_1(x), z_2(x) \in H^2(\mathbb{R}^3)$ which solve (2.4). But $z_1(x) = z_2(x)$ a.e. by the argument above. Therefore, via (3.8) $u_1(x) = u_2(x)$ a.e.

□

4 Appendix

If the fluid is divergence free, that is div $v = 0$, then the equation for the pressure writes

$$\Delta p = 0.$$

Its only bounded solutions in $\mathbb{R}^3$ are identical constants. Hence $v = 0$. If there are sources of mass and the fluid has constant density, then

$$\text{div } v = F$$

with some given function $F(x)$. Therefore

$$-\Delta p = F. \tag{4.1}$$

We have the following lemma concerning the properties of the solutions of the equation above.

Lemma A1. *Let $F(x) : \mathbb{R}^3 \to \mathbb{R}$ and $F(x) \in L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$. Then equation (4.1) admits a solution $p(x) \in W^{2,\,\infty}(\mathbb{R}^3)$.*

Proof. For a function $\psi(x) : \mathbb{R}^3 \to \mathbb{C}$ we denote its standard Fourier transform as

$$\widehat{\psi}(q) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \psi(x) e^{-iqx} dx, \ q \in \mathbb{R}^3$$

such that

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \widehat{\psi}(q) e^{iqx} dq, \ x \in \mathbb{R}^3$$

Thus for equation (4.1) we have

$$\widehat{p}(q) = \widehat{p}_1(q) + \widehat{p}_2(q),$$

where $\widehat{p}_1(q) := \dfrac{\widehat{F}(q)}{q^2} \chi_{\{|q| \le 1\}}$ and $\widehat{p}_2(q) := \dfrac{\widehat{F}(q)}{q^2} \chi_{\{|q| > 1\}}$. Here and further down χ_A stands for the characteristic function of a set $A \subseteq \mathbb{R}^3$. By means of the Schwarz inequality we easily estimate

$$\int_{\mathbb{R}^3} |\widehat{p}_2(q)| dq \le \|\widehat{F}(q)\|_{L^2(\mathbb{R}^3)} \sqrt{\int_{\mathbb{R}^3} \frac{1}{|s|^4} \chi_{\{|s|>1\}} ds} = \sqrt{4\pi} \|F(x)\|_{L^2(\mathbb{R}^3)} < \infty,$$

which implies $\widehat{p}_2(q) \in L^1(\mathbb{R}^3)$. Since $\|\widehat{F}\|_{L^\infty(\mathbb{R}^3)} \le \dfrac{1}{(2\pi)^{\frac{3}{2}}} \|F\|_{L^1(\mathbb{R}^3)}$ we obtain

$$\int_{\mathbb{R}^3} |\widehat{p}_1(q)| dq = \int_{\mathbb{R}^3} \frac{|\widehat{F}(q)|}{q^2} \chi_{\{|q| \le 1\}} dq \le \sqrt{\frac{2}{\pi}} \|F\|_{L^1(\mathbb{R}^3)} < \infty,$$

which implies $\widehat{p}_1(q) \in L^1(\mathbb{R}^3)$. Thus

$$|p(x)| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |\widehat{p}(q)| dq < \infty, \ x \in \mathbb{R}^3$$

and $p(x) \in L^\infty(\mathbb{R}^3)$. We estimate the second derivatives of the pressure as

$$\left| \frac{\partial^2 p}{\partial x_i \partial x_j} \right| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} q^2 |\widehat{p}(q)| dq = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |\widehat{F}(q)| dq,$$

which can be bounded above by means of the Schwarz inequality by

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\int_{\mathbb{R}^3}|\widehat{F}(q)|^2(1+q^2)^2dq}\sqrt{\int_{\mathbb{R}^3}\frac{ds}{(1+s^2)^2}}=C\|F\|_{H^2(\mathbb{R}^3)}<\infty,$$

such that $\dfrac{\partial^2 p}{\partial x_i \partial x_j}\in L^\infty(\mathbb{R}^3),\ 1\le i,j\le 3$. For the first derivatives of the pressure we have the upper bound

$$\Big|\frac{\partial p}{\partial x_j}\Big|\le\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|q||\widehat{p}(q)|\chi_{\{|q|\le 1\}}dq+\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|q||\widehat{p}(q)|\chi_{\{|q|>1\}}dq\le$$

$$\le\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|\widehat{p}(q)|dq+\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|q|^2|\widehat{p}(q)|dq\le\frac{1}{(2\pi)^{\frac{3}{2}}}\|\widehat{p}\|_{L^1(\mathbb{R}^3)}+C\|F\|_{H^2(\mathbb{R}^3)}<\infty,$$

such that $\dfrac{\partial p}{\partial x_j}\in L^\infty(\mathbb{R}^3),\ 1\le j\le 3$.

□

Remark. *If equation (4.1) possesses two solutions with the properties established in the lemma above, then their difference will be a bounded solution of the Laplace equation, which is a constant. This does not affect the statement and the proof of Theorem 3.*

Instead of equation (4.1) we can also consider the equation

$$\text{div } v = F - bp,$$

where b is some positive constant. In cell population dynamics, F describes the rate of cell proliferation, and the second term in the right-hand side shows how this rate decreases with pressure. Obviously, we arrive at

$$-\Delta p + bp = F,\ b>0. \tag{4.2}$$

The lemma below describes the properties of the solutions of such an equation.

Lemma A2. *Let $F(x):\mathbb{R}^3\to\mathbb{R}$ and $F(x)\in H^2(\mathbb{R}^3)$. Then equation (4.2) admits the unique solution $p(x)\in H^2(\mathbb{R}^3)\cap W^{2,\infty}(\mathbb{R}^3)$.*

Proof. The operator $-\Delta+b:\ H^2(\mathbb{R}^3)\to H^2(\mathbb{R}^3)$ has a bounded inverse for $b>0$, such that equation (4.2) admits the unique solution

$$p(x)=(-\Delta+b)^{-1}F=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{b}|x-y|}}{|x-y|}F(y)dy\in H^2(\mathbb{R}^3)$$

and $p(x) \in L^\infty(\mathbb{R}^3)$ by means of the Sobolev embedding. We estimate the first derivatives of the pressure

$$\Big|\frac{\partial p}{\partial x_j}\Big| \le \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\Big|\frac{q}{q^2+b}\Big||\widehat{F}(q)|dq \le C\int_{\mathbb{R}^3}|\widehat{F}(q)|dq,$$

which can be bounded above by $C\|F\|_{H^2(\mathbb{R}^3)} < \infty$ by the same argument as in the proof of the previous lemma, which yields $\dfrac{\partial p}{\partial x_j} \in L^\infty(\mathbb{R}^3),\ 1 \le j \le 3$. For the second derivatives of the pressure we have the inequality

$$\Big|\frac{\partial^2 p}{\partial x_i \partial x_j}\Big| \le \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\Big|\frac{q^2\widehat{F}(q)}{q^2+b}\Big|dq \le C\|F\|_{H^2(\mathbb{R}^3)} < \infty,$$

such that $\dfrac{\partial^2 p}{\partial x_i \partial x_j} \in L^\infty(\mathbb{R}^3),\ 1 \le i, j \le 3$.

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