An Adiabatic Theorem for Resonances

Alexander Elgart *
and
George A. Hagedorn †
Department of Mathematics, and
Center for Statistical Mechanics, Mathematical Physics,
and Theoretical Chemistry,
Virginia Polytechnic Institute and State University,
Blacksburg, Virginia 24061-0123, U.S.A.

Dedicated to the memory of Pierre Duclos.

Abstract

We prove a robust extension of the quantum adiabatic theorem. The theorem applies to systems that have resonances instead of bound states, and to systems for which just an approximation to a bound state is known. To demonstrate the theorem's usefulness in a concrete situation, we apply it to shape resonances.

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1 Introduction

The goal of this paper is to present a quantum adiabatic theorem that is general enough to apply to situations in which Hamiltonians have resonances instead of bound states or if just an approximation to a bound state is known. The hypotheses of our main result, Theorem 1.1, do not specifically mention resonances, so we demonstrate how one applies the result by considering the specific situation of shape resonances. We plan to apply our theorem to other resonance situations in the future.

Our application to shape resonances has considerable overlap with the work of Abou–Salem and Fröhlich [1], although many of the details are quite different. In some instances, we obtain sharper estimates.

The adiabatic theorem of quantum mechanics describes the long time behavior of solutions to the time–dependent Schrödinger equation when the Hamiltonian generating the evolution depends slowly on time. The theorem relates these solutions to spectral information of the instantaneous Hamiltonian.

The traditional quantum adiabatic theorem applies to Hamiltonians that have an eigenvalue which is separated from the rest of the spectrum by a gap. Some more recent versions do not require the gap condition. All one really needs for the adiabatic theorem is a spectral projection for the Hamiltonian that depends smoothly on time. This allows situations where an eigenvalue is embedded in the absolutely continuous spectrum of the Hamiltonian. Since embedded eigenvalues are intrinsically unstable, they usually become resonances once the system is perturbed. It is intuitively clear that on the time scales which are shorter than the resonance lifetime, there should not be much of the difference between a proper bound state and the corresponding resonance. This intuition leads to the question whether an adiabatic theorem holds if there is a *nearly* spectral projection for the Hamiltonian that depends smoothly on time. The main abstract theorem of this paper, Theorem 1.1, provides an affirmative answer to this question.

The paper is organized as follows: We state our abstract result in Section 1.1. We then describe its application to shape resonances in Section 1.2. Section 2 contains the proofs. Technical details are collected in the Appendix.

1.1 The Extended Adiabatic Theorem

To state an adiabatic theorem precisely, it is convenient to replace the physical time t by the rescaled time $s = \epsilon\, t$. One is then concerned with the solution of the initial value problem

$$i\,\epsilon\,\dot{\psi}_\epsilon(s) \;=\; H(s)\;\psi_\epsilon(s), \qquad \text{with} \qquad \psi_\epsilon(0) \;=\; \psi_0, \tag{1.1}$$

for small values of ϵ. The Hamiltonian $H(s)$ is self–adjoint for each s and depends sufficiently smoothly on s in an appropriate sense, and ψ_ϵ takes values in the Hilbert space. We shall be more specific about what we mean by smoothness below. Typically, s is kept in a fixed interval, so that the physical time t belongs to an interval of length $O(\epsilon^{-1})$.

Our main result, Theorem 1.1, hinges on three assumptions given below. The first is a static condition. The second is a dynamic condition imposed on the family $H(s)$. The third controls the relative boundness of the rate at which $H(s)$ changes.

Definition We say that a projection $P(s)$ is *nearly spectral* for $H(s)$ if it is self-adjoint and

$$\|\,(H(s)\;-\;E(s))\;\;P(s)\,\|\;\leq\;\delta/2, \tag{1.2}$$

for all $s \in [0,\,1]$, where δ is a small parameter.

Definition We say that a projection $P(s)$ is *smoothly nearly spectral* for $H(s)$ if it is self-adjoint, $P(0)$ is nearly spectral for $H(0)$, and

$$\left\|\,\frac{d}{ds}\left\{(H(s)\;-\;E(s))\;P(s)\right\}\right\|\;\leq\;\delta/2, \tag{1.3}$$

for all $s \in [0,\,1]$.

Remarks

1. If $P(s)$ is smoothly nearly spectral, then it is nearly spectral.

2. If $P(s)$ is the spectral projection for energy $E(s)$, then it is smoothly nearly spectral with $\delta = 0$.

Assumption 1. *We assume that there exists either a smoothly nearly spectral or a nearly spectral projection $P(s)$ for $H(s)$.*

Remarks

1. As we have already mentioned, it is reasonable to expect adiabatic behavior of the system whenever ϵ is small, but $\epsilon \gg \delta$.

2. From the Weyl Criterion (Theorem VII.12 of [12]), a non-trivial nearly spectral projection exists for any point E in the spectrum of a self-adjoint operator. However, we shall impose further dynamical assumptions on $P(s)$ below. In general, the dynamical assumptions limit the set of suitable $E(s)$ to eigenvalues or resonances.

3. The notion of a nearly spectral projection is also related to the ideas of Spectral Concentration. See, *e.g.*, Section XII.5 of [14].

Assumption 1 is a static condition imposed on the family $H(s)$. Our results require a dynamic hypothesis as well: We let g_a be a smoothed characteristic function that takes the value 1 at 0. More precisely, we assume

$$g_a(x) = 1 \quad \text{if} \quad |x| < a/2, \qquad g_a(x) = 0 \quad \text{if} \quad |x| > a, \qquad \text{and} \qquad g_a \in C^4(I\!\!R). \tag{1.4}$$

Assumption 2. *There exists* $a \in (0,\, 1)$ *such that*

$$\left\| \, g_a\Big(H(s) \, - \, E(s)\Big) \; \dot{P}(s) \; P(s) \right\| \; \leq \; \delta', \tag{1.5}$$

uniformly for $s \in [0,\, 1]$.

Remarks

1. Here δ' is another small parameter, while a should be thought of as an auxiliary tuning mechanism, which eventually could be optimized. In our application to shape resonances δ' is roughly of the same order of magnitude as δ.

2. Intuitively, condition (1.5) quantifies the rate at which the wave function “leaks” from the range of $P(s)$ to energetically close states. One can also think of Assumption 2 as requiring a bound on the spectral density in the vicinity of the resonance or bound state. Indeed, if there are no bound states with energies close to that of the resonance, the expression in (1.5) tends to zero as a tends to zero. The second possibility is that there are bound states nearby (*e.g.*, pure point spectrum), but their overlap with $\dot{P}(s)$ is small. The latter occurs in the Anderson localization problem.

3. We remarked earlier that non-trivial nearly spectral projections exist for any self-adjoint operator. However, if the operator depends on a parameter s, it may be impossible to construct a corresponding family of nearly spectral projection $P(s)$ that satisfies Assumption 2. Theorem 1.2 shows that both Assumptions 1 and 2 can be satisfied for shape resonances.

Finally, because we are dealing with unbounded perturbations, we impose the following technical requirement:

Assumption 3. *We assume that the operators $H(s)$ are self-adjoint on a common domain $\mathcal{D}$, and that the derivatives $\dot{H}(s)$ and $\ddot{H}(s)$ exist as operators from $\mathcal{D}$ to the Hilbert space $\mathcal{H}$. Furthermore, we assume there exists C, such that for all $s \in [0, 1]$,*

$$\left\| \dot{H}(s)\ (H(s) - i)^{-1} \right\| \le C \qquad \text{and} \qquad \left\| \ddot{H}(s)\ (H(s) - i)^{-1} \right\| \le C.$$

Here and throughout the text, C stands for a generic constant.

Our main abstract theorem is the following:

Theorem 1.1. *Suppose $P(s)$ is a nearly spectral projection for the Hamiltonian $H(s)$ that satisfies Assumptions 1 – 3. Then there exists a unitary propagator for (1.1).*

Let $\psi_\epsilon(0) \in \operatorname{Range} P(0)$.

If $P(s)$ is nearly spectral, then for all $s \in [0,1]$,

$$\operatorname{dist}\{\,\psi_\epsilon(s),\ \operatorname{Range} P(s)\,\} \;\le\; 2\,\delta' + 2\,\|\dot{P}(s)\|\ |||g_a|||_3\,\delta + K_a\,\epsilon + \delta/\epsilon\,. \tag{1.6}$$

If $P(s)$ is smoothly nearly spectral, then for all $s \in [0,1]$,

$$\operatorname{dist}\{\,\psi_\epsilon(s),\ \operatorname{Range} P(s)\,\} \;\le\; 2\,\delta' + 2\,\|\dot{P}(s)\|\ |||g_a|||_3\,\delta + \frac{C\,\delta}{a^2} + C\,\tilde{K}_a\,\epsilon + \delta/\epsilon\,. \tag{1.7}$$

The a–dependent constants K_a and $\tilde{K}_a$ in these expressions are given in (2.8) *(respectively* (2.10)*) below. The norm $|||g_a|||_3$ is one of the norms described in [6] and Appendix B of [11]. These norms have the form*

$$|||g_a|||_{n+2} \;=\; C \sum_{k=0}^{n+2} \left\| g_a^{(k)} \right\|_{k-n-1}, \qquad \text{with} \qquad \|f\|_l \;=\; \int (1+x^2)^{l/2}\ |f(x)|\ dx,$$

where the C depends only on n.

Remarks

1. If $P(s)$ is a spectral projection, we can take $\delta = 0$, and (1.7) corresponds to the (slightly improved) adiabatic theorem of [3]. Even in the case of a bound state it is often technically simpler to construct an approximant $P(s)$ for the bound state projection rather than the exact eigenprojection. The error in the approximation then contains the parameter δ.

2. If $P(s)$ corresponds to an eigenprojection for an isolated eigenvalue of $H(s)$, then it is typically relatively easy to verify its smoothness (with respect to the s variable), as it is "inherited" from the smoothness of $H(s)$. Otherwise, checking this is usually highly non-trivial (*c.f.* [9] where this task is carried for the ground state of the atom in a QED picture). One of the main obstacles in the implementation of the quantum adiabatic algorithm [7] is controlling the norms of the derivatives of $P(s)$. From this point of view, (1.7) is a better result than (1.6), as it only requires a bound on the first derivative of $P(s)$. We note that even if $P(s)$ is only nearly spectral, one can get a bound in the adiabatic theorem that depends only on the L^1 norm of $\|\dot{P}(s)\|$, but not on $\ddot{P}(s)$. That can be achieved using a mollifier argument. In our application to shape resonances, we have the good control on the smoothness of $P(s)$. (See Lemma 2.3.)

1.2 The Application to Shape Resonances

Theorem 1.1 applies to Schrödinger operators with shape resonances. The analysis is not much more complicated for certain magnetic fields, so we include them. We consider Schrödinger operators $H(s) := \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} + V(s)$ on $I\!R^d$, where $\boldsymbol{\mathcal{P}}_{\boldsymbol{A}} = -i\,\hbar\,\nabla - \boldsymbol{A}(x)$.

Assumption 4. *We assume the components of the vector potential* $\boldsymbol{A}$ *and their first derivatives* $\partial_i \boldsymbol{A}$ *for* $i = 1, ..., d$ *are bounded on* $I\!R^d$.

Shape resonances (see, *e.g.* [4]) are resonances of $H(s)$ that arise because the particle can be confined to a region of space that is bounded by a classically forbidden region. If the resonance has energy near E, then we define the classically forbidden region to be

$$J \;:=\; \{\, x \in I\!R^d :\; V(x) \;>\; E \,+\, b \,\}$$

for some $b > 0$. One usually assumes that J separates $\mathbb{R}^d$ into a bounded interior and an unbounded exterior component. The intuition is that the particle spends a long time in the interior component, but can eventually tunnel to the exterior component.

We examine this situation where the energy $E(s)$, potential $V(x, s)$, and classically forbidden region $J(s)$ all depend on s.

For simplicity, we assume that for an appropriate value of $E(s)$, $J(s)$ separates $\mathbb{R}^d$ into an exterior region $O(s)$ and a single connected interior region $I(s)$, so that

$$\mathbb{R}^d \;=\; J(s) \bigcup O(s) \bigcup I(s).$$

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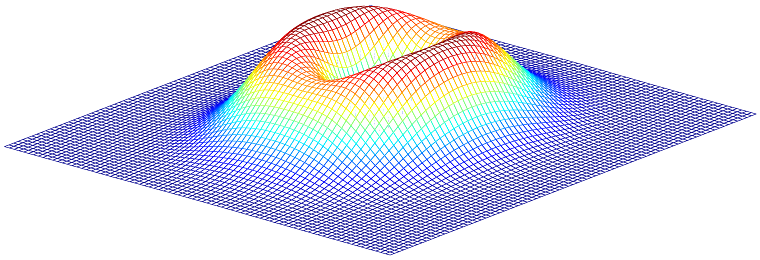

Figure 1. A two dimensional potential that has shape resonances.

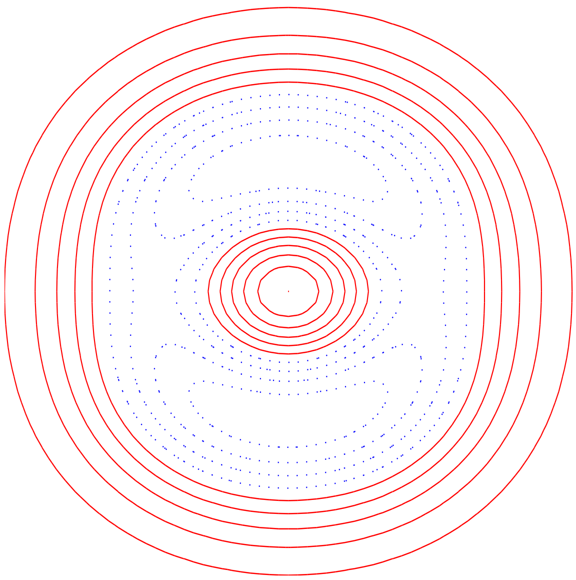

Figure 2. Contour plot for the same potential showing the classically allowed regions (red, solid curves) and classically forbidden region (blue, dotted curves) for a particular energy.

We assume that $H(s)$ has the following properties for each $s \in [0,\, 1]$:

Assumption 5.

1. *For every* $s \in [0,\, 1]$, *we assume* $H(s)$ *is a self-adjoint operator on an* s*–independent domain* $\mathcal{D}$.

2. *For every* $s \in [0,\, 1]$, *there exist* $c > 0$ *and an open set* $\Omega(s) \subset I\!\!R^d$, *such that*

$$\operatorname{dist}\{\, O(s),\ \Omega(s)\,\} \ \geq\ c, \qquad \textit{and} \qquad \operatorname{dist}\{I(s),\ \Omega^c(s)\,\} \ \geq\ c.$$

   Moreover, the Friedrichs extenstion $H_\Omega(s)$ *of the Dirichlet restriction of* $H(s)$ *to* $\Omega(s)$ *is bounded from below.*

3. *The operator* $H_\Omega(s)$ *has a discrete eigenvalue* $E(s)$ *that depends smoothly on* s.

4. $E(s)$ *is separated by a spectral gap* $\Delta(s) > \Delta > 0$ *from the rest of the spectrum of* $H_\Omega(s)$. *For convenience, we assume* $\Delta < 1$.

5. *The potential* $V(s)$ *satisfies* $\|\, V_J(s)\,\| \ \leq\ C$,

$$\left\| \dot{V}(s)\ \left(H_\Omega(s) - i\right)^{-1} \right\| \ \leq\ C, \qquad \textit{and} \qquad \left\| \ddot{V}(s)\ \left(H_\Omega(s) - i\right)^{-1} \right\| \ \leq\ C\,,$$

   where V_J *stands for the Dirichlet restriction to the set* J.
   We further assume that $\operatorname{supp} \dot{V}(s) \ \subset\ I(s)$.

Remarks

1. In typical situations, part 4 of this assumption forces Δ to be $O(\hbar)$.

2. Assumption 5 implies a "Combes–Thomas" estimate for $H_{J(s)}(s)$. The (improved) Combes–Thomas estimate [8] is usually stated for the operators acting on the whole space $I\!\!R^d$. The extension to the sub-domain case is presented as Theorem 3.4 in the Appendix.

The following theorem and its corollary are our main results concerning shape resonances.

Theorem 1.2. *Suppose that Assumption 5 is satisfied. Then there exists a family* $P(s)$ *of projections that satisfies Assumptions 1 – 3 for energy* $E(s)$. *The values of the corresponding parameters are*

$$\delta \ =\ C\ \hbar\ e^{-\eta/\hbar}, \qquad a \ =\ \Delta/2, \qquad \textit{and} \qquad \delta' \ =\ C\ \hbar\ e^{-\eta/\hbar}\ |||g_a|||_4/\Delta\,, \tag{1.8}$$

where $\eta > 0$.

Combining Theorems 1.1 and 1.2 we obtain

Corollary 1.3. *In the context of a shape resonance, the adiabatic theorem holds for* $\epsilon \gg \hbar^{-3} e^{-\eta/\hbar}$, *where* $\eta > 0$. *The error term in the adiabatic theorem is bounded by a constant times*

$$\hbar\,\Delta^{-4}\,e^{-\eta/\hbar} \;+\; \hbar\,\Delta^{-3}\,e^{-\eta/\hbar} \;+\; \Delta^{-4}\,\epsilon \;+\; \epsilon^{-1}\,\hbar\,e^{-\eta/\hbar}.$$

Proof The only nontrivial part of the proof of the corollary is estimating factors in (1.6) that come from $\dot{P}(s)$ and $\ddot{P}(s)$, some of which occur in K_a. However, $P(s)$ from Theorem 1.2 is constructed in Section 2.2 from $P_\Omega(s)$.

We begin by estimating derivatives of $P_\Omega(s)$ by writing

$$P_\Omega(s) \;=\; -\,\frac{1}{2\pi i}\,\oint_{|z-E(s)|=\Delta/2} (H_\Omega(s)-z)^{-1}\,dz.$$

The length of the contour is $\pi\,\Delta$, while self-adjointness and the gap condition assure that the norm of the integrand is bounded by $2/\Delta$.

The derivative is

$$\dot{P}_\Omega(s) \;=\; -\,\frac{1}{2\pi i}\,\oint_{|z-E(s)|=\Delta/2} (H_\Omega(s)-z)^{-1}\,\dot{V}(s)\,(H_\Omega(s)-z)^{-1}\,dz.$$

By Assumption 5 and the first resolvent formula, the norm of the integrand is bounded by C/Δ^2, so $\|\dot{P}_\Omega(s)\| \le C\pi/\Delta$. Similarly, $\|\ddot{P}_\Omega(s)\| \le C\pi/\Delta^2$.

The corollary is an immediate consequence of these estimates and Lemma 2.3. ∎

Remarks

1. As remarked earlier, Δ is typically bounded above and below by constants times $\hbar$. When this is the case, the error in the adiabatic theorem is bounded by a constant times
$$\hbar^{-3}\,e^{-\eta/\hbar} \;+\; \hbar^{-4}\,\epsilon \;+\; \epsilon^{-1}\,\hbar\,e^{-\eta/\hbar}$$
for small $\hbar$.

2. There are situations in which eigenvalues of H_Ω can cross as $\hbar$ is decreased. In those situations, we obtain no error estimate.

3. As we have already commented, the Weyl Criterion guarantees existence of nearly spectral projections at any point of the spectrum of a quantum Hamiltonian. However, for general Schrödinger operators, the ranges of those projections contain only functions are very delocalized in space. The nearly spectral projections we construct for shape resonances are localized in a region that can be chosen independent of δ and $\hbar$.

4. There are numerous definitions of resonances for a quantum mechanical Hamiltonian H. One is as follows: Since H is self-adjoint, the complex valued function

$$f(z) \;:=\; \langle\, \psi,\, (H-z)^{-1}\, \psi\, \rangle$$

is analytic in the upper half-plane for any vector ψ. For ψ in some appropriate set, f has an analytic continuation $\tilde{f}$ to some portion of the lower half-plane. A pole of $\tilde{f}$ at $E - i\Gamma$ corresponds to a resonance near energy E with lifetime proprotional to $1/\Gamma$. This definition is often not practically tractable.

2 Proofs

2.1 Proof of Theorem 1.1

Using Assumption 3, we obtain existence of the unitary propagator $U_\epsilon(s)$ by applying Theorem X.70 of [13].

Without loss of generality we can assume that $E(s) \equiv 0$ throughout the proof. Indeed, the dynamics generated by $H(s)$ differs from the one generated by $H(s) - E(s)$ only by the (dynamical) phase $\exp\big(-i\int_0^s E(t)\,dt/\epsilon\,\big)$.

The first step in many proofs of the adiabatic theorem (*e.g.*, [3]) is the construction of the so-called adiabatic evolution. This is a unitary family $U_a(s)$, such that

$$P(s) \;=\; U_a(s)\; P(0)\; U_a(s)^*\,. \tag{2.1}$$

The second step of the proof is to verify that U_a stays close to the true evolution U_ϵ, determined by

$$i\,\epsilon\,\dot{U}_\epsilon(s) \;=\; H(s)\,U_\epsilon(s), \quad \text{with} \quad U_\epsilon(0) \;=\; I. \tag{2.2}$$

We follow this strategy with one modification: Since $P(s)$ is *not* a spectral projection, it is hard to construct an evolution that satisfies (2.1) and is close to U_ϵ. Instead, we construct a *nearly* adiabatic evolution $U_n(s)$ and replace (2.1) by

$$\| P(s)\, U_n(s) \;-\; U_n(s)\, P(0) \| \;=\; \| P(s) \;-\; U_n(s)\, P(0)\, U_n(s)^* \| \;\leq\; \delta/\epsilon. \tag{2.3}$$

With this modification, the second step is the same as in the traditional adiabatic theorem.

Specifically, we let $U_n(s)$ be the solution to the initial value problem

$$i\, \epsilon\, \dot{U}_n(s) \;=\; H_n(s)\, U_n(s), \qquad \text{with} \qquad U_n(0) \;=\; I\,, \tag{2.4}$$

where the generator is $H_n(s) \;=\; H(s) \,+\, i\, \epsilon\, [\dot{P}(s),\, P(s)]$. Existence of $U_n(s)$ is guaranteed by Theorem X.70 of [13].

In order to check that the estimate (2.3) holds, we compute

$$\frac{d}{ds}\left(U_n(s)^*\; P(s)\; U_n(s) \right) \;=\; \frac{i}{\epsilon}\; U_n(s)^*\; [H(s),\, P(s)]\; U_n(s), \tag{2.5}$$

where we have used

$$\dot{P}(s) \;=\; \dot{P}(s)\; P(s) \;+\; P(s)\; \dot{P}(s) \qquad \text{and} \qquad P(s)\; \dot{P}(s)\; P(s) \;=\; 0.$$

The condition (1.2) and the unitarity of $U_n(s)$ guarantee that the right hand side of (2.5) is bounded in norm by δ/ϵ. We integrate both sides of (2.5) and use the fundamental theorem of calculus to see that

$$\| U_n(s)^*\; P(s)\; U_n(s) \;-\; P(0) \| \;\leq\; \delta/\epsilon, \tag{2.6}$$

for all $s \in [0,\, 1]$. The estimate (2.3) follows from the unitarity of U_n.

Next we show that the physical evolution U_ϵ stays close to U_n for all $s \in [0,\, 1]$.

We claim that for a nearly spectral projection,

$$\begin{aligned} \| U_n(s) \;-\; U_\epsilon(s) \| \;&=\; \| U_\epsilon(s)^*\; U_n(s) \;-\; I \| \\ &\leq\; 2\, \delta' \;+\; 2\, \|\dot{P}(s)\|\; |||g_a|||_3\; \delta \;+\; K_a\; \epsilon, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} K_a \;:=\; & 2\; |||g_a|||_3\; \max_r\; \|\dot{P}(r)\|^2 \;+\; 2\; |||g_a|||_3\; \max_r\; \|\dot{P}(r)\| \\ & +\;\; C\; \left(|||g_a|||_3\; \max_r\; \|\ddot{P}(r)\| + |||g_a|||_4\; \max_r\; \|\dot{P}(r)\| \right). \end{aligned} \tag{2.8}$$

For a smoothly nearly spectral projection, we can improve this bound to

$$\| U_n(s) \; - \; U_\epsilon(s) \| \;\; \leq \;\; 2\, \delta' \; + \; 2\, \|\dot{P}(s)\| \; |||g_a|||_3 \, \delta \; + \; C\, \delta/a^2 \; + \; C\, \tilde{K}_a \, \epsilon, \tag{2.9}$$

where

$$\tilde{K}_a \;\; := \;\; \max_s \left\{ |||g_a|||_5 \;\; + \;\; |||g_a|||_4 \; \|\dot{P}(s)\| \;\; + \;\; \frac{1}{a^2} \left(1 \; + \; \|\dot{P}(s)\| \right) \right\} .$$

To prove these estimates, we note that the unitary operator $\Omega(s) \; := \; U_\epsilon(s)^* \, U_n(s)$ satisfies the differential equation

$$\dot{\Omega}(s) \;\; = \;\; K(s) \; \Omega(s), \qquad \text{where} \qquad K(s) \;\; := \;\; U_\epsilon(s)^* \; [\dot{P}(s), \, P(s)] \; U_\epsilon(s).$$

We cast this in its integral form

$$\Omega(s) \;\; - \;\; I \;\; = \;\; \int_0^s \; K(r) \; \Omega(r) \; dr. \tag{2.10}$$

To show that the right hand side is small, we use the following lemmas that we prove below:

Lemma 2.1. *For a nearly spectral projection $P(s)$, there exist a pair $X(s) = X_1(s)$ and $Y(s) = Y_1(s)$, such that*

$$[\dot{P}(s), \, P(s)] \;\; = \;\; [X_1(s), \, H(s)] \;\; + \;\; Y_1(s), \tag{2.11}$$

where

$$\|X_1(s)\| \;\; \leq \;\; |||g_a|||_3 \; \|\dot{P}(s)\|, \tag{2.12}$$

$$\|\dot{X}_1(s)\| \;\; \leq \;\; C \; \left(|||g_a|||_3 \; \|\ddot{P}(s)\| \;\; + \;\; |||g_a|||_4 \; \|\dot{P}(s)\| \right), \tag{2.13}$$

and

$$\|Y_1(s)\| \;\; \leq \;\; 2\, \delta' \; + \; 2\, \|\dot{P}(s)\| \; |||g_a|||_3 \, \delta. \tag{2.14}$$

Lemma 2.2. *For a smooth nearly spectral projection $P(s)$, there exist a pair $X(s) = X_2(s)$ and $Y(s) = Y_2(s)$, such that*

$$[\dot{P}(s), \, P(s)] \;\; = \;\; [X_2(s), \, H(s)] \;\; + \;\; Y_2(s), \tag{2.15}$$

where

$$\|X_2(s)\| \;\leq\; C/a^2\,, \tag{2.16}$$

$$\|\dot{X}_2(s)\| \;\leq\; C\;\Big(|||g_a|||_5 \;+\; |||g_a|||_4\;\|\dot{P}(s)\|\Big)\,, \tag{2.17}$$

and

$$\|Y_2(s)\| \;\leq\; 2\;\delta' \;+\; 2\;\|\dot{P}(s)\|\;|||g_a|||_3\;\delta \;+\; C\,\delta/a^2. \tag{2.18}$$

If we substitute the representation (2.11) (respectively (2.15)) into the right hand side of (2.10) we observe that

$$\begin{aligned}\left\|\int_0^s K(r)\;\Omega(r)\;dr\right\| \;&\leq\; \int_0^s \|Y(r)\|\;dr \\ &+\; \left\|\int_0^s U_\epsilon(r)^*\;[X(r),\,H(r)]\;U_\epsilon(r)\;\Omega(r)\;dr\right\|.\end{aligned} \tag{2.19}$$

However,

$$U_\epsilon(r)^*\;[X(r),\,H(r)]\;U_\epsilon(r) \;=\; i\;\epsilon\;\frac{d}{dr}\left(U_\epsilon(r)^*\;X(r)\;U_\epsilon(r)\right) \;-\; i\;\epsilon\;U_\epsilon(r)^*\;\dot{X}(r)\;U_\epsilon(r).$$

So, the second contribution in (2.19) can be integrated by parts:

$$\begin{aligned}\int_0^s U_\epsilon(r)^*\;[X(r),\,H(r)]\;U_\epsilon(r)\;\Omega(r)\;dr \;&=\; -\,i\;\epsilon\;\int_0^s U_\epsilon(r)^*\;\dot{X}(r)\;U_\epsilon(r)\;\Omega(r)\;dr \\ &+\;\; i\;\epsilon\;U_\epsilon(r)^*\;X(r)\;U_\epsilon(r)\;\Omega(r)\Big|_0^s \\ &-\;\; i\;\epsilon\;\int_0^s U_\epsilon(r)^*\;X(r)\;U_\epsilon(r)\;\dot{\Omega}(r)\;dr.\end{aligned}$$

The norm of the first term on the right hand side is bounded by $\epsilon\;\max_s\|\dot{X}(s)\|$. The norm of the second term is bounded by $2\,\epsilon\;\max_s\|X(s)\|$. Finally, since $\Omega(r)$ is unitary, $\dot{\Omega}(r) = K(r)\,\Omega(r)$, and $\|K(r)\| \,\leq\, 2\,\|\dot{P}(r)\|$, we can bound the last term by $2\;\epsilon\;\max_s\|X(s)\|\;\|\dot{P}(s)\|$. Combining these estimates, we get (2.7) and (2.9).

We now let $\psi_0 \in \mathrm{Ran}\,P(0)$ be a unit vector and set $\psi_\epsilon(s) \;=\; U_\epsilon(s)\;\psi_0$. Then using

(2.3) and (2.6), we see that

$$\begin{aligned}
&\| \psi_\epsilon(s) \;-\; P(s)\, U_n(s)\, \psi_\epsilon(0) \| \\
\le\;& \| \psi_\epsilon(s) \;-\; U_n(s)\, \psi_\epsilon(0) \| \;+\; \| U_n(s)\, P(0)\, \psi_\epsilon(0) \;-\; P(s)\, U_n(s)\, \psi_\epsilon(0) \| \\
\le\;& \| U_n(s) \;-\; U_\epsilon(s) \| \;+\; \delta/\epsilon.
\end{aligned}$$

The theorem now follows from (2.7) ((2.9) respectively). ∎

Proof of Lemmas 2.1 and 2.2 The operator valued function $g_a(H(s))$ admits the Helffer–Sjöstrand representation

$$g_a(H(s)) \;=\; \int_{\mathbb{C}} (\partial_{\bar z}\widetilde{g}_a)(z)\; R_s(z)\; dz\, d\bar z, \tag{2.20}$$

where $R_s(z) := (H(s) - z)^{-1}$, $\widetilde{g}_a(z)$ is supported in the disc $|z| \le a$, and

$$\int_{\mathbb{C}} |\, \partial_{\bar z}\widetilde{g}_a(z)\, |\;\; |\, \mathrm{Im}\, z\, |^{-n-1}\; dz\, d\bar z \;\le\; |||g_a|||_{n+2}\,. \tag{2.21}$$

See *e.g.* [6] or Appendix B of [11]. The norm here is

$$|||g_a|||_{n+2} \;=\; C \sum_{k=0}^{n+2} \left\| g_a^{(k)} \right\|_{k-n-1}, \qquad \text{with} \qquad \|f\|_l \;=\; \int_{-\infty}^{\infty} (1+x^2)^{l/2}\; |f(x)|\; dx,$$

where the C depends only on n.

Our first goal is to show that (1.2) implies

$$\left\| \Big(g_a(H(s)) \;-\; I \Big)\; P(s) \right\| \;\le\; |||g_a|||_3\; \delta/2. \tag{2.22}$$

To prove this, we recall that we have taken $E(s) = 0$ and note that for any $z \in \mathbb{C}$, inequality (1.2) implies

$$\| \,(H(s) - z)\;\; P(s) \;+\; z\; P(s)\, \| \;\le\; \delta/2.$$

Thus,

$$\left\| \;\frac{1}{z}\; P(s) \;+\; (H(s) \;-\; z)^{-1}\;\; P(s) \;\right\| \;\le\; \frac{\delta}{2\; |\, \mathrm{Im}\, z\, |^2}.$$

Substituting this relation into (2.20) and using the bound (2.21) for the error term, we get

$$\|\; g_a(0)\; P(s) \;-\; g_a(H(s))\; P(s)\; \| \;\le\; |||g_a|||_3\; \delta/2.$$

Since $g_a(0) = 1$, inequality (2.22) follows.

We now define $X_1(s)$ in (2.11) and $X_2(s)$ in (2.15) to be

$$X_1(s) \;=\; -\int_{\mathbb{C}} (\partial_{\bar z}\widetilde{g}_a)(z)\; R_s(z)\; \dot P(s)\; R_s(z)\; dz\, d\bar z\,, \tag{2.23}$$

and

$$X_2(s) \;=\; (1-g_a(H(s)))\;\; R_s^2(0)\; \dot H(s)\; P(s) \;+\; h.\,c. \tag{2.24}$$

Since

$$\frac{d}{ds}\,R_s(z) \;=\; -\,R_s(z)\;\dot H(s)\;R_s(z),$$

it follows from (2.21) and Assumption 3 that $X_1(s)$ satisfies the bounds (2.12) and (2.13).

Since

$$\big\|\,(1-g_a(H(s)))\;\; R_s^2(0)\;(H(s)+i)\,\big\| \;\le\; C/a^2\,, \tag{2.25}$$

and

$$\Big\|\,R_s(-i)\;\dot H(s\,)\Big\| \;\le\; C$$

by Assumption 3, $X_2(s)$ satisfies the bound (2.16).

To get the bound (2.17) we first note that by partial fractions,

$$\frac{1}{z^2}\,R_s(z) \;=\; R_s(0)^2\;R_s(z) \;+\; \frac{1}{z}\,R_s(0)^2 \;+\; \frac{1}{z^2}\,R_s(0).$$

Second, since $(\alpha - z)^{-1}$ is the resolvent of multiplication by the constant α,

$$g_a'(\alpha) \quad=\quad -\,\frac{d}{d\alpha}\int_{\mathbb{C}}\frac{(\partial_{\bar z}\widetilde{g}_a)(z)}{(z-\alpha)}\,dz\,d\bar z \quad=\quad \int_{\mathbb{C}}\frac{(\partial_{\bar z}\widetilde{g}_a)(z)}{(z-\alpha)^2}\,dz\,d\bar z$$

When $\alpha = 0$, this is zero. Using these facts and $g_a(0) = 1$, we can write

$$(1-g_a(H(s)))\;\;R_s^2(0) \;=\; -\int_{\mathbb{C}}\frac{(\partial_{\bar z}\widetilde{g}_a)(z)}{z^2}\;R_s(z)\;dz\,d\bar z\,.$$

We then proceed as in the $X_1(s)$ case.

To conclude the proof, we need to verify (2.14) and (2.18). However, we explicitly have

$$[X_1(s),\;H(s)] \;=\; [\dot P(s),\;g_a(H(s))].$$

Thus,

$$Y_1(s) \;=\; [\,\dot P(s),\; (P(s) \,-\, g_a(H(s)))\,]. \tag{2.26}$$

Since $P(s) + (1 - P(s)) = I$, estimate (2.14) follows from (2.22) and (1.5).

On the other hand,

$$\begin{aligned}[X_2(s),\, H(s)] \;=\; &\Big\{(1 - g_a(H(s)))\;\; R_s^2(0)\;\dot H(s)\; P(s)\; H(s)\\ &\quad -\;(1 - g_a(H(s)))\;\; R_s(0)\;\dot H(s)\; P(s)\Big\} \;-\; h.\,c.\end{aligned}$$

From Assumption 1 with $P(s)$ smoothly nearly spectral and $E(s) = 0$, we have $\left\|\frac{d}{ds}\,(H(s)\,P(s))\right\| \;\leq\; \delta/2$. So, Assumption 3, and the bound (2.17) imply that the norm of the first contribution is $O\,(\delta/a^2)$, while the second contribution is equal to

$$\begin{aligned}&-\;\;(1 - g_a(H(s)))\;\; R_s(0)\;H(s)\;\dot P(s)\;+\;O\,(\delta/a)\\ &\quad=\;\;-\;(1 - g_a(H(s)))\;\;\dot P(s)\;+\;O\,(\delta/a)\;.\end{aligned}$$

Putting everything together and using $a < 1$, we get

$$[\,X_2(s),\, H(s)\,] \;=\; [\,\dot P(s),\; g_a(H(s))\,] \;+\; O\left(\delta/a^2\right)\;.$$

So, $Y_2(s)$ defined by (2.15) satisfies

$$Y_2(s) \;=\; [\,\dot P(s),\; (P(s) \,-\, g_a(H(s))\,] \;+\; O\left(\delta/a^2\right)\;. \tag{2.27}$$

The rest of the argument is the same as for the $Y_1(s)$ case. ■

2.2 Proof of Theorem 1.2

Existence of the propagator is a consequence of part (5) of Assumption 5.

We begin the proof of Theorem 1.2 by constructing a suitable family $P(s)$. We note that one reasonable candidate for $P(s)$ is a finite rank spectral projection $P_\Omega(s)$ for the Dirichlet restriction $H_\Omega(s)$ (extended by zero to the whole $\mathbb{R}^d$):

$$P_\Omega(s) \;=\; -\,\frac{1}{2\,\pi\,i}\,\oint_\Gamma\;(H_\Omega(s)\,-\,z)^{-1}\;dz, \tag{2.28}$$

where $\Gamma := \{\,z \in \mathbb{C} :\; |z - E(s)| = \Delta/2\,\}$.

Indeed, for x in the complement of $\partial\Omega(s)$ and any vector ϕ, we have

$$(H(s)\ P_\Omega(s)\ \phi)\,(x)\ =\ E(s)\ (P_\Omega(s)\ \phi)\,(x)$$

as desired. Unfortunately, the range of $P_\Omega(s)$ is not in the domain of $H(s)$, since it is not twice differentiable at the boundary $\partial\Omega(s)$ of the set $\Omega(s)$. So, we must modify this family. The key estimate that will enable us to control the errors introduced by this modification process is encoded in Lemma 3.5 (*c.f.* with the related result in [10]).

To define the desired family of projections, let $\{\,\psi_i\,\}_{i=1}^n$ be an orthonormal basis for Range $P_\Omega(s)$, and define

$$\widetilde{\psi}_i\ =\ (I\ -\ \chi_s)\ \psi_i,$$

where χ_s is a smoothed characteristic function of the set $\partial\Omega(s)$. By this we mean that χ_s is twice differentiable as a function of $x\in I\!R^d$, and if $x\in\partial\Omega(s)$, then $\chi_s(x)\,=\,1$, and if $\operatorname{dist}\{\,\partial\Omega(s),\,x\,\}\,>\,c/2$, then $\chi_s(x)=0$. We can assume χ_s has been chosen so that $\|\chi_s\|_{2,2}\,\leq\,c^{-2}$, where $\|\chi_s\|_{2,2}^2=\int\ |(1-\Delta_x)\chi_s(x)|^2\ dx$.

We define $P(s)$ to be the orthogonal projection onto the span of $\{\,\widetilde{\psi}_i\,\}_{i=1}^n$.

Lemma 2.3. *Let*

$$Q^{(n)}(s)\ :=\ \frac{d^n}{ds^n}\ (P_\Omega(s)\ -\ P(s))\ .$$

Then

$$\|\,Q^{(n)}(s)\,\|\ \leq\ C_n\ e^{-\eta/\hbar}/\Delta^n\,, \tag{2.29}$$

for $n=0,\,1,\,2$.

Proof Let $R_w(s)\ =\ \Big((I-\chi_s)\,P_\Omega(s)\,(I-\chi_s)-w\,I\Big)^{-1}$ and $\hat{R}_w(s)\ =\ \Big(P_\Omega(s)-w\,I\Big)^{-1}$, for $w\in\mathbb{C}$. Since $\sigma(P_\omega(s))=\{0,\,1\}$, Lemma 3.5 shows that

$$P_\Omega(s)\ -\ (I\ -\ \chi_s)\,P_\Omega(s)\,(I\ -\ \chi_s)\ =\ \chi_s\,P_\Omega(s)\ +\ P_\Omega(s)\,\chi_s\ -\ \chi_s\,P_\Omega(s)\,\chi_s\ =\ O(e^{-\eta/\hbar}).$$

Analytic perturbation theory then shows that for sufficiently small $\hbar$ and $|w-1|=1/2$, we have

$$\| R_w(s) \| \ \leq\ 3 \tag{2.30}$$

$$\Big\| R_w(s) - \hat{R}_w(s) \Big\| \ =\ O(e^{-\eta/\hbar}) \tag{2.31}$$

$$P(s) \ =\ -\,\frac{1}{2\pi i}\oint_\gamma R_w(s)\,dw \qquad \text{and} \qquad P_\Omega(s) \ =\ -\,\frac{1}{2\pi i}\oint_\gamma \hat{R}_w(s)\,dw\,, \tag{2.32}$$

where $\gamma := \{w \in \mathbb{C} :\ |w-1| = 1/2\}$. Combining these bounds, we get the estimate (2.29) for the $n=0$ case.

For the $n=1$ case, we note that

$$\frac{d}{ds}\hat{R}_w(s) \ =\ -\,\hat{R}_w(s)\ \dot{P}_\Omega(s)\ \hat{R}_w(s)\,;$$

$$\frac{d}{ds}R_w(s) \ =\ -\,R_w(s)\ \frac{d}{ds}\,\{\,(I\,-\,\chi_s)\ P_\Omega(s)\ (I\,-\,\chi_s)\,\}\ R_w(s)\,.$$

Using (2.31), Lemma 3.5 and smoothness of χ_s, we obtain that

$$\frac{d}{ds}\,\Big(\hat{R}_w(s)\,-\,R_w(s)\Big) \ =\ O\,\big(\,e^{-\eta/\hbar}/\Delta\,\big)\ ,$$

and the bound (2.29) for $n=1$ follows from the contour representations (2.32). The $n=2$ case is handled in the same way. ■

Proposition 2.4. *Assume the hypotheses of Theorem 1.2 and define $P(s)$ as above. Then Assumption 1 holds with $\delta\ =\ C\,\hbar\,e^{-\eta/\hbar}$.*

Proof It is clear from the definition of $P_\Omega(s)$ that $(1-\chi_s)\,(H(s)-E(s))\,P_\Omega(s) = 0$. We therefore have

$$(H(s)-E(s))\ (1-\chi_s)\ P_\Omega(s) \ =\ -\,[H(s),\,\chi_s]\ P_\Omega(s) \ =\ -\,[H(s),\,F_s]\ P_\Omega(s)\,,$$

where $F_s = \chi_s \cdot \chi_{\Omega(s)}$ is a smooth function, supported in $\Omega(s)$. Lemma 3.5 then shows that

$$\|\,(H(s)-E(s))\,(1-\chi_s)\,P_\Omega(s)\,(1-\chi_s)\,\| \ \leq\ C\,\hbar\,e^{-\eta/\hbar}\,.$$

The result now follows from the identity

$$P(s) \ =\ -\,\frac{1}{2\pi i}\oint_\gamma R_w(s)\,dw \ =\ -\,\frac{1}{2\pi i}\,(1-\chi_s)\ P_\Omega(s)\ (1-\chi_s)\ \oint_\gamma w^{-1}\,R_w(s)\,dw\,,$$

and the bound (2.30). ■

Our next goal is to show that Assumption 2 is fulfilled. Our proof relies on the geometric resolvent identity (Lemma 3.3 in Appendix).

Proposition 2.5. *Assume the hypotheses of Theorem 1.2, and define $P(s)$ as above. Then Assumption 2 holds with $\delta' = C\,\hbar\, e^{-\eta/\hbar}\,|||g_a|||_4/\Delta$, for $a<\Delta/2$.*

Proof We first observe that if $a<\Delta$, then $g_a(H_\Omega(s)-E(s))$ coincides with $P_\Omega(s)$. Since

$$P_\Omega(s)\,(1-P(s))\,\dot P(s) = (P_\Omega(s)-P(s))\,(1-P(s))\,\dot P(s) = O\left(e^{-\eta/\hbar}\right),$$

by Lemma 2.3, we only need to verify that

$$\begin{aligned}&(g_a(H(s)-E(s))-g_a(H_\Omega(s)-E(s)))\;(1-P(s))\,\dot P(s)\\ &\qquad = (g_a(H(s)-E(s))-g_a(H_\Omega(s)-E(s)))\;\dot P(s)\,P(s)\end{aligned}\tag{2.33}$$

is exponentially small. We note that by construction

$$\chi_{B(s)}\,\dot P(s) = 0, \quad \text{where} \quad B := \{x\in I\!R^d:\ \chi_s(x)=1\}\cap\Omega(s)\,.$$

So, we can multiply $\dot P(s)$ in (2.33) by $\chi_{\Omega(s)\setminus B(s)}$ from the left for free. We now use Lemma 3.3 with the choices $\Omega = I\!R^d$, $\Lambda_1 = \Omega(s)\setminus B(s)$ and $\Lambda=\Omega(s)$ with $\partial_x\Theta$ supported on the set $K\subset\Omega(s)$ with $\mathrm{dist}\,\{K,\Lambda_1\} > c'$. By taking adjoints, this yields

$$\begin{aligned}&(H(s)-z)^{-1}\;\chi_{\Omega(s)\setminus B(s)} - (H_\Omega(s)-z)^{-1}\;\chi_{\Omega(s)\setminus B(s)}\\ &= (H(s)-z)^{-1}\;[H(s),\,\Theta]\;(H_\Omega(s)-z)^{-1}\;\chi_{\Omega(s)\setminus B(s)}\,.\end{aligned}\tag{2.34}$$

Hence

$$\begin{aligned}&\left\|\left\{(H(s)-z)^{-1}\;\chi_{\Omega(s)\setminus B(s)} - (H_\Omega(s)-z)^{-1}\;\chi_{\Omega(s)\setminus B(s)}\right\}\;\dot P(s)\right\|\\ &= \left\|(H(s)-z)^{-1}\;[H(s),\,\Theta]\;(H_\Omega(s)-z)^{-1}\;\chi_{\Omega(s)\setminus B(s)}\;\dot P(s)\right\|,\\ &\le \left\|(H(s)-z)^{-1}\;[H(s),\,\Theta]\right\|\;\left\|\chi_{B(s)}\,(H_\Omega(s)-z)^{-1}\;\dot P(s)\right\|,\end{aligned}\tag{2.35}$$

where in the last step we have used $[H(s),\,\Theta] = [H(s),\,\Theta]\,\chi_{B(s)}$ and $\chi_{\Omega(s)\setminus B(s)}\ \dot{P}(s) = \dot{P}(s)$. We bound the first norm by $C\,\hbar\,|\mathrm{Im}z|^{-1}$ with C that depends only on $|z|$, using Lemma 3.2. To estimate the second norm, we bound

$$\begin{aligned}
&\left\|\,\chi_{B(s)}\ \left(H_\Omega(s)-z\right)^{-1}\ \dot{P}(s)\,\right\| \\
\leq\ &\left\|\,\chi_{B(s)}\ \left(H_\Omega(s)-z\right)^{-1}\ \dot{P}_\Omega(s)\,\right\|\ +\ \left\|\,\chi_{B(s)}\ \left(H_\Omega(s)-z\right)^{-1}\ \left(\dot{P}_\Omega(s)-\dot{P}(s)\right)\,\right\|\,.
\end{aligned}$$

By Lemma 2.3, the second contribution is bounded by $C\,e^{-\eta/\hbar}/|\Delta|$ for $|z-E(s)|<\Delta/2$. To estimate the first term, we compute

$$\begin{aligned}
\left(H_\Omega(s)-z\right)^{-1}\ \dot{P}_\Omega(s)\ &=\ \frac{d}{ds}\left\{\left(H_\Omega(s)-z\right)^{-1}\ P_\Omega(s)\right\}\ -\ \frac{d}{ds}\left\{\left(H_\Omega(s)-z\right)^{-1}\right\}\ P_\Omega(s) \\
&=\ \frac{d}{ds}\left\{\left(E(s)-z\right)^{-1}\right\}\ P_\Omega(s)\ +\ \left(E(s)-z\right)^{-1}\ \dot{P}_\Omega(s) \\
&\quad +\ \ \left(E(s)-z\right)^{-1}\ \left(H_\Omega(s)-z\right)^{-1}\ \dot{H}_\Omega(s)\ P_\Omega(s)\,.
\end{aligned}$$

Note that $\dot{H}_\Omega(s)=\dot{V}(s)$ by Assumption 5, with $\mathrm{supp}\,\dot{V}(s)\subset I(s)$. We can estimate

$$\begin{aligned}
&\left\|\,\chi_{B(s)}\ \left(H_\Omega(s)-z\right)^{-1}\ \dot{P}_\Omega(s)\,\right\| \\
\leq\ &\frac{|\dot{E}(s)|}{|E(s)-z|^2}\ \left\|\,\chi_{B(s)}\,P_\Omega(s)\,\right\|\ +\ \frac{1}{|E(s)-z|}\ \left\|\chi_{B(s)}\,\dot{P}_\Omega(s)\right\| \\
&+\ \ \frac{1}{|E(s)-z|}\ \left\|\,\chi_{B(s)}\ \left(H_\Omega(s)-z\right)^{-1}\ \chi_I\,\right\|\ \left\|\,\dot{V}(s)\ P_\Omega(s)\,\right\|\,.
\end{aligned} \tag{2.36}$$

Assumption 5, and the first resolvent formula show that

$$\left\|\,\dot{V}(s)\ \left(H_\Omega(s)-z\right)^{-1}\,\right\|\ \leq\ C/\Delta \qquad \text{when} \qquad |z-E(s)|=\Delta/2.$$

So, using the contour representation (2.28), we can bound $\left\|\,\dot{V}(s)\ P_\Omega(s)\,\right\|\leq C$. All other terms in (2.36) can be bounded using Lemma 3.5 (for $|z-E(s)|<\Delta/2$). We thus see that

$$\left\|\,\chi_{B(s)}\ \left(H_\Omega(s)-z\right)^{-1}\ \dot{P}_\Omega(s)\,\right\|\ \leq\ C\ \frac{e^{-\eta/\hbar}}{\Delta\ |\mathrm{Im}z|^2}\,.$$

Putting everything together in (2.35), we obtain

$$\begin{aligned}
&\left\|\,\left\{\left(H(s)-z\right)^{-1}\ \chi_{\Omega(s)\setminus B(s)}\ -\ \left(H_\Omega(s)-z\right)^{-1}\ \chi_{\Omega(s)\setminus B(s)}\right\}\ \dot{P}(s)\,\right\| \\
\leq\ \ &C\ \frac{\hbar\ e^{-\eta/\hbar}}{|\mathrm{Im}z|^3\ \Delta}\,.
\end{aligned} \tag{2.37}$$

The lemma now follows from the Helffer-Sjöstrand representation (2.20). ■

3 Appendix

Here we collect a number of the technical statements used throughout the text. Many of these are well-known results that we have generalized to include magnetic fields and/or restricted domains.

Let $H := \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} + V$ on $I\!R^d$, where $\boldsymbol{\mathcal{P}}_{\boldsymbol{A}} = -i\,\hbar\,\nabla - \boldsymbol{A}(q)$. Let H_Ω denote its Dirichlet restriction to the set Ω. We assume H is self-adjoint and that H_Ω is bounded from below, so that it admits the Friedrichs extension. Let Θ and $\tilde{\Theta}$ be a pair of smoothed characteristic functions supported inside Ω, and taking values in $[0,\,1]$, such that $\tilde{\Theta}\,\Theta = \Theta$ (which means that $\tilde{\Theta}$ is "fatter" than Θ). Throughout this Appendix, we assume that $\boldsymbol{A}$ and V satisfy the following hypotheses:

Assumption 6.

$\mathcal{A}$**1** *The components of the vector potential* $\boldsymbol{A}$ *and their first derivatives* $\partial_i \boldsymbol{A}$ *for* $i = 1, ..., d$ *are bounded on* $I\!R^d$.

$\mathcal{A}$**2** *For x in the support of* $\nabla\Theta$, $|V(x)| \leq C$.

We frequently estimate the norm of an operator A by looking at $A^* A$. The following bound will be used throughout the Appendix:

Lemma 3.1. *Let* $H_0 := \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} + 1$ *and let* $H_0^{\#}$ *be either* H_0 *or the Friedrichs extension of its restriction to* Ω. *Similarly, let* $\boldsymbol{\mathcal{P}}_i^{\#}$ *denote either i-th component of* $\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}$ *or its restriction to* Ω *(which is not self-adjoint). If* G *is a smooth bounded function with support inside* Ω, *then for* $\hbar \leq 1$, *we have*

$$\left\| \boldsymbol{\mathcal{P}}_i^{\#}\; G\; \left(H_0^{\#}\right)^{-1/2} \right\| \;\leq\; \|\, G \,\|_\infty \,+\, C\,\hbar\; \|\, G \,\|_{2,\infty}\;. \tag{3.1}$$

Remark Note that in the lemma, $\boldsymbol{\mathcal{P}}_i^{\Omega}\; G \;=\; \boldsymbol{\mathcal{P}}_i\; G$.

Proof The non-trivial part of (3.1) is the bound for the Ω restriction. (See *e.g.*, [15] for the $I\!R^d$ case). The first step is to show that for any C^1 function ζ supported inside Ω,

$$\left\| \boldsymbol{\mathcal{P}}_i^{\Omega}\; \zeta\, \left(H_0^{\Omega} + t^2\right)^{-1} \right\| \;\leq\; \frac{4\, \|\, \zeta \,\|_{1,\infty}}{|t| \,+\, 1} \tag{3.2}$$

for any value of the parameter $t \in I\!R$. To prove this, we write

$$\begin{aligned}
&\left(H_0^\Omega + t^2\right)^{-1} \ \zeta\, \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}\, \zeta \left(H_0^\Omega + t^2\right)^{-1} \\
= \ &\left(H_0^\Omega + t^2\right)^{-1} \ \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}\, \zeta^2 \left(H_0^\Omega + t^2\right)^{-1} \\
+ \ &\left(H_0^\Omega + t^2\right)^{-1} \ [\zeta,\, \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}]\, \zeta \left(H_0^\Omega + t^2\right)^{-1}
\end{aligned} \tag{3.3}$$

The first term here is bounded by $\|\zeta\|_\infty^2\, (1+t^2)^{-1}$. To bound the second term, note that

$$[\zeta,\, \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}]\, \zeta \ = \ [\zeta^2,\, \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}] \ + \ [\,[\zeta,\, \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}],\, \zeta\,]\,.$$

The second contribution, $[\,[\zeta,\, \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}],\, \zeta]$, is equal to $2\,\hbar^2\, (\nabla\zeta)^2$. We use this to see that by making an error whose norm is bounded by $2\,\hbar^2\, \|\zeta\|_{1,\infty}^2 / (1+t^2)^2$, the second term in (3.3), which equals

$$\left(H_0^\Omega + t^2\right)^{-1} \ [\zeta,\, H_0^\Omega]\, \zeta \ \left(H_0^\Omega + t^2\right)^{-1},$$

can be replaced by

$$\left(H_0^\Omega + t^2\right)^{-1} \ [\zeta^2,\, H_0^\Omega] \ \left(H_0^\Omega + t^2\right)^{-1}.$$

However,

$$\left(H_0^\Omega + t^2\right)^{-1} \ [\zeta^2,\, H_0^\Omega] \ \left(H_0^\Omega + t^2\right)^{-1} \ = \ \left(H_0^\Omega + t^2\right)^{-1} \ \zeta^2 \ - \ \zeta^2 \ \left(H_0^\Omega + t^2\right)^{-1}$$

has norm bounded by $(1+t^2)^{-1}$. Putting the various pieces together yields (3.2) by some simple estimates and $\hbar \le 1$.

Now we derive the bound (3.1). First, we have

$$\begin{aligned}
\left\| \boldsymbol{\mathcal{P}}_i^\Omega \ G \ \left(H_0^\Omega\right)^{-1/2} \right\|^2 \ &\le \ \left\| \left(H_0^\Omega\right)^{-1/2} \ G\, \boldsymbol{\mathcal{P}}_{\boldsymbol{A}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}\, G \ \left(H_0^\Omega\right)^{-1/2} \right\| \\
&\le \ \left\| \left(H_0^\Omega\right)^{1/2} \ G \ \left(H_0^\Omega\right)^{-1/2} \right\|^2
\end{aligned}$$

We use the representation

$$\left(H_0^\Omega\right)^{-1/2} \ = \ \frac{1}{\pi} \int \ \left(H_0^\Omega + t^2\right)^{-1} \ dt\,,$$

to see that

$$\begin{aligned}
\left(H_0^\Omega\right)^{1/2} \ G \ \left(H_0^\Omega\right)^{-1/2} \ &= \ G \ + \ \left(H_0^\Omega\right)^{1/2} \ \left[G,\, \left(H_0^\Omega\right)^{-1/2}\right] \\
&= \ G \ + \ \frac{1}{\pi} \left(H_0^\Omega\right)^{1/2} \int \ \left(H_0^\Omega + t^2\right)^{-1} [H_0^\Omega,\, G] \left(H_0^\Omega + t^2\right)^{-1} \ dt.
\end{aligned} \tag{3.4}$$

Since

$$[H_0^\Omega,\, G] \;=\; -\,2\,i\,\hbar\,\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\Omega}\cdot(\nabla\, G) \;+\; \hbar^2\,(\Delta\, G)\,,$$

we can use the bound (3.2) with $\zeta = \partial_j G$ to estimate

$$\left\| \, [H_0^\Omega,\, G] \;\left(H_0^\Omega + t^2 \right)^{-1} \right\| \;\leq\; \frac{C\,\hbar\, \|\, G\|_{2,\infty}}{|t| \,+\, 1}\,.$$

On the other hand,

$$\left\| \left(H_0^\Omega \right)^{1/2} \;\left(H_0^\Omega + t^2 \right)^{-1} \right\| \;\leq\; \frac{2}{|t| + 1}\,,$$

so the right hand side of (3.4) is bounded in norm by $\|\, G\,\|_\infty \,+\, C\,\hbar\,\|\, G\,\|_{2,\infty}$. ∎

Our next step is to establish the following uniform bounds:

Lemma 3.2. *For the setup above, we have*

$$\left\|\, [H,\,\Theta] \;\; (H - z)^{-1} \,\right\| \;\leq\; C\,\hbar \left(1 \;+\; \frac{|z|}{|\mathrm{Im}\, z|} \right)\,, \tag{3.5}$$

with C depending only on Θ. Furthermore, if $E \in \mathbb{R}$ satisfies $\mathrm{dist}\,(\sigma(H_\Omega),\, E) \geq \Delta > 0$, *then*

$$\left\|\, [H_\Omega,\,\Theta] \;\; (H_\Omega - z)^{-1} \,\right\| \;\leq\; C\,\hbar \left(1 \;+\; \frac{1 \,+\, |E|}{\Delta} \right)\,, \tag{3.6}$$

for any $z \in \mathbb{C}$, such that $|z - E| < \Delta/2$. Here C depends again only on Θ.

Remark Note that if ψ is smooth with support in Ω, then $[H,\,\Theta]\,\psi \;=\; [H_\Omega,\,\Theta]\,\psi$.

Proof Let $H_\#$ denote either H or H_Ω, and similarly for the other operators that appear. We observe that

$$[H_\#,\,\Theta] \;=\; \hbar^2\;(\Delta\,\Theta) \;-\; 2\;i\;\hbar\;\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\cdot(\nabla\,\Theta). \tag{3.7}$$

Let $R_z^\#$ denote the resolvent in (3.5) (or in 3.6 accordingly), then

$$\left\|\, [H_\#,\,\Theta]\;\, R_z^\# \,\right\| \;\leq\; \hbar^2\,\|\Delta\Theta\|_\infty \;\left\|\, R_z^\# \,\right\| \;+\; 2\;\hbar\;\left\| \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\cdot(\nabla\,\Theta)\,R_z^\# \right\|. \tag{3.8}$$

To bound the second term here, we write

$$\begin{aligned} \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\cdot(\nabla\,\Theta)\;R_z^\# \;&=\; \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\;\tilde\Theta\;\left(H_0^\#\right)^{-1}\;H_0^\#\;\cdot\;(\nabla\,\Theta)\;R_z^\# \\ &=\; \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\;\tilde\Theta\;\left(H_0^\#\right)^{-1}\;\cdot\;(\nabla\,\Theta)\;H_0^\#\;R_z^\# \\ &\qquad +\;\; \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\;\tilde\Theta\;\left(H_0^\#\right)^{-1}\;\cdot\;[H_0^\#,\,\nabla\,\Theta]\;R_z^\#\,, \end{aligned} \tag{3.9}$$

where H_0 is defined as in Lemma 3.1 and $\tilde{\Theta}$ is defined as in the beginning of this Appendix.

Note now that

$$H_0^{\#}\; R_z^{\#} \;=\; I \;+\; (\,(1+z)I \,-\, V_{\#}\,)\; R_z^{\#}. \tag{3.10}$$

Using this, Assumption 6, and Lemma 3.1, we see that the first term on the right hand side of (3.9) is bounded by

$$\begin{aligned}&\left\| \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\, \tilde{\Theta}\, \left(H_0^{\#}\right)^{-1} \right\|\; \|\,\nabla\,\Theta\,\|_{1,\infty} \;+\; \left\| \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\, \tilde{\Theta}\, \left(H_0^{\#}\right)^{-1} \right\|\; \|\,(\nabla\,\Theta)\cdot((1+z)I - V_{\#})\,\|\; \left\|R_z^{\#}\right\| \\ &\leq\; C \;+\; C\;(1+|z|)\; \left\|\,R_z^{\#}\,\right\|\,, \end{aligned} \tag{3.11}$$

where we have absorbed the Sobolev norms of Θ into C.

To bound the second term in (3.9), we write

$$\begin{aligned} [\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\cdot\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#},\; \partial_i\Theta] \;&=\; -\,\hbar^2\;(\Delta\,\partial_i\Theta) -\; 2\;i\;\hbar\;\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\cdot(\nabla\,\partial_i\Theta) \\ &=\; -\,\hbar^2\;(\Delta\,\partial_i\Theta) -\; 2\;i\;\hbar\;\tilde{\Theta}\;\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\;\tilde{\Theta}\;\cdot\;(\nabla\,\partial_i\Theta)\,. \end{aligned}$$

Using this, we see that

$$\begin{aligned} &\left\| \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\, \tilde{\Theta}\, \left(H_0^{\#}\right)^{-1}\; [H_0^{\#},\, \partial_i\Theta] \right\| \\ \leq\;& C\;\hbar^2\; \left\| \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\, \tilde{\Theta}\, \left(H_0^{\#}\right)^{-1} \right\| \;+\; 2\;\hbar\; \left\| \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\, \tilde{\Theta}\, \left(H_0^{\#}\right)^{-1}\; \tilde{\Theta}\;\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\;\tilde{\Theta} \right\|\; \|\nabla\,\partial_i\Theta\| \\ \leq\;& C\;\hbar^2 \;+\; C\;\hbar\; \left\| \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\, \tilde{\Theta}\, \left(H_0^{\#}\right)^{-1/2} \right\|\; \left\| \left(H_0^{\#}\right)^{-1/2}\; \tilde{\Theta}\;\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\;\tilde{\Theta} \right\| \\ \leq\;& C\;\hbar, \end{aligned}$$

where we have repeatedly used Lemma 3.1 with $G = \tilde{\Theta}$ and the identity

$$\tilde{\Theta}\;\left(\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\right)^{*}\;\tilde{\Theta} \;=\; \tilde{\Theta}\;\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\;\tilde{\Theta}.$$

We can consequently bound the second term in (3.9) by

$$\left\| \boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\, \tilde{\Theta}\, \left(H_0^{\#}\right)^{-1}\cdot\; [\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#}\cdot\boldsymbol{\mathcal{P}}_{\boldsymbol{A}}^{\#},\; \nabla\,\Theta]\; R_z^{\#} \right\| \;\leq\; C\;\hbar\;\left\|\,R_z^{\#}\,\right\|\,.$$

Putting everything together into (3.8), the bounds (3.5) and (3.6) follow from
$\|\,R_z\,\| \leq 1/|\mathrm{Im} z|$, and $\left\|\,R_z^{\Omega}\,\right\| \leq 2/\Delta$ whenever $|E-z| < \Delta/2$. ∎

Many of the technical results throughout the paper rely on the following general result (Lemma 4.2 in [2]) that is known as the geometric resolvent identity.

Lemma 3.3. *[The Geometric Resolvent Identity] Let H be a Schrödinger operator on $I\!R^d$. Consider four open sets Λ_1, Λ_2, Λ, and Ω that satisfy $\Lambda_1 \subset \Lambda$, $\Lambda_2 \subset \Lambda$, $\Lambda \subset \Omega$, and $dist\{\Lambda_1 \cup \Lambda_2, \Lambda^c\} > 0$. Let Θ be a smooth function which is identically 1 on a neighborhood of $\Lambda_1 \cup \Lambda_2$ and identically 0 on a neighborhood of Λ^c. Given any restrictions H_Ω and H_Λ of H to Ω and Λ, respectively, we have*

$$\chi_{\Lambda_1}\ (H_\Omega - z)^{-1} \ = \ \chi_{\Lambda_1}\ (H_\Lambda - z)^{-1}\ \Theta \qquad (3.12)$$

$$+ \ \ \chi_{\Lambda_1}\ (H_\Lambda - z)^{-1}\ [H, \Theta]\ (H_\Omega - z)^{-1}$$

for any z for which both resolvents exist. Also,

$$\chi_{\Lambda_1}\ (H_\Omega - z)^{-1}\ \chi_{\Lambda_2} \ = \ \chi_{\Lambda_1}\ (H_\Lambda - z)^{-1}\ \chi_{\Lambda_2} \qquad (3.13)$$

$$+ \ \ \chi_{\Lambda_1}\ (H_\Lambda - z)^{-1}\ [H, \Theta]\ (H_\Omega - z)^{-1}\ \chi_{\Lambda_2}\,,$$

under the same conditions.

Proof Since the function Θ has support strictly contained within Λ, multiplication by Θ satisfies $H_\Lambda\, \Theta = H_\Omega\, \Theta$ on $\mathcal{D}(H_\Omega)$. Thus, on $\mathcal{D}(H_\Omega)$,

$$[H, \Theta] \ = \ (H_\Lambda - z)\ \Theta \ - \ \Theta\ (H_\Omega - z)\ .$$

We multiply this on the left by $(H_\Lambda - z)^{-1}$ and on the right by $(H_\Omega - z)^{-1}$ to see that

$$\Theta\ (H_\Omega - z)^{-1} \ = \ (H_\Lambda - z)^{-1}\ \Theta \ + \ (H_\Lambda - z)^{-1}\ [H, \Theta]\ (H_\Omega - z)^{-1}.$$

Multiplying both sides of the above equation by χ_{Λ_1} from the left and using $\chi_{\Lambda_1} \cdot \Theta = \chi_{\Lambda_1}$ gives (3.12). Multiplying both sides of (3.12) by χ_{Λ_2} on the right gives (3.13). ■

Armed with this tool, we can prove

Theorem 3.4. *[The Combes–Thomas estimate] Let $H = \boldsymbol{\mathcal{P}_A} \cdot \boldsymbol{\mathcal{P}_A} + V$ be as above. Suppose that on the domain J, the potential V is greater than $E+b$, for some $\hbar$–independent $b > 0$. Then there exists an $\hbar$–independent $\eta > 0$, such that the resolvent of the Dirichlet restriction H_J of H to J satisfies*

$$\left\|\, \chi_{J_i}\ (H_J - z)^{-1}\ \chi_{J_j} \,\right\| \ \leq \ K\ \left(\hbar^{-1} + \frac{1\,+\,|E|}{b} \right)\ e^{-\eta/\hbar} \qquad (3.14)$$

for $z \in \mathbb{C}$, *such that* $|E - z| < b/2$, *and any* $J_{i,j} \subset J$ *that satisfy*

$$\text{dist}\,\{J_i,\ J_j\} \ \geq\ c/2.$$

Here K *is a constant that depends only on* c.

Proof of Theorem 3.4 Consider the operator $\tilde{H} := \hbar^2\,\boldsymbol{\mathcal{P}_A}\cdot\boldsymbol{\mathcal{P}_A} + \tilde{V}(x)$ acting on $\mathbb{R}^d$, where

$$\tilde{V}(x) \ = \ \begin{cases} V(x) & \text{if } x \in J \\ E + b & \text{otherwise.} \end{cases}$$

Then the (improved) Combes-Thomas estimate [8] is applicable for $\tilde{H}$ and implies

$$\left\| \chi_{J_i} \left(\tilde{H} - z\right)^{-1} \chi_{J_j} \right\| \ \leq \ \tilde{K}\ \hbar^{-1}\ e^{-\eta/\hbar}. \tag{3.15}$$

for $|E - z| < b/2$ and $\text{dist}\,\{J_i(s),\ J_j(s)\} \geq c/8$, and some generic constant $\tilde{K}$.

If we now use the geometric resolvent identity (3.13), with $\chi_{\Lambda_i} = \chi_{J_i}$, $\Omega = \mathbb{R}^d$, and $\Lambda = J$, where Θ satisfies

$$\text{dist}\,\{\text{supp}\ (\nabla\Theta)\,,\, \chi_{J_i}\} \ \geq\ c/8\,, \quad \text{for} \quad i = 1,\,2\,,$$

we get

$$\begin{aligned} \left\| \chi_{J_1}\ (H_J - z)^{-1}\ \chi_{J_2} \right\| \ \leq\ & \left\| \chi_{J_1} \left(\tilde{H} - z\right)^{-1} \chi_{J_2} \right\| \\ & +\ \left\| \chi_{J_1}\ (H_J - z)^{-1}\ [H,\,\Theta] \right\| \ \left\| \chi_{J_3} \left(\tilde{H} - z\right)^{-1} \chi_{J_2} \right\|, \end{aligned} \tag{3.16}$$

where J_3 can be chosen in such a way that

$$[H,\,\Theta]\ \chi_{J_3} \ = \ [H,\,\Theta]\,, \quad \text{and} \quad \text{dist}\,\{J_3,\ J_2\} \ \geq\ c/8\,.$$

Using (3.6) and (3.15), we bound the right hand side of (3.16) by

$$K\left(\hbar^{-1} \ + \ \frac{1\ +\ |E|}{b}\right)\ e^{-\eta/\hbar}\,,$$

since the gap Δ in (3.6) in the situation at hand is b. ∎

The Combes-Thomas estimate leads to the following result, where we suppress the s–dependence whenever possible for the sake of brevity.

Lemma 3.5. *[Exponential bounds for the spectral projection] Let H_Ω satisfy Assumption 5. Suppose F and its first and second derivatives are smooth and bounded, and that F vanishes on $B := \{x \in I\!R^d : \operatorname{dist}\{x, \partial\Omega\} > c/8\}$. Then*

$$\| F\ P_\Omega \| \ \leq\ C\ e^{-\eta/\hbar}, \tag{3.17}$$

$$\left\| F\ \dot{P}_\Omega(s) \right\| \ \leq\ C\ e^{-\eta/\hbar}/\Delta\,, \qquad \left\| F\ \ddot{P}_\Omega(s) \right\| \ \leq\ C\ e^{-\eta/\hbar}/\Delta^2\,, \tag{3.18}$$

$$\| [H_\Omega,\, F]\, P_\Omega(s) \| \ \leq\ C\ \hbar\ e^{-\eta/\hbar}\,, \tag{3.19}$$

and

$$\| F\ \ (H_\Omega - z)^{-1}\ \chi_{\widetilde{I}} \| \ \leq\ C\ e^{-\eta/\hbar}/\Delta, \quad \textit{for} \quad |z - E| \,<\, \Delta/2. \tag{3.20}$$

Here $\widetilde{I}$ is a set that is slightly larger than I and satisfies $I \subset \widetilde{I}$ and

$$\operatorname{dist}\{I,\, \widetilde{I}^c\} \ \geq\ c'\,, \qquad \operatorname{dist}\{B,\, \widetilde{I}^c\} \ \geq\ 3\,c/4$$

for some $\hbar$–independent $c' > 0$. The constant C depends only on c, c' and E.

Proof of Lemma 3.5 Assertions (3.17) – (3.18) are consequences of (3.20), so we prove (3.20) first. To do so, we make use of Lemma 3.3. We let $\Lambda = \Omega \bigcap J$ and $\Lambda_o \subset \Lambda$ be a neighborhood of B. We choose Θ so that $\partial_x \Theta$ is supported in a set $S \subset J \backslash \widetilde{I}$, such that $\operatorname{dist}\{S,\, B\} \geq c/2$. Lemma 3.3 then yields

$$F\ \ (H_\Omega - z)^{-1}\ \chi_{\tilde{I}} \ =\ F\,(H_\Lambda - z)^{-1}\ [H,\, \Theta]\ \ (H_\Omega - z)^{-1}\ \chi_{\tilde{I}}. \tag{3.21}$$

Since $\chi_S\ [H,\, \Theta] \ =\ [H,\, \Theta]$, we obtain

$$F\ \ (H_\Omega - z)^{-1}\ \chi_{\widetilde{I}} \ =\ \left(F\ \ (H_\Lambda - z)^{-1}\ \chi_S\right)\ \left([H,\, \Theta]\ \ (H_\Omega - z)^{-1}\ \chi_{\widetilde{I}}\right). \tag{3.22}$$

Note that for $|z - E| < \Delta/2$, the first term in the parentheses on the right hand side enjoys the property (3.14). Putting together (3.22), (3.14), and (3.6), we get (3.20).

Note now that $\chi_{\tilde{I}}\ \dot{H}(s) \ =\ \dot{H}(s)$,

$$\frac{d}{ds}\ (H_\Omega(s) - z)^{-1} \ =\ -\ (H_\Omega(s) - z)^{-1}\ \dot{H}(s)\ \ (H_\Omega(s) - z)^{-1}$$

and

$$\left\| \dot{H}(s)\ (H_\Omega(s) - z)^{-1} \right\| \ \leq\ C/\Delta \qquad \text{for} \qquad |z - E(s)| = \Delta/2$$

by Assumption 5. It follows from the integral representation (2.28) and the bound (3.20) that the first assertion in (3.18) holds. The second assertion in (3.18) is established in the same way by taking the second derivative of the resolvent (with respect to the s variable).

We now prove the bound (3.17). The integral representation (2.28) and the bound (3.20) show that

$$\| F\ P_\Omega\ \chi_{\tilde{I}} \| \ \leq\ C\ e^{-\eta/\hbar}.$$

The desired bound (3.17) will be obtained if we can show that

$$\| F\ P_\Omega\ \chi_{\tilde{I}} \| \ \leq\ C\ e^{-\eta/\hbar} \qquad \text{implies} \qquad \| F\ P_\Omega \| \ \leq\ \tilde{C}\ e^{-\eta/\hbar}.$$

To prove this assertion, it suffices to show that for every $\psi \in \mathrm{Range}\, P_\Omega$ with $\|\psi\| = 1$, we have

$$\| \chi_{\tilde{I}}\, \psi \| \ \geq\ K\ >\ 0. \tag{3.23}$$

Indeed, it follows from (3.23) that

$$P_\Omega\ \chi_{\tilde{I}}^2\ P_\Omega\ >\ K^2\, P_\Omega\,,$$

so

$$F\ P_\Omega\ \chi_{\tilde{I}}^2\ P_\Omega\ F\ >\ K^2\ (F\ P_\Omega\ F)\ ,$$

and the result follows.

Inequality (3.23) is a consequence of the bound

$$\| F_I\, \psi \| \ \geq\ K\,, \tag{3.24}$$

for some smoothed characteristic function F_I of the set I that satisfies

$$F_I\ \chi_{\tilde{I}}\ =\ F_I\,, \qquad \text{and} \qquad F_I(x) = 1 \quad \text{for} \quad x \in I\,.$$

To show that (3.24) holds, let

$$\psi_1\ :=\ F_I\ \psi, \qquad \psi_2\ :=\ (I\ -\ F_I)\ \psi, \qquad \text{and} \qquad \psi\ =\ \psi_1\ +\ \psi_2\,.$$

Then

$$\langle\, \psi_2,\ H_\Omega\ \psi_2 \,\rangle\ \geq\ (E + b)\ \|\psi_2\|^2$$

by the definition of the region J. On the other hand,

$$\begin{aligned} E &= \langle \psi, \; H_\Omega \, \psi \rangle \\ &= \langle \psi_1, \; H_\Omega \, \psi_1 \rangle \; + \; \langle \psi_2, \; H_\Omega \, \psi_2 \rangle \; + \; 2 \, \mathrm{Re} \, \langle \psi_1, \; H_\Omega \, \psi_2 \rangle \\ &\leq E \, \|\psi_1\|^2 \, + \, (E + b) \, \|\psi_2\|^2 \, - \, E \, \langle \psi_1, \, \psi_2 \rangle \, + \, \langle [H_\Omega, \, F_I] \, \psi, \, \psi_1 \rangle \\ &= E \, + \, b \, \|\psi_2\|^2 \, + \, \langle [H_\Omega, \, F_I] \, \psi, \, \psi_1 \rangle. \end{aligned}$$

We now use the integral representation (2.28) together with the bound (3.6) to see that

$$\| [H, \, F_I] \, \psi \| \; \leq \; C \, \hbar \, \| F_I \|_{2,\infty} \, ,$$

where C is independent of the choice of $\psi \in \mathrm{Range} \, P_\Omega$. So, with the appropriate choice of F_I, we see that $\| \psi_2 \|^2 \; \leq \; C \, \hbar$, and (3.24) follows.

Finally, we show that (3.19) follows from (3.17). The proof closely follows the proof of Lemma 3.1. We use (3.7) and (3.17) to bound

$$\| [H_\Omega, \, F] \, P_\Omega \| \; \leq \; 2 \, \hbar \, \left\| \boldsymbol{\mathcal{P}_A} \, \tilde{F} \, P_\Omega \right\| \; + \; C \, \hbar^2 \, e^{-\eta/\hbar},$$

where $\tilde{F} := |\nabla \, F|$. Now, we have

$$P_\Omega \, \tilde{F} \, \boldsymbol{\mathcal{P}_A} \cdot \boldsymbol{\mathcal{P}_A} \, \tilde{F} \, P_\Omega \; = \; P_\Omega \, \tilde{F}^2 \, \boldsymbol{\mathcal{P}_A} \cdot \boldsymbol{\mathcal{P}_A} \, P_\Omega \; + \; P_\Omega \, [[\tilde{F}, \, \boldsymbol{\mathcal{P}_A} \cdot \boldsymbol{\mathcal{P}_A}], \, \tilde{F}] \, P_\Omega \, . \tag{3.25}$$

The first term is bounded by

$$\left\| P_\Omega \, \tilde{F} \right\| \quad \left\| \tilde{F} \, H_0^\Omega \, P_\Omega \right\| \; \leq \; C \, e^{-\eta/\hbar} \, ,$$

using the bound (3.17), the identity (3.10) with the estimate in (3.11), as well as the contour representation (2.28). Since $\left[\, (\tilde{F}, \boldsymbol{\mathcal{P}_A} \cdot \boldsymbol{\mathcal{P}_A}), \, \tilde{F} \right] \; = \; 2 \, \hbar^2 \, (\nabla \Theta)^2$, the second term in (3.25) is bounded by $C \, e^{-\eta/\hbar}$ using (3.17). ■

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