ON THE STRUCTURE OF THE ESSENTIAL SPECTRUM OF ELLIPTIC OPERATORS ON METRIC SPACES

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ABSTRACT. We give a description of the essential spectrum of a large class of operators on metric measure spaces in terms of their localizations at infinity. These operators are analogues of the elliptic operators on Euclidean spaces and our main result concerns the ideal structure of the C^*-algebra generated by them.

1. INTRODUCTION

The question we consider in this paper is whether the essential spectrum of an operator can be described in terms of its "localizations at infinity". Later on we shall give a general and precise mathematical meaning to this notion, but for the moment let's stick to the naive interpretation of localizations at infinity of an operator H as "asymptotic operators" obtained as limits of translates at infinity of H. However, we stress that translations have no meaning for the class of spaces of interest here and very soon we shall abandon this interpretation.

We begin with the simplest situation when $X = \mathbb{R}^d$. Note that we are interested only in operators H which are self-adjoint (quantum Hamiltonians). Denote U_a the unitary operator of translation by $a \in X$ in $L^2(X)$, so that $(U_a f)(x) = f(x + a)$, and say that $H_\varkappa$ is an asymptotic Hamiltonian of H if there is a sequence $a_n \in X$ with $|a_n| \to \infty$ such that $U_{a_n} H U_{a_n}^*$ converges in strong resolvent sense to $H_\varkappa$. Then $\mathrm{Sp}_{\mathrm{ess}}(H) = \overline{\cup}_\varkappa \mathrm{Sp}(H_\varkappa)$ holds for very large classes of Schrödinger operators. We refer to the paper [HM] of Helffer and Mohamed as one of the first dealing with this question in a general setting and to that of Last and Simon [LaS] for the most recent results obtained by similar techniques (geometric methods) and for a complete list of references. On the other hand, the importance of asymptotic operators (or "limit operators", as they call them) has been emphasized in a series of papers in the nineties by Rabinovich, Roch, and Silbermann and summarized in their book [RRS]. They are especially concerned with the case $X = \mathbb{Z}^d$ and they do not use geometric methods, but their results can be applied to the case of differential operators on $L^p(\mathbb{R}^d)$ with the help of a discretization method.

Results of this nature have also been obtained in [GI1, GI3] by a quite different method where the description of localizations at infinity in terms of asymptotic operators is not so natural and rather looks like an accident. To explain this point, we recall one result. Let X be an abelian locally compact non-compact group, define U_a as above, and for any character k of X let V_k be the operator of multiplication by k on $L^2(X)$. Let $\mathscr{E} \equiv \mathscr{E}(X)$ be the set of bounded operators T on $L^2(X)$ such that $\|V_k^* T V_k - T\| \to 0$ and $\|(U_a - 1)T^{(*)}\| \to 0$ when $k \to 1$ and $a \to 0$. A self-adjoint operator H satisfying $(H - i)^{-1} \in \mathscr{E}$ is said to be affiliated to $\mathscr{E}$; it is easy to see that this class of operators is very large. Let $\delta \equiv \delta(X)$ be the set of ultrafilters on X finer than the Frechét filter. If H is affiliated to $\mathscr{E}$ then for each $\varkappa \in \delta$ the limit $\lim_{a\to\varkappa} U_a H U_a^* = H_\varkappa$ exists in the strong resolvent sense and we have $\mathrm{Sp}_{\mathrm{ess}}(H) = \overline{\cup}_{\varkappa\in\delta} \mathrm{Sp}(H_\varkappa)$. Thus the essential spectrum of an operator affiliated to $\mathscr{E}$ is determined by its asymptotic operators.

The proof goes as follows. The space $\mathscr{E}$ is in fact a C^*-algebra canonically associated to X, namely the crossed product of the algebra of bounded uniformly continuous functions on X by the natural action of X. Moreover, the space $\mathscr{K} \equiv \mathscr{K}(X)$ of compact operators on $L^2(X)$ is an ideal of $\mathscr{E}$. Note that by ideal

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in a C^*-algebra we mean "closed bilateral ideal" and we call morphism a $*$-homomorphism between two $*$-algebras. It is easy to see that for each $\varkappa \in \delta$ and each $T \in \mathscr{E}$ the strong limit $\tau_\varkappa := \lim_{a\to\varkappa} U_a T U_a^*$ exists and that the so defined $\tau_\varkappa$ is an endomorphism of $\mathscr{E}$ so its kernel $\ker \tau_\varkappa$ is an ideal of $\mathscr{E}$ which clearly contains $\mathscr{K}$. The main fact is $\cap_{\varkappa\in\delta} \ker \tau_\varkappa = \mathscr{K}$ and the proof is not so easy. But from here we immediately deduce the preceding formula for the essential spectrum of the operators affiliated to $\mathscr{E}$. Indeed, it suffices to recall that the essential spectrum of an operator in a C^*-algebra like $\mathscr{E}$ which contains $\mathscr{K}$ is equal to the spectrum of the image of the operator in the quotient algebra $\mathscr{E}/\mathscr{K}$.

We shall call $\mathscr{E}(X)$ the *elliptic C^*-algebra of the group* X. It is probably not clear that this has something to do with the elliptic operators so we justify now the terminology. The C^*-algebra generated by a set of self-adjoint operators on a given Hilbert space is by definition the smallest C^*-algebra which contains the resolvents of these operators. Let $X = \mathbb{R}^d$ and let P be a real elliptic polynomial of order m on X. Then $\mathscr{E}(X)$ is the C^*-algebra generated by the self-adjoint operators of the form $P(\mathrm{i}\nabla) + S$ where S runs over the set of symmetric differential operators of order $< m$ whose coefficients are C^∞ functions which are bounded together with all their derivatives.

We stress that although $\mathscr{E}(X)$ is generated by a small class of elliptic differential operators, the class of self-adjoint operators affiliated to it is quite large and contains many singular perturbations of the usual elliptic operators. This is obvious from the description of $\mathscr{E}(X)$ we gave before and many explicit examples may be found in [DG1, GI3].

The main object of this paper, the C^*-algebra $\mathscr{E}(X)$ defined in (2.4), plays the same role as the preceding algebra for the case of a general class of metric spaces (for which the notion of differential operator is not defined). In Section 6 we show that if X is a unimodular amenable group then $\mathscr{E}(X)$ is the crossed product of the algebra of bounded uniformly continuous functions on X by the left action of X. Thus we may recover as a corollary of Theorem 2.1, our main result, the results of [GI1, GI3] for locally compact abelian groups and those due to Roe [Ro2] in the case of finitely generated discrete (non-abelian) groups (see also [RRR]). We mention that amenability is not necessary if we work with the reduced crossed product, an analogue of Yu's Property A is sufficient.

At an abstract level, the main point of the approach sketched above is to shift attention from one operator to an algebra of operators. Instead of studying the essential spectrum (or other qualitative spectral properties, like the Mourre estimate) of a self-adjoint operators H on a Hilbert space $\mathcal{H}$, we consider a C^*-algebra $\mathscr{E}$ of operators on $\mathcal{H}$ which contains $\mathscr{K} = K(\mathcal{H})$ and such that H is affiliated to it and try to find an "efficient" description of the quotient C^*-algebra $\mathscr{E}/\mathscr{K}$. For this, we look for a family of ideals $\mathscr{J}_\varkappa$ of $\mathscr{E}$ such that $\bigcap_\varkappa \mathscr{J}_\varkappa = \mathscr{K}$ because then we have a natural embedding

$$\mathscr{E}/\mathscr{K} \hookrightarrow \textstyle\prod_\varkappa \mathscr{E}/\mathscr{J}_\varkappa \tag{1.1}$$

and we think of this as an efficient representation of $\mathscr{E}/\mathscr{K}$ if the family $\{\mathscr{J}_\varkappa\}$ is rather small and, in our concrete situation, the ideals $\mathscr{J}_\varkappa$ have a geometrically simple interpretation. This is in an important point and we get back to it later on. For the moment note that any representation like (1.1) has important consequences in the spectral theory of the operators $T \in \mathscr{E}$, for example if T is normal and $T_\varkappa$ is the projection of T in $\mathscr{E}/\mathscr{J}_\varkappa$ then its essential spectrum is given by

$$\mathrm{Sp_{ess}}(T) = \overline{\textstyle\bigcup}_\varkappa \mathrm{Sp}(T_\varkappa). \tag{1.2}$$

We make some more comments on the role of ideals in the spectral analysis of the operators $T \in \mathscr{E}$. Consider an arbitrary ideal $\mathscr{J} \subset \mathscr{E}$ and denote $T/\mathscr{J}$ the image of T in the quotient algebra $\mathscr{E}/\mathscr{J}$. Clearly $\mathrm{Sp}(T/\mathscr{J}) \subset \mathrm{Sp}(T)$ and if $\mathscr{J}$ contains the compacts then $\mathrm{Sp}(T/\mathscr{J}) \subset \mathrm{Sp_{ess}}(T)$. It is natural in our framework to call $T/\mathscr{J}$ *localization of T at $\mathscr{J}$* (this is justified in the abelian case in Section 4.4).

We refer to [ABG, BG1, BG2, DG2, Geo] for a general discussion concerning the operation of localization with respect to an ideal and for applications in the spectral theory of many-body systems and quantum field theory but we shall mention here an example which clarifies (we hope) our point of view. Let H be the Hamiltonian of a system of N non-relativistic particles interacting through two-body potentials and let V_{jk} be the potential linking particles j and k. For each partition σ of the system of particles let H_σ

be the Hamiltonian obtained by replacing the V_{jk} such that j, k belong to different clusters of σ by zero. Then the HVZ theorem says that $\mathrm{Sp}_{\mathrm{ess}}(H) = \bigcup_\sigma \mathrm{Sp}(H_\sigma)$ where the one-cluster partition is not included in the union. In fact, this is an immediate consequence of the preceding algebraic formalism: the N-body C^*-algebra is easy to describe and H_σ is the localization of H at a certain ideal which is easy to describe. The point is that we do not have to take some limit at infinity to get H_σ, although this could be done (this would mean that we use "geometric methods").

Now let's get back to our problem. Assuming we have chosen the "correct" algebra $\mathscr{E}(X)$, we must find the relevant ideals. In the abelian group case, this is easy, because there is a natural class of ideals associated to translation invariant filters finer than the Frechét filter [GI1]. In trying to find the analog of such filters for arbitrary metric spaces one easily finds Proposition 6.6 and see that what we call coarse filters are good candidates. This explains our definition (2.5) where we introduce the ideals $\mathscr{J}_\xi$ which play the main role in our constructions. Note that they are defined by the behavior of the operators at certain regions at infinity.

One should note that this strategy denotes a certain bias toward the role played by the behavior at infinity in X (thought as physical or configuration space): we think that it has a dominant role since we hope that our choices of ideals is sufficient to describe the quotient $\mathscr{E}/\mathscr{K}$. There is no a priori reason for this to be true: there are physically natural situations in which ideals defined in terms of behavior at infinity in momentum or phase space must be taken into account [GI1]. However, it does not seem so clear to us how to defined such physically meaningful objects in the present context.

Now comes a crucial point: for a general metric space these geometrically defined ideals do not suffice to compute $\mathscr{K}(X)$, i.e. *we do not have* $\bigcap_{\varkappa\in\delta}\mathscr{E}_{(\varkappa)}(X) = \mathscr{K}(X)$ with a notation introduced in (5.28). After several unsuccessful attempts to prove equality here I accidentally learned that this is not true, in fact an ideal strictly larger than the compacts appears naturally in the algebra $\mathscr{E}(X)$, the so-called *ghost ideal*. The counter-example is due to Higson, Laforgue and Skandalis [HLS] and is important in the context of the Baum-Connes conjecture. And this happens in the simplest case of discrete metric spaces with bounded geometry (the number of points in a ball of radius r is bounded independently of the center of the ball) when $\mathscr{E}(X)$ is the *uniform Roe algebra*. The ideal structure of the uniform Roe algebra [Ro1] is studied in detail in a series of papers by Chen and Wang [CW1, CW2, Wa] and we adapted to our case their idea of kernel truncation with the help of positive type functions in case X has Yu's Property A.

Our interest in the case of general metric spaces was roused by a recent paper of E. B. Davies [Dav] in which a C^*-algebra $\mathscr{C}(X)$ (much) larger than $\mathscr{E}(X)$ is introduced and studied, cf. our Remark 3.3. Davies points out a class of ideals of $\mathscr{C}(X)$ and describes their role in understanding the essential spectrum of the operators affiliated to it. In Section 4.2 we present the filters which are implicitly used in [Dav] and in Section 6.4 we give a simple characterization of the algebra $\mathscr{C}(X)$ in the case of abelian groups.

We refer to [GI2] for a detailed discussion and historical comments in relation with our approach but emphasize the previous work of J. Bellissard, who was one of the first to stress the necessity of considering C^*-algebras generated by Hamiltonians in the context of solid state physics [Be1, Be2], and that of H. O. Cordes [Cor] who, already in the seventies, studied C^*-algebras of pseudo-differential operators on manifolds and computed their quotient with respect to the compacts in various situations.

2. Main results

A metric space $X = (X, d)$ is *proper* if each closed ball $B_x(r) = \{y \mid d(x, y) \le r\}$ is a compact set. This implies the local compactness of the topological space X but is much more because local compactness means only that the small balls are compact. In particular, if X is not compact, then the metric cannot be bounded. We are interested in proper non-compact metric spaces equipped with Radon measures μ with support equal to X, so $\mu(B_x(r)) > 0$ for all $x \in X$ and all $r > 0$, and which satisfy (at least) the following condition

$$V(r) := \sup_{x\in X} \mu(B_x(r)) < \infty \text{ for all real } r > 0. \tag{2.3}$$

To simplify the notations we set $\mathrm{d}\mu(x) = \mathrm{d}x$, $L^2(X) = L^2(X,\mu)$, and $B_x = B_x(1)$. We denote $\mathscr{B}(X)$ the C^*-algebra of all bounded operators on $L^2(X)$ and $\mathscr{K}(X)$ the ideal of $\mathscr{B}(X)$ consisting of compact operators. For $A \subset X$ we denote 1_A its characteristic function and if A is measurable then we use the same notation for the operator of multiplication by 1_A in $L^2(X)$.

Since X is locally compact the spaces $\mathcal{C}_{\mathrm{o}}(X)$ and $\mathcal{C}_{\mathrm{c}}(X)$ of continuous functions on X which tend to zero at infinity or have compact support respectively are well defined. We use the slightly unusual notation $\mathcal{C}(X)$ for the set of *bounded uniformly continuous* functions on X equipped with the sup norm. Then $\mathcal{C}(X)$ is a C^*-algebra and $\mathcal{C}_{\mathrm{o}}(X)$ is an ideal in it. We embed $\mathcal{C}(X) \subset \mathscr{B}(X)$ by identifying $\varphi \in \mathcal{C}$ with the operator $\varphi(Q)$ of multiplication by φ (this is an embedding because the support of μ is equal to X). We shall however use the notation $\varphi(Q)$ if we think that this is necessary for the clarity of the text.

Functions $k : X^2 \to \mathbb{C}$ on the product space $X^2 = X \times X$ are also called kernels on X. We say that k is a *controlled kernel* if there is a real number r such that $d(x,y) > r \Rightarrow k(x,y) = 0$. With the terminology of [HPR], a kernel is controlled if it is supported by an entourage of the bounded coarse structure on X coming from the metric. We denote $\mathcal{C}_{\mathrm{trl}}(X^2)$ the set of *bounded uniformly continuous controlled kernels* and to each $k \in \mathcal{C}_{\mathrm{trl}}(X^2)$ we associate an operator $Op(k)$ on $L^2(X)$ by $(Op(k)f)(x) = \int_X k(x,y)f(x)\mathrm{d}x$. It is easy to check (see Section 3) that the set of such operators is a $*$-subalgebra of $\mathscr{B}(X)$. Hence

$$\mathscr{E}(X) \equiv \mathscr{E}(X,d,\mu) = \text{ norm closure of } \{Op(k) \mid k \in \mathcal{C}_{\mathrm{trl}}(X^2)\} \tag{2.4}$$

is a C^*-algebra of operators on $L^2(X)$. We shall say that $\mathscr{E}(X)$ is the *elliptic algebra* of X.

There is a natural $\mathcal{C}(X)$-bimodule structure on $\mathscr{E}(X)$ because $\mathcal{C}(X)\mathscr{E}(X) = \mathscr{E}(X)\mathcal{C}(X) = \mathscr{E}(X)$ and it is easy to check that $\mathscr{K}(X) = \mathcal{C}_{\mathrm{o}}(X)\mathscr{E}(X) = \mathscr{E}(X)\mathcal{C}_{\mathrm{o}}(X) \subset \mathscr{E}(X)$. For reasons explained above we are interested in giving a “geometrically meaningful” representation of the quotient C^*-algebra $\mathscr{E}(X)/\mathscr{K}(X)$. For this purpose we introduce the class of “coarse ideals” described below.

If $F \subset X$ and $r > 0$ is real we denote $F^{(r)}$ the set of points x which belong to the interior of F and are at distance larger than r from the boundary, more precisely $\inf_{y \notin F} d(x,y) > r$. A filter ξ of subsets of X will be called *coarse* if $F \in \xi \Rightarrow F^{(r)} \in \xi$ for all r. Note that the set of complements of a coarse filter is a coarse ideal of subsets of X in the sens of [HPR]. There is a trivial coarse filter, namely $\xi = \{X\}$, which is of no interest for us. The *Fréchet filter*, by which we mean the set of sets with relatively compact complement, is clearly coarse, we denote it ∞. All the other coarse filters are finer that ∞.

To each coarse filter ξ on X we associate an ideal of $\mathscr{E}(X)$ by defining

$$\mathscr{J}_\xi(X) = \{T \in \mathscr{E}(X) \mid \inf_{F\in\xi} \|1_F T\| = 0\} = \{T \in \mathscr{E}(X) \mid \inf_{F\in\xi} \|T 1_F\| = 0\} \tag{2.5}$$

where the $\inf$ is taken only over measurable $F \in \xi$. We shall see that the set $\mathcal{I}_\xi(X)$ of $\varphi \in \mathcal{C}(X)$ such that $\lim_\xi \varphi = 0$ is an ideal of $\mathcal{C}(X)$ and $\mathscr{J}_\varkappa(X) = \mathcal{I}_\varkappa(X)\mathscr{E}(X) = \mathscr{E}(X)\mathcal{I}_\varkappa(X)$.

Let $\beta(X)$ be the set of all ultrafilters of X (this is the Stone-Cěch compactification of the *discrete* space X) and let $\delta(X)$ be the set of ultrafilters finer than the Fréchet filter. For each $\varkappa \in \beta(X)$ we denote $\mathrm{co}(\varkappa)$ the maximal coarse filter contained in $\varkappa$ and we set $\mathcal{C}_{(\varkappa)}(X) = \mathcal{I}_{\mathrm{co}(\varkappa)}(X)$ and $\mathscr{E}_{(\varkappa)}(X) = \mathscr{J}_{\mathrm{co}(\varkappa)}(X)$. These are ideals in $\mathcal{C}(X)$ and $\mathscr{E}(X)$ respectively and we have

$$\mathscr{E}_{(\varkappa)}(X) = \mathcal{C}_{(\varkappa)}(X)\mathscr{E}(X) = \mathscr{E}(X)\mathcal{C}_{(\varkappa)}(X). \tag{2.6}$$

Then to each ultrafilter $\varkappa \in \delta(X)$ we associate the quotient C^*-algebra

$$\mathscr{E}_\varkappa(X) = \mathscr{E}(X)/\mathscr{E}_{(\varkappa)}(X) \tag{2.7}$$

and call it *localization of $\mathscr{E}(X)$ at $\varkappa$* We denote $\varkappa.T$ the image of $T \in \mathscr{E}(X)$ through the canonical morphism $\mathscr{E}(X) \to \mathscr{E}_\varkappa(X)$ and we say that $\varkappa.T$ is the *localization of T at $\varkappa$*.

We may now state our main result. Note that condition (ii) is a version of *Property A* of Guoliang Yu.

Theorem 2.1. *Let (X,d) be a proper non-compact metric space and μ a Borel measure on X such that*

(i) $\mu(B_x(r)) > 0$ *and* $\sup_x \mu(B_x(r)) < \infty$ *if* $r > 0$*; moreover,* $\inf_x \mu(B_x(1/2)) > 0$*;*

(ii) *for each $\varepsilon, r > 0$ there is a Borel map $\phi : X \to L^2(X)$ with $\|\phi(x)\| = 1$, $\operatorname{supp}\phi(x) \subset B_x(s)$ for some number s independent of x, and such that $\|\phi(x) - \phi(y)\| < \varepsilon$ if $d(x, y) < r$.*

Then $\bigcap_{\varkappa\in\delta(X)} \mathscr{E}_{(\varkappa)}(X) = \mathscr{K}(X)$. Equivalently, we have a canonical embedding

$$\mathscr{E}(X)/\mathscr{K}(X) \hookrightarrow \prod_{\varkappa\in\delta(X)} \mathscr{E}_{\varkappa}. \tag{2.8}$$

In particular, the essential spectrum of any normal operator $T \in \mathscr{E}(X)$ is equal to the closure of the union of the spectra of its localizations at infinity:

$$\mathrm{Sp}_{\mathrm{ess}}(T) = \overline{\bigcup}_{\varkappa\in\delta(X)} \mathrm{Sp}(\varkappa.T). \tag{2.9}$$

The choice of $1/2$ in (i) is, of course, rather arbitrary, and an assumption of the form $\inf_x \mu(B_x(r)) > 0$ for all $r > 0$ would be more natural. Note that a large part of the theory can be developed assuming only the first part of condition (i), so each time we use the second part of (i) or (ii) we shall say it explicitly.

Several versions of Yu's Property A appear in the literature (see [Tu] for the discrete case), we have chosen that which was easier to state and use in our context. Later on we shall state and use a more abstract version which can easily be reformulated in terms of positive type functions on X^2.

In view of applications to self-adjoint operators affiliated to $\mathscr{E}(X)$, we recall [ABG] that an *observable affiliated to a C^*-algebra $\mathscr{A}$* is a morphism $H : \mathcal{C}_{\mathrm{o}}(\mathbb{R}) \to \mathscr{A}$. We set $\varphi(H) := H(\varphi)$. If $\mathcal{P} : \mathscr{A} \to \mathscr{B}$ is a morphism between two C^*-algebras then $\varphi \mapsto \mathcal{P}(\varphi(H))$ is an observable affiliated to $\mathscr{B}$ denoted $\mathcal{P}(H)$. So $\mathcal{P}(\varphi(H)) = \varphi(\mathcal{P}(H))$. If $\mathscr{A}$ and $\mathscr{B}$ are realized on Hilbert spaces $\mathcal{H}_a, \mathcal{H}_b$, then any self-adjoint operator H on $\mathcal{H}_a$ affiliated to $\mathscr{A}$ defines an observable affiliated to $\mathscr{A}$, but the observable $\mathcal{P}(H)$ is not necessarily associated to a self-adjoint operator on $\mathcal{H}_b$ because the natural operator associated to it could be non-densely defined (in our context, it often has domain equal to $\{0\}$). The spectrum and essential spectrum of an observable are defined in an obvious way [ABG].

Now clearly, if H is an observable affiliated to $\mathscr{E}(X)$ then $\varkappa.H$ defined by $\varphi(\varkappa.H) = \varkappa.\varphi(H)$ is an observable affiliated to $\mathscr{E}_{\varkappa}(X)$. This is the *localization of H at $\varkappa$* and we have

$$\mathrm{Sp}_{\mathrm{ess}}(H) = \overline{\bigcup}_{\varkappa\in\delta(X)} \mathrm{Sp}(\varkappa.H). \tag{2.10}$$

We shall not give in this paper affiliation criteria specific to the algebra $\mathscr{E}(X)$ but the results of Section 6 and the examples form [GI3] should convince the reader that the class of operators affiliated to $\mathscr{E}(X)$ is very large. On the other hand, if H is a positive self-adjoint operator such that $\mathrm{e}^{-H} \in \mathscr{E}(X)$ then H is affiliated to $\mathscr{E}(X)$. Or this is condition is certainly satisfied by the Laplace operator associated to a large class of Riemannian manifolds due to known estimates on the heat kernel of the manifold. We thank Thierry Coulhon for an e-mail exchange on this question.

3. The elliptic C^*-algebra

In this section $X = (X, d, \mu)$ is a metric space (X, d) equipped with a measure μ and such that:

- (X, d) is a locally compact not compact metric space and each closed ball is a compact set,
- μ is a Radon measure on X with support equal to X and $\sup_x \mu(B_x(r)) = V(r) < \infty\ \forall r > 0$.

The other assumptions of Theorem 2.1 are not used for the moment. If k is a controlled kernel then the least number r with the property $d(x, y) > r \Rightarrow k(x, y) = 0$ is denoted $d(k)$. We recall that

$$\mathcal{C}_{\mathrm{trl}}(X^2) = \{k : X^2 \to \mathbb{C} \mid k \text{ is a bounded uniformly continuous controlled kernel}\}. \tag{3.11}$$

If $k \in \mathcal{C}_{\mathrm{trl}}(X^2)$ then $Op(k)$ is the operator on $L^2(X)$ given by $(Op(k)f)(x) = \int_X k(x, y)f(x)\mathrm{d}x$. From

$$\|Op(k)\|^2 \le \sup_x \int |k(x, y)|\mathrm{d}y \cdot \sup_y \int |k(x, y)|\mathrm{d}x, \tag{3.12}$$

which is the Schur estimate, we get

$$\|Op(k)\| \leq V(d(k)) \sup |k|. \tag{3.13}$$

If $k, l \in \mathcal{C}_{\rm trl}(X^2)$ then we denote $k^*(x,y) = \bar{k}(y,x)$ and $(k \star l)(x,y) = \int k(x,z) l(z,y) \mathrm{d}z$. Clearly $Op(k)^* = Op(k^*)$ and $Op(k)Op(l) = Op(k \star l)$. The following simple fact is useful.

Lemma 3.1. *If $k, l \in \mathcal{C}_{\rm trl}(X^2)$ then $k \star l \in \mathcal{C}_{\rm trl}(X^2)$, we have $d(k \star l) \leq d(k) + d(l)$, and*

$$\sup |k \star l| \leq \sup |k| \cdot \sup |l| \cdot \min\{V(d(k)), V(d(l))\}.$$

Proof: If we set $s = d(k)$ and $t = d(l)$ then clearly

$$|(k \star l)(x,y)| \leq \sup |k| \cdot \sup |l| \cdot \mu\left(B_x(s) \cap B_y(t)\right)$$

which gives both estimates from the statement of the lemma. To prove the uniform continuity we use

$$\begin{aligned} |(k \star l)(x,y) - (k \star l)(x',y)| &\leq \sup_z |k(x,z) - k(x',z)| \int |l(z,y)| \mathrm{d}z \\ &\leq \sup_z |k(x,z) - k(x',z)| \cdot \sup |l| \cdot V(t) \end{aligned}$$

and a similar inequality for $|(k \star l)(x,y) - (k \star l)(x,y')|$. □

Thus $\mathcal{C}_{\rm trl}(X^2)$, when equipped with the usual linear structure and the operations k^* and $k \star l$, becomes a $*$-algebra and $k \mapsto Op(k)$ is a morphism into $\mathscr{B}(X)$ hence its range is a $*$-subalgebra of $\mathscr{B}(X)$. Hence the elliptic algebra $\mathscr{E}(X)$ defined in (2.4) is a C^*-algebra of operators on $L^2(X)$.

The uniform continuity assumption involved in the definition (3.11) of $\mathcal{C}_{\rm trl}(X)$ hence in that of $\mathscr{E}(X)$ is important because thanks to it we have $\mathscr{E}(X) = \mathcal{C}(X) \rtimes_{\rm r} X$ if X is a unimodular locally compact group, cf. Section 6. Here $\mathcal{C}(X)$ is the C^*-algebra of left uniformly continuous functions on X on which X acts by left translations and $\rtimes_{\rm r}$ denotes the reduced crossed product. In particular, the equality $\mathcal{C}(X) \rtimes_{\rm r} X = \mathscr{E}(X)$ gives a description of the crossed product independent of the group structure of X. The following example shows the role played by the uniform continuity condition.

Remark 3.2. From the results of Section 6 we see that if $X = \mathbb{R}$ one can describe the elliptic algebra in very simple terms. Let U_a, V_a be the unitary operators in $L^2(\mathbb{R})$ given by $(U_a f)(x) = f(x-a)$ and $(V_a f)(x) = \mathrm{e}^{\mathrm{i}ax} f(x)$. Then $\mathscr{E}(\mathbb{R})$ is the set of operators $T \in \mathscr{B}(\mathbb{R})$ such that $\|(U_a - 1)T^{(*)}\| \to 0$ and $\|V_a T V_a^* - T\| \to 0$ as $a \to 0$. Here $T^{(*)}$ means that the relation holds for T and T^*. We clearly may take $k(x,y) = \varphi(x)\theta(x-y)$ with $\varphi \in \mathcal{C}(\mathbb{R})$ and $\theta \in \mathcal{C}_{\rm c}(\mathbb{R})$ and then $Op(k) = \varphi(Q)\psi(P) \in \mathscr{E}(\mathbb{R})$ with ψ the Fourier transform (conveniently normalized) of θ. The advantage now is that we can see what happens if φ is only bounded and continuous. Then it is easy to check that $\varphi(Q)\psi(P) \in \mathscr{E}(\mathbb{R})$ if and only if $\|(\varphi(Q+a) - \varphi(Q))\psi(P)\| \to 0$ when $a \to 0$. For example, if $\varphi(x) = \mathrm{e}^{\mathrm{i}x^2}$ the last condition is equivalent to $\|(\mathrm{e}^{\mathrm{i}aQ} - 1)\psi(P)\| \to 0$, which is equivalent to $\psi(P) = \eta(Q)S$ for some $\eta \in \mathcal{C}_{\rm o}(\mathbb{R})$ and $S \in \mathscr{B}(\mathbb{R})$. But then $\psi(P)$ is compact as a norm limit of operators of the form $\zeta(Q)\psi(P)$ with $\zeta \in \mathcal{C}_{\rm o}(\mathbb{R})$, which is not true if $\psi \neq 0$. Thus, *the operator associated to a kernel of the form $k(x,y) = \mathrm{e}^{\mathrm{i}x^2}\theta(x-y)$ with $\theta \in \mathcal{C}_{\rm c}^\infty(\mathbb{R})$ and not zero does not belong to $\mathscr{E}(\mathbb{R})$.*

Remark 3.3. We recall that $T \in \mathscr{B}(X)$ is a *controlled operator* [Ro1] if there is $r > 0$ such that if F, G are closed subsets of X with $d(F,G) > r$ then $1_F T 1_G = 0$. The class of controlled operators has also been isolated in [Dav] and in [GG2] (under the name "finite range operators"). Observe that the $Op(k)$ with $k \in \mathcal{C}_{\rm trl}(X^2)$ are controlled operators but if X is not discrete then there are many others and most of them do not belong to $\mathscr{E}(X)$ (cf. Remark 3.2). Then the norm closure of the set of controlled operators is the C^*-algebra $\mathscr{C}(X)$ of *pseudo-local* operators, which clearly contains $\mathscr{E}(X)$. If X is a proper metric space this is the "standard algebra" from [Dav]. If X is a discrete metric space with bounded geometry then $\mathscr{C}(X) = \mathscr{E}(X)$ is the "uniform Roe C^*-algebra" from [CW1, CW2, Wa]. Anticipating on some of our later results, note that *if ξ is a coarse filter on X then the set of $T \in \mathscr{C}(X)$ such that $\inf_{F \in \xi} \|1_F T\| = 0$ is an ideal of $\mathscr{C}(X)$* (see the proof of Lemma 5.1). But if X is not discrete this class of ideals is too small to allow one to describe the quotient $\mathscr{C}(X)/\mathscr{K}(X)$ even in the simplest cases (see Proposition 6.14).

Since the kernel of $\varphi(Q)Op(k)$ is $\varphi(x)k(x,y)$ and that of $Op(k)\varphi(Q)$ is $k(x,y)\varphi(y)$, we clearly have

$$\mathcal{C}(X)\mathscr{E}(X) = \mathscr{E}(X)\mathcal{C}(X) = \mathscr{E}(X).$$

This defines a $\mathcal{C}(X)$-bimodule structure on $\mathscr{E}(X)$. We note that, as a consequence of the Cohen-Hewitt theorem, *if $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{C}(X)$ then the set $\mathcal{A}\mathcal{E}(X)$ consisting of products AT of elements $A \in \mathcal{A}$ and $T \in \mathscr{E}(X)$ is equal to the closed linear subspace of $\mathscr{E}(X)$ generated by these products.*

Proposition 3.4. *We have $\mathscr{K}(X) = \mathcal{C}_{\rm o}(X)\mathscr{E}(X) = \mathscr{E}(X)\mathcal{C}_{\rm o}(X) \subset \mathscr{E}(X)$.*

Proof: If $\varphi \in \mathcal{C}_{\rm c}$ and $k \in \mathcal{C}_{\rm trl}$ then the operator $\varphi Op(k)$ has kernel $\varphi(x)k(x,y)$ which is a continuous function with compact support on X^2, hence $\varphi Op(k)$ is a Hilbert-Schmidt operator. Thus we have $\mathcal{C}_{\rm o}(X)\mathscr{E}(X) \subset \mathscr{K}(X)$ and by taking adjoints we also get $\mathscr{E}(X)\mathcal{C}_{\rm o}(X) \subset \mathscr{K}(X)$. Conversely, an operator with kernel in $\mathcal{C}_{\rm c}(X^2)$ clearly belongs to $\mathcal{C}_{\rm c}(X)\mathscr{E}(X)$ for example. □

$\mathscr{E}(X)$ is a non-degenerate $\mathcal{C}_{\rm o}(X)$-bimodule and there is a natural topology associated to such a structure, we call it the local topology on $\mathscr{E}(X)$. Its utility will be clear from Section 6.

Definition 3.5. *The* local topology *on $\mathscr{E}(X)$ is the topology associated to the family of seminorms $\|T\|_\theta = \|T\theta(Q)\| + \|\theta(Q)T\|$ with $\theta \in \mathcal{C}_{\rm o}(X)$.*

This is the analog of the topology of local uniform convergence on $\mathcal{C}(X)$. Obviously one may replace the θ with 1_Λ where Λ runs over the set of compact subsets of X. If $T \in \mathscr{E}(X)$ and $\{T_\alpha\}$ is a net of operators in $\mathscr{E}(X)$ we write $T_\alpha \to T$ or $\lim_\alpha T_\alpha = T$ *locally* if the convergence takes place in the local topology. Since X is σ-compact there is $\theta \in \mathcal{C}_{\rm o}(X)$ with $\theta(x) > 0$ for all $x \in X$ and then $\|\cdot\|_\theta$ is a norm on $\mathscr{E}(X)$ which induces on bounded subsets of $\mathscr{E}(X)$ the local topology.

The local topology is finer than the $*$-strong operator topology inherited from the embedding $\mathscr{E}(X) \subset \mathscr{B}(X)$. We may also consider on $\mathscr{E}(X)$ the (intrinsically defined) strict topology associated to the smallest essential ideal $\mathscr{K}(X)$; this is weaker than the local topology and finer than the $*$-strong operator topology, but coincides with the last one on bounded sets.

Lemma 3.6. *The involution $T \mapsto T^*$ is locally continuous on $\mathscr{E}(X)$. The multiplication is locally continuous on bounded sets.*

Proof: Since $\|T^*\|_\theta = \|T\|_{\bar\theta}$ the first assertion is clear. Now assume $S_\alpha \to S$ locally and $\|S_\alpha\| \le C$ and $T_\alpha \to T$ locally. If $\theta \in \mathcal{C}_{\rm o}$ then $T\theta$ is a compact operator so there is $\theta' \in \mathcal{C}_{\rm o}$ such that $T\theta = \theta' K$ for some compact operator K. Then we write $(S_\alpha T_\alpha - ST)\theta = S_\alpha(T_\alpha - T)\theta + (S_\alpha - S)\theta' K$. □

Clearly $\mathscr{K}$ is the smallest ideal of $\mathscr{E}$ but there is a second ideal which appears quite naturally in the theory. This is the *ghost ideal* defined as follows:

$$\mathscr{G}(X) := \{T \in \mathscr{E}(X) \mid \lim_{x\to\infty} \|1_{B_x(r)}T\| = 0\ \forall r\} = \{T \in \mathscr{E}(X) \mid \lim_{x\to\infty} \|T1_{B_x(r)}\| = 0\ \forall r\}. \quad (3.14)$$

The fact that $\mathscr{G}$ is an ideal of $\mathscr{E}$ follows from the equality stated above which in turn is proved as follows: for each $\varepsilon > 0$ there is a controlled kernel k such that $\|T - Op(k)\| < \varepsilon$ hence if $R = r + d(k)$ we have

$$\|T1_{B_x(r)}\| < \varepsilon + \|Op(k)1_{B_x(r)}\| = \varepsilon + \|1_{B_x(R)}Op(k)1_{B_x(r)}\| < 2\varepsilon + \|1_{B_x(R)}T\|$$

which is less than 3ε for large x.

It is known that $\mathscr{K}(X) \subset \mathscr{G}(X)$ *strictly* in general [HLS, p. 349] and the role of the Property A is exactly to exclude this possibility. We refer to [CW1, CW2, Wa] for a detailed study of this question in the case of discrete spaces with bounded geometry and in the rest of this section we consider it in the present framework.

Lemma 3.7. *If $\inf_x \mu(B_x(1/2)) > 0$ then there is a subset $Z \subset X$ and for each real $r \ge 1$ there is a number $N(r) \in \mathbb{N}$ such that $X = \cup_{z\in Z} B_z(r)$ and for any $x \in X$ the number of $z \in Z$ such that $B_z(r) \cap B_x(r) \neq \emptyset$ is at most $N(r)$.*

Proof: Let Z be a maximal subset of X such that $d(a,b) > 1$ if a,b are distinct points in Z. Then we have $X = \cup_{z\in Z} B_z$ (the contrary would contradict the maximality of Z). Now fix $r \geq 1$, let $x \in X$, denote Z_x the set of $z \in Z$ such that $B_z(r) \cap B_x(r) \neq \emptyset$, and let N_x be the number of elements of Z_x. Choose $a \in Z$ such that $x \in B_a$. Then $B_x(r) \subset B_a(r+1)$ hence if $z \in Z_x$ then $B_z(r) \cap B_a(r+1) \neq \emptyset$ so $d(z,a) \leq 2r+1$. Since the balls $B_z(1/2)$ corresponding to these z are pairwise disjoint and included in $B_a(2r+2)$, the volume of their union is larger than νN_x, where $\nu = \inf_{y\in X} \mu(B_y(1/2))$, and smaller than $V(2r+2)$, hence $N_x \leq V(2r+2)/\nu$. Thus we may take $N(r) = V(2r+2)/\nu$. □

Lemma 3.8. *Assume that* $\inf_x \mu(B_x(1/2)) > 0$. *Then for* $T \in \mathscr{E}(X)$ *we have*

$$T \in \mathscr{G}(X) \Leftrightarrow \lim_{x\to\infty} \|1_{B_x} T\| = 0 \Leftrightarrow \lim_{x\to\infty} \|T 1_{B_x}\| = 0. \tag{3.15}$$

Proof: Let $T \in \mathscr{G}(X)$ with $\|1_{B_x} T\| \to 0$ as $x \to \infty$ and let $r > 1$; we prove that $\|1_{B_x(r)} T\| \to 0$ if $x \to \infty$. Let $\varepsilon > 0$ and, with the notations of Lemma 3.7, let F be a finite subset of Z such that $\|1_{B_z} T\| < \varepsilon/N(r)$ if $z \in Z \setminus F$. We consider points x such that $d(x,F) > r+1$ and denote $Z(x,r)$ the set of $z \in Z$ such that $B_z \cap B_x(r) \neq \emptyset$. Then $Z(x,r)$ has at most $N(r)$ elements and $B_x(r) \subset \cup_{z\in Z(x,r)} B_z$ hence $\|1_{B_x(r)} T\| \leq N(r) \max_{z\in Z(x,r)} \|1_{B_z} T\| < \varepsilon$ because $F \cap Z(x,r) = \emptyset$. □

An operator $T \in \mathscr{B}(X)$ is called *locally compact* [Ro1] if for any compact set K the operators $1_K T$ and $T 1_K$ are compact. Clearly *any operator in* $\mathscr{E}(X)$ *is locally compact*.

Lemma 3.9. *If* $T \in \mathscr{B}(X)$ *is a controlled locally compact operator and* $\|1_{B_x} T\| \to 0$ *as* $x \to \infty$, *then* T *is compact.*

Proof: Assume that $T \in \mathscr{B}(X)$ is locally compact and has the property $1_F T 1_G = 0$ if $F, G \subset X$ satisfy $d(F,G) > r$ for some fixed r. To prove the compactness of T it suffices to show that $\|1_R T\| \to 0$ as $R \to \infty$, where 1_R is the characteristic function of the set of points x such that $d(x,o) > R$ for some fixed $o \in X$. We set $|x| = d(x,o)$ and below denote by z points in Z. Then

$$\|1_{R+1} T f\|^2 \leq \sum_{|z|>R} \|1_{B_z} T f\|^2 = \sum_{|z|>R} \|1_{B_z} T 1_{B_z(r+1)} f\|^2 \leq \sup_{|z|>R} \|1_{B_z} T\|^2 \sum_z \|1_{B_z(r+1)} f\|^2$$

and the last sum is $\leq C(r)^2 \|f\|^2$ by Lemma 3.7. Thus $\|1_{R+1} T\| \leq C(r) \sup_{|z|>R} \|1_{B_z} T\|$. □

Our next purpose is to show that under the conditions of Theorem 2.1 we have $\mathscr{K}(X) = \mathscr{G}(X)$. For this we use an idea from [CW1] (truncation of kernels with the help of functions of positive type) and the technique of the proof of Theorem 5.1 from [Pi].

Let $\mathcal{H}$ be an arbitrary separable Hilbert space (in Theorem 2.1 we take $\mathcal{H} = L^2(X)$) and let $\phi : X \to \mathcal{H}$ be a Borel function such that $\|\phi(x)\| = 1$ for all x. Define $M_\phi : L^2(X) \to L^2(X;\mathcal{H}) = L^2(X) \otimes \mathcal{H}$ by $(M_\phi f)(x) = f(x)\phi(x)$. Then M_ϕ is a linear operator with $\|M_\phi\| = 1$ and its adjoint $M_\phi^* : L^2(X;\mathcal{H}) \to L^2(X)$ acts as follows: $(M_\phi^* F)(x) = \langle \phi(x) | F(x) \rangle$. Let $T \mapsto T_\phi$ be the linear continuous map on $\mathscr{B}(X)$ given by $T_\phi = M_\phi^* (T \otimes 1) M_\phi$. Clearly $\|T_\phi\| \leq \|T\|$.

Let $k : X^2 \to \mathbb{C}$ be a locally integrable function. We say that an operator $T \in \mathscr{B}(X)$ has integral kernel k if $\langle f | T g \rangle = \int_{X^2} k(x,y) \bar{f}(x) g(x) \mathrm{d}x \mathrm{d}y$ for all $f, g \in \mathcal{C}_\mathrm{c}(X)$. If k is a Schur kernel, i.e. $\sup_x \int_X (|k(x,y)| + |k(y,x)|) \mathrm{d}y < \infty$, then we say that T is a Schur operator and we have the estimate (3.12) for its norm. And T is a Hilbert-Schmidt operator if and only if $k \in L^2(X^2)$. From the relation $\langle f | T_\phi g \rangle = \langle f\phi | T \otimes 1 g\phi \rangle$ valid for $f, g \in \mathcal{C}_\mathrm{c}(X)$ we easily get:

Lemma 3.10. *If* T *has kernel* k *then* T_ϕ *has kernel* $k_\phi(x,y) = \langle \phi(x) | \phi(y) \rangle k(x,y)$. *In particular, if* T *is a Schur or Hilbert-Schmidt operator then* T_ϕ *is a Schur or Hilbert-Schmidt operator respectively. And if* T *is compact then* T_ϕ *is compact too.*

This follows easily from the relation $\langle f | T_\phi g \rangle = \langle f\phi | (T \otimes 1) g\phi \rangle$ valid for $f, g \in \mathcal{C}_\mathrm{c}(X)$.

Lemma 3.11. *Assume that* $\langle \phi(x) | \phi(y) \rangle = 0$ *if* $d(x,y) > r$. *Then for each* $T \in \mathscr{B}(X)$ *the operator* T_ϕ *is controlled, more precisely: if* F, G *are closed subsets of* X *with* $d(F,G) > r$ *then* $1_F T_\phi 1_G = 0$.

Proof: We have to prove that $\langle 1_F f|T_\phi 1_G g\rangle = 0$ for all $f, g \in L^2(X)$ and $T \in \mathscr{B}(X)$. The map $T \mapsto T_\phi$ is continuous for the weak operator topology and the set of finite range operators is dense in $\mathscr{B}(X)$ for this topology. Thus it suffices to assume that T is Hilbert-Schmidt (or even of rank one) and then the assertion is clear by Lemma 3.10. □

Observe that if $\theta : X \to \mathbb{C}$ is a bounded Borel function then $M_\phi \theta(Q) = (\theta(Q) \otimes 1)M_\phi$ hence $\theta T_\phi = (\theta T)_\phi$ and $T_\phi \theta = (T\theta)_\phi$ with the usual abbreviation $\theta = \theta(Q)$. In particular, Lemma 3.10 implies:

Lemma 3.12. *Let $T \in \mathscr{B}(X)$. If T is locally compact then T_ϕ is locally compact. If $\|1_{B_x(r)}T\| \to 0$ as $x \to \infty$, then $\|1_{B_x(r)}T_\phi\| \to 0$ as $x \to \infty$.*

Proposition 3.13. *Under the conditions of Theorem 2.1 we have $\mathscr{K}(X) = \mathscr{G}(X)$.*

Proof: Let $T \in \mathscr{G}(X)$ and ϕ as above. Then T is locally compact hence T_ϕ is locally compact, and we have $\|1_{B_x}T_\phi\| \to 0$ as $x \to \infty$ by Lemma 3.12. Moreover, if ϕ is as in Lemma 3.11 then T_ϕ is controlled so, by Lemma 3.9, T_ϕ is compact. Thus it suffices to show that any $T \in \mathscr{E}(X)$ is a norm limit of operators T_ϕ with ϕ of the preceding form. Since $T \mapsto T_\phi$ is a linear contraction, it suffices to show this for operators of the form $T = Op(k)$ with $k \in \mathcal{C}_{\rm trl}(X^2)$. But then $T - T_\phi$ is an operator with kernel $k(x,y)(1 - \langle\phi(x)|\phi(y)\rangle)$ hence, if we denote $M = \sup|k|, d = d(k)$, from (3.12) we get

$$\|T - T_\phi\| \le M \sup_x \int_{B_x(d)} |1 - \langle\phi(x)|\phi(y)\rangle| \mathrm{d}y.$$

Until now we did not use the fact that $\mathcal{H} = L^2(X)$ in Theorem 2.1. If we are in this situation note that we may replace $\phi(x)$ by $|\phi(x)|$ without loss of generality and then $\langle\phi(x)|\phi(y)\rangle$ is real. More generally, assume that the $\phi(x)$ belong to a real subspace of the (abstract) Hilbert space $\mathcal{H}$ so that $\langle\phi(x)|\phi(y)\rangle$ is real for all x, y. Then $1 - \langle\phi(x)|\phi(y)\rangle = \|\phi(x) - \phi(y)\|^2/2$ so we have

$$\|T - T_\phi\| \le (M/2) \sup_x \int_{B_x(d)} \|\phi(x) - \phi(y)\|^2 \mathrm{d}y.$$

Under the conditions of Theorem 2.1 it is clear that one may choose ϕ such that this be smaller than any given number. □

4. Coarse filters on X and ideals of $\mathcal{C}(X)$

4.1. **Filters.** We recall some elementary facts; for the moment X is an arbitrary set. A *filter* on X is a nonempty set ξ of subsets of X which is stable under finite intersections, does not contain the empty set, and has the property: $G \supset F \in \xi \Rightarrow G \in \xi$. If Y is a topological space and $\phi : X \to Y$ then $\lim_\xi \phi = y$ or $\lim_{x\to\xi} \phi(x) = y$ means that $y \in Y$ and if V is a neighborhood of y then $\phi^{-1}(V) \in \xi$.

The set of filters on X is equipped with the order relation given by inclusion. Then the trivial filter $\{X\}$ is the smallest filter and the lower bound of any nonempty set $\mathcal{F}$ of filters exists: $\inf \mathcal{F} = \cap_{\xi\in\mathcal{F}}\xi$. A set $\mathcal{F}$ of filters is called *admissible* if $\cap_{\xi\in\mathcal{F}}F_\xi \neq \emptyset$ if $F_\xi \in \xi$ for all ξ and $F_\xi = X$ but for a finite number of indices ξ. If $\mathcal{F}$ is admissible then the upper bound $\sup \mathcal{F}$ exists: this is the set of sets of the form $\cap_{\xi\in\mathcal{F}}F_\xi$ where $F_\xi \in \xi$ for all ξ and $F_\xi = X$ but for a finite number of indices ξ.

Let $\beta(X)$ be the set of ultrafilters on X. If ξ is a filter let $\xi^\dagger$ be the set of ultrafilters finer than it. Then $\xi = \inf \xi^\dagger$. We equip $\beta(X)$ with the topology defined by the condition: a nonempty subset of $\beta(X)$ is closed if and only if it is of the form $\xi^\dagger$ for some filter ξ. Note that for the trivial filter consisting of only one set we have $\{X\}^\dagger = \beta(X)$. Then $\beta(X)$ becomes a compact topological space, this is the Stone-Čech compactification of the *discrete* space X, and is naturally identified with the spectrum of the C^*-algebra of all bounded complex functions on X. There is an obvious dense embedding $X \subset \beta(X)$, any bounded function $\varphi : X \to \mathbb{C}$ has a unique continuous extension $\beta(\varphi)$ to $\beta(X)$, and any map $\phi : X \to X$ has a unique extension to a continuous map $\beta(\phi) : \beta(X) \to \beta(X)$.

More generally, if Y is a compact topological space, each map $\phi : X \to Y$ has a unique extension to a continuous map $\beta(\phi) : \beta(X) \to Y$. The following simple fact should be noticed: if ξ is a filter and o is a point in Y then $\lim_\xi \phi = o$ is equivalent to $\beta(\phi)|\xi^\dagger = o$. Indeed, $\lim_\xi \phi = o$ is equivalent to $\lim_\varkappa \phi = o$ for any $\varkappa \in \xi^\dagger$ (for the proof, observe that if this last relation holds then for each neighborhood V of o the set $\phi^{-1}(V)$ belongs to $\varkappa$ for all $\varkappa \in \xi^\dagger$, hence $\phi^{-1}(V) \subset \cap_{\varkappa\in\xi^\dagger} \varkappa = \xi$).

Now assume that X is a locally compact non-compact topological space. Then the *Fréchet filter* is the set of complements of relatively compact sets; we denote it ∞, so that $\lim_{x\to\infty} \phi(x) = y$ has the standard meaning. Let $\delta(X) = \infty^\dagger$ be the set of ultrafilters finer than it. Thus $\delta(X)$ is a compact subset of $\beta(X)$ and we have $\delta(X) \subset \beta(X) \setminus X$ (strictly in general):

$$\delta(X) = \{\varkappa \in \beta(X) \mid \text{ if } K \subset X \text{ is relatively compact then } K \notin \varkappa\}.$$

Indeed, if $\varkappa$ is an ultrafilter then for any set K either $K \in \varkappa$ or $K^c \in \varkappa$. If we interpret $\varkappa$ as a character of $\ell_\infty(X)$ then $\varkappa \in \delta(X)$ means $\varkappa(\varphi) = 0$ for all $\varphi \in \mathcal{C}_o(X)$.

4.2. **Coarse filters.** Now assume that X is a metric space. If $F \subset X$ then $\bar{F}$ is its closure and $F^c = X\setminus F$ its complement. We set $d_F(x) := \inf_{y\in F} d(x, y)$. Note that $d_F = d_{\bar{F}}$ and $|d_F(x) - d_F(y)| \le d(x, y)$. If $r > 0$ let $F^{(r)}$ be the set of points x such that $d(x, F^c) > r$, this is an open subset of F at distance r from the boundary. Let $F_{(r)} := \{x \mid d(x, F) \le r\}$ be the neighborhood "of order r" of F.

We say that a filter ξ is *coarse* if for any $F \in \xi$ and $r > 0$ we have $F^{(r)} \in \xi$. We emphasize that this should hold for *all* $r > 0$. If for each $F \in \xi$ there is $r > 0$ such that $F^{(r)} \in \xi$ then the filter is called *round*. Equivalently, ξ is coarse if for each $F \in \xi$ and $r > 0$ there is $G \in \xi$ such that $G_{(r)} \subset F$ and ξ is round if for each $F \in \xi$ there are $G \in \xi$ and $r > 0$ such that $G_{(r)} \subset F$.

Our terminology is related to the notion of coarse ideal introduced in [HPR] (our space X being equipped with the bounded metric coarse structure). More precisely, a *coarse ideal* is a set $\mathcal{I}$ of subsets of X such that $B \subset A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$ and $A \in \mathcal{I} \Rightarrow A_{(r)} \in \mathcal{I}$ for all $r > 0$. Clearly $\mathcal{I} \mapsto \mathcal{I}^c := \{A^c \mid A \in \mathcal{I}\}$ is a one-one correspondence between coarse ideals and filters.

Coarse filters on groups are very natural objects: *if X is a group, then a round filter is coarse if and only if it is translation invariant* (Proposition 6.6).

The Fréchet filter is coarse because if K is relatively compact then $K_{(r)}$ is compact for any r (the function d_K is proper under our assumptions on X). The trivial filter $\{X\}$ is coarse.

More general examples of coarse filters are constructed as follows [Dav, GI1]. Let $L \subset X$ be a set such that $L_{(r)} \neq X$ for all $r > 0$. Then the filter generated by the sets $L^c_{(r)} = \{x \mid d(x, L) > r\}$ when r runs over the set of positive real numbers is coarse (indeed, it is clear that the $L_{(r)}$ generate a coarse ideal). If L is compact the associated filter is ∞. If $X = \mathbb{R}$ and $L =]-\infty, 0]$ then the corresponding filter consists of neighborhoods of $+\infty$ and this example has obvious n-dimensional versions. If L is a sparse set (i.e. the distance between $a \in L$ and $L \setminus \{a\}$ tends to infinity as $a \to \infty$) then the ideal in $\mathcal{C}(X)$ associated to it (cf. below) and its crossed product by the action of X (if X is a group) are quite remarkable objects, cf. [GI1]. It should be clear however that most coarse filters are not associated to any set L.

Let X be an Euclidean space and let $G(X)$ be the set of finite unions of strict vector subspaces of X. The sets $L^c_{(r)}$ when L runs over $G(X)$ and r over $\mathbb{R}_+$ form a filter basis and the filter generated by it is the *Grassmann filter* γ of X. This is a translation invariant hence coarse filter which plays a role in a general version of the N-body problem, see [GI3, Section 6.5]. The relation $\lim_\gamma \varphi = 0$ means that the function φ vanishes when we are far from any strict affine subspace.

Lemma 4.1. *If $\mathcal{F}$ is a nonempty set of coarse filters then* $\inf \mathcal{F}$ *is a coarse filter. If $\mathcal{F}$ is admissible then* $\sup \mathcal{F}$ *is a coarse filter.*

Proof: If $F \in \inf \mathcal{F} = \cap_{\xi\in\mathcal{F}}\xi$ then for any $r > 0$ and ξ we have $F^{(r)} \in \xi$ and so $F \in \cap_{\xi\in\mathcal{F}}\xi$. Now assume for example that $F \in \xi$ and $G \in \chi$ with $\xi, \chi \in \mathcal{F}$ and let $r > 0$. Then there are $F' \in \xi$ and $G' \in \chi$ such that $F'_{(r)} \subset F$ and $G'_{(r)} \subset G$ hence $(F' \cap G')_{(r)} \subset F'_{(r)} \cap G'_{(r)} \subset F \cap G$. The argument for sets of the form $\cap_\xi F_\xi$ with $F_\xi = X$ but for a finite number of indices ξ is similar. $\square$

Lemma 4.2. *A coarse filter is either trivial, and then $\xi^\dagger = \beta(X)$, or finer than the Fréchet filter, and then $\xi^\dagger \subset \delta(X)$.*

Proof: Assume that $\xi \neq \emptyset$ is not finer than the Fréchet filter. Then there is a compact set K such that $K^c \notin \xi$. Hence for any $F \in \xi$ we have $F \not\subset K^c$ so $F \cap K \neq \emptyset$. Note that the closed sets in ξ form a basis of ξ (if $F \in \xi$ then the closure of $F^{(2)}$ belongs to ξ and is included in $F^{(1)}$ hence in F). The set $\{F \cap K \mid F \in \xi \text{ and is closed}\}$ is a filter basis consisting of closed sets in the compact set K hence there is $a \in K$ such that $a \in F$ for all $F \in \xi$. Then if $F \in \xi$ and $r > 0$ there is $G \in \xi$ such that $G_{(r)} \subset F$ and since $a \in G$ we have $B_a(r) \subset G_{(r)} \subset F$. But $X = \cup_r B_a(r)$ so $X \subset F$. □

4.3. **Coarse ideals of $\mathcal{C}(X)$.** We now recall some facts concerning the relation between filters on X and ideals of $\mathcal{C}(X)$. To each filter ξ on X we associate an ideal $\mathcal{I}_\xi(X)$ of $\mathcal{C}(X)$:

$$\mathcal{I}_\xi(X) := \{\varphi \in \mathcal{C}(X) \mid \lim_\xi \varphi = 0\} \tag{4.16}$$

If ξ is the Fréchet filter then $\lim_\xi \varphi = 0$ means $\lim_{x\to\infty} \varphi(x) = 0$ in the usual sense and so the corresponding ideal is $\mathcal{C}_\mathrm{o}(X)$. The ideal associated to the trivial filter clearly is $\{0\}$. We also have:

$$\xi \subset \eta \Rightarrow \mathcal{I}_\xi(X) \subset \mathcal{I}_\eta(X) \tag{4.17}$$

$$\mathcal{I}_{\xi\cap\eta}(X) = \mathcal{I}_\xi(X) \cap \mathcal{I}_\eta(X) = \mathcal{I}_\xi(X)\mathcal{I}_\eta(X) \tag{4.18}$$

The *round envelope* ξ° of ξ is the finer round filter included in ξ. Clearly this is the filter generated by the sets $F_{(r)}$ when F runs over ξ and r over $\mathbb{R}_+$. Note that $\mathcal{I}_\xi(X) = \mathcal{I}_{\xi^\circ}(X)$, i.e. for $\varphi \in \mathcal{C}(X)$ we have $\lim_\xi \varphi = 0$ if and only if $\lim_{\xi^\circ} \varphi = 0$. Indeed, if $\varepsilon > 0$ let F be the set of points were $|\varphi(x)| < \varepsilon/2$ and let $r > 0$ be such that $|\varphi(x) - \varphi(y)| < \varepsilon/2$ if $d(x,y) \leq r$. Then $|\varphi(x)| < \varepsilon$ if $x \in F_{(r)}$.

We recall a well-known description of the spectrum of the algebra $\mathcal{C}(X)$ in terms of round filters.

Proposition 4.3. *The map $\xi \mapsto \mathcal{I}_\xi(X)$ is a bijection between the set of all round filters on X and the set of all ideals of $\mathcal{C}(X)$.*

An ideal $\mathcal{I}$ of $\mathcal{C}(X)$ will be called *coarse* if for each positive $\varphi \in \mathcal{I}$ and $r > 0$ there is a positive $\psi \in \mathcal{I}$ such that

$$d(x,y) \leq r \text{ and } \psi(y) < 1 \Rightarrow \varphi(x) < 1. \tag{4.19}$$

Lemma 4.4. *Let F, G be subsets of X such that $G_{(r)} \subset F$. Then the function $\theta = d_{F^c}\left(d_{F^c} + d_G\right)^{-1}$ belongs to $\mathcal{C}(X)$ and satisfies the estimates $1_G \leq \theta \leq 1_F$ and $|\theta(x) - \theta(y)| \leq 3r^{-1}d(x,y)$. In particular, a filter ξ is coarse if and only if for any $F \in \xi$ and any $\varepsilon > 0$ there is $G \in \xi$ and a function θ such that $1_G \leq \theta \leq 1_F$ and $|\theta(x) - \theta(y)| \leq \varepsilon d(x,y)$.*

Proof: If $a \in G$ and $b \notin F$ then $r < d(a,b) \leq d(x,a) + d(x,b)$ for any x. By taking the lower bound of the right hand side over a, b we get $r \leq d_G(x) + d_{F^c}(x) \equiv D(x)$. Hence if $d(x) \equiv d_{F^c}(x)$ then

$$|\theta(x) - \theta(y)| \leq \frac{|d(x) - d(y)|}{D(x)} + d(y)\frac{|D(x) - D(y)|}{D(x)D(y)} \leq \frac{d(x,y)}{r} + |D(x) - D(y)| \leq \frac{d(x,y)}{3r}.$$

To prove the last assertion, notice that if such a θ exists for some $\varepsilon < 1/r$ and if $x \in G$ and $d(x,y) \leq r$ then $\theta(x) = 1$ and $|\theta(x) - \theta(y)| < 1$ hence $\theta(y) > 0$ so $y \in F$. Thus $G_{(r)} \subset F$. □

Proposition 4.5. *The filter ξ is coarse if and only if the ideal $\mathcal{I}_\xi(X)$ is coarse.*

Proof: Assume ξ is not trivial and coarse and let $\varphi \in \mathcal{I}_\xi$ positive and $r > 0$. Then $\mathcal{O}_\varphi := \{\varphi < 1\} \in \xi$ hence there is $G \in \xi$ such that $G_{(2r)} \subset \mathcal{O}_\varphi$. By using Lemma 4.4 we construct $\psi \in \mathcal{C}$ such that $0 \leq \psi \leq 1$, $\psi|_G = 0$, and $\psi|_{G^c_{(r)}} = 1$. Clearly $\psi \in \mathcal{I}_\xi$. If $\psi(y) < 1$ then $y \in G_{(r)}$ hence if $d(x,y) \leq r$ then $x \in G_{(2r)}$ so $\varphi(x) < 1$. Thus $\mathcal{I}_\xi$ is coarse. Reciprocally, assume that $\mathcal{I}_\xi$ is a coarse ideal and let $F \in \xi$ and $r > 0$. There is $\varphi \in \mathcal{I}_\xi$ positive such that $\mathcal{O}_\varphi \subset F$ and there is a positive function $\psi \in \mathcal{I}_\xi$ such that (4.19) holds. But then $\mathcal{O}_\psi \in \xi$ and $(\mathcal{O}_\psi)_{(r)} \subset \mathcal{O}_\varphi$ so ξ is coarse. □

4.4. **Coarse envelope.** If ξ is a filter then the family of coarse filters included in ξ is admissible, hence there is a largest coarse filter included in ξ. We denote it $\mathrm{co}(\xi)$ and call it *coarse envelope of* ξ. Clearly, a set F belongs to $\mathrm{co}(\xi)$ if and only if for any $r>0$ there is $G\in\xi$ such that $F\supset G_{(r)}$.

By Lemma 4.2 we have only two possibilities: either $\mathrm{co}(\xi)=\{X\}$ or $\mathrm{co}(\xi)\supset\infty$. Since $\mathrm{co}(\xi)\subset\xi$, we see that either ξ is finer than Fréchet, and then $\mathrm{co}(\xi)\supset\infty$, or not, and then $\mathrm{co}(\xi)=\{X\}$.

To each ultrafilter $\varkappa\in\beta(X)$ we associate a compact subset of $\widehat{\varkappa}\subset\beta(X)$ by the rule

$$\widehat{\varkappa}:=\mathrm{co}(\varkappa)^\dagger=\text{ set of ultrafilters finer than the coarse cover of }\varkappa. \tag{4.20}$$

Thus we have either $\varkappa\in\delta(X)$ and then $\widehat{\varkappa}\subset\delta(X)$, or $\varkappa\notin\delta(X)$ and then $\widehat{\varkappa}=\beta(X)$. On the other hand, we have $\bigcup_{\varkappa\in\delta(X)}\widehat{\varkappa}=\delta(X)$ because $\varkappa\in\widehat{\varkappa}$.

We introduce now the ideals which play the main role in our analysis of $\mathscr{E}(X)$: *to each ultrafilter $\varkappa$ on X we associate the coarse ideal $\mathcal{C}_{(\varkappa)}(X)$ of $\mathcal{C}(X)$* defined by

$$\mathcal{C}_{(\varkappa)}(X):=\mathcal{I}_{\mathrm{co}(\varkappa)}=\{\varphi\in\mathcal{C}(X)\mid \lim\nolimits_{\mathrm{co}(\varkappa)}\varphi=0\}. \tag{4.21}$$

The quotient C^*-algebra $\mathcal{C}_\varkappa(X):=\mathcal{C}(X)/\mathcal{C}_{(\varkappa)}(X)$ will be called *localization of $\mathcal{C}(X)$ at $\varkappa$*. If $\varphi\in\mathcal{C}(X)$ then its image in the quotient is denoted $\varkappa.\varphi$ and is called *localization of φ at $\varkappa$*. The next comments give another description of these objects and will make clear that *localization means extension followed by restriction*.

Observe that $\varphi\in\mathcal{C}(X)$ belongs to $\mathcal{C}_{(\varkappa)}(X)$ if and only if the restriction of $\beta(\varphi)$ to $\widehat{\varkappa}$ is zero. Hence two bounded uniformly continuous functions are equal modulo $\mathcal{C}_{(\varkappa)}(X)$ if and only if their restrictions to $\widehat{\varkappa}$ are equal. Thus $\varphi\mapsto\beta(\varphi)|\widehat{\varkappa}$ *induces an embedding $\mathcal{C}_\varkappa(X)\hookrightarrow C(\widehat{\varkappa})$ which allows us to identify $\mathcal{C}_\varkappa(X)$ with an algebra of continuous functions on $\widehat{\varkappa}$*. From this we deduce

$$\bigcap\nolimits_{\varkappa\in\delta(X)}\mathcal{C}_{(\varkappa)}(X)=\mathcal{C}_{\mathrm{o}}(X). \tag{4.22}$$

Indeed, φ belongs to the left hand side if and only if $\beta(\varphi)|\widehat{\varkappa}=0$ for all $\varkappa\in\delta(X)$. But the union of the sets $\widehat{\varkappa}$ is equal to $\delta(X)$ hence this means $\beta(\varphi)|\delta(X)=0$ which is equivalent to $\varphi\in\mathcal{C}_{\mathrm{o}}(X)$.

A *maximal coarse filter* is a coarse filter which is maximal in the set of coarse filters equipped with inclusion as order relation. This set is inductive (the union of an increasing set of coarse filters is a coarse filter) hence each coarse filter is majorated by a maximal one. Dually, we say that a subset $T\subset\delta(X)$ is *coarse* if it is of the form $T=\varkappa^\dagger$ for some coarse filter $\varkappa$. Note that if T is a minimal coarse set then $T=\widehat{\varkappa}$ for any ultrafilter $\varkappa\in T$. In general the coarse sets of the form $\widehat{\varkappa}$ with $\varkappa\in\delta(X)$ are not minimal.

5. IDEALS OF $\mathscr{E}(X)$

For any filter ξ on X we define

$$\mathscr{J}_\xi(X)=\{T\in\mathscr{E}(X)\mid \inf_{F\in\xi}\|1_FT\|=0\}. \tag{5.23}$$

Here $\inf_{F\in\xi}\|1_FT\|$ is the lower bound of the numbers $\|1_FT\|$ when F runs over the set of measurable $F\in\xi$ and we define $\inf_{F\in\xi}\|T1_F\|$ similarly. Note that $\|1_FT\|\le\|1_GT\|$ and $\|T1_F\|\le\|T1_G\|$ if $F\subset G$ are measurable.

Lemma 5.1. *If $T\in\mathscr{E}$ and ξ is a coarse filter then* $\inf_{F\in\xi}\|1_FT\|=\inf_{F\in\xi}\|T1_F\|$.

Proof: If $\inf_{F\in\xi}\|1_FT\|=a$ and $\varepsilon>0$ then there is $F\in\xi$ such that $\|1_FT\|<a+\varepsilon$. We may choose $k\in\mathcal{C}_{\mathrm{trl}}$ such that $\|T-Op(k)\|<\varepsilon$ and then $\|1_FOp(k)\|<a+2\varepsilon$. Assume that $k(x,y)=0$ if $d(x,y)\ge r$ and let $G\in\xi$ such that $G_{(r)}\subset F$. Then $k(x,y)1_G(y)=1_{G_{(r)}}(x)k(x,y)1_G(y)$ hence $Op(k)1_G=1_{G_{(r)}}Op(k)1_G=1_{G_{(r)}}1_FOp(k)1_G$ so $\|Op(k)1_G\|\le\|1_FOp(k)\|<a+2\varepsilon$ and so $\|T1_G\|<a+3\varepsilon$. ∎

Proposition 5.2. *If ξ is a coarse filter on X then $\mathscr{J}_\xi(X)$ is an ideal of $\mathscr{E}(X)$ and we have*

$$\mathscr{J}_\xi(X) = \mathcal{I}_\xi(X)\mathscr{E}(X) = \mathscr{E}(X)\mathcal{I}_\xi(X). \tag{5.24}$$

For the Fréchet filter we have $\mathscr{J}_\infty(X) = \mathscr{K}(X)$.

Proof: Since $\mathscr{J}_\xi$ a closed right ideal, the fact that it is an ideal follows from Lemma 5.1. That $\mathscr{J}_\infty = \mathscr{K}$ follows from the fact that $1_K T$ is compact if K is compact (or use (5.24) and Proposition 3.4).

We now prove the first equality in (5.24) (the second one follows by taking adjoints). Clearly $\varphi \in \mathcal{I}_\xi$ if and only if for each $\varepsilon > 0$ there is $F \in \xi$ such that $\|1_F\varphi\| < \varepsilon$ hence if and only if $\inf_{F\in\xi} \|1_F\varphi\| = 0$. This implies $\mathcal{I}_\xi\mathscr{E} \subset \mathscr{J}_\xi$ and so it remains to be shown that for each $T \in \mathscr{J}_\xi$ there are $\varphi \in \mathcal{I}_\xi$ and $S \in \mathscr{E}$ such that $T = \varphi S$. If ξ is trivial this is clear, so we may suppose that ξ is finer than ∞.

Choose a point $o \in X$ and let $K_n = B_o(n)$ for $n \geq 1$ integer. We get an increasing sequence of compact sets such that $\cup_n K_n = X$ and $K_n^c \in \xi$. We construct by induction a sequence $F_1 \supset G_1 \supset F_2 \supset G_2 \ldots$ of sets in ξ such that:

$$F_n \subset K_n^c, \quad \|1_{F_n}T\| \leq n^{-2}, \quad d(G_n, F_n^c) > 1, \quad d(F_{n+1}, G_n^c) > 1.$$

We start with $F_1' \in \xi$ such that $\|1_{F_1'}T\| \leq 1$, we set $F_1 = F_1' \cap K_1^c$ and then we choose $G_1 \in \xi$ such that $d(G_1, F_1^c) > 1$. Next, we choose $F_2' \in \xi$ with $\|1_{F_2'}T\| \leq 1/4$ and $G_1' \in \xi$ with $G_1' \subset G_1$ and $d(G_1', G_1^c) > 1$. We take $F_2 = F_2' \cap G_1' \cap K_2^c$, so $d(F_2, G_1^c) > 1$, and then we choose $G_2 \in \xi$ with $G_2 \subset F_2$ such that $d(G_2, F_2^c) > 1$, and so on.

Now we use Lemma 4.4 and for each n we construct a function $\theta_n \in \mathcal{C}$ such that $1_{G_n} \leq \theta_n \leq 1_{F_n}$ and $|\theta_n(x) - \theta_n(y)| \leq 3d(x,y)$. Let $B_a = B_a(1)$. Then it is clear that either $B_a \cap F_1 = \emptyset$ or there is a unique m such that $B_a \cap F_m \neq \emptyset$ and $B_a \cap F_{m+1} = \emptyset$ and in this case $\theta_n = 1$ on B_a if $n < m$ and $\theta_n = 0$ on B_a if $n > m$. Let $\theta(x) = \sum_n \theta_n(x)$. Then $\theta(x) = 0$ on F_1^c and if $a \in X$ is such that $B_a \cap F_m \neq \emptyset$ and $B_a \cap F_{m+1} = \emptyset$ we get

$$\theta(x) = \sum_{n\leq m} \theta_n(x) = m - 1 + \theta_m(x). \tag{5.25}$$

Thus $\theta : X \to \bar{\mathbb{R}}_+$ is well defined and for $d(x,y) < 1$ and a conveniently chosen m we have

$$|\theta(x) - \theta(y)| = |\theta_m(x) - \theta_m(y)| \leq 3d(x,y).$$

On the other hand

$$\|\theta_n T\| \leq \|1_{F_n}T\| \leq \frac{1}{n^2}.$$

Thus if $\theta_0 = 1$ then the limit of $\sum_{n\leq m} \theta_n T$ as $m \to \infty$ exists in norm and so defines an element S of $\mathscr{E}$. Then

$$T = \left(\textstyle\sum_{n\leq m}\theta_n\right)^{-1}\left(\textstyle\sum_{n\leq m}\theta_n\right)T \to (1+\theta)^{-1}S$$

because $\left(\sum_{n\leq m}\theta_n\right)^{-1} \to (1+\theta)^{-1}$ strongly on $L^2(X)$. If $\varphi := (1+\theta)^{-1}$ then $0 \leq \varphi \leq 1$ and

$$|\varphi(x) - \varphi(y)| \leq |\theta(x) - \theta(y)| \leq 3d(x,y) \quad \text{if } d(x,y) < 1.$$

Thus $\varphi \in \mathcal{C}$. If $x \in B_a$ with $B_a \cap F_m \neq \emptyset$ and $B_a \cap F_{m+1} = \emptyset$ then (5.25) gives

$$\varphi(x) = (1 + m - 1 + \theta_m(x))^{-1} \leq 1/m$$

hence $\varphi(x) \leq 1/m$ on F_m. Thus $\lim_\xi \varphi = 0$ and $T = \varphi S$ with $\varphi \in \mathcal{I}_\varkappa$ and $S \in \mathscr{E}$. □

Lemma 5.3. *If ξ is a coarse filter and $T \in \mathscr{J}_\xi(X)$ then*

$$\lim_{x\to\xi} \|1_{B_x(r)}T\| = \lim_{x\to\xi} \|T1_{B_x(r)}\| = 0 \quad \forall r > 0. \tag{5.26}$$

If $\inf_x \mu(B_x(1/2)) > 0$, $T \in \mathscr{E}(X)$ is controlled, and $\lim_{x\to\xi} \|T1_{B_x}\| = 0$ then $T \in \mathscr{J}_\xi(X)$.

Proof: Recall that $\lim_{x\to\xi}\|T1_{B_x(r)}\|=0$ means: for each $\varepsilon>0$ there is $G\in\xi$ such that $\|T1_{B_x(r)}\|<\varepsilon$ for all $x\in G$. If $T\in\mathscr{J}_\xi(X)$ and $\varepsilon>0$ then there is $F\in\xi$ such that $\|T1_F\|<\varepsilon$ and for any r there is $G\in\xi$ such that $G_{(r)}\subset F$. Hence for $x\in G$ we have $\|T1_{B_x(r)}\|\le\|T1_{G_{(r)}}\|\le\|T1_F\|<\varepsilon$ hence $\lim_{x\to\xi}\|T1_{B_x(r)}\|=0$. Replacing T by T^* we also get $\lim_{x\to\xi}\|1_{B_x(r)}T\|=0$.

Now assume $\inf_x\mu(B_x(1/2))>0$ and let $T\in\mathscr{B}(X)$ be a controlled operator. Then there is a set Z as in Lemma 3.7 and there is $r>0$ such that $T1_{B_x}=1_{B_x(r)}T1_{B_x}$ for all x. If F is a measurable set and if we denote $Z(F)$ the set of $z\in Z$ such that $B_z\cap F\neq\emptyset$ then for any $f\in L^2(X)$ we have

$$\begin{aligned}\|1_FT^*f\|^2 &\le \sum_{z\in Z(F)}\|1_{B_z}T^*f\|^2=\sum_{z\in Z(F)}\|1_{B_z}T^*1_{B_z(r)}f\|^2\\ &\le \sup_{z\in Z(F)}\|1_{B_z}T^*\|^2\sum_{z\in Z(F)}\|1_{B_z(r)}f\|^2\le\sup_{x\in F_{(1)}}\|1_{B_x}T^*\|^2C(r)^2\|f\|^2\end{aligned}$$

hence $\|T1_F\|\le C(r)\sup_{x\in F_{(1)}}\|T1_{B_x}\|$ where $C(r)$ is a number which depends only on $N(r)$, cf. Lemma 3.7. Thus for any controlled operator we have $\inf_{F\in\xi}\|T1_F\|=0$ if $\lim_{x\to\xi}\|T1_{B_x}\|=0$. And if $T\in\mathscr{E}(X)$ this means $T\in\mathscr{J}_\xi(X)$. □

Proposition 5.4. *Under the conditions of Theorem 2.1, if ξ is a coarse filter and $T\in\mathscr{E}(X)$ then*

$$T\in\mathscr{J}_\xi(X)\Leftrightarrow\lim_{x\to\xi}\|T1_{B_x}\|=0\Leftrightarrow\lim_{x\to\xi}\|1_{B_x}T\|=0. \tag{5.27}$$

Proof: We use the same techniques as in the proof of Proposition 3.13. Let $T\in\mathscr{E}(X)$ such that $\lim_{x\to\xi}\|T1_{B_x}\|=0$. Then as we saw in Section 3 we have $(T1_{B_x})_\phi=T_\phi1_{B_x}$ hence for conveniently chosen ϕ the operator $T_\phi\in\mathscr{E}(X)$ is controlled and $\lim_{x\to\xi}\|T_\phi1_{B_x}\|=0$. From Lemma 5.3 we get $T_\phi\in\mathscr{J}_\xi(X)$ which is closed, so since $T_\phi\to T$ in norm as $\phi\to1$, we get $T\in\mathscr{J}_\xi(X)$. □

Remark 5.5. The relation (5.27) is not true in general if Property A is not satisfied. Indeed, if we take $\xi=\infty$ then this would mean $\mathscr{K}(X)=\mathscr{G}(X)$, which does not hold generally.

The ideals of $\mathscr{E}(X)$ which are of real interest in our context are defined as follows

$$\varkappa\in\delta(X)\Rightarrow\mathscr{E}_{(\varkappa)}(X):=\mathscr{J}_{\mathrm{co}(\varkappa)}(X)=\{T\in\mathscr{E}(X)\mid\inf_{F\in\mathrm{co}(\varkappa)}\|1_FT\|=0\}. \tag{5.28}$$

By Proposition 5.2 this can be expressed in terms of the ideals of $\mathcal{C}(X)$ introduced in (4.21) as follows:

$$\mathscr{E}_{(\varkappa)}(X)=\mathcal{C}_{(\varkappa)}(X)\mathscr{E}(X)=\mathscr{E}(X)\mathcal{C}_{(\varkappa)}(X). \tag{5.29}$$

Prof of Theorem 2.1: Assume that $T\in\mathscr{E}_{(\varkappa)}$ for all $\varkappa\in\delta(X)$; we have to show that T is a compact operator (the converse being obvious). If $\varkappa\in\delta(X)$ and $r>0$ then for any $\varepsilon>0$ there is $F\in\mathrm{co}(\varkappa)$ such that $\|1_FT\|<\varepsilon$ and there is $G\in\varkappa$ such that $G_{(r)}\subset F$, hence for any $x\in G$ we have $\|1_{B_x(r)}T\|<\varepsilon$. This proves that $\lim_{x\to\varkappa}\|1_{B_x(r)}T\|=0$. Now define $\theta(x)=\|1_{B_x(r)}T\|$, we obtain a bounded function on X such that $\lim_\varkappa\theta=0$ for any $\varkappa\in\delta(X)$. The continuous extension $\beta(\theta):\beta(X)\to\mathbb{R}$ has the property $\beta(\theta)(\varkappa)=\lim_\varkappa\theta$ thus $\beta(\theta)$ is zero on the compact subset $\delta(X)=\infty^\dagger$ of $\beta(X)$ hence we have $\lim_\infty\theta=0$ according to a remark from Section 4.1. Thus we have $\lim_{x\to\infty}\|1_{B_x(r)}T\|=0$, which means that T belongs to the ghost ideal $\mathscr{G}$. Now the compactness of T follows from Proposition 3.13. □

6. Locally compact groups

6.1. **Crossed products.** In this section we assume that X is a locally compact topological group with neutral element e and μ is a left Haar measure. We write $\mathrm{d}\mu(x)=\mathrm{d}x$ and denote Δ the modular function defined by $\mathrm{d}(xy)=\Delta(y)\mathrm{d}x$ or $\mathrm{d}x^{-1}=\Delta(x)^{-1}\mathrm{d}x$ (with slightly formal notations). There are left and right actions of X on functions φ defined on X given by $(a.\varphi)(x)=\varphi(a^{-1}x)$ and $(\varphi.a)(x)=\varphi(xa)$.

The left and right regular representation of X are defined by $\lambda_af=a.f$ and $\rho_af=\sqrt{\Delta(a)}f.a$ for $f\in L^2(X)$. Then λ_a and ρ_a are unitary operators on $L^2(X)$ which induce unitary representation of X

on $L^2(X)$. These representations commute: $\lambda_a\rho_b = \rho_b\lambda_a$ for all $a, b \in X$. Moreover, for $\varphi \in L^\infty(X)$ we have $\lambda_a\varphi(Q)\lambda_a^* = (a.\varphi)(Q)$ and $\rho_a\varphi(Q)\rho_a^* = (\varphi.a)(Q)$.

The convolution of two functions f, g on X is defined by

$$(f * g)(x) = \int f(y)g(y^{-1}x)\mathrm{d}y = \int f(xy^{-1})\Delta(y)^{-1}g(y)\mathrm{d}y.$$

For $\psi \in L^1(X)$ let $\lambda_\psi = \int \psi(y)\lambda_y \mathrm{d}y \in \mathscr{B}(X)$. Then $\|\lambda_\psi\| \le \|\psi\|_{L^1}$ and $\psi * g = \lambda_\psi g$ for $g \in L^2$.

We recall the definition of the $*$-algebra $L^1(X)$: the product is the convolution product $f * g$ and the involution is given by $f^*(x) = \Delta(x)^{-1}\bar{f}(x^{-1})$; the factor Δ^{-1} ensures that $\|f^*\|_{L^1} = \|f\|_{L^1}$. The enveloping C^*-algebra of $L^1(G)$ is the *group C^*-algebra* $\mathcal{C}^*(X)$. The norm closure in $\mathscr{B}(X)$ of the set of operators λ_ψ with $\psi \in L^1(X)$ is the *reduced group C^*-algebra* $\mathcal{C}^*_{\mathrm{r}}(X)$. There is a canonical surjective morphism $\mathcal{C}^*(X) \to \mathcal{C}^*_{\mathrm{r}}(X)$ which is injective if and only if X is amenable.

Lemma 6.1. *If $T \in \mathcal{C}^*_{\mathrm{r}}(X)$ then $\rho_a T = T\rho_a \forall a \in X$. If X is not compact then $\mathcal{C}^*_{\mathrm{r}}(X) \cap \mathscr{K}(X) = \{0\}$.*

Proof: The first assertion is clear because $\rho_a\lambda_b = \lambda_b\rho_a$. If X is not compact, then $\rho_a \to 0$ weakly on $L^2(X)$ hence if $T \in \mathcal{C}^*_{\mathrm{r}}(X)$ is compact $\|Tf\| = \|T\rho_a f\| \to 0$ hence $\|Tf\| = 0$ for all $f \in L^2(X)$. □

In what follows by uniform continuity we mean "right uniform continuity", so $\varphi : X \to \mathbb{C}$ is uniformly continuous if for any $\varepsilon > 0$ there is a neighborhood V of e such that $xy^{-1} \in V \Rightarrow |\varphi(x) - \varphi(y)| < \varepsilon$ (see page 60 in [RS]). Let $\mathcal{C}(X)$ be the C^*-algebra of bounded uniformly continuous complex functions. If $\varphi : X \to \mathbb{C}$ is bounded measurable then $\varphi \in \mathcal{C}(X)$ if and only if $\|\lambda_a\varphi(Q)\lambda_a^* - \varphi(Q)\| \to 0$ as $a \to e$.

We consider now crossed products of the form $\mathcal{A} \rtimes X$ where $\mathcal{A} \subset \mathcal{C}(X)$ is a C^*-subalgebra stable under (left) translations (so $a.\phi \in \mathcal{A}$ if $\phi \in \mathcal{A}$; only the case $\mathcal{A} = \mathcal{C}(X)$ is of interest later). We refer to [Wil] for generalities on crossed products. The C^*-algebra $\mathcal{A} \rtimes X$ is the enveloping C^*-algebra of the Banach $*$-algebra $L^1(X; \mathcal{A})$, where the algebraic operations are defined as follows:

$$(f * g)(x) = \int f(y)\, y.g(y^{-1}x)\mathrm{d}y, \quad f^*(x) = \Delta(x)^{-1}\, x.\bar{f}(x^{-1}).$$

Thus $\mathcal{C}^*(X) = \mathbb{C} \rtimes X$. If we define $\Lambda : L^1(X; \mathcal{A}) \to \mathscr{B}(X)$ by $\Lambda(\phi) = \int \phi(a)\lambda_a \mathrm{d}a$ it is easy to check that this is a continuous $*$-morphism hence it extends uniquely to a morphism $\mathcal{A} \rtimes X \to \mathscr{B}(X)$ for which we keep the same notation Λ. A short computation gives for $\phi \in \mathcal{C}_{\mathrm{c}}(X; \mathcal{A})$ and $f \in L^2(X)$

$$(\Lambda(\phi)f)(x) = \int \phi(x, xy^{-1})\Delta(y)^{-1}f(y)\mathrm{d}y$$

where for an element $\phi \in \mathcal{C}_{\mathrm{c}}(X; \mathcal{A})$ we set $\phi(x, a) = \phi(a)(x)$. Thus $\Lambda(\phi)$ is an integral operator with kernel $k(x, y) = \phi(x, xy^{-1})\Delta(y)^{-1}$ or $\Lambda(\phi) = Op(k)$ with our previous notation.

The next simple characterization of Λ follows from the density in $\mathcal{C}_{\mathrm{c}}(X; \mathcal{A})$ of the algebraic tensor product $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{C}_{\mathrm{c}}(X)$: *there is a unique morphism $\Lambda : \mathcal{A} \rtimes X \to \mathscr{B}(X)$ such that $\Lambda(\varphi \otimes \psi) = \varphi(Q)\lambda_\psi$ for $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{C}_{\mathrm{c}}(X)$*. Here we take $\phi = \varphi \otimes \psi$ with $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{C}_{\mathrm{c}}(X)$, so $\phi(a) = \varphi\psi(a)$. Note that the kernel of the operator $\varphi(Q)\lambda_\psi$ is $k(x, y) = \varphi(x)\psi(xy^{-1})\Delta(y)^{-1}$.

The reduced crossed product $\mathcal{A} \rtimes_{\mathrm{r}} X$ is a quotient of the full crossed product $\mathcal{A} \rtimes X$, the precise definition is of no interest here. Below we give a description of it which is more convenient in our setting. As usual, we embed $\mathcal{A} \subset \mathscr{B}(X)$ by identifying $\varphi = \varphi(Q)$ and if $\mathscr{M}, \mathscr{N}$ are subspaces of $\mathscr{B}(X)$ then $\mathscr{M} \cdot \mathscr{N}$ is the closed linear subspace generated by the operators MN with $M \in \mathscr{M}$ and $N \in \mathscr{N}$.

Theorem 6.2. *The kernel of Λ is equal to that of $\mathcal{A} \rtimes X \to \mathcal{A} \rtimes_{\mathrm{r}} X$, hence Λ induces a canonical embedding $\mathcal{A} \rtimes_{\mathrm{r}} X \subset \mathscr{B}(X)$ whose range is $\mathcal{A} \cdot \mathcal{C}^*_{\mathrm{r}}(X)$. This allows us to identify $\mathcal{A} \rtimes_{\mathrm{r}} X = \mathcal{A} \cdot \mathcal{C}^*_{\mathrm{r}}(X)$.*

We thank Georges Skandalis for showing us that this is an easy consequence of results from the thesis of Athina Mageira. Indeed, it suffices to take $A = \mathcal{A}$ and $B = \mathcal{C}_{\mathrm{o}}(X)$ in [Mag, Proposition 1.3.12] by taking into account that the multiplier algebra of $\mathcal{C}_{\mathrm{o}}(X)$ is $\mathcal{C}_{\mathrm{b}}(X)$, and then to use $\mathcal{C}_{\mathrm{o}}(X) \rtimes X = \mathscr{K}(X)$ (Takai's theorem, cf. [Mag, Example 1.3.4]) and the fact that the multiplier algebra of $\mathscr{K}(X)$ is $\mathscr{B}(X)$.

The crossed product of interest here is $\mathcal{C}(X) \rtimes_{\mathrm{r}} X = \mathcal{C}(X) \cdot \mathcal{C}^*_{\mathrm{r}}(X)$. Obviously we have $\mathscr{K}(X) = \mathcal{C}_{\mathrm{o}}(X) \rtimes_{\mathrm{r}} X \subset \mathcal{C}(X) \rtimes_{\mathrm{r}} X$, the first equality being a consequence of Takai's theorem but also of the following trivial argument: if $\varphi, \psi \in \mathcal{C}_{\mathrm{c}}(X)$ then the kernel $\varphi(x)\psi(xy^{-1})\Delta(y)^{-1}$ of the operator $\varphi(Q)\lambda_\psi$ belongs to $\mathcal{C}_{\mathrm{c}}(X^2)$ hence $\varphi(Q)\lambda_\psi$ is a Hilbert-Schmidt operator.

We recall that the *local topology* on $\mathcal{C}(X) \rtimes_{\mathrm{r}} X$ (see Definition 3.5 here and [GI3, page 447]) is defined by the family of seminorms of the form $\|T\|_\Lambda = \|1_\Lambda T\| + \|T 1_\Lambda\|$ with $\Lambda \subset X$ compact.

The following is an extension of [GI3, Proposition 5.9] in the present context (see also pages 30–31 in the preprint version of [GI1] and [Ro2]). Recall that any bounded function $\varphi : X \to \mathbb{C}$ extends to a continuous function $\beta(\varphi)$ on $\beta(X)$. If $\varkappa \in \beta(X)$ we define $\varphi_\varkappa : X \to \mathbb{C}$ by

$$\varphi_\varkappa(x) = \beta(x^{-1}\varphi)(\varkappa) = \lim_{a\to\varkappa} \varphi(xa). \tag{6.30}$$

Lemma 6.3. *If $\varphi \in \mathcal{C}(X)$ then for any $\theta \in \mathcal{C}_{\mathrm{o}}(X)$ the set $\{\theta\varphi.a \mid a \in X\}$ is relatively compact in $\mathcal{C}_{\mathrm{o}}(X)$ and the map $a \mapsto \theta\varphi_a \in \mathcal{C}_{\mathrm{o}}(X)$ is norm continuous. In particular, for any $\varkappa \in \beta(X)$ the limit in* (6.30) *exists locally uniformly in x and we have $\varphi_\varkappa \in \mathcal{C}(X)$.*

Proof: By the Ascoli-Arzela theorem, to show the relative compactness of the set of functions of the form $\theta\varphi.a$ it suffices to show that the set is equicontinuous. For each $\varepsilon > 0$ there is a neighborhood V of e such that $|\varphi(x)-\varphi(y)| < \varepsilon$ if $xy^{-1} \in V$. Then $|\varphi(xa)-\varphi(ya)| < \varepsilon$ for all $a \in X$, which proves the assertion. In particular, $\lim_{a\to\varkappa} \theta\varphi.a$ exists in norm in $\mathcal{C}_{\mathrm{o}}(X)$, hence the limit in (6.30) exists locally uniformly in x. Moreover, we shall have $|\varphi_\varkappa(x) - \varphi_\varkappa(y)| < \varepsilon$ so $\varphi_\varkappa$ belongs to $\mathcal{C}(X)$. Finally, we show that for any compact set K and any $\varepsilon > 0$ there is a neighborhood V of e such that $\sup_K |\varphi(xa) - \varphi(x)| < \varepsilon$ for all $a \in V$. For this, let $\mathcal{U}$ be an open cover of K such that the oscillation of φ over any $U \in \mathcal{U}$ is $< \varepsilon$ and note that there is an neighborhood V of e such that for any $x \in K$ there is $U \in \mathcal{U}$ such that $xV \subset U$ (use the Lebesgue property for the left uniform structure). □

Proposition 6.4. *For each $T \in \mathcal{C}(X) \rtimes_{\mathrm{r}} X$ and each $a \in X$ we have $\tau_a(T) := \rho_a T \rho_a^* \in \mathcal{C}(X) \rtimes_{\mathrm{r}} X$ and the map $a \mapsto \tau_a(T)$ is locally continuous on X and has locally relatively compact range. For each ultrafilter $\varkappa \in \beta(X)$ and each $T \in \mathcal{C}(X) \rtimes_{\mathrm{r}} X$ the limit $\tau_\varkappa(T) := \lim_{a\to\varkappa} \tau_a(T)$ exists in the local topology of $\mathcal{C}(X) \rtimes_{\mathrm{r}} X$. The so defined map $\tau_\varkappa : \mathcal{C}(X) \rtimes_{\mathrm{r}} X \to \mathcal{C}(X) \rtimes_{\mathrm{r}} X$ is a morphism uniquely determined by the property $\tau_\varkappa(\varphi(Q)\lambda_\psi) = \varphi_\varkappa(Q)\lambda_\psi$.*

Proof: If $T = \varphi(Q)\lambda_\psi$ then $\rho_a T \rho_a^* = (\varphi.a)(Q)\lambda_\psi$ is an element of $\mathcal{C}(X) \rtimes_{\mathrm{r}} X$ and so τ_a is an automorphism of $\mathcal{C}(X) \rtimes_{\mathrm{r}} X$. If we take ψ with compact support and Λ is a compact set then $\lambda_\psi 1_\Lambda = 1_K \lambda_\psi 1_\Lambda$ where $K = (\mathrm{supp}\,\psi)\Lambda$ is also compact. Then $\tau_a(T)1_\Lambda = (\varphi.a)(Q)1_K\lambda_\psi 1_\Lambda$ and the map $a \mapsto (\varphi.a)(Q)1_K$ is norm continuous, cf. Lemma 6.3. This implies that $a \mapsto \tau_a(T)$ is locally continuous on X for any T. To show that the range is relatively compact, it suffices again to consider the case $T = \varphi(Q)\lambda_\psi$ with ψ with compact support and to use $\tau_a(T)1_\Lambda = (\varphi.a)(Q)1_K\lambda_\psi 1_\Lambda$ and the relative compactness of the $\{(\varphi.a)(Q)1_K \mid a \in X\}$ established in Lemma 6.3. The other assertions of the proposition follow easily from these facts. □

6.2. **Elliptic C^*-algebra.** From now on X is a locally compact non-compact topological group. Since we do not require that X be metrizable, we have to adapt some of the notions used in the metric case to this context. Of course, we could use the more general framework of coarse spaces [Ro1] to cover both situations, but we think that the case of metric groups is already sufficiently general. So the reader may assume that X is equipped with an invariant proper distance d. Our leftist bias in Section 6.1 forces us to take d right invariant, i.e. $d(x,y) = d(xz,yz)$ for all x,y,z. If we set $|x| = d(x,e)$ then we get a function $|\cdot|$ on X such that $|x^{-1}| = |x|$, $|xy| \le |x| + |y|$, and $d(x,y) = |xy^{-1}|$. The balls $B(r)$ defined by relations of the form $|x| \le r$ are a basis of compact neighborhoods of e, a function on X is d-uniformly continuous if and only if it is right uniformly continuous, etc.

Note that $B_x(r) = B(r)x$ so in the non-metrizable case the role of the balls $B_x(r)$ is played by the sets Vx with V compact neighborhoods of e. Recall that the range of the modular function Δ is a subgroup of the multiplicative group $]0,\infty[$ hence it is either $\{1\}$ or unbounded. Since $\mu(Vx)=\mu(V)\Delta(x)$ our assumption (2.3) is satisfied only if X is unimodular and in this case we have $\mu(Vx)=\mu(V)$ for all x.

We emphasize the importance of the condition that the metric be proper. Fortunately, it has been proved in [HP] that a locally compact group is second countable if and only if its topology is generated by a proper right invariant metric.

For coherence, in the non metrizable case we are forced to say that a kernel $k:X^2\to\mathbb{C}$ is *controlled* if there is a compact set $K\subset X$ such that $k(x,y)=0$ if $xy^{-1}\notin K$. The symbol $d(k)$ should be defined now as the smallest compact set K with the preceding property. On the other hand, k is uniformly continuous if it is right uniformly continuous, i.e. if for any $\varepsilon>0$ there is a neighborhood V of e such that $|k(ax,by)-k(x,y)|<\varepsilon$ for all $a,b\in V$ and $x,y\in X$. Then the Schur estimate (3.12) gives $\|Op(k)\|\leq \sup|k|\sup_a\mu(Ka)$ so only if X is unimodular we have a simple estimate $\|Op(k)\|\leq\mu(K)\sup|k|$.

To summarize, if X is unimodular then $\mathcal{C}_{\mathrm{trl}}(X^2)$ is well defined and Lemma 3.1 remains valid if we set $V(d(k))=\mu(d(k))$ so we may *define the elliptic algebra* $\mathscr{E}(X)$ *as in* (2.4). But in fact, what we get is just a description of the crossed product $\mathcal{C}(X)\rtimes_{\mathrm{r}}X$ independent of the group structure of X:

Proposition 6.5. *If X is unimodular then $\mathscr{E}(X)=\mathcal{C}(X)\rtimes_{\mathrm{r}}X=\mathcal{C}(X)\cdot\mathcal{C}^*_{\mathrm{r}}(X)$.*

Proof: From the results presented in Section 6.1 and the fact that $\Delta=1$ we get that $\mathcal{C}(X)\rtimes X$ is the closed linear space generated by the operators $Op(k)$ with kernels $k(x,y)=\varphi(x)\psi(xy^{-1})$, where $\varphi\in\mathcal{C}(X)$ and $\psi\in\mathcal{C}_{\mathrm{c}}(X)$. Thus $\mathcal{C}(X)\rtimes X\subset\mathscr{E}(X)$. To show the converse, let $k\in\mathcal{C}_{\mathrm{trl}}(X^2)$ and let $\widetilde{k}(x,y)=k(x,y^{-1}x)$ hence $k(x,y)=\widetilde{k}(x,xy^{-1})$. If $K=K^{-1}\subset X$ is a compact set such that $k(x,y)\neq0\Rightarrow xy^{-1}\in K$ then $\mathrm{supp}\widetilde{k}\subset X\times K$. Fix $\varepsilon>0$ and let V be a neighborhood of the origin such that $|\widetilde{k}(x,y)-\widetilde{k}(x,z)|<\varepsilon$ if $yz^{-1}\in V$. Then let $Z\subset K$ be a finite set such that $K\subset\cup_{z\in Z}Vz$ and let $\{\theta_z\}$ be a partition of unity subordinated to this covering. If $\widetilde{l}(x,y)=\sum_{z\in Z}\widetilde{k}(x,z)\theta_z(y)$ or $\widetilde{l}=\sum_{z\in Z}\widetilde{k}(\cdot,z)\otimes\theta_z$ then

$$|\widetilde{k}(x,y)-\widetilde{l}(x,y)|=|\sum_{z\in Z}(\widetilde{k}(x,y)-\widetilde{k}(x,z))\theta_z(y)|\leq\sum_{z\in Z}|\widetilde{k}(x,y)-\widetilde{k}(x,z)|\theta_z(y)\leq\varepsilon$$

because $\mathrm{supp}\theta_z\subset Vz$. Now let us set $l(x,y)=\widetilde{l}(x,xy^{-1})=\sum_{z\in Z}\widetilde{k}(x,z)\theta_z(xy^{-1})$. If $l(x,y)\neq0$ then $\theta_z(xy^{-1})\neq0$ for some z hence $xy^{-1}\in Vz\subset VK$. In this construction we may choose $V\subset U$ where U is a fixed compact neighborhood of the origin. Then we will have $l(x,y)\neq0\Rightarrow xy^{-1}\subset UK$ which is a compact set independent of l and from (3.13) we get $\|Op(k)-Op(l)\|\leq C\sup|k-l|\leq C\varepsilon$ for some constant C independent of ε. But clearly $Op(l)\in\mathcal{C}(X)\rtimes_{\mathrm{r}}X$. □

Thus if X is a unimodular group then we may apply Proposition 6.4 and get endomorphisms $\tau_\varkappa$ of $\mathscr{E}(X)$ indexed by $\varkappa\in\delta(X)$. These will play an important role in the next subsection.

We make now some comments on the relation between amenability and Property A in the case of groups. First, the Property A is much more general than amenability, cf. the discussion in [NY] for the case of discrete groups. To show that amenability implies Property A we choose from the numerous known equivalent descriptions that which is most convenient in our context [Pat, page 128]: *X is amenable if and only if for any $\varepsilon>0$ and any compact subset K of X there is a positive function $\varphi\in\mathcal{C}_{\mathrm{c}}(X)$ with $\|\varphi\|=1$ such that $\|\rho_a\varphi-\varphi\|<\varepsilon$ for all $a\in K$.* Now let us set $\phi(x)=\rho^*_x\varphi$, so $\phi(x)(z)=\Delta(x)^{-1/2}\varphi(zx^{-1})$. We get a strongly continuous function $\phi:X\to L^2(X)$ such that $\|\phi(x)\|=1$, $\mathrm{supp}\phi(x)=(\mathrm{supp}\varphi)x$, and $\|\phi(x)-\phi(y)\|=\|\rho_{xy^{-1}}\varphi-\varphi\|\leq\varepsilon$ if $xy^{-1}\in K$. In the metric case we get a function as in condition (ii) of Theorem 2.1, so the metric version of the Property A is satisfied.

6.3. **Coarse filters in groups.** A filter ξ on a locally compact non-compact group X is called *round* if the sets of the form $VG=\{xy\mid x\in V,y\in G\}$, where V runs over the set of neighborhoods of e and G over ξ, are a basis of ξ. And ξ is (left) *invariant* if $x\in X,F\in\xi\Rightarrow xF\in\xi$. Naturally, ξ is *coarse* if for any $F\in\xi$ and any compact set $K\subset X$ there is $G\in\xi$ such that $KG\subset F$.

The simplicity of the next proof owes much to a discussion with H. Rugh. In our initial argument Proposition 6.6 was a corollary of Proposition 4.5.

Proposition 6.6. *A filter is coarse if and only if it is round and invariant.*

Proof: Note first that ξ is invariant if and only if for each $H \in \xi$ and each finite $N \subset X$ there is $G \in \xi$ such that $H \supset NG$. This is clear because $NG \subset H$ is equivalent to $G \subset \cap_{x\in N} x^{-1}H$. Now assume that ξ is also round. Then for any $F \in \xi$ there is a neighborhood V of e and a set $H \in \xi$ such that $F \supset VH$. If K is any compact set then there is a finite set N such that $VN \supset K$. Then there is $G \in \xi$ such that $H \supset NG$. So $F \supset VNG \supset KH$. □

Proposition 6.7. *Let X be unimodular and let ξ be a coarse filter. Then for any $T \in \mathscr{J}_\xi(X)$ we have $\lim_{a\to\xi} \tau_a(T) = 0$ locally. If X is amenable then the converse assertion holds, so*

$$\mathscr{J}_\xi(X) = \{T \in \mathscr{E}(X) \mid \lim_{a\to\xi} \tau_a(T) = 0 \text{ locally}\} = \{T \in \mathscr{E}(X) \mid \tau_\varkappa(T) = 0\ \forall \varkappa \in \xi^\dagger\}. \tag{6.31}$$

Moreover, if X is amenable then for any compact neighborhood V of e and any $T \in \mathscr{E}(X)$ we have:

$$T \in \mathscr{J}_\xi(X) \Leftrightarrow \lim_{a\to\xi} \|T1_{Va}\| = 0 \Leftrightarrow \lim_{a\to\xi} \|\tau_a(T)1_V\| = 0 \tag{6.32}$$

Proof: We have $1_{Va}(Q) = \rho_a^* 1_V(Q)\rho_a$ hence $\|T1_{Va}\| = \|T\rho_a^* 1_V(Q)\rho_a\| = \|\tau_a(T)1_V(Q)\|$ hence for $T \in \mathscr{J}_\xi(X)$ we have $\lim_{a\to\xi} \tau_a(T) = 0$ locally. If X is amenable then Proposition 5.4 in the metric case and a suitable modification in the non-metrizable group case gives (6.31). Then (6.32) is easy. □

Theorem 6.8. *Let X be a unimodular amenable locally compact group. Then for each $\varkappa \in \delta(X)$ and for each $T \in \mathscr{E}(X)$ the limit $\tau_\varkappa(T) := \lim_{a\to\varkappa} \rho_a T \rho_a^*$ exists in the local topology of $\mathscr{E}(X)$, in particular in the strong operator topology of $\mathscr{B}(X)$. The maps $\tau_\varkappa$ are endomorphisms of $\mathscr{E}(X)$ and $\bigcap_{\chi\in\delta(X)} \ker\tau_\chi = \mathscr{K}(X)$. In particular, the map $T \mapsto (\tau_\varkappa(T))$ is a morphism $\mathscr{E}(X) \to \prod_{\varkappa\in\delta(X)} \mathscr{E}(X)$ with $\mathscr{K}(X)$ as kernel, hence the essential spectrum of any normal operator $H \in \mathscr{E}(X)$ or any observable H affiliated to $\mathscr{E}(X)$ is given by $\mathrm{Sp_{ess}}(H) = \overline{\bigcup}_\varkappa \mathrm{Sp}(\tau_\varkappa(H))$.*

Proof: We have seen in Section 4.4 that $\bigcup_{\varkappa\in\delta(X)} \widehat{\varkappa} = \delta(X)$ and from (6.31) we get

$$\mathscr{E}_{(\varkappa)}(X) = \textstyle\bigcap_{\chi\in\widehat{\varkappa}} \ker\tau_\chi \quad \text{for each } \varkappa \in \delta(X). \tag{6.33}$$

On the other hand, we have shown before that $\cap_{\varkappa\in\delta(X)}\mathscr{E}_{(\varkappa)}(X) = \mathscr{K}(X)$ is a consequence of Property A, hence of amenability. □

Remark 6.9. Recall that after (2.7) we defined the localization $\varkappa.T$ at $\varkappa \in \delta(X)$ of some $T \in \mathscr{E}$ as the quotient of T in $\mathscr{E}_\varkappa = \mathscr{E}/\mathscr{E}_{(\varkappa)}$. If T is normal then from (6.33) we get $\mathrm{Sp}(\varkappa.T) = \overline{\bigcup}_{\chi\in\widehat{\varkappa}} \mathrm{Sp}(\tau_\chi(T))$ but many of the operators $\tau_\chi(T)$ which appear here are unitary equivalent, in particular have the same spectrum. Indeed, note that there is a natural (left) action of X on $\beta(X)$ which leaves $\delta(X)$ invariant and $\widehat{\varkappa}$ is the minimal closed invariant subset of $\delta(X)$ which contains $\varkappa$. And if $\chi \in \delta(X)$ and $a \in X$ then by using $a\chi = \lim_{b\to\chi} ab$ we get $\tau_{a\chi}(T) = \rho_a \tau_\chi(T)\rho_a^*$.

6.4. **Pseudo-local and quasi-local operators.** We describe briefly other C^*-algebras of operators which are analogs of $\mathscr{E}(X)$. We need an analogue of Lemma 3.7 in the group context.

Lemma 6.10. *Let ω be a compact neighborhood of e and Z a maximal ω-separated subset of X (i.e. if a, b are distinct elements of Z then $(a\omega) \cap (b\omega) = \emptyset$). Then for any compact set $K \supset \omega\omega^{-1}$ we have $ZK = X$ and for any $a \in Z$ the number of $z \in Z$ such that $(zK) \cap (aK) \neq \emptyset$ is at most $\mu(KK^{-1}\omega)/\mu(\omega)$.*

Proof: That such maximal Z exist follows from Zorn lemma. By maximality, $(x\omega)\cap(Z\omega) \neq \emptyset$ for any x, hence $x \in Z\omega\omega^{-1}$, so $X = ZK$ if $K \supset \omega\omega^{-1}$. Now fix $a \in Z$ and let N be the number of points $z \in Z$ such that $(zK) \cap (aK) \neq \emptyset$. For each such z we have $z \in aKK^{-1}$ hence $z\omega \subset aKK^{-1}\omega$. But the sets $z\omega$ are pairwise disjoint and have the same measure $\mu(\omega)$ so $N\mu(\omega) \leq \mu(aKK^{-1}\omega) = \mu(KK^{-1}\omega)$. □

Remark 6.11. By using the homeomorphism $x \mapsto x^{-1}$, or just by taking μ a right Haar measure in the proof, we get a "left" version of this result: if we define ω-separation by $(\omega a) \cap (\omega b) = \emptyset$, then for any compact $K \supset \omega^{-1}\omega$ we have $X = KZ$ and $\#\{z \in Z \mid (Kz) \cap (Ka) \neq \emptyset\} \leq \mu(\omega K^{-1} K)/\mu(\omega)$

By analogy with the notion of controlled operator introduced in Remark 3.3, we say that $S \in \mathscr{B}(X)$ is *controlled* if there is a compact set $\Lambda \subset X$ such that if F, G are closed subsets of X with $F \cap (\Lambda G) = \emptyset$ then $1_F S 1_G = 0$. If X is a metric group this is equivalent to: there is a number $r > 0$ such that if F, G are closed subsets of X with $d(F, G) > r$ then $1_F S 1_G = 0$, which is the definition of Remark 3.3. This version of the definition is clearly independent of the group structure of X. As before, we denote $\mathscr{C}(X)$ the norm closure of the set of controlled operators, this is the *C^*-algebra of pseudo-local operators*.

Now assume that X is an abelian group. Let X^* be the dual group and to each $p \in X^*$ let us associate the unitary operator ν_p on $L^2(X)$ defined by $(\nu_p f)(x) = p(x) f(x)$. Then (see [GI3]):

Proposition 6.12. *If X is an abelian group then $\mathscr{E}(X) = \mathcal{C}(X) \rtimes X = \mathcal{C}(X) \rtimes_{\mathrm{r}} X$ is the set of operators $T \in \mathscr{B}(X)$ such that $\|\nu_p T \nu_p^* - T\| \to 0$ and $\|(\lambda_a - 1)T^{(*)}\| \to 0$ if $p \to e$ in X^* and $a \to e$ in X.*

Remark 6.13. The relation $\mathscr{E}(X) \subset \mathcal{C}(X) \rtimes X$ follows easily from Proposition 6.12 if X is abelian. The operators $\nu_p Op(k) \nu_p^*$ and $\lambda_a Op(k)$ have kernels $p(x)k(x,y)\bar{p}(y) = p(xy^{-1})k(x,y)$ and $k(xa^{-1}, y)$. Hence from (3.13) we get $\|\nu_p Op(k)\nu_p^* - Op(k)\| \leq \sup_{xy^{-1} \in K} |p(xy^{-1}) - 1||k(x,y)|\mu(K)$ which tends to zero as $p \to e$ in X^* by the definition of the topology on X^*. Similarly $\|(\lambda_a - 1)Op(k)\| \to 0$ as $a \to e$ in X. Hence $Op(k) \in \mathcal{C}(X) \rtimes X$ for each $k \in \mathcal{C}_{\mathrm{trl}}(X^2)$.

If X is an abelian group then the set of *Q-regular operators*, more precisely the operators $T \in \mathscr{B}(X)$ which satisfy only the first condition from Proposition 6.12, i.e. such that the map $p \mapsto \nu_p T \nu_p^*$ is norm continuous, is a C^*-algebra which contains $\mathscr{E}(X)$, strictly if X is not discrete, which seems to depend on the group structure of X. But in fact this is not the case, it depends only on the coarse structure of X.

Proposition 6.14. *If X is an abelian group then $\mathscr{C}(X) = \{T \in \mathscr{B}(X) \mid \lim_{p \to e} \|\nu_p T \nu_p^* - T\| = 0\}$.*

For the proof, it suffices to use [GG2, Propositions 4.11 and 4.12] (arXiv version) and Lemma 6.10.

Finally, we mention another C^*-algebra which is of a similar nature to $\mathscr{C}(X)$ and makes sense and is useful in the context of arbitrary locally compact spaces X and arbitrary geometric Hilbert X-modules, see [GG2, Ro1]. Let us say that $S \in B(\mathcal{H})$ is *quasilocal* (or "decay preserving") if for each $\varphi \in \mathcal{C}_{\mathrm{o}}(X)$ there are operators $S_1, S_2 \in B(\mathcal{H})$ and functions $\varphi_1, \varphi_2 \in \mathcal{C}_{\mathrm{o}}(X)$ such that $S\varphi(Q) = \varphi_1(Q) S_1$ and $\varphi(Q) S = S_2 \varphi_2(Q)$. The set of quasilocal operators is a C^*-algebra which contains strictly $\mathscr{C}(X)$ if X is a locally compact non-compact abelian group. Indeed, if $u \in L^\infty(X^*)$ has compact support then $u(P)$ is quasilocal (because $u(P)\varphi(Q)$ and $\varphi(Q)u(P)$ are compact) but it belongs to $\mathscr{C}(X)$ if and only if u is continuous. Here $u(P) = \mathcal{F}^{-1} M_u \mathcal{F}$ where M_u is the operator of multiplication by u on $L^2(X^*)$ and $\mathcal{F}$ is the Fourier transformation.

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