On the Borel summable solutions of multi-dimensional heat equation

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Abstract. We give a new characterisation of Borel summability of formal power series solutions to the n-dimensional heat equation in terms of holomorphic properties of the integral means of the Cauchy data. We also derive the Borel sum for the summable formal solutions.

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1. Introduction

We consider the initial value problem for complex n-dimensional heat equation

$$\partial_t u = \Delta u, \qquad u(0,z) = \varphi(z), \tag{1}$$

where $t \in \mathbb{C}$, $z \in \mathbb{C}^n$, $\Delta = \sum_{i=1}^n \partial_{z_i}^2$ is the complex Laplace operator and φ is holomorphic in a complex neighbourhood of the origin. The unique formal power series solution of (1) is given by

$$\hat{u}(t,z) = \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{k!} t^k. \tag{2}$$

In the dimension $n = 1$ the problem of convergence of formal solution (2) was already solved by Kowalevskaya [3]. She showed that $\hat{u}$ is convergent if and only if the Cauchy data φ is an entire function of exponential order at most 2. In the multidimensional case Aronszajn at all [1] solved the problem of convergence of $\hat{u}$ in terms of the growth of $\Delta^k \varphi(z)$ for $k \in \mathbb{N}_0$. Another approach was given by Łysik [5]. He proved that $\hat{u}$ is convergent if and only if the integral mean of φ over the closed ball $B(x,t)$ or the sphere $S(x,t)$ as a function of a radius t extends to an entire function of exponential order at most 2.

If $\hat{u}$ diverges, it is natural to ask when it is Borel summable. In the one-dimensional case the solution was given by Lutz at all [4]. They proved that $\hat{u}$ is 1-summable in a direction d if and only if φ can be analytically continued to infinity in directions $d/2$ and $\pi + d/2$ and the continuation is of exponential order at most 2. In the multidimensional case the author [6] proved that $\hat{u}$ is Borel summable in a direction d if and only if the function

$$\Phi_n(t,z) = \begin{cases} \int_{S(0,1)} \varphi(z+tx)\, dS(x) & \text{if} \quad n \text{ is odd} \\ \int_{B(0,1)} \frac{\varphi(z+tx)\, dx}{\sqrt{1-|x|^2}} & \text{if} \quad n \text{ is even} \end{cases}$$

is analytically continued to infinity in directions $d/2$ and $\pi+d/2$ (with respect to t) and to some ball with a centre at the origin (with respect to z) and this continuation is of exponential order at most 2 as $t \to \infty$.

In the present paper we show that for arbitrary dimension n, we may replace the functions $\Phi_n(t,z)$ in the above characterisation by the integral mean of φ over the closed ball $B(x,t)$ or the sphere $S(x,t)$. The result is based upon the mean-value formulas for analytic functions (see [5, Theorem 3.1]). As an application, we use the procedure of Borel summability to find the Borel sum u of the formal solution $\hat{u}$. In this way we obtain the representation of the solution u of the heat equation given by a complex version of the convolution of the heat kernel and the Cauchy data φ.

2. Preliminaries

In the paper we use the following notation. The real closed ball (respectively sphere) with a centre at $x \in \mathbb{R}^n$ and a radius $t > 0$ is denoted by $B(x,t)$ (respectively $S(x,t)$). Moreover, the complex disc in $\mathbb{C}^n$ with a centre at the origin and a radius $r > 0$ is denoted by $D_r^n := \{z \in \mathbb{C}^n : \ |z| < r\}$. If the radius r is not essential, then we write it D^n for short.

Let $a \in \mathbb{R}$. The Pochhammer symbol is defined by $(a)_0 := 1$ and $(a)_k := a(a+1)\cdots(a+k-1)$ for $k \in \mathbb{N}$.

A sector in a direction $d \in \mathbb{R}$ with an opening $\varepsilon > 0$ in the universal covering space $\widetilde{\mathbb{C}}$ of $\mathbb{C} \setminus \{0\}$ is defined by

$$S(d,\varepsilon) := \{z \in \widetilde{\mathbb{C}} : \ z = re^{i\theta}, \ d - \varepsilon/2 < \theta < d + \varepsilon/2, \ r > 0\}.$$

Moreover, if the value of opening angle ε is not essential, then we denote it briefly by S_d. We denote by $\hat{S}_d$ the set $S_d \cup D^1$. By $\mathcal{O}(G)$ we understand the space of holomorphic functions on a domain $G \subseteq \mathbb{C}^n$.

Let us also recall some definitions and fundamental facts about the Borel summability. For more details we refer the reader to [2].

Definition 1. A function $u(t,z) \in \mathcal{O}(S(d,\varepsilon) \times D_r^n)$ is of *exponential growth of order at most $s > 0$ as $t \to \infty$ in $S(d,\varepsilon)$* if and only if for any $r_1 \in (0,r)$ and any $\varepsilon_1 \in (0,\varepsilon)$ there exist $A, B < \infty$ such that

$$\max_{|z| \le r_1} |u(t,z)| \le Ae^{B|t|^s} \quad \text{for} \quad t \in S(d,\varepsilon_1).$$

The space of such functions is denoted by $\mathcal{O}^s(S(d,\varepsilon)\times D^n_r)$. We also write $\mathcal{O}^s(\hat{S}_d\times D^n)$ for the space $\mathcal{O}^s(S_d\times D^n)\cap\mathcal{O}(\hat{S}_d\times D^n)$.

Analogously, a function $\varphi\in\mathcal{O}(S(d,\varepsilon))$ is of *exponential growth of order at most $s>0$ as $z\to\infty$ in $S(d,\varepsilon)$* if and only if for any $\varepsilon_1\in(0,\varepsilon)$ there exist $A,B<\infty$ such that

$$|\varphi(z)|\le Ae^{B|z|^s}\quad\text{for}\quad z\in S(d,\varepsilon_1).$$

The space of such functions is denoted by $\mathcal{O}^s(S(d,\varepsilon))$. We also set $\mathcal{O}^s(\hat{S}_d):=\mathcal{O}^s(S_d)\cap\mathcal{O}(\hat{S}_d)$.

Definition 2. Let $d\in\mathbb{R}$. A formal series

$$\hat{u}(t,z)=\sum_{j=0}^{\infty}\frac{u_j(z)}{j!}t^j\quad\text{with}\quad u_j(z)\in\mathcal{O}(D^n)\tag{3}$$

is called *Borel summable in a direction* d if and only if its *Borel transform* $\hat{\mathcal{B}}\hat{u}$ satisfies

$$(\hat{\mathcal{B}}\hat{u})(s,z):=\sum_{j=0}^{\infty}\frac{u_j(z)}{(j!)^2}s^j\in\mathcal{O}^1(\hat{S}_d\times D^n).$$

The *Borel sum of $\hat{u}$ in the direction d* is represented by the Laplace transform of $v(s,z):=(\hat{\mathcal{B}}\hat{u})(s,z)$

$$u^\theta(t,z):=\frac{1}{t}\int_0^{\infty(\theta)}e^{-s/t}v(s,z)\,ds,$$

where the integration is taken over any ray $e^{i\theta}\mathbb{R}_+:=\{re^{i\theta}:\ r\ge0\}$ with $\theta\in(d-\varepsilon/2,d+\varepsilon/2)$.

According to the general theory of moment summability (see [2, Section 6.5]), a formal series (3) is Borel summable in a direction d if and only if the same holds for the series

$$\sum_{j=0}^{\infty}u_j(z)\frac{j!}{(2j)!}t^j.$$

Consequently, we obtain a characterisation of Borel summability, which is analogous to Definition 2 (see also [2, Theorem 38 and Section 11])

Proposition 1. *Let $d\in\mathbb{R}$. A formal series (3) is Borel summable in a direction d if and only if its modified Borel transform $\tilde{\mathcal{B}}\hat{u}$ satisfies*

$$(\tilde{\mathcal{B}}\hat{u})(s,z)=\sum_{j=0}^{\infty}\frac{u_j(z)}{(2j)!}s^j\in\mathcal{O}^1(\hat{S}_d\times D^n).$$

The Borel sum of $\hat{u}$ in the direction d is represented by the Ecalle acceleration operator acting on $\tilde{v}(s,z):=(\tilde{\mathcal{B}}\hat{u})(s,z)$ as follows

$$u^\theta(t,z)=\frac{1}{\sqrt{t}}\int_0^{\infty(\theta)}\tilde{v}(s,z)C_2(\sqrt{s/t})\,d\sqrt{s}$$

with $\theta \in (d-\varepsilon, d+\varepsilon)$. Here integration is taken over the ray $e^{i\theta}\mathbb{R}_+$ and C_2 is defined by

$$C_2(\zeta) := \frac{1}{2\pi i}\int_\gamma \frac{e^{u-\zeta\sqrt{u}}}{\sqrt{u}}\,du \tag{4}$$

with a path of integration γ as in the Hankel integral for the inverse gamma function (from ∞ along $\arg u = -\pi$ to some $u_0 < 0$, then on the circle $|u| = |u_0|$ to $\arg u = \pi$, and back to ∞ along this ray).

3. Integral means

In this section we recall the notion of integral means. To this end we take a continuous function φ on a domain $\Omega \subset \mathbb{R}^n$, $x \in \Omega$ and $0 < t < \operatorname{dist}(x, \partial\Omega)$. Then we denote by $M(\varphi; t, x)$ and $N(\varphi; t, x)$ the integral means of φ over the closed ball $B(x,t)$ and the sphere $S(x,t)$ respectively, i.e.

$$M(\varphi; t, x) = \fint_{B(x,t)} \varphi(y)\,dy := \frac{1}{\alpha(n)t^n}\int_{B(x,t)} \varphi(y)\,dy$$

$$N(\varphi; t, x) = \fint_{S(x,t)} \varphi(y)\,dS(y) := \frac{1}{n\alpha(n)t^{n-1}}\int_{S(x,t)} \varphi(y)\,dS(y),$$

where $\alpha(n) := \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of the n-dimensional unit ball $B(0,1)$. Moreover, since

$$M(\varphi; t, x) = \fint_{B(0,1)} \varphi(x+ty)\,dy \quad \text{and} \quad N(\varphi; t, x) = \fint_{S(0,1)} \varphi(x+ty)\,dS(y),$$

we may also consider $M(\varphi; t, z)$ and $N(\varphi; t, z)$ for complex variables t and z. Hence, according to the mean-value properties for analytic functions we have

Proposition 2 (see [5, Theorem 3.1]**).** *Let $\varphi \in \mathcal{O}(G)$, $G \subseteq \mathbb{C}^n$ and $z \in G$. Then $M(\varphi; t, z)$ and $N(\varphi; t, z)$ are holomorphic functions at the origin as functions of t and for t small enough*

$$M(\varphi; t, z) = \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{4^k(\frac{n}{2}+1)_k k!} t^{2k} \quad \textit{and} \quad N(\varphi; t, z) = \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{4^k(\frac{n}{2})_k k!} t^{2k}. \tag{5}$$

Using the above proposition we find the relation between the series $\sum_{k=0}^{\infty} \frac{\Delta^k\varphi(z)}{(2k)!} t^{2k}$ and $\sum_{k=0}^{\infty} \frac{\Delta^k\varphi(z)}{(k!)^2} t^{2k}$ and the integral means $M(\varphi; t, z)$ and $N(\varphi; t, z)$. Namely, we have

Lemma 1. *Assume that $\varphi \in \mathcal{O}(G)$, $G \subseteq \mathbb{C}^n$, $z \in G$ and t is small enough. Then it holds*

$$\sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{(2k)!} t^{2k} = \frac{1}{n!!}\partial_t (t^{-1}\partial_t)^{\frac{n-1}{2}} t^n M(\varphi; t, z)$$

$$= \frac{1}{(n-2)!!}\partial_t (t^{-1}\partial_t)^{\frac{n-3}{2}} t^{n-2} N(\varphi; t, z)$$

for $n = 2m+1$*;*

$$\sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{(k!)^2} t^{2k} = \frac{1}{n!!} (t^{-1}\partial_t)^{\frac{n}{2}} t^n M(\varphi; 2t, z)$$

$$= \frac{1}{(n-2)!!} (t^{-1}\partial_t)^{\frac{n-2}{2}} t^{n-2} N(\varphi; 2t, z)$$

for $n = 2m$*.*

Proof. First, note that

$$4^k (\frac{n}{2}+1)_k k! = (2k)!! \frac{(n+2k)!!}{n!!} \quad \text{and} \quad 4^k (\frac{n}{2})_k k! = (2k)!! \frac{(n+2k-2)!!}{(n-2)!!}. \tag{6}$$

If n is an odd number then by (5) and (6) we obtain

$$\frac{1}{n!!} \partial_t (t^{-1}\partial_t)^{\frac{n-1}{2}} t^n M(\varphi; t, z) = \frac{1}{n!!} \partial_t (t^{-1}\partial_t)^{\frac{n-1}{2}} \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{4^k (\frac{n}{2}+1)_k k!} t^{2k+n}$$

$$= \frac{1}{n!!} \sum_{k=0}^{\infty} \frac{(2k+n)(2k+n-2)\cdots(2k+1)\Delta^k \varphi(z)}{\frac{(2k)!!(2k+n)!!}{n!!}} t^{2k} = \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{(2k)!} t^{2k}$$

and

$$\frac{1}{(n-2)!!} \partial_t (t^{-1}\partial_t)^{\frac{n-3}{2}} t^{n-2} N(\varphi; t, z)$$

$$= \frac{1}{(n-2)!!} \partial_t (t^{-1}\partial_t)^{\frac{n-3}{2}} \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{4^k (\frac{n}{2}+1)_k k!} t^{2k+n}$$

$$= \frac{1}{(n-2)!!} \sum_{k=0}^{\infty} \frac{(2k+n-2)(2k+n-4)\cdots(2k+1)\Delta^k \varphi(z)}{\frac{(2k)!!(2k+n-2)!!}{(n-2)!!}} t^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{(2k)!} t^{2k},$$

which proves the first part of the lemma.

Analogously, if n is an even number then by (5) and (6) we have

$$\frac{1}{n!!} (t^{-1}\partial_t)^{\frac{n}{2}} t^n M(\varphi; 2t, z) = \frac{1}{n!!} (t^{-1}\partial_t)^{\frac{n}{2}} \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z) 4^k}{4^k (\frac{n}{2}+1)_k k!} t^{2k+n}$$

$$= \frac{1}{n!!} \sum_{k=0}^{\infty} \frac{(2k+n)(2k+n-2)\cdots(2k+2)\Delta^k \varphi(z) 4^k}{\frac{(2k)!!(2k+n)!!}{n!!}} t^{2k} = \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{(k!)^2} t^{2k}.$$

and

$$\begin{aligned}
&\frac{1}{(n-2)!!}(t^{-1}\partial_t)^{\frac{n-2}{2}}t^{n-2}N(\varphi;2t,z)\\
&=\frac{1}{(n-2)!!}(t^{-1}\partial_t)^{\frac{n-2}{2}}\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)4^k}{4^k(\frac{n}{2})_k k!}t^{2k+n-2}\\
&=\frac{1}{(n-2)!!}\sum_{k=0}^{\infty}\frac{(2k+n-2)(2k+n-4)\cdots(2k+2)\Delta^k\varphi(z)4^k}{\frac{(2k)!!(2k+n-2)!!}{(n-2)!!}}t^{2k}\\
&=\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(k!)^2}t^{2k},
\end{aligned}$$

which proves the second part of the lemma. □

4. Summability of formal solutions

Now, we are ready to state the main result of the paper.

Theorem 1. *Let $\hat{u}$ be a formal solution of the n-dimensional complex heat equation*

$$\partial_t u = \Delta u, \qquad u(0,z)=\varphi(z)\in\mathcal{O}(D^n). \tag{7}$$

Then the following conditions are equivalent:

- *$\hat{u}$ is Borel summable in a direction d;*
- *$M(\varphi;t,z)\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$;*
- *$N(\varphi;t,z)\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$.*

Proof. The formal solution $\hat{u}$ of (7) is given by (2). Applying the modified Borel transform $\tilde{\mathcal{B}}$ and the Borel transform $\hat{\mathcal{B}}$ to $\hat{u}$ and replacing s by t^2 we have

$$(\tilde{\mathcal{B}}\hat{u})(t^2,z)=\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(2k)!}t^{2k} \quad\text{and}\quad (\hat{\mathcal{B}}\hat{u})(t^2,z)=\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(k!)^2}t^{2k}.$$

By Proposition 1 and by Definition 2, the formal solution $\hat{u}$ is Borel summable in a direction d if and only if $(\tilde{\mathcal{B}}\hat{u})(t^2,z)\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$ or, equivalently if and only $(\hat{\mathcal{B}}\hat{u})(t^2,z)\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$. On the other hand, by Lemma 1, $M(\varphi;t,z)\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$ if and only if $N(\varphi;t,z)\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$ and, moreover, if and only if $\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(2k)!}t^{2k}\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$ in odd dimensions and $\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(k!)^2}t^{2k}\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$ in even dimension. Hence we obtain our assertion. □

Using the representation of the Borel transform $\hat{\mathcal{B}}\hat{u}$ and the modified Borel transform $\tilde{\mathcal{B}}\hat{u}$, we derive the Borel sum u for Borel summable formal solution $\hat{u}$. To this end, we calculate the function C_2 defined by (4). Using the power series expansion (see [2, p. 175]) of C_2 we have

$$C_2(\zeta) = \sum_{n=0}^{\infty} \frac{(-\zeta)^n}{n!\Gamma(1-(n+1)/2)}.$$

Since the Gamma function $\Gamma(z)$ has the simple poles for $z = 0, -1, -2, ..$ and

$$\Gamma(-k+1/2) = \frac{(-1)^k k! 4^k \sqrt{\pi}}{(2k)!} \quad \text{for} \quad k \in \mathbb{N}_0,$$

we obtain

$$C_2(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{(2k)!\Gamma(-k+1/2)} = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \zeta^{2k}}{4^k k!} = \frac{1}{\sqrt{\pi}} e^{-\frac{\zeta^2}{4}}. \tag{8}$$

Now we are ready to prove that the procedure of Borel summability gives us the solution u of the heat equation as the convolution of the heat kernel and the Cauchy data. Namely, we have

Theorem 2. *Let $\hat{u}$ be a formal solution of (7), which is Borel summable in a direction d. Then the Borel sum of $\hat{u}$ in the direction d is given by*

$$u^\theta(t,z) = \frac{1}{(4\pi t)^{n/2}} \int_{(e^{i\theta/2}\mathbb{R})^n} e^{-\frac{e^{i\theta}|x|^2}{4t}} \varphi(z+x)\, dx,$$

where $\theta \in (d-\varepsilon/2, d+\varepsilon/2)$.

Proof. Fix $\theta \in (d-\varepsilon/2, d+\varepsilon/2)$. First, let us assume that $n = 2m+1$. By Proposition 1, (8) and Lemma 1, we have

$$\begin{aligned} u^\theta(t,z) &= \frac{1}{\sqrt{t}} \int_0^{\infty(\theta)} (\tilde{\mathcal{B}}\hat{u})(s,z) \frac{1}{\sqrt{\pi}} e^{-\frac{s}{4t}}\, d\sqrt{s} \\ &\overset{s=\tau^2}{=} \frac{1}{\sqrt{\pi t}} \int_0^{\infty(\theta/2)} (\tilde{\mathcal{B}}\hat{u})(\tau^2,z) e^{-\frac{\tau^2}{4t}}\, d\tau \\ &= \frac{1}{\sqrt{\pi t}} \int_0^{\infty(\theta/2)} e^{-\frac{\tau^2}{4t}} \frac{1}{(n-2)!!} \partial_\tau (\tau^{-1}\partial_\tau)^{\frac{n-3}{2}} \tau^{n-2} N(\varphi;\tau,z)\, d\tau. \end{aligned}$$

Now, by $(1+\frac{n-3}{2})$-fold integration by parts we have

$$\begin{aligned} u^\theta(t,z) &= \frac{1}{\sqrt{\pi t}} \int_0^{\infty(\theta/2)} \frac{\tau}{2t} e^{-\frac{\tau^2}{4t}} \frac{1}{(n-2)!!} (\tau^{-1}\partial_\tau)^{\frac{n-3}{2}} \tau^{n-2} N(\varphi;\tau,z)\, d\tau \\ &= \frac{1}{(n-2)!!(2t)^{\frac{n-1}{2}} \sqrt{\pi t}} \int_0^{\infty(\theta/2)} e^{-\frac{\tau^2}{4t}} \tau^{n-1} N(\varphi;\tau,z)\, d\tau. \end{aligned}$$

Finally, using the definition of the integral means over the sphere we have

$$u^\theta(t,z) = \frac{1}{(n-2)!!(2t)^{\frac{n-1}{2}}\sqrt{\pi t}} \int_0^{\infty(\theta/2)} e^{-\frac{\tau^2}{4t}} \tau^{n-1} \fint_{S(0,1)} \varphi(z+\tau y)\, dS(y)\, d\tau$$

$$\stackrel{\tau y = x}{=} \frac{1}{(4\pi t)^{n/2}} \int_{(e^{i\theta/2}\mathbb{R})^n} e^{-\frac{e^{i\theta}|x|^2}{4t}} \varphi(z+x)\, dx,$$

since

$$\frac{1}{n\alpha(n)} = \frac{\Gamma(1+n/2)}{n\pi^{n/2}} = \frac{n!!\pi^{1/2}}{2^{\frac{n+1}{2}} n\pi^{n/2}} = \frac{(n-2)!!}{2^{\frac{n+1}{2}}\pi^{\frac{n-1}{2}}}.$$

Analogously, for $n = 2m$ we apply Definition 2, (8) and Lemma 1 to calculate

$$u^\theta(t,z) = \frac{1}{t}\int_0^{\infty(\theta)} e^{-s/t}(\mathcal{B}\hat{u})(s,z)\, ds \stackrel{s=\tau^2}{=} \frac{1}{t}\int_0^{\infty(\theta/2)} e^{-\tau^2/t}(\mathcal{B}\hat{u})(\tau^2,z) 2\tau\, d\tau$$

$$= \frac{2}{t}\int_0^{\infty(\theta/2)} e^{-\tau^2/t}\tau \frac{1}{(n-2)!!}(\tau^{-1}\partial_\tau)^{\frac{n-2}{2}} \tau^{n-2} N(\varphi; 2\tau, z)\, d\tau$$

By $\frac{n-2}{2}$-fold integration by parts and by the definition of the integral means over the sphere we have

$$u^\theta(t,z) = \frac{2^{n/2}}{t^{n/2}(n-2)!!}\int_0^{\infty(\theta/2)} e^{-\tau^2/t}\tau^{n-1} \fint_{S(0,1)} \varphi(z+2\tau y)\, dS(y)\, d\tau$$

$$\stackrel{2\tau=\sigma}{=} \frac{1}{(2t)^{n/2}(n-2)!!}\int_0^{\infty(\theta/2)} e^{-\frac{\sigma^2}{4t}}\sigma^{n-1} \fint_{S(0,1)} \varphi(z+\sigma y)\, dS(y)\, d\sigma$$

$$\stackrel{\sigma y = x}{=} \frac{1}{(4\pi t)^{n/2}} \int_{(e^{i\theta/2}\mathbb{R})^n} e^{-\frac{e^{i\theta}|x|^2}{4t}} \varphi(z+x)\, dx,$$

since

$$\frac{1}{n\alpha(n)} = \frac{\Gamma(1+n/2)}{n\pi^{n/2}} = \frac{n!!}{2^{n/2}n\pi^{n/2}} = \frac{(n-2)!!}{2^{n/2}\pi^{n/2}}.$$

□

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