Analyticity of density of states

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Abstract

In this paper we consider the Anderson type model and show that the density of states is analytic in an interval for large disorder. The examples we have allow for the single site distributions to have some singular components. We also allow non-nearest neighbour hopping terms.

1 Introduction

The Anderson tight binding model describes a quantum mechanical particle such as an electron moving in a random potential. The model is defined by (the $N = 1$ case of the) operator

$$H^\omega_\lambda = H_0 + \lambda V^\omega \quad \text{on } \ell^2(\mathbf{Z}^d), \tag{1}$$ OP

with H_0 and V^ω given as follows.

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Let e_j denote the unit vector in the j-th direction in $\mathbb{Z}^d$. Then let $(T_j^{\pm}u)(n) = u(n \pm e_j)$, in $\ell^2(\mathbb{Z}^d)$. With this notation,

$$(H_0 u)(x) = \sum_{j=1}^{d} \sum_{k=1}^{N} (T_j^k + T_j^{-k} u)(x). \tag{2}$$

The collection $\{V^\omega(x)\}$ are i.i.d. random variables with common probability distribution μ and λ is a real coupling constant. The operator H_0 is seen to be multiplication by the function

$$\sum_{j=1}^{d} \sum_{k=1}^{N} 2\cos(k\theta_j) = \sum_{j=1}^{d} \left(\frac{\sin((N+\frac{1}{2})\theta_j)}{\sin(\frac{\theta_j}{2})} - 1 \right), \text{ on } L^2(\mathbb{T}^d).$$

Therefore the spectrum of H_0 is given by $\sigma(H_0) = [E_0, 2Nd]$ with

$$-2Nd \leq E_0 \leq \begin{cases} 0, \text{ if } N \text{ is } \quad \text{even}, \\ -2d, \text{ if } N \text{ is } \quad \text{odd}. \end{cases} \tag{3}$$ eqn1

Hence we see that H_0 is a bounded self-adjoint operator with $\|H_0\| = 2Nd$.

In general it is not clear how to determine E_0 for any given N (except for the case $N = 1$).

Then, H_λ^ω are self-adjoint operators. If Γ is a finite box of $\mathbf{Z}^d$, we will denote by $H_{\lambda,\Gamma}^\omega$ the operator H_λ^ω restricted to $\ell^2(\Gamma)$ with Dirichlet boundary conditions. The integrated density of states(IDS for short), $\mathcal{N}(E)$, is defined by

$$\mathcal{N}(E) = \lim_{\Gamma \to \mathbf{Z}^d} \frac{1}{\#\Gamma} \#\{\text{eigenvalues of } H_{\lambda,\Gamma}^\omega \leq E\}.$$

It is a consequence of ergodic theorem for almost every ω the limit exists for all $E \in \mathbf{R}$ and is independent of ω. Moreover $\text{supp}(d\mathcal{N}) = \sigma(H_\lambda^\omega)$ a.e. ω. The basic facts about the density of states is found in any of the standard books in the area for example Cycon-Froese-Kirsch-Simon [4], Carmona-Lacroix [3] and Figotin-Pastur [7]. It is a result of Pastur [11] that $\mathcal{N}(E)$ is always continuous. The IDS $\mathcal{N}(E)$ is positive, non-decreasing and bounded (by 1) function satisfying $\mathcal{N}(\infty) = 1$. So it is the distribution function of a probability measure. In the case when this measure is absolutely continuous, the density $n(E)$ of this measure is called the "the density of states". One of the questions of interest is the degree of smoothness of the function n, which is also often referred to as the smoothness of IDS, which we do in the following.

There are many results on the smoothness of IDS for one-dimensional case. For example, $\mathcal{N}(E)$ is differentiable, even infinitely differentiable under some regularity assumptions on μ (Companino-Klein [5] and Simon-Taylor [12]). Moreover the smoothness of IDS in the Anderson model on a strip are considered, for example, Klein-Speis[9], Klein-Lacroix-Speis[8], Glaffig[6] and Klein-Speis[10].

On the other hand, there are very few results on the smoothness of IDS for multi-dimentional case. Using Molchanov formula (of expressing the matrix elements of $e^{-itH_\lambda^\omega}$ in terms of a random walk on the lattice), Carmona showed (see section VI.3 [3]) that for the Cauchy distribution the IDS is C^∞.

Among the most important other results in the multi-dimensional case are Bovier-Campanino-Klein-Perez [1] and Constantinescu-Fröhlich-Spencer[2] and all the available results require that the disorder parameter λ is large or the region of energy considered is away from the middle of the spectrum.

A typical result in Bovier-Companino-Klein-Perez [1] is that $\mathcal{N}(E)$ is $(n+1)$-times continuously differentiable under the condition that the Fourier transform $h(t)$ of $d\mu$ satisfies $(1+t)^{d+n}h(t) \in L^1$. On the other hand Constantinescu-Fröhlich-Spencer [2] show that $\mathcal{N}(E)$ is real analytic in E, for $|\Re E|$ large enough if the density of μ is analytic in the strip $\{V : |\Im V| < 2(d+\epsilon)\}$ for arbitrarily small, but positive ϵ.

We consider the question of smoothness of the IDS for the operators given in equation (1). Our main theorem is the following.

`maintheorem` **Theorem 1.** *Consider H_λ^ω and let (a,b) be an interval $(a,b) \subset \Sigma = \sigma(H_\lambda^\omega)$ a.e. ω such that*

$$B_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{x-z}$$

is analytic in $z \in (a,b)$ so that for some interval $(c,d) \subset (a,b)$,

$$\sup_{\Re z \in (c,d)} \left| \int_{-\infty}^{\infty} \frac{d\mu(x)}{(x-z)^k} \right|^{1/k} \frac{(2Nd)}{\lambda} := \alpha < 1, \tag{4}$$ hyp

holds for every $k \geq 1$. Then, for each $\lambda > \lambda_0 := (2Nd)/(b-a)$ there exists an interval $(E_-(\lambda), E_+(\lambda)) \subset \sigma(H_\lambda^\omega)$ such that $n(E)$ is analytic in $(E_-(\lambda), E_+(\lambda))$.

Example 1. *Let (a,b) be an interval and let P be a polynomial which is positive in (a,b) and let $\int_a^b P(x)dx = I_P$. Let $(c,d) \subset (a,b)$ and consider the measure*

$$d\mu = \alpha d\nu + \frac{\beta}{I_P} \chi_{(a,b)} P(x) dx$$

where ν is an arbitrary probability measure such that $\mathrm{supp}\nu \cap (c,d) = \emptyset$. Then our theorem is valid for this collection of probability measures. We note that in the case when $P = 1$ and $\alpha = 0$, we get the uniform measure.

Our theorem is valid for this class of measures, if they further satisfy the hypothesis (4). Clearly ν being arbitrary it can have any component.

It is simple to see that $B_\mu(z)$ is analytic for $\Re z \in (a,b)$ (we do an integration by parts for the absolutely continuous part of μ to get this).

Remark 1.

1. *It is easy to show that $\Sigma = \sigma(H_0) + \mathrm{supp}\mu$, with the proof following the $N = 1$ case. This together with equation 3 shows that our theorem is non-trivial.*

2. *Our method of proof involves using a Neumann series expansion and controlling the terms, a method in some sense similar to that employed by Constantinescu-Fröhlich-Spencer [2].*

   However, our assumptions on ν allow for singular (even atomic components) in contrast to the theorems in [2] and [1]. Both these works require the measures ν to be absolutely continuous.

   We show in our proofs that we can localize the analysis using Neumann series expressions.

2 Proof of Theorem

We begin with stating several well-known properties on Borel transforms, where given a real-valued probability measure σ we denote $B_\sigma(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\sigma(x)$.

1. $B_\sigma(x + i0) = \lim_{\epsilon \to 0} B_\sigma(x + i\epsilon)$ exists for Lebesge-a.e. $x \in \mathbf{R}$.

2. The absolutely continuous part σ_{ac} of the measure σ has the derivative given by
$$\frac{d\sigma_{ac}}{dx} = \frac{1}{\pi} \Im B_\sigma(x + i0). \tag{5}$$
$\boxed{\texttt{IDS1}}$

3. The singular part σ_{sing} of the measure σ is supported by the set $\{E \in \mathbf{R} : \lim_{\epsilon \to 0} \Im B_\sigma(x + i\epsilon) = \infty\}$.

Note that the density of states $n(E)$ is given as the following by using Borel transform $B_{\mathcal{N}}$ of $\mathcal{N}$.

$$n(E) = \frac{1}{\pi} \Im B_{\mathcal{N}}(E + i0) = \frac{1}{\pi} \lim_{\epsilon \to 0} \mathbf{E}(\Im \langle \delta_0, (H_\lambda^\omega - E - i\epsilon)^{-1} \delta_0 \rangle). \tag{6}$$
$\boxed{\texttt{IDS2}}$

Proof of Theorem

We first consider the Neumann series expansion of $\mathbf{E}\langle \delta_0, (H_\lambda^\omega - z)^{-1} \delta_0 \rangle$.

$$\begin{aligned}
& \mathbf{E}\langle \delta_0, (H_\lambda^\omega - z)^{-1} \delta_0 \rangle \\
= \ & \mathbf{E}\left[\sum_{m=0}^{\infty} \langle \delta_0, (\lambda V^\omega - z)^{-1} H_0)^m (\lambda V^\omega - z)^{-1} \delta_0 \rangle \right] \\
= \ & \sum_{m=0}^{\infty} \mathbf{E}\langle \delta_0, (\lambda V^\omega - z)^{-1} H_0)^m (\lambda V^\omega - z)^{-1} \delta_0 \rangle & (7) \\
= \ & \sum_{m=0}^{\infty} \gamma_m(z). & (8)
\end{aligned}$$

which converges for $|\Im z| > \|H_0\|$. We used Fubini's theorem in the last equality.

Let

$$\begin{aligned}\Lambda_N(n) &= \bigcup_{j=1}^{d}\{k \in \mathbb{Z}^d : 1 \leq |(n-k)_j| \leq N, (n-k)_{j'} = 0,\ j \neq j'\}, \\ S_m &= \{(0, n_1, n_2, \ldots, n_m, 0) \in (\mathbb{Z}^d)^{m+2}, n_1 \in \Lambda_N(0), \\ &\quad n_2 \in \Lambda_N(n_1), \ldots, n_{m-1} \in \Lambda_N(n_{m-2}), n_m \in \Lambda_N(n_{m-1}) \cap \Lambda_N(0)\}.\end{aligned} \tag{9}$$

It is then clear that the cardinality of $\Lambda_N(n)$ is equal to $2Nd$ for any $n \in \mathbb{Z}^d$ and therefore the cardinarity of S_m satisfies the bound $\#S_m \leq (2Nd)^m$. We will denote below points of S_m by $\vec{n}$ and the matrix elements $H_0(k,l) = \langle \delta_k, H_0\delta_l\rangle,\ k,l \in \mathbb{Z}^d$.

Now to see the convergence of the series in equation (7), we will start by looking at a typical summand, namely $\gamma_m(z)$ and expand it explicitly as,

$$\begin{aligned} & \gamma_m(z) \\ = \;& \mathbf{E}\langle \delta_0, ((\lambda V^\omega - z)^{-1}H_0)^m(\lambda V^\omega - z)^{-1}\delta_0\rangle \\ = \;& \mathbf{E}\sum_{\vec{n}\in S_m}\langle \delta_0, \left[\prod_{j=1}^{m}(\lambda V^\omega(n_j) - z)^{-1}H_0(n_j, n_{j+1})\right](\lambda V^\omega(0) - z)^{-1}\delta_0\rangle \\ = \;& \sum_{\vec{n}\in S_m}\mathbf{E}\langle \delta_0, \left[\prod_{j=1}^{m}(\lambda V^\omega(n_j) - z)^{-1}\right](\lambda V^\omega(0) - z)^{-1}\delta_0\rangle \\ = \;& \sum_{\vec{n}\in S_m} I_m(\vec{n}, z).\end{aligned}$$

In the penultimate equality was obtained using the fact that $H_0(n_j, n_{j+1}) = 1$ for $\vec{n} = (0, n_2, n_3, \cdots, n_m, 0) \in S_m$.

We next consider each of the terms $I_m(\vec{n}, z)$ in the sum above.

$$I_m(\vec{n}, z) = \mathbf{E}\langle \delta_0, \left[\prod_{j=1}^{m}(\lambda V^\omega(n_j) - z)^{-1}\right](\lambda V^\omega(0) - z)^{-1}\delta_0\rangle,$$

where $\vec{n} = (0, n_2, n_3, \cdots, n_m, 0) \in S_m$. Using the independence of the random variables $V^\omega(n_j)$ for distinct n_j's and the fact that in the above term some of the n_j's may coincide, we see that the expectations are of the form

$$I_m(\vec{n}, z) = \prod_{i=1}^{l} \int_{-\infty}^{\infty} \frac{d\mu(x)}{(\lambda x - z)^{k_i}} = \frac{1}{\lambda^{m+1}} \prod_{i=1}^{l} \int_{-\infty}^{\infty} \frac{d\mu(x)}{(x - z/\lambda)^{k_i}}, \qquad (10) \quad \boxed{\texttt{g\_func}}$$

for some collection of numbers $(k_1, k_2, \cdots, k_l)$ satisfying $\sum_{i=1}^{l} k_i = m+1, 1 \leq k_i \; \forall \; 1 \leq i \leq l$, where the index l is at least one but does not exceed m. The exact value of l is not easy to get and we do not need this information in what we do below.

Let $(E_-(\lambda), E_+(\lambda)) = (\lambda c, \lambda d)$, where (c, d) is given in the assumption on μ given before equation (4). Then, by the assumption on μ, we see that each of the factors occuring in equation (10), for $I_m(\vec{n}, z)$ is analytic in z in the region $\Re z \in I(\lambda) = (E_-(\lambda), E_+(\lambda))$ and we have

$$\begin{aligned}
\sup_{\Re z \in I(\lambda)} |I_m(\vec{n}, z)| &\leq \sup_{\Re z \in I(\lambda)} \frac{1}{\lambda^{m+1}} \prod_{i=1}^{l} \left(\left| \int_{-\infty}^{\infty} \frac{d\mu(x)}{(x - \frac{z}{\lambda})^{k_i}} \right|^{1/k_i} \right)^{k_i} \\
&\leq \frac{1}{\lambda^{m+1}} \prod_{i=1}^{l} \left(\sup_{\Re z \in I(\lambda)} \left| \int_{-\infty}^{\infty} \frac{d\mu(x)}{(x - \frac{z}{\lambda})^{k_i}} \right|^{1/k_i} \right)^{k_i} \\
&\leq \frac{1}{\lambda^{m+1}} \prod_{i=1}^{l} \left(\frac{\alpha \lambda}{2Nd} \right)^{k_i} \\
&\leq \frac{1}{\lambda^{m+1}} \left(\frac{\alpha \lambda}{2Nd} \right)^{m+1} = \left(\frac{\alpha}{2Nd} \right)^{m+1}.
\end{aligned}$$

Therefore it follows that,

$$\begin{aligned}
\sup_{\Re z \in I(\lambda)} |\gamma_m(z)| &\leq \sum_{\vec{n} \in S_m} \sup_{\Re z \in I(\lambda)} |I_m(\vec{n}, z)| \\
&\leq (\# S_m) \left(\frac{\alpha}{2Nd} \right)^{m+1} \\
&\leq (2Nd)^m \frac{\alpha^{m+1}}{(2Nd)^{(m+1)}} \\
&= \frac{\alpha^{m+1}}{2Nd}.
\end{aligned}$$

Hence, the series in equation (7) converges uniformly in any compact subset of $\{z : \Re z \in (E_-(\lambda), E_+(\lambda))\}$. The assumption on μ shows that the functions

$\int \frac{1}{(x-z)^k}\ d\mu(x)$ are analytic in $\Re z \in (a,b)$ for any k, these being constant multiples of the k-th derivatives of the function $B_\mu(z)$ which is analytic there. Therefore finite products of such functions are also analytic in the same region. Hence each of the terms $I_m(\vec{n}, z)$ and hence also $\gamma_m(z)$ are analytic in $\Re z \in (E_-(\lambda), E_+(\lambda))$, thus showing the analyticity of the left hand side there. $\square$

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