Direct integrals and spectral averaging

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Abstract

A one parameter family of selfadjoint operators gives rise to a corresponding direct integral. We show how to use the Putnam Kato theorem to obtain a new method for the proof of a spectral averaging result.

1 Introduction

To us the basic issue of spectral averaging is to derive continuity properties of an integral of spectral measures; thus we consider a selfadjoint operator A in a separable Hilbert space $\mathcal{H}$ as well as a bounded operator B on $\mathcal{H}$, $B \geq 0$ and denote $H(t) := A + tB$. We write $\rho^{\Phi}_{H(t)}$ for the spectral measure of $H(t)$ with respect to the vector $\Phi \in \mathcal{H}$. Our main result is

Theorem 1.1. *Let $H(\cdot)$ be as above and let $\Phi \in \overline{Range(B)}$. Then the measures*

$$\nu = \int \rho^{\Phi}_{H(t)} h(t) dt$$

are absolutely continuous (with respect to Lebesgue measure) for any $h \in L^1(\mathbb{R})$.

Results of this type have quite a history and due to their importance for random operators, the interest has been steady. We refer to [9, 11, 7] and the references in there for early results, partly building on even older work [6] and to [3, 4, 12] for the more recent state of matters. Note however that we concentrate on one part of the intrigue, the continuity of the integrated spectral measures, while the emphasis in the cited works is somewhat different. There the main point is to deduce the spectral type of the single operators $H(t)$ the integral is made of. Clearly, in the setting of our main result nothing can be said about that.

The main improvement that had happened during the last 20 years of development is the generality of the operator B that appears, a feature that is of prime importance for applications to random operators. One of the main ideas that enter the usual proof, as presented, e.g. in [12], has also been fundamental in the adaptation of the fractional moment method to continuum models, cf. [1]. It uses the fact that a maximally accretive operator can always be obtained as the dilation of a selfadjoint operator. In contrast, in the early papers B was merely a rank one projection which already turned out to be extremely useful for discrete random models.

Our proof of the above theorem is quite different: we consider

$$\mathbf{H} := \int_{\mathbb{R}}^{\oplus} H(t)dt \text{ in } \int_{\mathbb{R}}^{\oplus} \mathcal{H}dt$$

and apply the Kato-Putnam theorem to this operator to show that some of its spectral measures are absolutely continuous. (In the next section we recall the necessary notions from the theory of direct integrals of Hilbert spaces.) We should like to point out that the idea to apply Mourre theory to obtain spectral averaging results can be found in [5], leading to a somewhat different proof that is nevertheless quite related to what we have done here. A major point in the present paper is the simplicity of the method.

2 Spectral averaging and direct integrals

What we need about direct integrals can be found in [10], p. 280 ff.

As we remarked above, we are dealing with a separable Hilbert space $\mathcal{H}$ and consider the constant fibre direct integral

$$\mathcal{K} = L^2(\mathbb{R}, \mathcal{H}) = \int_{\mathbb{R}}^{\oplus} \mathcal{H} dt,$$

with the inner product $\langle f, g\rangle_{\mathcal{K}} = \int \langle \overline{f(t)}, g(t)\rangle_{\mathcal{H}}\, dt$. The direct integral of a selfadjoint operator function is described in:

Remark 2.1. *Let $H(t)$ be selfadjoint in $\mathcal{H}$ for $t \in \mathbb{R}$. Then $D(H) := \{f \in L^2(\mathbb{R}, \mathcal{H}) \mid f(t) \in D(H(t))$ for a.e. $t \in \mathbb{R}, \int_{\mathbb{R}} \|H(t)f(t)\|^2 dt < \infty\}$, $Hf := \int_{\mathbb{R}}^{\oplus} H(t)f(t)dt$ defines a selfadjoint operator. It follows that $\phi(H)$ is decomposable for any bounded measurable $\phi : \mathbb{R} \to \mathbb{C}$ and*

$$\phi(H) = \int_{\mathbb{R}}^{\oplus} \phi(H(t))dt.$$

In particular,

$$\langle E_H(I) f \otimes g, f \otimes g\rangle = \int_{\mathbb{R}} \langle E_{H(t)}(I) f, f\rangle |g(t)|^2 dt$$

for the spectral projections and

$$\rho_H^{f\otimes g} = \int_{\mathbb{R}} \rho_{H(t)}^{f} |g(t)|^2 dt$$

for the spectral measures.

See [10], p. 280 ff, in particular Thm XIII.85. The latter formula makes the connection to spectral averaging clear.

Note that the obvious isometric isomorphism gives

$$\mathcal{K} = \mathcal{H} \otimes L^2(\mathbb{R}).$$

We will use this additional structure and write, e.g.

$$\mathbf{A} := A \otimes 1$$

for the canonical extension of A (which is a selfadjoint operator in $\mathcal{H}$) to $\mathcal{K}$. In much the same way we extend the position operator Q. Using some ideas from [7] we introduce the following: $T = \tanh Q$ the maximal multiplication operator in $L^2(\mathbb{R})$ with tanh, as well as $D := \arctan(P)$, where $P = -i\frac{d}{dt}$ is the momentum operator in $L^2(\mathbb{R})$.

Proposition 2.2. ([7, Lemma 2.9]) *On $L^2(\mathbb{R})$, consider the operators T and D above. Then $i[T,D] = C$ is positive definite.*

We next infer the following result of Putnam and Kato [8, 10]:

Proposition 2.3. *Let H and D be selfadjoint and D be bounded. If $C = i[T,D] \geq 0$, then H is absolutely continuous on $Range(C)$.*

Corollary 2.4. *The operator $\hat{\mathbf{H}} = \int_{\mathbb{R}}^{\oplus}(A + \tanh tB)dt$ is absolutely continuous on $\overline{Range(B)} \otimes L^2(\mathbb{R})$.*

Proof. By what we know from above,

$$i[\hat{\mathbf{H}}, \mathbf{D}] = \mathbf{BC} = B \otimes C \geq 0.$$

Since C is positive definite it follows that $Range(C)$ is dense in $L^2(\mathbb{R})$. □

Proof of Theorem 1.1. **Step 1:** The preceding Corollary and the above Remark 2.1 give that for any $\Phi \in \overline{Range(B)}$, $g \in L^2$,

$$\int \rho^{\Phi}_{A+\tanh tB}|g(t)|^2 dt << dt,$$

where the latter indicates absolute continuity with respect to Lebesgue measure.
Step 2: By specializing and change of variables: For any $\Phi \in \overline{Range(B)}$, $g \in L^\infty$ with compact support:

$$\int \rho^{\Phi}_{A+tB}|g(t)|^2 dt << dt,$$

Now by approximation, we get arbitrary positive $h \in L^1$ and, by linearity, the assertion of the Theorem. □

A standard extension formulated in a way that is suited for the application we have in mind is the following Corollary from the proof of Theorem 1.1.

Corollary 2.5. *Let A and B be as above and assume that $\overline{\{\varphi(A)Bf \mid f \in \mathcal{H}\}} = \mathcal{H}$. Then, for any $h \in L^1$ and any $\phi \in \mathcal{H}$:*

$$\int_{\mathbb{R}} \langle E_{A+tB}(\cdot)\phi, \phi\rangle h(t) dt << dt.$$

Proof. We consider the operators $\hat{\mathbf{H}}$, $\mathbf{A}$ and $\mathbf{B}$ on $\mathcal{K}$ as above. By what we proved above, the absolutely continuous subspace of $\hat{\mathbf{H}}$ contains $Range(B) \otimes L^2(\mathbb{R})$. Moreover it is cyclic for $\hat{\mathbf{H}}$ and closed. Therefore, the arguments from [7], Proof of Theorem 2.7, p.61 give that the absolutely continuous subspace of $\hat{\mathbf{H}}$ is all of $\mathcal{K}$. As in the above proof this implies the asserted absolute continuity. □

It is time to compare what we have shown so far with what is known by other methods, see [5, 3, 4, 12].

Remark 2.6.

- *Strictly speaking, the results of [3, 4, 12] and our Corollary above are not comparable, but the latter can be used to deduce what we have shown here. More precisely:*
- *In [3, 4, 12] instead of $h(t)dt$ more general measures are allowed. The continuity of $\nu := \int_{\mathbb{R}} \langle E_{A+tB}(\cdot)\phi, \phi\rangle d\mu(t)$ as well as that of μ is measured in terms of the modulus of continuity $s(\mu, \varepsilon) := \sup\{\mu([a, b]) \mid a, b \in \mathbb{R}, b - a = \varepsilon\}$ and the conclusion is that $s(\nu, \varepsilon) \leq Cs(\mu, \varepsilon)$, provided $\phi \in Range(B^{\frac{1}{2}})$.*
- *Clearly, the latter estimate directly does not give anything in the case of our result above: for absolutely continuous $\mu = h(t)dt$ the modulus of continuity does not need to decay at a certain rate as ε tends to zero. But, we can approximate h by bounded h_n in a suitable way. At the same time, we can approximate any $\phi \in \overline{Range(B)}$ by a sequence $\phi_n \in Range(B^{\frac{1}{2}})$ the resulting $\nu_n := \int_{\mathbb{R}} \langle E_{A+tB}(\cdot)\phi_n, \phi_n\rangle h_n(t)dt$ will converge to ν and all the ν_n are absolutely continuous with respect to dt, thus giving the assertion of our Theorem 1.1.*
- *In [5] the method of proof is pretty much similar to our strategy here. There, a direct application of Mourre estimates is used to prove spectral averaging and Wegner estimates. While their result concerns a more general setup it requires even differentiability of the density h; see Thm 1.1, Cor. 1.2 and 1.4 in the cited paper for results analogous to ours.*

3 Absolute continuity of the IDS; a very short proof.

We consider $L^2(\mathbb{R}^d)$ and the operators

$$H^\omega = -\Delta + \sum_{n\in\mathbb{Z}^d} \omega_n u(\cdot - n) \tag{1}$$

where u is a non-negative bounded measurable function that is positive on some open set. Let $\omega_n, n \in \mathbb{Z}^d$ be i.i.d random variables with a probability distribution μ which is absolutely continuous and has a compactly supported, integrable density h. We denote by $\mathbb{P} := \bigotimes_{n\in\mathbb{Z}} \mu$ the product measure and by $\mathbb{E}$ the corresponding expectation.

By $\Lambda(0)$ we denote the unit unit cube. In view of the Pastur-Shubin trace formula we can express the integrated density of states, IDS, in terms of

$$\mathcal{N}(I) = \mathbb{E}\left[Tr(\chi_{\Lambda(0)} E_{H^\omega}(I)\chi_{\Lambda(0)})\right], \tag{2}$$

for any bounded Borel set I. See [13] for an extensive bibligraphy on the IDS and the proof of the trace formula in a more general situation. The IDS is quite often also expressed as the distribution function $N(E) : \mathcal{N}(-\infty, E]$ of the measure $\mathcal{N}$ defined above.

Corollary 3.1. *In the situation above, $\mathcal{N} << dt$.*

Proof. Note that in the situation given we can apply the cyclicity result of [2], Prop. A2.2, and know that for

$$A := -\Delta + \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \omega_n u(\cdot - n) \qquad B := u(\cdot),$$

the assumptions of Corollary 2.5 are met. We fix an orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ of $\mathcal{H} = L^2(\mathbb{R}^d)$ and write

$$\begin{aligned}
\mathcal{N}(I) &= \mathbb{E}\left[Tr(\chi_{\Lambda(0)} E_{H^\omega}(I)\chi_{\Lambda(0)})\right] \\
&= \mathbb{E}\left[\sum_{k\in\mathbb{N}} \langle \chi_{\Lambda(0)} E_{H^\omega}(I)\chi_{\Lambda(0)}\phi_k \mid \phi_k\rangle\right] \\
&= \sum_{k\in\mathbb{N}} \mathbb{E}\left[E_{H^\omega}(I)\chi_{\Lambda(0)}\phi_k \mid \chi_{\Lambda(0)}\phi_k\rangle\right].
\end{aligned}$$

It suffices to show that every sum in the term is absolutely continuous with respect to dt and this works as follows:

$$\mathbb{E}\left[E_{H^\omega}(I)\chi_{\Lambda(0)}\phi_k \mid \chi_{\Lambda(0)}\phi_k\rangle\right] =$$

$$= \mathbb{E}\left[E_{(-\Delta+\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\omega_n u(\cdot-n)+\omega_0 B)}(I)\chi_{\Lambda(0)}\phi_k \mid \chi_{\Lambda(0)}\phi_k\rangle\right] =$$

$$= \mathbb{E}\left[\int_{\mathbb{R}} E_{(-\Delta+\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\omega_n u(\cdot-n)+tB)}(I)\chi_{\Lambda(0)}\phi_k \mid \chi_{\Lambda(0)}\phi_k\rangle dt\right].$$

For fixed $\omega' := (\omega_n)_{n\in\mathbb{Z}^d\setminus\{0\}}$ the inner integral is seen to give an absolutely continuous measure: set A as above and apply Cor. 2.5. The expectation preserves the absolute continuity and that establishes the claim. $\square$

Of course, an additional periodic background potential V_0 would not change the proof; all the ingredients we cited are valid in this case as well.

Acknowledgement. This work was started during a visit of P.S. to India; he wants to thank for generous support by the IMSC, Chennai and the hospitality of the staff there as well as for the generous support by the DFG (German Science Foundation).

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