On complex singularity analysis of holomorphic solutions of linear partial differential equations

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Abstract

We construct formal series solutions of linear partial differential equations as linear combinations of powers of solutions of first order nonlinear differential equations, following the classical tanh method. We give sufficient conditions under which the constructed formal series define holomorphic functions on some punctured polydiscs of $\mathbb{C}^2$. Moreover, we study the rate of growth of these solutions near their singular points.

Key words: linear Cauchy problems, Banach spaces of entire functions, formal power series, ordinary differential equations. 2000 MSC: 35C10, 35C20.

1 Introduction

In the literature, the *tanh method*, which has been initiated by W. Malfliet in [11], is an effective algebraic method for finding explicit exact solutions of certain nonlinear partial differential equations with constant coefficients of the form

$$H(u(t,x), \partial_t u(t,x), \partial_x u(t,x), \partial_x^2 u(t,x), \ldots) = 0$$

where H is a polynomial. The method consists in looking for solutions of the form

$$u(t,x) = \sum_{j \in J} u_j(\varphi(\kappa(x - wt)))^j$$

where J is a finite subset of $\mathbb{Z}$, u_j, κ, w are constants and φ is a solution of a Ricatti equation

$$\varphi'(\xi) = a + b\varphi(\xi) + c(\varphi(\xi))^2,$$

where a, b, c are well chosen real numbers. Notice that this approach has been generalized in several directions and encounters a growing success these recent years, see for instance [3], [4], [13].

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In this paper, we will consider special solutions of linear partial differential equations of the form

$$\partial_z^S X(t,z) = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}} a_{\mathbf{k}}(t,z)\partial_t^{k_0}\partial_z^{k_1}X(t,z)$$

with holomorphic coefficients $a_{\mathbf{k}}(t,z)$ on some domain $\mathcal{D}\subset\mathbb{C}$ with respect to t and near the origin in $\mathbb{C}$ with respect to z. Inspired by the tanh method, we are looking for solutions in the form of *infinite* sums

$$X(t,z) = \sum_{j\geq 0} X_j(t,z)(\phi(t))^j \tag{1}$$

where $X_j(t,z)$ are holomorphic functions on $\mathcal{D}$ with respect to t and near the origin in z and $\phi(t)$ is a solution of a general algebraic first order differential equation

$$\phi'(t) = P(t,\phi(t)) \tag{2}$$

where $P(t,X)$ is a polynomial in X of degree larger than 2, with holomorphic coefficients on $\mathcal{D}$. These kind of series are also called transasymptotic expansions in the terminology of J. Écalle, see [8]. In the framework of differential equations, similar series have been used to study the formation of complex singularities and local behaviour of the holomorphic solutions near the singular points along Stokes directions for systems of non linear differential equations with irregular singularity at 0 of the form

$$t^2y'(t) = F(t,y(t)),$$

where F is a holomorphic function from $\mathbb{R}\times\mathbb{R}^n$ into $\mathbb{R}^n$, for $n\geq 1$. The constructed holomorphic solutions are of the form $y(t)=\sum_{l\geq 0}x_l(t)(\varphi(t))^l$, where $x_l(t)$ are the $k-$sums of formal series $\hat{x}_l(t)\in\mathbb{C}^n[[t]]$ in some direction d, see [6], and where $\varphi(t)$ is a solution of the differential equation $-t^2\varphi'(t)=\varphi(t)$. More recently, a similar study has been investigated for $q-$difference-differential equations by the first author, see [10].

From the result of P. Painlevé, see [9], we know that the only movable singularities in $\mathbb{C}$ of the solutions $\phi(t)$ of (2) are poles and/or algebraic branch points. In this work, we will show that the special solutions (1) define holomorphic functions with respect to t near a movable singularity $t_0\in\mathcal{D}$ of ϕ and near the origin with respect to z. Moreover, we will analyse the local behaviour of the constructed solutions as t tends to t_0 in $\mathcal{D}$.

In the section 2, we consider a Cauchy problem in $\mathbb{C}^3$ with holomorphic solutions $V(t,z,T)$, which extends both the classical Cauchy-Kowalevskii problem in holomorphic functions spaces near the origin in $\mathbb{C}^2$ with respect to t,z and a global version in entire functions spaces with exponential growth with respect to T given by J. Dubinskii, see [7]. The difficulty to combine these two results comes from the fact that the growth rate of the solution $V(t,z,T)$ in the local variable t and the global variable T both depend on the variable z near the origin as observed in [12], [7].

In the section 3, we first construct the special solutions (1) in a formal sense. Then, in the section 4, using a majorazing series method, we reduce the problem of the description of their convergence domain to the construction of a solution of an auxiliary Cauchy problem in $\mathbb{C}^3$ to which the results of the section 2 can be applied. Finally, in the Theorem 2, we show that the solutions (1) have at most an exponential growth rate as t approaches t_0 in $\mathcal{D}$.

2 A Cauchy problem in a class of entire functions with exponential growth and finite order.

2.1 Weighted Banach spaces of holomorphic functions in $\mathbb{C}^3$.

Definition 1 *Let m, σ, δ_1, δ_2, q and b be positive real numbers such that $q \geq 1$ and $b > 1$. We define a vector space $\mathcal{G}_q(\delta_1, \delta_2; \sigma)$ which is a subspace of the vector space $\mathcal{A}(\mathbb{C})\{t, z\}$ of entire functions $V(t, z, T)$ with respect to T which are holomorphic on a neighbourhood of the origin in $\mathbb{C}^2$ with respect to (t, z). A function*

$$V(t, z, T) = \sum_{n, \beta \geq 0} v_{n,\beta}(T) \frac{t^n}{n!} \frac{z^\beta}{\beta!}$$

belongs to $\mathcal{G}_q(\delta_1, \delta_2; \sigma)$, if the series

$$\sum_{n, \beta \geq 0} ||v_{n,\beta}(T)||_{\beta;\sigma} \frac{\delta_1^n \delta_2^\beta}{(n+\beta)!}$$

converges, where

$$||v_{n,\beta}(T)||_{\beta;\sigma} = \sup_{T \in \mathbb{C}} |v_{n,\beta}(T)| \left(1 + |T|\right)^{-m} \exp\left(-\sigma r_b(\beta) |T|^q\right)$$

with

$$r_b(\beta) = \sum_{n=0}^{\beta} \frac{1}{(n+1)^b}.$$

We also define a weighted L^1 norm on $\mathcal{G}_q(\delta_1, \delta_2; \sigma)$ as

$$||V(t, z, T)||_{\delta_1, \delta_2; \sigma} = \sum_{n, \beta \geq 0} ||v_{n,\beta}(T)||_{\beta;\sigma} \frac{\delta_1^n \delta_2^\beta}{(n+\beta)!}.$$

One can easily show that $\left(\mathcal{G}_q(\delta_1, \delta_2; \sigma), ||.||_{\delta_1, \delta_2; \sigma}\right)$ is a Banach space.

Remark: We have the following continuous inclusions $\mathcal{G}_q(\delta_1', \delta_2'; \sigma') \hookrightarrow \mathcal{G}_q(\delta_1, \delta_2; \sigma)$ if $\delta_1' \geq \delta_1$, $\delta_2' \geq \delta_2$, $\sigma' \leq \sigma$ and $r_b(\beta)$ is such that $\lim_{\beta \to +\infty} r_b(\beta) = \zeta(b)$ where ζ is the Riemann zeta function.

In the next proposition, we study the rate of growth of the entire functions belonging to the constructed Banach spaces.

Proposition 1 *Let $V(t, z, T) \in \mathcal{G}_q(\delta_1, \delta_2; \sigma)$. Then, there exists a positive constant C_1 such that $\forall T \in \mathbb{C}$, $\forall t \in \mathbb{C}$ with $|t| < \frac{\delta_1}{2}$ and $\forall z \in \mathbb{C}$ with $|z| < \frac{\delta_2}{2}$, we have*

$$|V(t, z, T)| \leq C_1 \left(1 - \frac{2|t|}{\delta_1}\right)^{-1} \left(1 - \frac{2|z|}{\delta_2}\right)^{-1} (1 + |T|)^m \exp\left(\sigma \zeta(b) |T|^q\right).$$

Proof We expand $V(t, z, T)$ in powers of t and z

$$V(t, z, T) = \sum_{\beta \geq 0} \sum_{n \geq 0} v_{n,\beta}(T) \frac{t^n}{n!} \frac{z^\beta}{\beta!}.$$

As $V(t,z,T) \in \mathcal{G}_q(\delta_1,\delta_2;\sigma)$, there exists a positive constant c_1 such that

$$||v_{n,\beta}(T)||_{\beta;\sigma} \frac{\delta_1^n \delta_2^\beta}{(n+\beta)!} \leq c_1$$

for all n, $\beta \geq 0$. So that for all $T \in \mathbb{C}$, we have

$$\frac{|v_{n,\beta}(T)|}{n!\beta!} \leq c_1 (1+|T|)^m \exp(\sigma r_b(\beta)|T|^q) \frac{1}{\delta_1^n \delta_2^\beta} \frac{(n+\beta)!}{n!\beta!} \tag{3}$$

for all n, $\beta \geq 0$.

Lemma 1 *We have $(n+\beta)!/(n!\beta!) \leq 2^{n+\beta}$ for all n, $\beta \geq 0$.*

The lemma is a consequence of the classical binomial formula. From the estimates (3) we deduce that

$$\frac{|v_{n,\beta}(T)|}{n!\beta!} \leq c_1 (1+|T|)^m \exp(\sigma r_b(\beta)|T|^q) \left(\frac{2}{\delta_1}\right)^n \left(\frac{2}{\delta_2}\right)^\beta \tag{4}$$

for all $T \in \mathbb{C}$, and for all n, $\beta \geq 0$. From (4), we get that $\forall T \in \mathbb{C}$, $\forall t \in \mathbb{C}$ with $|t| < \frac{\delta_1}{2}$ and $\forall z \in \mathbb{C}$ with $|z| < \frac{\delta_2}{2}$

$$\begin{aligned} |V(t,z,T)| &\leq \sum_{\beta \geq 0} \sum_{n \geq 0} \frac{|v_{n,\beta}(T)|}{n!\beta!} |t|^n |z|^\beta \\ &\leq c_1 (1+|T|)^m \frac{1}{1-\frac{2|t|}{\delta_1}} \sum_{\beta \geq 0} \exp(\sigma r_b(\beta)|T|^q) \left(\frac{2|z|}{\delta_2}\right)^\beta . \end{aligned}$$

Now, as $r_b(\beta) \leq \zeta(b)$ for all $\beta \in \mathbb{N}$, we get that there exists a positive constant c_1, such that $\forall T \in \mathbb{C}$, $\forall t \in \mathbb{C}$ with $|t| < \frac{\delta_1}{2}$ and $\forall z \in \mathbb{C}$ with $|z| < \frac{\delta_2}{2}$ we have

$$|V(t,z,T)| \leq c_1 (1+|T|)^m \frac{1}{1-\frac{2|t|}{\delta_1}} \exp(\sigma \zeta(b)|T|^q) \frac{1}{1-\frac{2|z|}{\delta_2}}.$$

This proves the proposition.

2.2 Linear operators defined on the weighted Banach spaces.

We define the integration operator $\partial_z^{-1} : \mathcal{A}(\mathbb{C})\{t,z\} \to \mathcal{A}(\mathbb{C})\{t,z\}$ as

$$\left(\partial_z^{-1} V\right)(t,z,T) := \int_0^z V(t,\xi,T) d\xi$$

and recall the classical estimates

$$\frac{\Gamma(x+\gamma)}{\Gamma(x)} \sim x^\gamma \tag{5}$$

for all $\gamma \geq 0$, as x tends to $+\infty$.
In this subsection, we study the continuity property of some linear operators defined on the spaces $\mathcal{G}_q(\delta_1,\delta_2;\sigma)$. In the next proposition, we analyse some integro-differential operator.

Proposition 2 *Let s, ν, κ and S be non negative integers satisfying*

$$\frac{b\left(s+\kappa(q-1)\right)}{q}+\nu<S.$$

Then the operator $T^s\partial_t^\nu\partial_T^\kappa\partial_z^{-S}$ is a bounded linear operator from $\left(\mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right),||.||_{\delta_1,\delta_2;\sigma}\right)$ into itself. Moreover, there exists a constant $C_5>0$ such that

$$\left|\left|\left(T^s\partial_t^\nu\partial_T^\kappa\partial_z^{-S}V\right)(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma}\leq C_5\delta_1^{-\nu}\delta_2^S\,||V(t,z,T)||_{\delta_1,\delta_2;\sigma}$$

for all $V(t,z,T)\in\mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right)$.

Proof For

$$V(t,z,T)=\sum_{\beta\geq 0}\sum_{n\geq 0}v_{n,\beta}(T)\frac{t^n}{n!}\frac{z^\beta}{\beta!}\in\mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right),$$

we have

$$\left(T^s\partial_t^\nu\partial_T^\kappa\partial_z^{-S}V\right)(t,z,T)=\sum_{\beta\geq S}\sum_{n\geq 0}\left(T^s\partial_T^\kappa v_{n+\nu,\beta-S}\right)(T)\frac{t^n}{n!}\frac{z^\beta}{\beta!}.$$

Therefore, by definition, we obtain that

$$\left|\left|\left(T^s\partial_t^\nu\partial_T^\kappa\partial_z^{-S}V\right)(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma}=\sum_{\beta\geq S}\sum_{n\geq 0}||\left(T^s\partial_T^\kappa v_{n+\nu,\beta-S}\right)(T)||_{\beta;\sigma}\frac{\delta_1^n\delta_2^\beta}{(n+\beta)!}$$

where

$$||\left(T^s\partial_T^\kappa v_{n+\nu,\beta-S}\right)(T)||_{\beta;\sigma}=\sup_{T\in\mathbb{C}}\left|\left(T^s\partial_T^\kappa v_{n+\nu,\beta-S}\right)(T)\right|\left(1+|T|\right)^{-m}\exp\left(-\sigma r_b(\beta)|T|^q\right).$$

From Cauchy's theorem, for $T\in\mathbb{C}$, we have

$$\left(T^s\partial_T^\kappa v_{n+\nu,\beta-S}\right)(T)=\frac{\kappa!}{2i\pi}\int_{|\xi-T|=a}\frac{T^s v_{n+\nu,\beta-S}(\xi)}{(\xi-T)^{\kappa+1}}d\xi$$

where $a>0$ can be chosen arbitrarily. We replace ξ by $T+a\exp(i\theta)$, where $\theta\in[0,2\pi[$ and obtain

$$\begin{aligned}\left|\left(T^s\partial_T^\kappa v_{n+\nu,\beta-S}\right)(T)\right|\leq\ &\frac{\kappa!|T|^s}{2\pi a^\kappa}\,||v_{n+\nu,\beta-S}(T)||_{\beta-S;\sigma}\\ &\times\int_0^{2\pi}\left(1+|T+a\exp(i\theta)|\right)^m\exp\left(\sigma r_b(\beta-S)|T+a\exp(i\theta)|^q\right)d\theta.\end{aligned}$$

Using the triangle inequality, we get

$$\begin{aligned}\left|\left(T^s\partial_T^\kappa v_{n+\nu,\beta-S}\right)(T)\right|\leq\ &\frac{\kappa!}{a^\kappa}|T|^s\,||v_{n+\nu,\beta-S}(T)||_{\beta-S;\sigma}\\ &\times\left(1+|T|+a\right)^m\exp\left(\sigma r_b(\beta-S)(|T|+a)^q\right).\end{aligned}$$

We denote by $I_{a,\beta}(T)$ a part of the right hand side of inequality below

$$I_{a,\beta}(T):=\frac{\kappa!}{a^\kappa}|T|^s\left(1+|T|+a\right)^m\exp\left(\sigma r_b(\beta-S)(|T|+a)^q\right). \tag{6}$$

Now, our aim is to estimate $||(T^s\partial_T^\kappa v_{n+\nu,\beta-S})\,(T)||_{\beta;\sigma}$

$$||(T^s\partial_T^\kappa v_{n+\nu,\beta-S})\,(T)||_{\beta;\sigma} \leq \left(\sup_{T\in\mathbb{C}} I_{a,\beta}(T)(1+|T|)^{-m}\exp\left(-\sigma r_b(\beta)|T|^q\right)\right)$$
$$\times\, ||v_{n+\nu,\beta-S}(T)||_{\beta-S;\sigma}\,.$$

So we have to prove the following lemma.

Lemma 2 *There exists a contant $C_6 > 0$ such that*

$$\sup_{T\in\mathbb{C}} I_{a,\beta}(T)(1+|T|)^{-m}\exp\left(-\sigma r_b(\beta)|T|^q\right) \leq C_6\beta^{\frac{b(s+\kappa(q-1))}{q}}$$

for all $\beta \geq S$.

Proof We denote by $J_{a,\beta}$ the supremum to estimate

$$J_{a,\beta} := \sup_{T\in\mathbb{C}} I_{a,\beta}(T)(1+|T|)^{-m}\exp\left(-\sigma r_b(\beta)|T|^q\right).$$

By definition of $I_{a,\beta}(T)$, we have

$$J_{a,\beta} = \kappa!\sup_{T\in\mathbb{C}}\frac{|T|^s}{a^\kappa}\left(\frac{1+|T|+a}{1+|T|}\right)^m\exp\left(\sigma\left\{r_b(\beta-S)(|T|+a)^q - r_b(\beta)|T|^q\right\}\right).$$

As $|T| \in [0,+\infty[$ and a, $m \geq 0$,

$$\left(\frac{1+|T|+a}{1+|T|}\right)^m = \left(1+\frac{a}{1+|T|}\right)^m \leq (1+a)^m,$$

we have

$$J_{a,\beta} \leq \kappa!\sup_{T\in\mathbb{C}}\frac{|T|^s}{a^\kappa}(1+a)^m\exp\left(\sigma\left\{r_b(\beta-S)-r_b(\beta)\right\}(|T|+a)^q\right)$$
$$\times\exp\left(\sigma r_b(\beta)\left\{(|T|+a)^q - |T|^q\right\}\right),$$

using the equality $\gamma_1\gamma_2 - \gamma_3\gamma_4 = \gamma_1(\gamma_2-\gamma_3)+\gamma_3(\gamma_1-\gamma_4)$ in the exponential.
After, we shall also need the estimates

$$r_b(\beta) - r_b(\beta-S) = \sum_{n=\beta-S+1}^{\beta}\frac{1}{(n+1)^b} \leq \frac{S}{(\beta+1)^b}.$$

Now, remember that the radius a in the Cauchy's theorem is still arbitrarily. As suggested by Y. Dubinskiĭ in [7], let us choose $a > 0$ such that $(|T|+a)^q = |T|^q + 1$, thus

$$a = (|T|^q+1)^{1/q} - |T|.$$

Then, we obtain that

$$J_{a,\beta} \leq \kappa!\exp\left(\sigma r_b(\beta)\right)\exp\left(-\frac{\sigma S}{(\beta+1)^b}\right)$$

$$\times \sup_{T\in\mathbb{C}} |T|^s \frac{\left(1+(|T|^q+1)^{1/q}-|T|\right)^m}{\left((|T|^q+1)^{1/q}-|T|\right)^\kappa} \exp\left(-\frac{\sigma S}{(\beta+1)^b}|T|^q\right).$$

To estimate the supremum on the right-hand side, we distinguish two cases. For the first case, using Taylor's formula at the origin

$$(1+x)^{1/q} = 1+\frac{x}{q}+x\varepsilon(x), \qquad |x|<1$$

while $\varepsilon(x)$ in the remainder term satisfies $\lim_{x\to 0}\varepsilon(x)=0$, there exists a constant $0\leq \varsigma<1$ such that

$$|\varepsilon(x)|<\frac{1}{2q}$$

for $|x|<\varsigma$. Therefore, for $|T|>\varsigma^{-1/q}$, there exists a constant $c_1>0$, independent of β such that

$$\frac{\left(1+(|T|^q+1)^{1/q}-|T|\right)^m}{\left((|T|^q+1)^{1/q}-|T|\right)^\kappa} = |T|^{\kappa(q-1)}\frac{\left(1+\frac{1}{q|T|^{q-1}}+\frac{1}{|T|^{q-1}}\varepsilon\left(\frac{1}{|T|^q}\right)\right)^m}{\left(\frac{1}{q}+\varepsilon\left(\frac{1}{|T|^q}\right)\right)^\kappa} \leq c_1|T|^{\kappa(q-1)}.$$

Let A, B positive numbers, a study of the function $f(x)=x^A\exp(-Bx)$ shows that its supremum yields at $x=A\backslash B$, thus

$$\sup_{T\in\mathbb{C},|T|^q>\varsigma^{-1}} |T|^{s+\kappa(q-1)}\exp\left(-\frac{\sigma S}{(\beta+1)^b}|T|^q\right)$$

$$\leq \left(\frac{s+\kappa(q-1)}{q\sigma S}(\beta+1)^b\right)^{\frac{s+\kappa(q-1)}{q}}\exp\left(-\frac{s+\kappa(q-1)}{q}\right).$$

It turns out that there exists a constant $c_2>0$, independent of β, such that

$$\sup_{T\in\mathbb{C},|T|^q>\varsigma^{-1}} |T|^s \frac{\left(1+(|T|^q+1)^{1/q}-|T|\right)^m}{\left((|T|^q+1)^{1/q}-|T|\right)^\kappa} \exp\left(-\frac{\sigma S}{(\beta+1)^b}|T|^q\right) \leq c_2\beta^{\frac{b(s+\kappa(q-1))}{q}}.$$

For the second case $|T|\leq\varsigma^{-1/q}$, it is easy to see that there exists a constant $c_3>0$, independent of β, such that

$$\sup_{T\in\mathbb{C},|T|^q\leq\varsigma^{-1}} |T|^s \frac{\left(1+(|T|^q+1)^{1/q}-|T|\right)^m}{\left((|T|^q+1)^{1/q}-|T|\right)^\kappa} \exp\left(-\frac{\sigma S}{(\beta+1)^b}|T|^q\right) \leq c_3$$

for all $\beta\geq S$. Hence, we have two constants c_4, $c_5>0$, independent of β, such that

$$J_{a,\beta}\leq c_4\kappa!\exp\left(\sigma\zeta(b)\right)\exp\left(-\frac{\sigma S}{(\beta+1)^b}\right)\beta^{b(s+\kappa(q-1))/q}\leq c_5\beta^{\frac{b(s+\kappa(q-1))}{q}}$$

for all $\beta\geq S$. This proves the lemma 2.

Now, we return to the proof of proposition 2. Using Lemma 2, we obtain that there exists a constant $c_1>0$, independent of β, such that

$$||\left(T^s\partial_T^\kappa v_{n+\nu,\beta-S}\right)(T)||_{\beta;\sigma}\leq c_1\beta^{\frac{b(s+\kappa(q-1))}{q}}\,||v_{n+\nu,\beta-S}(T)||_{\beta-S;\sigma}$$

for all $n \geq 0$ and for all $\beta \geq S$, and hence

$$\left|\left|\left(T^{s}\partial_t^{\nu}\partial_T^{\kappa}\partial_z^{-S}V\right)(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma}$$

$$\leq c_1\delta_1^{-\nu}\delta_2^{S}\sum_{\beta\geq S}\sum_{n\geq 0}\beta^{\frac{b(s+\kappa(q-1))}{q}}\frac{(n+\nu+\beta-S)!}{(n+\beta)!}\left|\left|v_{n+\nu,\beta-S}(T)\right|\right|_{\beta-S;\sigma}\frac{\delta_1^{n+\nu}\delta_2^{\beta-S}}{(n+\nu+\beta-S)!}.$$

Using the estimates (5) and our hypotheses that $\frac{b(s+\kappa(q-1))}{q}+\nu<S$, there exists a constant $c_2>0$, independent of β, such that

$$\beta^{\frac{b(s+\kappa(q-1))}{q}}\frac{(n+\nu+\beta-S)!}{(n+\beta)!}\leq c_2\frac{\beta^{\frac{b(s+\kappa(q-1))}{q}}}{\beta^{S-\nu}}\leq c_2$$

for all $\beta \geq S$. To conclude the proof of Proposition 2, there exists a constant $c_3>0$ such that

$$\left|\left|\left(T^{s}\partial_t^{\nu}\partial_T^{\kappa}\partial_z^{-S}V\right)(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma}\leq c_3\delta_1^{-\nu}\delta_2^{S}\left|\left|V(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma}$$

for all $V(t,z,T)\in\mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right)$.

In the next proposition, we study the continuity of the operator of multiplication by a holomorphic function.

Proposition 3 *Let* $a(t,z)=\sum_{\beta\geq 0}\sum_{n\geq 0}a_{n,\beta}\frac{t^n}{n!}\frac{z^\beta}{\beta!}$ *a holomorphic function near the origin in* $\mathbb{C}^2$. *We define*

$$|a|(t,z)=\sum_{\beta\geq 0}\sum_{n\geq 0}\left|a_{n,\beta}\right|\frac{t^n}{n!}\frac{z^\beta}{\beta!}.$$

Then there exists a constant $C_7>0$*, such that*

$$\left|\left|a(t,z)V(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma}\leq C_7|a|(\delta_1,\delta_2)\left|\left|V(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma}$$

for all $V(t,z,T)\in\mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right)$.

Proof For

$$V(t,z,T)=\sum_{\beta\geq 0}\sum_{n\geq 0}v_{n,\beta}(T)\frac{t^n}{n!}\frac{z^\beta}{\beta!}\in\mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right),$$

we have

$$\frac{t^{n_0}}{n_0!}\frac{z^{\beta_0}}{\beta_0!}V(t,z,T)=\sum_{\beta\geq\beta_0}\sum_{n\geq n_0}\frac{n!\beta!}{n_0!(n-n_0)!\beta_0!(\beta-\beta_0)!}v_{n-n_0,\beta-\beta_0}(T)\frac{t^n}{n!}\frac{z^\beta}{\beta!}.$$

Therefore, by definition, we obtain that

$$\left|\left|\frac{t^{n_0}}{n_0!}\frac{z^{\beta_0}}{\beta_0!}V(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma}=\sum_{\beta\geq\beta_0}\sum_{n\geq n_0}\left|\left|\frac{n!\beta!}{n_0!(n-n_0)!\beta_0!(\beta-\beta_0)!}v_{n-n_0,\beta-\beta_0}(T)\right|\right|_{\beta;\sigma}\frac{\delta_1^n\delta_2^\beta}{(n+\beta)!},$$

where

$$\left|\left|\frac{n!\beta!}{n_0!(n-n_0)!\beta_0!(\beta-\beta_0)!}v_{n-n_0,\beta-\beta_0}(T)\right|\right|_{\beta;\sigma}=\frac{n!\beta!}{n_0!(n-n_0)!\beta_0!(\beta-\beta_0)!}\left|\left|v_{n-n_0,\beta-\beta_0}(T)\right|\right|_{\beta;\sigma}$$

and

$$||v_{n-n_0,\beta-\beta_0}(T)||_{\beta;\sigma} = \sup_{T\in\mathbb{C}} |v_{n-n_0,\beta-\beta_0}(T)| (1+|T|)^{-m} \exp(-\sigma r_b(\beta)|T|^q)$$

for all $n \geq n_0$ and for all $\beta \geq \beta_0$.
Observe that

$$||v_{n-n_0,\beta-\beta_0}(T)||_{\beta;\sigma} \leq \sup_{T\in\mathbb{C}} |v_{n-n_0,\beta-\beta_0}(T)| (1+|T|)^{-m} \exp(-\sigma r_b(\beta-\beta_0)|T|^q)$$

since $r_b(\beta) \geq r_b(\beta-\beta_0)$ for all $\beta \geq \beta_0$.
Therefore, we have

$$||v_{n-n_0,\beta-\beta_0}(T)||_{\beta;\sigma} \leq ||v_{n-n_0,\beta-\beta_0}(T)||_{\beta-\beta_0;\sigma}$$

for all $\beta \geq \beta_0$.
Using the upper mentioned inequalities, we obtain

$$\left|\left|\frac{t^{n_0}}{n_0!}\frac{z^{\beta_0}}{\beta_0!}V(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma} \leq \frac{\delta_1^{n_0}}{n_0!}\frac{\delta_2^{\beta_0}}{\beta_0!}\sum_{\beta\geq\beta_0}\sum_{n\geq n_0}\frac{n!\beta!}{(n-n_0)!(\beta-\beta_0)!}\frac{(n-n_0+\beta-\beta_0)!}{(n+\beta)!}$$
$$\times ||v_{n-n_0,\beta-\beta_0}(T)||_{\beta-\beta_0;\sigma}\frac{\delta_1^{n-n_0}\delta_2^{\beta-\beta_0}}{(n-n_0+\beta-\beta_0)!}.$$

On the other hand, using the estimates (5), there exists a constant $c_1 > 0$ (independent on n and β), such that

$$\frac{n!\beta!}{(n-n_0)!(\beta-\beta_0)!}\frac{(n-n_0+\beta-\beta_0)!}{(n+\beta)!} \leq c_1$$

for all $n \geq n_0$ and for all $\beta \geq \beta_0$, therefore

$$\left|\left|\frac{t^{n_0}}{n_0!}\frac{z^{\beta_0}}{\beta_0!}V(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma} \leq c_1\frac{\delta_1^{n_0}}{n_0!}\frac{\delta_2^{\beta_0}}{\beta_0!}||V(t,z,T)||_{\delta_1,\delta_2;\sigma}.$$

So that, there exists a constant $c_2 > 0$ such that

$$||a(t,z)V(t,z,T)||_{\delta_1,\delta_2;\sigma} \leq \sum_{\beta_0\geq 0}\sum_{n_0\geq 0}\left|\left|a_{n_0,\beta_0}\frac{t^{n_0}}{n_0!}\frac{z^{\beta_0}}{\beta_0!}V(t,z,T)\right|\right|_{\delta_1\delta_2;\sigma}$$
$$\leq \sum_{\beta_0\geq 0}\sum_{n_0\geq 0}|a_{n_0,\beta_0}|\left|\left|\frac{t^{n_0}}{n_0!}\frac{z^{\beta_0}}{\beta_0!}V(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma} \leq c_2\,|a|\,(\delta_1,\delta_2)\,||V(t,z,T)||_{\delta_1,\delta_2;\sigma}.$$

This proves proposition 3.

In the next proposition, we analyse the action of some differential operator on functions of $\mathcal{G}_q(\delta_1,\delta_2;\sigma)$ which are polynomial in the variable z.

Proposition 4 *Let s, ν, κ and θ non negative integers and let $W(t,z,T) \in \mathcal{G}_q(\delta_{1,0},\delta_2;\sigma_0)$ with $W(t,z,T)$ polynomial in the variable z. Then there exist $\delta_1 > 0$, small enough ($\delta_1 \leq \delta_{1,0}$) and σ, big enough ($\sigma > \sigma_0$) with*

$$\sigma_0 r_b(d) + \frac{1}{2} \leq \sigma,$$

such that the series $\left(T^s\partial_t^\nu\partial_T^\kappa\partial_z^\theta W\right)(t,z,T)$ belongs to $\mathcal{G}_q(\delta_1,\delta_2;\sigma)$. Moreover, there exists a constant $C_8 > 0$ such that

$$\left|\left|\left(T^s\partial_t^\nu\partial_T^\kappa\partial_z^\theta W\right)(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma} \leq C_8\delta_1^{-\nu}\delta_2^{-\theta}||W(t,z,T)||_{\delta_{1,0},\delta_2;\sigma_0}$$

for all $W(t,z,T) \in \mathcal{G}_q(\delta_{1,0},\delta_2;\sigma_0)$ such that $W(t,z,T)$ is a polynomial in the variable z.

Proof For $W(t,z,T) \in \mathcal{G}_q(\delta_{1,0},\delta_2;\sigma_0)$ and

$$W(t,z,T) = \sum_{\beta=0}^{d} \sum_{n\geq 0} w_{n,\beta}(T) \frac{t^n}{n!} \frac{z^\beta}{\beta!}$$

polynomial in the variable z, we have

$$\left(T^s \partial_t^\nu \partial_T^\kappa \partial_z^\theta W\right)(t,z,T) = \sum_{\beta=0}^{d-\theta} \sum_{n\geq 0} \left(T^s \partial_T^\kappa w_{n+\nu,\beta+\theta}\right)(T) \frac{t^n}{n!} \frac{z^\beta}{\beta!}.$$

Therefore, by definition, we obtain that

$$||\left(T^s \partial_t^\nu \partial_T^\kappa \partial_z^\theta W\right)(t,z,T)||_{\delta_1,\delta_2;\sigma} = \sum_{\beta=0}^{d-\theta} \sum_{n\geq 0} ||\left(T^s \partial_T^\kappa w_{n+\nu,\beta+\theta}\right)(T)||_{\beta;\sigma} \frac{\delta_1^n \delta_2^\beta}{(n+\beta)!} \tag{7}$$

where

$$||\left(T^s \partial_T^\kappa w_{n+\nu,\beta+\theta}\right)(T)||_{\beta;\sigma} = \sup_{T\in\mathbb{C}} |\left(T^s \partial_T^\kappa w_{n+\nu,\beta+\theta}\right)(T)| (1+|T|)^{-m} \exp\left(-\sigma r_b(\beta)|T|^q\right).$$

As in the proof of Proposition 2, using Cauchy's theorem, we have, for all $T \in \mathbb{C}$

$$\begin{aligned} |\left(T^s \partial_T^\kappa w_{n+\nu,\beta+\theta}\right)(T)| \leq{}& \frac{\kappa! |T|^s}{2\pi a^\kappa} ||w_{n+\nu,\beta+\theta}(T)||_{\beta+\theta;\sigma_0} \\ &\times \int_0^{2\pi} (1+|T+a\exp(i\tau)|)^m \exp\left(\sigma_0 r_b(\beta+\theta)|T+a\exp(i\tau)|^q\right) d\tau \end{aligned}$$

where $a > 0$ can be chosen arbitrarily and σ_0 will be chosen later. Using the triangular inequality, we get

$$\begin{aligned} |\left(T^s \partial_T^\kappa w_{n+\nu,\beta+\theta}\right)(T)| \leq{}& \frac{\kappa!}{a^\kappa} |T|^s ||w_{n+\nu,\beta+\theta}(T)||_{\beta+\theta;\sigma_0} \\ &\times (1+|T|+a)^m \exp\left(\sigma_0 r_b(\beta+\theta)(|T|+a)^q\right). \end{aligned}$$

As in (6), we define

$$I_{a,\beta;\sigma_0}(T) := \frac{\kappa!}{a^\kappa} |T|^s (1+|T|+a)^m \exp\left(\sigma_0 r_b(\beta+\theta)(|T|+a)^q\right).$$

Now, our aim is to estimate $||\left(T^s \partial_T^\kappa w_{n+\nu,\beta+\theta}\right)(T)||_{\beta;\sigma}$

$$\begin{aligned} ||\left(T^s \partial_T^\kappa w_{n+\nu,\beta+\theta}\right)(T)||_{\beta;\sigma} \leq{}& \left(\sup_{T\in\mathbb{C}} I_{a,\beta;\sigma_0}(T)(1+|T|)^{-m} \exp\left(-\sigma r_b(\beta)|T|^q\right)\right) \\ &\times ||w_{n+\nu,\beta+\theta}(T)||_{\beta+\theta;\sigma_0}. \end{aligned} \tag{8}$$

So we have to prove the following lemma.

Lemma 3 *There exists a contant $C_9 > 0$ such that*

$$\sup_{T\in\mathbb{C}} I_{a,\beta;\sigma_0}(T)(1+|T|)^{-m} \exp\left(-\sigma r_b(\beta)|T|^q\right) \leq C_9$$

for all $0 \leq \beta \leq d-\theta$.

Proof As in the proof of Lemma 2, we denote by $J_{a,\beta;\sigma_0,\sigma}$ the supremum to estimate

$$J_{a,\beta;\sigma_0,\sigma} := \sup_{T\in\mathbb{C}} I_{a,\beta;\sigma_0}(T)(1+|T|)^{-m} \exp\left(-\sigma r_b(\beta)|T|^q\right).$$

By definition of $I_{a,\beta;\sigma_0}(T)$, we have

$$J_{a,\beta;\sigma_0,\sigma} = \kappa! \sup_{T\in\mathbb{C}} \frac{|T|^s}{a^\kappa}\left(\frac{1+|T|+a}{1+|T|}\right)^m \exp\left(\sigma_0 r_b(\beta+\theta)(|T|+a)^q - \sigma r_b(\beta)|T|^q\right).$$

As in the proof of Lemma 2, as $|T| \in [0,+\infty[$ and a, $m \geq 0$,

$$\left(\frac{1+|T|+a}{1+|T|}\right)^m = \left(1+\frac{a}{1+|T|}\right)^m \leq (1+a)^m,$$

and we have

$$J_{a,\beta;\sigma_0,\sigma} \leq \kappa! \sup_{T\in\mathbb{C}} \frac{|T|^s}{a^\kappa}(1+a)^m \exp\left(\{\sigma_0 r_b(\beta+\theta) - \sigma r_b(\beta)\}(|T|+a)^q\right)$$

$$\times \exp\left(\sigma r_b(\beta)\{(|T|+a)^q - |T|^q\}\right)$$

using the equality $\gamma_1\gamma_2 - \gamma_3\gamma_4 = \gamma_1(\gamma_2-\gamma_3) + \gamma_3(\gamma_1-\gamma_4)$ in the exponential. Now, we will fix σ. It is necessary that $\sigma_0 \leq \sigma$. Therefore, we can show that for all $0 \leq \beta \leq d-\theta$

$$\sigma_0 r_b(\beta+\theta) - \sigma r_b(\beta) \leq \sigma_0 r_b(d) - \sigma r_b(0) = \sigma_0 r_b(d) - \sigma.$$

Let σ ($\sigma \geq \sigma_0$) such that

$$\sigma \geq \sigma_0 r_b(d) + \frac{1}{2},$$

then we have that

$$\sigma_0 r_b(\beta+\theta) - \sigma r_b(\beta) \leq -\frac{1}{2}$$

for all $0 \leq \beta \leq d-\theta$. It turns out that

$$J_{a,\beta;\sigma_0,\sigma} \leq \kappa! \sup_{T\in\mathbb{C}} \frac{|T|^s}{a^\kappa}(1+a)^m \exp\left(-\frac{1}{2}(|T|+a)^q\right) \exp\left(\sigma r_b(\beta)\{(|T|+a)^q - |T|^q\}\right).$$

With the same choice for the radius a as before

$$a = (|T|^q+1)^{1/q} - |T|,$$

we obtain that

$$J_{a,\beta;\sigma_0,\sigma} \leq \kappa! \exp\left(\sigma r_b(\beta)\right) \exp\left(-\frac{1}{2}\right) \sup_{T\in\mathbb{C}} |T|^s \frac{\left(1+(|T|^q+1)^{1/q}-|T|\right)^m}{\left((|T|^q+1)^{1/q}-|T|\right)^\kappa} \exp\left(-\frac{1}{2}|T|^q\right)$$

for all $0 \leq \beta \leq d-\theta$.
On the other hand, there exists a constant $c_1 > 0$, such that

$$\sup_{T\in\mathbb{C}} |T|^s \frac{\left(1+(|T|^q+1)^{1/q}-|T|\right)^m}{\left((|T|^q+1)^{1/q}-|T|\right)^\kappa} \exp\left(-\frac{1}{2}|T|^q\right) \leq c_1$$

and therefore, there exists a constant $c_2 > 0$, independent of β, such that

$$J_{a,\beta;\sigma_0,\sigma} \leq c_2$$

for all $0 \leq \beta \leq d - \theta$. This proves the lemma 3.

Now, we return to the proof of proposition 4. Using Lemma 3, we obtain for (8) that there exists a constant $c_1 > 0$, independent of β, such that

$$||(T^s \partial_T^\kappa w_{n+\nu,\beta+\theta})\,(T)||_{\beta;\sigma} \leq c_1\, ||w_{n+\nu,\beta+\theta}(T)||_{\beta+\theta;\sigma_0}$$

for all $n \geq 0$ and for all $0 \leq \beta \leq d - \theta$, and hence (7) becomes

$$\begin{aligned} &\left|\left|\left(T^s \partial_t^\nu \partial_T^\kappa \partial_z^\theta W\right)(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma} \\ &\qquad \leq c_1 \delta_1^{-\nu} \delta_2^{-\theta} \sum_{\beta=0}^{d-\theta} \sum_{n\geq 0} \frac{(n+\nu+\beta+\theta)!}{(n+\beta)!}\, ||w_{n+\nu,\beta+\theta}(T)||_{\beta+\theta;\sigma_0} \frac{\delta_1^{n+\nu}\delta_2^{\beta+\theta}}{(n+\nu+\beta+\theta)!}. \end{aligned}$$

Using the estimates (5)

$$\frac{(n+\nu+\beta+\theta)!}{(n+\beta)!} \sim (n+\beta)^{\nu+\theta}$$

as n tends to infinity and for $0 \leq \beta \leq d-\theta$, there exist two constants $c_2 \geq 1$, $c_3 > 0$, independent of n and β, such that

$$\frac{(n+\nu+\beta+\theta)!}{(n+\beta)!} \leq c_3 c_2^{n+\nu}$$

for all $n \geq 0$ and for all $0 \leq \beta \leq d - \theta$. We get that there exists a constant $c_4 > 0$ such that

$$\left|\left|\left(T^s \partial_t^\nu \partial_T^\kappa \partial_z^\theta W\right)(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma} \leq c_4 \delta_1^{-\nu} \delta_2^{-\theta} \sum_{\beta=0}^{d} \sum_{n\geq 0} ||w_{n,\beta}(T)||_{\beta;\sigma_0} \frac{(c_2\delta_1)^n\, \delta_2^\beta}{(n+\beta)!}.$$

Now, choosing $\delta_1 > 0$ such that $\delta_1 \leq \delta_{1,0}/c_2$, we conclude that

$$\left|\left|\left(T^s \partial_t^\nu \partial_T^\kappa \partial_z^\theta W\right)(t,z,T)\right|\right|_{\delta_1,\delta_2;\sigma} \leq c_4 \delta_1^{-\nu} \delta_2^{-\theta}\, ||W(t,z,T)||_{\delta_{1,0},\delta_2;\sigma_0}$$

for all $W(t,z,T) \in \mathcal{G}_q(\delta_{1,0}, \delta_2; \sigma_0)$. This proves the proposition 4.

2.3 A Cauchy problem in the weighted Banach spaces.

In the next two definitions, we introduce some linear operators.

Definition 2 *Let $\mathcal{M}$ the linear operator from $\mathcal{A}(\mathbb{C})\{t,z\}$ into $\mathcal{A}(\mathbb{C})\{t,z\}$ defined as*

$$\mathcal{M}\left(U(t,z,T)\right) = \partial_z^S U(t,z,T) - \sum_{\mathbf{q}=(q_0,q_1,q_2)\in\mathcal{Q}} B_{\mathbf{q}}(t,z,T) \partial_t^{q_0} \partial_T^{q_1} \partial_z^{q_2} U(t,z,T) \tag{9}$$

where $\mathcal{Q}$, S and $B_{\mathbf{q}}(t,z,T)$ satisfy the following assumptions.

(A1) S is a positive integer, $\mathcal{Q}$ is a finite subset of $\mathbb{N}^3$ and we denote by

$$\mathbf{q} = (q_0, q_1, q_2) \in \mathcal{Q}.$$

(A2) For all $\mathbf{q} \in \mathcal{Q}$, $B_{\mathbf{q}}(t,z,T)$ is polynomial in T

$$B_{\mathbf{q}}(t,z,T) = \sum_{j=0}^{\sigma(\mathbf{q})} b_j^{\mathbf{q}}(t,z) T^j$$

where $\sigma(\mathbf{q})$ is the degree of $B_{\mathbf{q}}(t,z,T)$ with respect to the variable T and the coefficients $b_j^{\mathbf{q}}(t,z)$ are holomorphic functions near the origin in $\mathbb{C}^2$.

(A3) For all $\mathbf{q} = (q_0, q_1, q_2) \in \mathcal{Q}$,

$$\frac{b\left(\sigma(\mathbf{q}) + q_1(q-1)\right)}{q} + q_0 + q_2 < S$$

where $q \geq 1$ and $b > 1$ are positive numbers defined from our Banach space $\mathcal{G}_q\left(\delta_1, \delta_2; \sigma\right)$.

Definition 3 *Let $\mathcal{N}$ be the linear operator from $\mathcal{A}(\mathbb{C})\{t,z\}$ into $\mathcal{A}(\mathbb{C})\{t,z\}$ defined as*

$$\mathcal{N}\left(U(t,z,T)\right) = \sum_{\mathbf{q}\in\mathcal{Q}} B_{\mathbf{q}}(t,z,T) \partial_t^{q_0} \partial_T^{q_1} \partial_z^{q_2 - S} U(t,z,T) \tag{10}$$

where $\mathcal{Q}$, S and $B_{\mathbf{q}}(t,z,T)$ satisfy the assumptions (**A1**), (**A2**) *and* (**A3**).

Using the definitions (9) and (10), we obtain the following relation between the operators $\mathcal{M}$ and $\mathcal{N}$

$$\mathcal{M} \circ \partial_z^{-S} = \mathrm{Id} - \mathcal{N} \tag{11}$$

where Id is the identity operator from $\mathcal{A}(\mathbb{C})\{t,z\}$ into itself.

In the next proposition, we show that $\mathcal{N}$ is a linear bounded operator on the constructed Banach spaces with small norm.

Proposition 5 *Let $\mathcal{N}$ be the operator defined by (10) and which satisfies the assumptions* (**A1**), (**A2**) *and* (**A3**). *Then, there exists a real number $\delta_2 > 0$, small enough (which depends on the coefficients $B_{\mathbf{q}}(t,z,T)$ for $\mathbf{q} = (q_0,q_1,q_2) \in \mathcal{Q}$), such that $\mathcal{N}$ is a bounded linear operator from $\left(\mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right), ||.||_{\delta_1,\delta_2;\sigma}\right)$ into itself. Moreover, there exists a constant C_{10}, $0 < C_{10} < 1$, such that*

$$||\mathcal{N}\left(U(t,z,T)\right)||_{\delta_1,\delta_2;\sigma} \leq C_{10} \, ||U(t,z,T)||_{\delta_1,\delta_2;\sigma}$$

for all $U(t,z,T) \in \mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right)$.

Proof By definition (10) of the operator $\mathcal{N}$ and the assumption (**A2**), we get

$$||\mathcal{N}\left(U(t,z,T)\right)||_{\delta_1,\delta_2;\sigma} \leq \sum_{\mathbf{q}\in\mathcal{Q}} \sum_{j=0}^{\sigma(\mathbf{q})} \left|\left| b_j^{\mathbf{q}}(t,z) T^j \partial_t^{q_0} \partial_T^{q_1} \partial_z^{q_2-S} U(t,z,T) \right|\right|_{\delta_1,\delta_2;\sigma}$$

for all $U(t,z,T) \in \mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right)$. From the fact that $b_j^{\mathbf{q}}(t,z)$, for $j = 0, \ldots, \sigma(\mathbf{q})$, are holomorphic functions near the origin in $\mathbb{C}^2$, we deduce from Proposition 3 that there exists a constant $c_1 > 0$, such that

$$||\mathcal{N}\left(U(t,z,T)\right)||_{\delta_1,\delta_2;\sigma} \leq c_1 \sum_{\mathbf{q}\in\mathcal{Q}} \sum_{j=0}^{\sigma(\mathbf{q})} \left| b_j^{\mathbf{q}} \right| (\delta_1,\delta_2) \left|\left| T^j \partial_t^{q_0} \partial_T^{q_1} \partial_z^{q_2-S} U(t,z,T) \right|\right|_{\delta_1,\delta_2;\sigma}$$

for all $U(t,z,T) \in \mathcal{G}_q(\delta_1,\delta_2;\sigma)$. Under the assumption (**A3**), we can apply proposition 2 and we obtain the existence of a constant $c_2 > 0$, such that

$$||\mathcal{N}(U(t,z,T))||_{\delta_1,\delta_2;\sigma} \leq c_2 \left(\sum_{\mathbf{q}\in\mathcal{Q}} \left(\sum_{j=0}^{\sigma(\mathbf{q})} \left| b_j^{\mathbf{q}} \right| (\delta_1,\delta_2) \right) \delta_1^{-q_0} \delta_2^{S-q_2} \right) ||U(t,z,T)||_{\delta_1,\delta_2;\sigma}$$

for all $U(t,z,T) \in \mathcal{G}_q(\delta_1,\delta_2;\sigma)$. Now, with $\delta_2 > 0$ small enough, we have

$$0 < c_2 \sum_{\mathbf{q}\in\mathcal{Q}} \left(\sum_{j=0}^{\sigma(\mathbf{q})} \left| b_j^{\mathbf{q}} \right| (\delta_1,\delta_2) \right) \delta_1^{-q_0} \delta_2^{S-q_2} =: C_6 < 1.$$

This proves the proposition. As a consequence of the proposition 5, we deduce the invertibility of the operator $\mathcal{M} \circ \partial_z^{-S}$.

Proposition 6 *Let $\mathcal{M}$ the operator defined by (9) and which satisfies the assumptions* (**A1**), (**A2**) *and* (**A3**). *Then, there exists a real number $\delta_2 > 0$, small enough (which depends on the coefficients $B_{\mathbf{q}}(t,z,T)$ for $\mathbf{q} = (q_0,q_1,q_2) \in \mathcal{Q}$), such that, $\mathcal{M} \circ \partial_z^{-S}$ is a bounded invertible operator from $\left(\mathcal{G}_q(\delta_1,\delta_2;\sigma), ||.||_{\delta_1,\delta_2;\sigma}\right)$ into itself. Moreover, there exists a constant $C_{11} > 0$, such that*

$$\left|\left| \left(\mathcal{M} \circ \partial_z^{-S}\right)^{-1} (V(t,z,T)) \right|\right|_{\delta_1,\delta_2;\sigma} \leq C_{11} ||V(t,z,T)||_{\delta_1,\delta_2;\sigma}$$

for $V(t,z,T) \in \mathcal{G}_q(\delta_1,\delta_2;\sigma)$.

Now, we are able to prove the main result of this section.

Theorem 1 *Consider a partial differential equation of the form*

$$\partial_z^S U(t,z,T) = \sum_{\mathbf{q}\in\mathcal{Q}} B_{\mathbf{q}}(t,z,T) \partial_t^{q_0} \partial_T^{q_1} \partial_z^{q_2} U(t,z,T) \tag{12}$$

which satisfies the assumptions (**A1**), (**A2**) *and* (**A3**). *We impose the following initial conditions. For all $0 \leq j \leq S-1$,*

$$\left(\partial_z^j U\right)(t,0,T) = \psi_j(t,T) \tag{13}$$

where the $\psi_j(t,T)$ belong to $\mathcal{G}_q(\delta_{1,0},\delta_{2,0};\sigma_0)$, for given $\delta_{1,0}$, $\delta_{2,0}$ and $\sigma_0 > 0$.

Then, there exists δ_1, δ_2 and $\sigma > 0$, such that the problem (12) and (13) has a unique solution $U(t,z,T) \in \mathcal{G}_q(\delta_1,\delta_2;\sigma)$.

Proof A formal series $U(t,z,T) \in \mathcal{A}(\mathbb{C})[[t,z]]$ which satisfies the equation (13) can be written in the form

$$U(t,z,T) = \partial_z^{-S} V(t,z,T) + W(t,z,T)$$

where

$$W(t,z,T) := \sum_{j=0}^{S-1} \psi_j(t,T) \frac{z^j}{j!}.$$

From the fact that the initial conditions (13) belong to $\mathcal{G}_q(\delta_{1,0},\delta_{2,0};\sigma_0)$ and Proposition 3, we get that $W(t,z,T)$ belongs to $\mathcal{G}_q(\delta_{1,0},\delta_{2,0};\sigma_0)$. Moreover, $U(t,z,T)$ is a solution of (12) and (13) if and only if $V(t,z,T)$ satisfies the equation

$$\left(\mathcal{M} \circ \partial_z^{-S}\right)(V(t,z,T)) = -\mathcal{M}(W(t,z,T)). \tag{14}$$

From the fact that $W(t,z,T)$ is a polynomial in the variable z, and that $W(t,z,T) \in \mathcal{G}_q(\delta_{1,0},\delta_{2,0};\sigma_0)$, using Proposition 3 and Proposition 4, we get that there exist δ_1 and $\sigma > 0$ such that

$$\mathcal{M}(W(t,z,T)) \in \mathcal{G}_q(\delta_1,\delta_{2,0};\sigma).$$

Therefore, by Proposition 6, for $\delta_2 > 0$ small enough, there exists a unique solution $V(t,z,T)$ of (14) which belongs to $\mathcal{G}_q(\delta_1,\delta_2;\sigma)$. To conclude, it remains to show that $U(t,z,T) \in \mathcal{G}_q(\delta_1,\delta_2;\sigma)$. Indeed, we have $U(t,z,T) = \partial_z^{-S}V(t,z,T) + W(t,z,T)$. Using Proposition 2, we get that $\partial_z^{-S}V(t,z,T) \in \mathcal{G}_q(\delta_1,\delta_2;\sigma)$. Moreover, we have also $W(t,z,T) \in \mathcal{G}_q(\delta_1,\delta_2;\sigma)$. In conclusion, $U(t,z,T)$ belongs to $\mathcal{G}_q(\delta_1,\delta_2;\sigma)$. This proves our theorem.

3 Formal solutions of a linear Cauchy problem.

Let us consider $\phi(t)$ a holomorphic solution of the ordinary differential equation

$$\phi'(t) = P(t,\phi(t)) \tag{15}$$

where $P(t,X)$ is a holomorphic function on some domain $\mathcal{D} \subset \mathbb{C}$ with respect to t and polynomial in the variable X of degree larger than 2. Consider the following linear partial differential equation

$$\partial_z^S X(t,z) = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}} a_{\mathbf{k}}(t,z)\partial_t^{k_0}\partial_z^{k_1}X(t,z) \tag{16}$$

where S is a positive integer and $\mathcal{J}$ is a finite set which satisfies

$$\mathcal{J} \subset \left\{\mathbf{k} = (k_0,k_1) \in \mathbb{N}^2 \mid k_1 \le S-1\right\}. \tag{17}$$

The coefficients $a_{\mathbf{k}}(t,z)$ are holomorphic functions on the domain $\mathcal{D}$ with respect to t and near the origin in $\mathbb{C}$ with respect to z. Our aim is to look for *transseries* solution $X(t,z)$ of (16) in powers of $\phi(t)$ and z

$$\hat{X}(t,z) = \sum_{\beta\ge 0}\sum_{\ell\ge 0} X_{\ell,\beta}(t)\frac{\phi(t)^\ell}{\ell!}\frac{z^\beta}{\beta!} \tag{18}$$

where $X_{\ell,\beta}(t)$ are holomorphic functions on the domain $\mathcal{D} \subset \mathbb{C}$. To fix the notations, for all integers $n \ge 1$, for all vectors $\mathbf{p} = (p_1,\ldots,p_n) \in \mathbb{N}^n$, we define

$$|\mathbf{p}|_1 := p_1 + p_2 + \ldots + p_n \quad \text{and} \quad |\mathbf{p}|_2 := p_1 + 2p_2 + \ldots + np_n.$$

We define the set $\mathcal{P}_2(n)$ as

$$\mathcal{P}_2(n) := \{\mathbf{p} = (p_1,\ldots,p_n) \in \mathbb{N}^n \mid |\mathbf{p}|_2 = n\}.$$

By convention, we put $\mathcal{P}_2(0) = \{0\}$.

Remark: If $\mathbf{p} \in \mathcal{P}_2(n)$, then $|\mathbf{p}|_1 \le n$. Indeed,

$$n - |\mathbf{p}|_1 = p_1 + 2p_2 + \ldots + np_n - (p_1 + p_2 + \ldots + p_n) = p_2 + 2p_3 + \ldots + (n-1)p_n \ge 0.$$

Proposition 7 *Let $X_{\ell,\beta}$, $\ell \geq 0$, $0 \leq \beta \leq S-1$, be given holomorphic functions on $\mathcal{D}$. Then the partial differential equation (16) has a formal solution $\hat{X}(t,z)$ of the form (18) for given initial conditions*

$$(\partial_z^j \hat{X})(t,0) = \sum_{\ell \geq 0} X_{\ell,j}(t) \frac{\phi(t)^\ell}{\ell!} \quad , \quad 0 \leq j \leq S-1. \tag{19}$$

Proof Our goal is to construct a sequence of holomorphic functions $X_{\ell,\beta}(t)$, $\ell, \beta \geq 0$, such that the formal series $\hat{X}(t,z)$ is a solution of the problem (16), (19).

In a first step, we plug $\hat{X}(t,z)$ into the equation (16) using the formal expansion (18) and we deduce a recursion for the coefficients $X_{\ell,\beta}(t)$ by identification of the powers of $\phi(t)$ and z. For the left-hand side term of (16), we obtain

$$\partial_z^S \hat{X}(t,z) = \sum_{\beta \geq 0} \sum_{\ell \geq 0} X_{\ell,\beta+S}(t) \frac{\phi(t)^\ell}{\ell!} \frac{z^\beta}{\beta!}.$$

For the right-hand side term, it will be more complicated. A first calculus shows us that

$$\partial_t^{k_0} \partial_z^{k_1} \hat{X}(t,z) = \sum_{\beta \geq 0} \sum_{\ell \geq 0} \partial_t^{k_0} \left\{ X_{\ell,\beta+k_1}(t) \frac{\phi(t)^\ell}{\ell!} \right\} \frac{z^\beta}{\beta!}. \tag{20}$$

Using Leibniz formula for

$$D_{\ell,\beta}^{\mathbf{k}}(t,\phi(t)) := \partial_t^{k_0} \left\{ X_{\ell,\beta+k_1}(t) \frac{\phi(t)^\ell}{\ell!} \right\},$$

we have

$$D_{\ell,\beta}^{\mathbf{k}}(t,\phi(t)) = \sum_{m=0}^{k_0} \frac{k_0!}{m!(k_0-m)!} \left(\partial_t^{k_0-m} X_{\ell,\beta+k_1}(t) \right) \left(\partial_t^m \frac{\phi(t)^\ell}{\ell!} \right).$$

To expand $\partial_t^m \frac{\phi(t)^\ell}{\ell!}$, we use Faà di Bruno formula

$$\partial_t^m \frac{\phi(t)^\ell}{\ell!} = \sum_{\substack{\mathbf{p} \in \mathcal{P}_2(m) \\ |\mathbf{p}|_1 \leq \ell}} \frac{m!}{p_1! \dots p_m!} \frac{\phi(t)^{\ell-|\mathbf{p}|_1}}{(\ell-|\mathbf{p}|_1)!} \prod_{k=1}^m \left(\frac{\phi^{(k)}(t)}{k!} \right)^{p_k}. \tag{21}$$

We introduce the notation

$$\Omega_\beta^{\mathbf{k}}(t,\phi(t)) := \sum_{\ell \geq 0} D_{\ell,\beta}^{\mathbf{k}}(t,\phi(t))$$

$$= \sum_{\ell \geq 0} \sum_{m=0}^{k_0} \frac{k_0!}{m!(k_0-m)!} \left(\partial_t^{k_0-m} X_{\ell,\beta+k_1}(t) \right) \sum_{\substack{\mathbf{p} \in \mathcal{P}_2(m_2) \\ |\mathbf{p}|_1 \leq \ell}} \frac{m!}{p_1! \dots p_m!} \frac{\phi(t)^{\ell-|\mathbf{p}|_1}}{(\ell-|\mathbf{p}|_1)!} \prod_{k=1}^m \left(\frac{\phi^{(k)}(t)}{k!} \right)^{p_k}.$$

Therefore, the equation (16) yields

$$\sum_{\beta \geq 0} \sum_{\ell \geq 0} X_{\ell,\beta+S}(t) \frac{\phi(t)^\ell}{\ell!} \frac{z^\beta}{\beta!} = \sum_{\mathbf{k}=(k_0,k_1) \in \mathcal{J}} a_{\mathbf{k}}(t,z) \sum_{\beta \geq 0} \Omega_\beta^{\mathbf{k}}(t,\phi(t)) \frac{z^\beta}{\beta!}. \tag{22}$$

Using for $a_{\mathbf{k}}(t,z)$ the Taylor expansion

$$a_{\mathbf{k}}(t,z)=\sum_{\beta\geq 0} a_\beta^{\mathbf{k}}(t)\frac{z^\beta}{\beta!},$$

where the coefficients $a_\beta^{\mathbf{k}}(t)$ are holomorphic functions on $\mathcal{D}$, we obtain from (22)

$$\sum_{\beta\geq 0}\left(\sum_{\ell\geq 0} X_{\ell,\beta+S}(t)\frac{\phi(t)^\ell}{\ell!}\right)\frac{z^\beta}{\beta!}$$

$$=\sum_{\beta\geq 0}\left(\sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}}\sum_{\beta_2=0}^{\beta}\frac{\beta!}{\beta_2!(\beta-\beta_2)!}a_{\beta-\beta_2}^{\mathbf{k}}(t)\Omega_{\beta_2}^{\mathbf{k}}\left(t,\phi(t)\right)\right)\frac{z^\beta}{\beta!}.$$

By identification of the coefficients of $\dfrac{z^\beta}{\beta!}$, we get

$$\sum_{\ell\geq 0} X_{\ell,\beta+S}(t)\frac{\phi(t)^\ell}{\ell!}=\sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}}\sum_{\beta_2=0}^{\beta}\frac{\beta!}{\beta_2!(\beta-\beta_2)!}a_{\beta-\beta_2}^{\mathbf{k}}(t)\Omega_{\beta_2}^{\mathbf{k}}\left(t,\phi(t)\right)\tag{23}$$

for all $\beta\in\mathbb{N}$. On the right-hand side, the term $\Omega_{\beta_2}^{\mathbf{k}}\left(t,\phi(t)\right)$ has the form

$$\Omega_{\beta_2}^{\mathbf{k}}\left(t,\phi(t)\right)=\sum_{\ell\geq 0}\sum_{m=0}^{k_0}\left(\partial_t^{k_0-m}X_{\ell,\beta_2+k_1}(t)\right)\omega_\ell^{\mathbf{k},m}\left(t,\phi(t)\right)\tag{24}$$

where

$$\omega_\ell^{\mathbf{k},m}\left(t,\phi(t)\right)=\frac{k_0!}{(k_0-m)!}\sum_{\substack{\mathbf{p}\,\in\,\mathcal{P}_2(m)\\ |\mathbf{p}|_1\,\leq\,\ell}}\frac{1}{p_1!\ldots p_m!}\frac{\phi(t)^{\ell-|\mathbf{p}|_1}}{(\ell-|\mathbf{p}|_1)!}\prod_{k=1}^{m}\left(\frac{\phi^{(k)}(t)}{k!}\right)^{p_k},$$

contains powers of $\phi(t)$ and powers of $\phi^{(k)}(t)$ for $k=1,\ldots,m$.

In the next two lemmas, we are going to specify the form of $\omega_\ell^{\mathbf{k},m}\left(t,\phi(t)\right)$.

Lemma 4 *Let us consider the ordinary differential equation (15) with*

$$P(t,X)=\sum_{j=0}^{d} p_j(t)\frac{X^j}{j!},\qquad d\geq 2$$

where the coefficients $p_j(t)$ are holomorphic on some domain $\mathcal{D}\subset\mathbb{C}$ and d is the degree of $P(t,X)$ with respect to the variable X (in particular $p_d(t)\not\equiv 0$). Then, for all positive integer n, there exists a function $Q_n(t,X)$ such that

$$\phi^{(n)}(t)=Q_n\left(t,\phi(t)\right)\tag{25}$$

with

$$Q_n(t,X)=\sum_{j=0}^{d_n} q_{n,j}(t)\frac{X^j}{j!}$$

where the coefficients $q_{n,j}(t)$ are holomorphic on $\mathcal{D}$. Moreover,

$$d_n := \deg_X Q_n(t,X) = n(d-1)+1$$

is the degree of $Q_n(t,X)$ with respect to the variable X and

$$q_{n,d_n}(t) = (p_d(t))^n \prod_{j=1}^{n} \frac{d_j!}{d!(d_j-d)!}.$$

Proof We prove this lemma by induction on n. The assertions are true for $n=1$, with $Q_1(t,X) = P(t,X)$, therefore

$$d_1 = d = (d-1)+1 \qquad \text{and} \qquad q_{1,d_1}(t) = p_d(t) = p_d(t) \prod_{j=1}^{1} \frac{d_1!}{d!(d_1-d)!}.$$

We suppose, that the assertions are true until $n \geq 1$. Differentating (25) gives

$$\phi^{(n+1)}(t) = \sum_{j=0}^{d_n} q'_{n,j}(t) \frac{\phi(t)^j}{j!} + \phi'(t) \sum_{j=0}^{d_n-1} q_{n,j+1}(t) \frac{\phi(t)^j}{j!}. \tag{26}$$

By (15), the higest power in (26) of $\phi(t)$ is in the second sum, it is

$$p_d(t) q_{n,d_n}(t) \frac{\phi(t)^{d-1+d_n}}{d!(d_n-1)!} = \frac{(d-1+d_n)!}{d!(d_n-1)!} p_d(t) q_{n,d_n}(t) \frac{\phi(t)^{d-1+d_n}}{(d-1+d_n)!}.$$

Therefore, the right-hand side of (26) can be written as

$$Q_{n+1}\left(t, \phi(t)\right) := \sum_{j=0}^{d_{n+1}} q_{n+1,j}(t) \frac{\phi(t)^j}{j!}$$

where the coefficients $q_{n+1,j}(t)$ are holomorphic on $\mathcal{D}$ which are sums and products of the $p_j(t)$, $q_{n,j}(t)$ and $q'_{n,j}(t)$. Moreover,

$$d_{n+1} := \deg_X Q_{n+1}(t,X) = d-1+d_n = (n+1)(d-1)+1$$

and

$$q_{n+1,d_{n+1}}(t) = \frac{d_{n+1}!}{d!(d_{n+1}-d)!} p_d(t) q_{n,d_n}(t) = (p_d(t))^{n+1} \prod_{j=1}^{n+1} \frac{d_j!}{d!(d_j-d)!}.$$

This proves our lemma.

Lemma 5 *Let $m \in \mathbb{N}$ with $m \geq 1$ and $\mathbf{p} \in \mathcal{P}_2(m)$. Under the assumptions of Lemma 4, there exist a positive integer D_m and holomorphic coefficients $Q_j^m(t)$ on $\mathcal{D}$, such that*

$$\prod_{k=1}^{m} \left(\frac{\phi^{(k)}(t)}{k!} \right)^{p_k} = \sum_{j=0}^{D_m} Q_j^m(t) \frac{\phi(t)^j}{j!}$$

where

$$D_m = m(d-1) + |\mathbf{p}|_1$$

and

$$Q_{D_m}^m(t) = D_m! \, (p_d(t))^m \left(\prod_{k=1}^{m} \frac{\prod_{j=1}^{k} \frac{d_j!}{d!(d_j-d)!}}{k! d_k!} \right)^{p_k}.$$

Proof Using Lemma 4, we have

$$\prod_{k=1}^{m}\left(\frac{\phi^{(k)}(t)}{k!}\right)^{p_k}=\prod_{k=1}^{m}\left(\frac{1}{k!}\sum_{j=0}^{d_k}q_{k,j}(t)\frac{\phi(t)^j}{j!}\right)^{p_k}.$$

In the right handside, the higest power of $\phi(t)$ is

$$D_m:=\sum_{k=1}^{m}d_kp_k=\sum_{k=1}^{m}\left(k(d-1)+1\right)p_k=(d-1)\sum_{k=1}^{m}kp_k+\sum_{k=1}^{m}p_k=m(d-1)+|\mathbf{p}|_1$$

because $\mathbf{p}\in\mathcal{P}_2(m)$. Also, the coefficient $Q^m_{D_m}(t)$ of $\dfrac{\phi(t)^{D_m}}{D_m!}$ is given by

$$Q^m_{D_m}(t)=D_m!\prod_{k=1}^{m}\left(\frac{q_{k,d_k}(t)}{k!d_k!}\right)^{p_k}.$$

By Lemma 4, we get

$$Q^m_{D_m}(t)=D_m!\,(p_d(t))^m\prod_{k=1}^{m}\left(\frac{\prod_{j=1}^{k}\frac{d_j!}{d!(d_j-d)!}}{k!d_k!}\right)^{p_k}.$$

This proves our lemma.

We return to the proof of our proposition 7. By Lemma 5, $\omega_\ell^{\mathbf{k},m}\left(t,\phi(t)\right)$ has the form

$$\omega_\ell^{\mathbf{k},m}\left(t,\phi(t)\right)=\frac{k_0!}{(k_0-m)!}\sum_{\substack{\mathbf{p}\,\in\,\mathcal{P}_2(m)\\|\mathbf{p}|_1\,\le\,\ell}}\frac{1}{p_1!\dots p_m!}\sum_{j=0}^{D_m}Q_j^m(t)\frac{\phi(t)^{\ell+j-|\mathbf{p}|_1}}{(\ell-|\mathbf{p}|_1)!j!}.$$

For fixed ℓ, $\omega_\ell^{\mathbf{k},m}\left(t,\phi(t)\right)$ is a polynomial in $\phi(t)$ with powers of $\phi(t)$ between $\ell-m$ and $\ell+D_m-|\mathbf{p}|_1=\ell+m(d-1)$. To avoid confusion for the identification of the coefficients of $\dfrac{\phi(t)^\ell}{\ell!}$, we will replace ℓ by L in $\omega_\ell^{\mathbf{k},m}\left(t,\phi(t)\right)$. Therefore (24) has the form

$$\Omega^{\mathbf{k}}_{\beta_2}\left(t,\phi(t)\right)=\sum_{L\ge0}\sum_{m=0}^{k_0}\left(\partial_t^{k_0-m}X_{L,\beta_2+k_1}(t)\right)\omega_L^{\mathbf{k},m}\left(t,\phi(t)\right)$$

and we obtain for equation (23)

$$(27)\quad\sum_{\ell\ge0}X_{\ell,\beta+S}(t)\frac{\phi(t)^\ell}{\ell!}=\sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}}\sum_{\beta_2=0}^{\beta}\frac{\beta!}{\beta_2!(\beta-\beta_2)!}a^{\mathbf{k}}_{\beta-\beta_2}(t)\sum_{L\ge0}\sum_{m=0}^{k_0}\left(\partial_t^{k_0-m}X_{L,\beta_2+k_1}(t)\right)$$

$$\times\frac{k_0!}{(k_0-m)!}\sum_{\substack{\mathbf{p}\,\in\,\mathcal{P}_2(m)\\|\mathbf{p}|_1\,\le\,L}}\frac{1}{p_1!\dots p_m!}\sum_{j=0}^{D_m}Q_j^m(t)\frac{(L+j-|\mathbf{p}|_1)!}{(L-|\mathbf{p}|_1)!j!}\frac{\phi(t)^{L+j-|\mathbf{p}|_1}}{(L+j-|\mathbf{p}|_1)!}$$

for all $\beta \in \mathbb{N}$.

By identification of the coefficients of $\dfrac{\phi(t)^\ell}{\ell!}$, we get the following recursion for the coefficients $X_{\ell,\beta}(t)$,

$$X_{\ell,\beta+S}(t) = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}} \sum_{\beta_2=0}^{\beta} \frac{\beta!}{\beta_2!(\beta-\beta_2)!} a^{\mathbf{k}}_{\beta-\beta_2}(t) \sum_{m=0}^{k_0} \frac{k_0!}{(k_0-m)!} \tag{28}$$
$$\times \sum_{(L,\mathbf{p},j)\in\Delta_{\ell,m}} \left(\partial_t^{k_0-m} X_{L,\beta_2+k_1}(t)\right) \frac{1}{p_1!\dots p_m!} Q_j^m(t) \frac{\ell!}{(L-|\mathbf{p}|_1)!j!}$$

where

$$\Delta_{\ell,m} = \{(L,\mathbf{p},j) \in \mathbb{N}\times\mathbb{N}^m\times\mathbb{N} \mid \mathbf{p}\in\mathcal{P}_2(m),\ j=0,\dots,D_m,\ L+j-|\mathbf{p}|_1=\ell\}$$

for all $(\ell,\beta)\in\mathbb{N}^2$.

In a second step of the proof, we consider now given holomorphic functions $X_{\ell,\beta}(t)$ on $\mathcal{D}$, for all $\ell\geq 0$, for $\beta=0,\dots,S-1$. From (17), we deduce that there exists a unique sequence $(X_{\ell,\beta}(t))_{\ell\geq 0,\beta\geq 0}$ of holomorphic functions on $\mathcal{D}$ which satisfies the recurrence (28) for the given initial holomorphic functions $X_{\ell,\beta}(t)$, for all $\ell\geq 0$, for $\beta=0,\dots,S-1$. From the first part of the proof, we deduce that the formal transseries

$$\hat{X}(t,z) = \sum_{\ell\geq 0,\beta\geq 0} X_{\ell,\beta}(t)\frac{\phi(t)^\ell}{\ell!}\frac{z^\beta}{\beta!}$$

is a solution of the Cauchy problem (16), (19). This ends the proof of Proposition 7.

4 Holomorphic solutions of the linear Cauchy problem.

In this section, we will give sufficient conditions under which the formal transasymptotic series solutions of the Cauchy problem (16), (19) constructed in the previous section are convergent in some punctured polydiscs of $\mathbb{C}^2$.

4.1 Classification of singularities for first order ordinary differential equations

Consider our first order non linear differential equation (2)

$$\phi'(t) = P\left(t,\phi(t)\right),$$

where $P(t,X)$ is polynomial in X of degree larger than 2, with holomorphic coefficients in $\mathcal{D}$. As a consequence of a result of P. Painlevé (see [9], Theorem 3.3.2), we know that the only movable singularities in $\mathbb{C}$ of the solutions $\phi(t)$ of (2) are poles and/or algebraic branch points.

In the following, we define $D_\theta(t,r)$ as an open disc $D(t,r)$ centered at $t\in\mathbb{C}$ with radius $r>0$, minus the segment $\left[t,re^{i\theta}\right)$, for $\theta\in\mathbb{R}$. Let $\phi(t)$ a solution of (2) defined on $D_\theta(t_0,r_0)\subset\mathcal{D}$, with $r_0>0$, and $\theta\in\mathbb{R}$, where $\phi(t)$ can be represented by a Puiseux series of the form

$$\phi(t) = \sum_{n\geq -n_0} f_n(t-t_0)^{n/\mu} \tag{29}$$

where $\mu\geq 1$ and $n_0\geq 1$ are positive integers such that $f_{-n_0}\neq 0$. If μ is equal to 1, the point t_0 is called a pole of order n_0 and then the function $\phi(t)$ is holomorphic on $\mathcal{D}(t_0,r)\setminus\{t_0\}$, otherwise the point t_0 is called an algebraic branch point.

4.2 An auxiliary linear Cauchy problem.

Let

$$v_{n_0,\ell,\beta} := \sup_{t\in\mathcal{D}} |\partial_t^{n_0} X_{\ell,\beta}(t)|$$

for all n_0, ℓ, $\beta \geq 0$, where $(X_{\ell,\beta}(t))_{\ell\geq 0,\beta\geq 0}$ is the sequence constructed in Proposition 7. In this subsection, we will construct a majorazing sequence $u_{n_0,\ell,\beta}$ of $v_{n_0,\ell,\beta}$ such that the formal series

$$U(t,z,T) = \sum_{\beta\geq 0}\sum_{\ell\geq 0}\sum_{n_0\geq 0} u_{n_0,\ell,\beta}\frac{t^{n_0}}{n_0!}\frac{T^\ell}{\ell!}\frac{z^\beta}{\beta!}$$

satisfies a linear Cauchy problem.

We recall the recursion formula (28) for the coefficients $X_{\ell,\beta}(t)$

$$X_{\ell,\beta+S}(t) = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}}\sum_{\beta_2=0}^{\beta}\frac{\beta!}{\beta_2!(\beta-\beta_2)!}\sum_{m=0}^{k_0}\frac{k_0!}{(k_0-m)!} \times \sum_{(L,\mathbf{p},j)\in\Delta_{\ell,m}}\frac{1}{p_1!\dots p_m!}\frac{\ell!}{j!(\ell-j)!}A^{\mathbf{k},m}_{j,\beta-\beta_2}(t)\left(\partial_t^{k_0-m}X_{L,\beta_2+k_1}(t)\right) \tag{30}$$

where $A^{\mathbf{k},m}_{j,\beta-\beta_2}(t) := a^{\mathbf{k}}_{\beta-\beta_2}(t)Q^m_j(t)$ (with $a^{\mathbf{k}}_\beta(t)$ the coefficient of $z^\beta/\beta!$ in the expansion of $a_{\mathbf{k}}(t,z)$ with respect to z and $Q^m_j(t)$ are defined in Lemma 5 and

$$\Delta_{\ell,m} = \{(L,\mathbf{p},j)\in\mathbb{N}\times\mathbb{N}^m\times\mathbb{N}\,|\, j=0,\dots,D_m;\ \mathbf{p}\in\mathcal{P}_2(m);\ L+j-|\mathbf{p}|_1=\ell\}$$

for all ℓ, $\beta\geq 0$. From the Leibniz formula, if we differentiate n_0 times equation (30), we have

$$\partial_t^{n_0}X_{\ell,\beta+S}(t) = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}}\sum_{\beta_2=0}^{\beta}\sum_{m=0}^{k_0}\sum_{(L,\mathbf{p},j)\in\Delta_{\ell,m}}\frac{\beta!}{\beta_2!(\beta-\beta_2)!}\frac{k_0!}{(k_0-m)!}\frac{1}{p_1!\dots p_m!} \times\frac{\ell!}{j!(\ell-j)!}\sum_{n_1=0}^{n_0}\frac{n_0!}{n_1!(n_0-n_1)!}\left(\partial_t^{n_0-n_1}A^{\mathbf{k},m}_{j,\beta-\beta_2}(t)\right)\left(\partial_t^{k_0-m+n_1}X_{L,\beta_2+k_1}(t)\right) \tag{31}$$

for all n_0, ℓ, $\beta\geq 0$. Therefore, we get the following inequalities for the $v_{n_0,\ell,\beta}$

$$v_{n_0,\ell,\beta+S} = \sup_{t\in\mathcal{D}}|\partial_t^{n_0}X_{\ell,\beta+S}(t)| \leq \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}}\sum_{\beta_2=0}^{\beta}\sum_{m=0}^{k_0}\sum_{(L,\mathbf{p},j)\in\Delta_{\ell,m}}\frac{\beta!}{\beta_2!(\beta-\beta_2)!}\frac{k_0!}{(k_0-m)!} \times\frac{1}{p_1!\dots p_m!}\frac{\ell!}{j!(\ell-j)!}\sum_{n_1=0}^{n_0}\frac{n_0!}{n_1!(n_0-n_1)!}\sup_{t\in\mathcal{D}}\left|\partial_t^{n_0-n_1}A^{\mathbf{k},m}_{j,\beta-\beta_2}(t)\right|\sup_{t\in\mathcal{D}}\left|\partial_t^{k_0-m+n_1}X_{L,\beta_2+k_1}(t)\right| \tag{32}$$

for all n_0, ℓ, $\beta\geq 0$. As $A^{\mathbf{k},m}_{j,\beta}(t)$ is holomorphic on $\mathcal{D}$, using Cauchy estimates, there exist two constants $M_{j,\beta}$ and r (r depends only on $\mathcal{D}$) such that

$$\sup_{t\in\mathcal{D}}\left|\partial_t^{n_0-n_1}A^{\mathbf{k},m}_{j,\beta-\beta_2}(t)\right| \leq \frac{M_{j,\beta-\beta_2}(r)}{r^{n_0-n_1}}(n_0-n_1)! =: A_{n_0-n_1,j,\beta-\beta_2}. \tag{33}$$

Therefore, we obtain from (32)

$$v_{n_0,\ell,\beta+S} \leq \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}} \sum_{\beta_2=0}^{\beta} \sum_{m=0}^{k_0} \sum_{(L,\mathbf{p},j)\in\Delta_{\ell,m}} \sum_{n_1=0}^{n_0} \frac{\beta!}{\beta_2!(\beta-\beta_2)!} \frac{k_0!}{(k_0-m)!} \times \frac{1}{p_1!\ldots p_m!} \frac{\ell!}{j!(\ell-j)!} \frac{n_0!}{n_1!(n_0-n_1)!} A_{n_0-n_1,j,\beta-\beta_2} v_{k_0-m+n_1,L,\beta_2+k_1} \tag{34}$$

for all n_0, ℓ, $\beta \geq 0$. Now, let $(u_{n_0,\ell,\beta})_{n_0\geq 0,\ell\geq 0,\beta\geq 0}$ the sequence satisfying

$$u_{n_0,\ell,\beta+S} = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}} \sum_{\beta_2=0}^{\beta} \sum_{m=0}^{k_0} \sum_{(L,\mathbf{p},j)\in\Delta_{\ell,m}} \sum_{n_1=0}^{n_0} \frac{\beta!}{\beta_2!(\beta-\beta_2)!} \frac{k_0!}{(k_0-m)!} \frac{1}{p_1!\ldots p_m!} \times \frac{\ell!}{j!(\ell-j)!} \frac{n_0!}{n_1!(n_0-n_1)!} A_{n_0-n_1,j,\beta-\beta_2} u_{k_0-m+n_1,L,\beta_2+k_1} \tag{35}$$

for all n_0, ℓ, $\beta \geq 0$.
From (17), we have $0 \leq k_1 \leq S-1$ for all $\mathbf{k} = (k_0, k_1) \in \mathcal{J}$. We deduce that the sequence $(u_{n_0,\ell,\beta})_{n_0\geq 0,\ell\geq 0,\beta\geq 0}$ is uniquely determined by (35) and the initial conditions

$$u_{n_0,\ell,\beta} := v_{n_0,\ell,\beta} = \sup_{t\in\mathcal{D}} |\partial_t^{n_0} X_{\ell,\beta}(t)| \tag{36}$$

for $\beta = 0, \ldots, S-1$ and for all n_0, $\ell \geq 0$. We deduce from (34), (35) and (36) that

$$v_{n_0,\ell,\beta} \leq u_{n_0,\ell,\beta} \tag{37}$$

for all n_0, ℓ, $\beta \geq 0$.

Now, we define the formal series

$$U(t,z,T) = \sum_{\beta\geq 0} \sum_{\ell\geq 0} \sum_{n_0\geq 0} u_{n_0,\ell,\beta} \frac{t^{n_0}}{n_0!} \frac{T^\ell}{\ell!} \frac{z^\beta}{\beta!} \tag{38}$$

where the coefficients $u_{n_0,\ell,\beta}$ satisfy (35) and (36) for all n_0, ℓ, $\beta \geq 0$. We have the following proposition.

Proposition 8 *The formal series (38) is the unique solution of the Cauchy problem*

$$\partial_z^S U(t,z,T) = \sum_{\mathbf{q}=(q_0,q_1,q_2)\in\mathcal{Q}} B_{\mathbf{q}}(t,z,T) \partial_t^{q_0} \partial_T^{q_1} \partial_z^{q_2} U(t,z,T) \tag{39}$$

for given initial conditions

$$(\partial_z^j U)(t,0,T) = \sum_{\ell,n_0,\geq 0} v_{n_0,\ell,j} \frac{t^{n_0}}{n_0!} \frac{T^\ell}{\ell!} \quad , \quad 0 \leq j \leq S-1, \tag{40}$$

where

$$\mathcal{Q} = \{\mathbf{q} = (q_0,q_1,q_2) \in \mathbb{N}^3 \mid \mathbf{k} = (k_0,k_1) \in \mathcal{J};\ m = 0,\ldots,k_0;\ \mathbf{p} \in \mathcal{P}_2(m); \\ q_0 = k_0 - m;\ q_1 = |\mathbf{p}|_1;\ q_2 = k_1\}$$

and

$$B_{\mathbf{q}}(t,z,T) = \frac{k_0!}{q_0!} \frac{1}{p_1!\ldots p_m!} \sum_{\beta\geq 0} \sum_{j=0}^{D_m} \sum_{n\geq 0} A_{n,j,\beta} \frac{t^n}{n!} \frac{T^j}{j!} \frac{z^\beta}{\beta!} \tag{41}$$

where the coefficients $A_{n,j,\beta}$ are defined by (33).

Proof We plug $U(t,z,T)$ into (39) using the expansion (38). For the left-hand side term of (39), we obtain

$$\partial_z^S U(t,z,T) = \sum_{\beta\geq 0}\sum_{\ell\geq 0}\sum_{n_0\geq 0} u_{n_0,\ell,\beta+S}\frac{t^{n_0}}{n_0!}\frac{T^\ell}{\ell!}\frac{z^\beta}{\beta!}. \tag{42}$$

For the right-hand side term, a first calculus shows us that

$$\partial_t^{q_0}\partial_T^{q_1}\partial_z^{q_2} U(t,z,T) = \sum_{\beta\geq 0}\sum_{\ell\geq 0}\sum_{n_0\geq 0} u_{n_0+q_0,\ell+q_1,\beta+q_2}\frac{t^{n_0}}{n_0!}\frac{T^\ell}{\ell!}\frac{z^\beta}{\beta!}.$$

Using for $B_{\mathbf{q}}(t,z,T)$ the expansion (41), we obtain

$$\begin{aligned} B_{\mathbf{q}}(t,z,T)\partial_t^{q_0}\partial_T^{q_1}\partial_z^{q_2} U(t,z,T) = \sum_{\beta\geq 0}\sum_{\ell\geq 0}\sum_{n_0\geq 0}\Bigg(\sum_{\beta_2=0}^{\beta}\sum_{\ell_1=0}^{\ell}\sum_{n_1=0}^{n_0}\frac{\beta!}{\beta_2!(\beta-\beta_2)!} \\ \times\frac{\ell!}{\ell_1!(\ell-\ell_1)!}\frac{n_0!}{n_1!(n_0-n_1)!}\frac{k_0!}{q_0!}\frac{1}{p_1!\dots p_m!}A_{n_0-n_1,\ell-\ell_1,\beta-\beta_2}u_{n_1+q_0,\ell_1+q_1,\beta_2+q_2}\Bigg)\frac{t^{n_0}}{n_0!}\frac{T^\ell}{\ell!}\frac{z^\beta}{\beta!}\end{aligned} \tag{43}$$

where $A_{n_0-n_1,\ell-\ell_1,\beta-\beta_2}=0$ for $\ell-\ell_1>D_m$.
By identification of the coefficients of $\dfrac{t^{n_0}}{n_0!}\dfrac{T^\ell}{\ell!}\dfrac{z^\beta}{\beta!}$ in (39), using (42) and (43), we get

$$\begin{aligned} u_{n_0,\ell,\beta+S} = \sum_{\mathbf{q}=(q_0,q_1,q_2)\in\mathcal{Q}}\sum_{\beta_2=0}^{\beta}\sum_{\ell_1=0}^{\ell}\sum_{n_1=0}^{n_0}\frac{\beta!}{\beta_2!(\beta-\beta_2)!}\frac{\ell!}{\ell_1!(\ell-\ell_1)!}\frac{n_0!}{n_1!(n_0-n_1)!} \\ \times\frac{k_0!}{q_0!}\frac{1}{p_1!\dots p_m!}A_{n_0-n_1,\ell-\ell_1,\beta-\beta_2}u_{n_1+q_0,\ell_1+q_1,\beta_2+q_2}\end{aligned}$$

for all n_0, ℓ, $\beta\geq 0$ and where $A_{n_0-n_1,\ell-\ell_1,\beta-\beta_2}=0$ for $\ell-\ell_1>D_m$. Using the definition of the set $\mathcal{Q}$, we have

$$\begin{aligned} u_{n_0,\ell,\beta+S} = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}}\sum_{m=0}^{k_0}\sum_{\mathbf{p}\in\mathcal{P}_2(m)}\sum_{\beta_2=0}^{\beta}\sum_{\ell_1=0}^{\ell}\sum_{n_1=0}^{n_0}\frac{\beta!}{\beta_2!(\beta-\beta_2)!}\frac{\ell!}{\ell_1!(\ell-\ell_1)!}\frac{n_0!}{n_1!(n_0-n_1)!} \\ \times\frac{k_0!}{(k_0-m)!}\frac{1}{p_1!\dots p_m!}A_{n_0-n_1,\ell-\ell_1,\beta-\beta_2}u_{n_1+k_0-m,\ell_1+|\mathbf{p}|_1,\beta_2+k_1}\end{aligned}$$

for all n_0, ℓ, $\beta\geq 0$ and where $A_{n_0-n_1,\ell-\ell_1,\beta-\beta_2}=0$ for $\ell-\ell_1>D_m$. Or with an other formulation, we obtain

$$\begin{aligned} u_{n_0,\ell,\beta+S} = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}}\sum_{m=0}^{k_0}\sum_{\mathbf{p}\in\mathcal{P}_2(m)}\sum_{\beta_2=0}^{\beta}\sum_{\ell_2=0}^{D_m}\sum_{n_1=0}^{n_0}\frac{\beta!}{\beta_2!(\beta-\beta_2)!}\frac{\ell!}{\ell_2!(\ell-\ell_2)!}\frac{n_0!}{n_1!(n_0-n_1)!} \\ \times\frac{k_0!}{(k_0-m)!}\frac{1}{p_1!\dots p_m!}A_{n_0-n_1,\ell_2,\beta-\beta_2}u_{n_1+k_0-m,\ell-\ell_2+|\mathbf{p}|_1,\beta_2+k_1}\end{aligned}$$

for all n_0, ℓ, $\beta\geq 0$, which is exactly equation (35). This proves our proposition.

4.3 The main result.

We are now in position to state the main result of the paper.

Theorem 2 *Consider an ordinary differential equation*

$$\phi'(t) = P(t, \phi(t)) \tag{44}$$

where $P(t, X)$ is a holomorphic function on some domain $\mathcal{D} \subset \mathbb{C}$ with respect to t and polynomial in the variable X of degree $d \geq 2$. Let $\phi(t)$ be a holomorphic solution of (44) on some punctured disc $D_\theta(t_0, r_0) \subset \mathcal{D}$, where an expansion (29) holds. Consider the partial differential equation

$$\partial_z^S X(t,z) = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}} a_{\mathbf{k}}(t,z)\partial_t^{k_0}\partial_z^{k_1} X(t,z) \tag{45}$$

where S is a positive integer and $\mathcal{J}$ is a finite set which satisfies

$$\mathcal{J} \subset \left\{\mathbf{k} = (k_0, k_1) \in \mathbb{N}^2 \mid k_1 \leq S-1\right\}.$$

The coefficients $a_{\mathbf{k}}(t,z)$ are holomorphic functions on the domain $\mathcal{D}$ with respect to t and near the origin in $\mathbb{C}$ with respect to z. We make the following assumptions.

(A4) *There exist two real constants $b > 1$ and $q \geq 1$ such that*

$$\max_{0\leq m\leq k_0} \max_{(p_1,\ldots,p_m)\in\mathcal{P}_2(m)} \frac{b(m(d-1)+q(p_1+\ldots+p_m))}{q} + k_0 - m + k_1 < S$$

for all $\mathbf{k} = (k_0, k_1) \in \mathcal{J}$, where $\mathcal{P}_2(m) = \{(p_1, \ldots, p_m) \in \mathbb{N}^m / p_1 + 2p_2 + \ldots + mp_m = m\}$, for all $m \geq 1$, and $\mathcal{P}_2(0) = \{0\}$ by convention.

(A5) *Let $X_{\ell,\beta}(t)$, $\ell \geq 0$, $0 \leq \beta \leq S-1$, be bounded holomorphic fonctions on $\mathcal{D}$, satisfying the estimates*

$$\text{for all } \beta = 0, \ldots, S-1 \qquad \sum_{n\geq 0} \left\| \sum_{\ell\geq 0} \sup_{t\in\mathcal{D}} |\partial_t^n X_{\ell,\beta}(t)| \frac{T^\ell}{\ell!} \right\|_{0;\sigma_0} \frac{\delta_{1,0}^n}{n!} < +\infty.$$

Then, the formal series (18)

$$X(t,z) = \sum_{\beta\geq 0}\sum_{\ell\geq 0} X_{\ell,\beta}(t) \frac{\phi(t)^\ell}{\ell!} \frac{z^\beta}{\beta!}$$

constructed in Proposition 7, solution of (45), defines a holomorphic function on the open set $D_\theta(t_0, r_0) \times D\left(0, \frac{\delta\delta_2}{2}\right)$, for a given $\delta \in (0,1)$ and $\delta_2 > 0$. Moreover, the function $X(t,z)$ has the following rate of growth on the set $D_\theta(t_0, r_0) \times D\left(0, \frac{\delta\delta_2}{2}\right)$. If r_0 is small enough, there exist positive constants C_{11} and C_{12} such that

$$\sup_{|z|\leq\frac{\delta\delta_2}{2}} |X(t,z)| \leq C_{11}(1-\delta)^{-1}|t-t_0|^{-n_0 m/\mu} \exp(C_{12}|t-t_0|^{-n_0 q/\mu}) \tag{46}$$

for all $t \in \mathcal{D}_\theta(t_0, r_0)$.

Proof From Proposition 8, the formal series (38)

$$U(t,z,T)=\sum_{\beta\geq 0}\sum_{\ell\geq 0}\sum_{n_0\geq 0}u_{n_0,\ell,\beta}\frac{t^{n_0}}{n_0!}\frac{T^\ell}{\ell!}\frac{z^\beta}{\beta!}$$

is solution of the auxiliary partial differential equation (39)

$$\partial_z^S U(t,z,T)=\sum_{\mathbf{q}=(q_0,q_1,q_2)\in\mathcal{Q}}B_{\mathbf{q}}(t,z,T)\partial_t^{q_0}\partial_T^{q_1}\partial_z^{q_2}U(t,z,T)$$

where

$$\mathcal{Q}=\{\mathbf{q}=(q_0,q_1,q_2)\in\mathbb{N}^3\mid \mathbf{k}=(k_0,k_1)\in\mathcal{J};\ m=0,\dots,k_0;\ \mathbf{p}\in\mathcal{P}_2(m);$$
$$q_0=k_0-m;\ q_1=|\mathbf{p}|_1;\ q_2=k_1\}$$

and

$$B_{\mathbf{q}}(t,z,T)=\frac{k_0!}{q_0!}\frac{1}{p_1!\dots p_m!}\sum_{\beta\geq 0}\sum_{j=0}^{D_m}\sum_{n\geq 0}A_{n,j,\beta}\frac{t^n}{n!}\frac{T^j}{j!}\frac{z^\beta}{\beta!}.$$

The formal series $U(t,z,T)$ satisfies also the initial conditions

$$\partial_z^\beta U(t,0,T)=\sum_{\ell\geq 0}\sum_{n_0\geq 0}u_{n_0,\ell,\beta}\frac{t^{n_0}}{n_0!}\frac{T^\ell}{\ell!}=:\psi_\beta(t,T)$$

for all $\beta=0,\dots,S-1$. From the assumptions (**A5**), we get that the series $\psi_\beta(t,T)$ belong to the Banach space $\mathcal{G}_q(\delta_{1,0},\delta_{2,0};\sigma_0)$, for all $\beta=0,\dots,S-1$, for some $\delta_{2,0}>0$. Moreover, the equation (39) satisfies the assumptions (**A1**) and (**A2**). From the assumption (**A4**) one can easily check that the equation (39) also satisfies the condition (**A3**). We get that all the assumptions of Theorem 1 are satisfied and we deduce the existence of δ_1, δ_2 and $\sigma>0$ such that the formal series $U(t,z,T)$ belongs to $\mathcal{G}_q(\delta_1,\delta_2;\sigma)$ for b,q given in the assumption (**A4**).

Now, consider the formal series in powers of T and z

$$W(t,z,T)=\sum_{\beta\geq 0}\sum_{\ell\geq 0}X_{\ell,\beta}(t)\frac{T^\ell}{\ell!}\frac{z^\beta}{\beta!}$$

where $(X_{\ell,\beta}(t))_{\ell\geq 0,\beta\geq 0}$ is the sequence constructed in Proposition 7. We have from Proposition 1 and (37), that there exist a positive constant c_1 such that, for $T\in\mathbb{C}$, $t\in\mathcal{D}$ and $z\in\mathbb{C}$, $|z|\leq\frac{\delta\delta_2}{2}$, where $\delta\in]0,1[$ is given,

$$\sup_{|z|\leq\frac{\delta\delta_2}{2}}\sup_{t\in\mathcal{D}}|W(t,z,T)|\leq\sum_{\beta\geq 0}\sum_{\ell\geq 0}v_{0,\ell,\beta}\frac{|T|^\ell}{\ell!}\frac{1}{\beta!}\left(\frac{\delta\delta_2}{2}\right)^\beta\leq\left|U\left(0,\frac{\delta\delta_2}{2},|T|\right)\right|$$
$$\leq c_1(1-\delta)^{-1}(1+|T|)^m\exp(\sigma\zeta(b)|T|^q) \tag{47}$$

So that $W(t,z,T)$ defines a holomorphic function on $\mathcal{D}$ with respect to the variable t, on $D\left(0,\frac{\delta\delta_2}{2}\right)$ with respect to the variable z and an entire function with respect to the variable T with at most an exponential growth of order q.

We deduce that the function

$$X(t,z)=W(t,z,\phi(t))$$

solution of (45) defines a holomorphic function on $D_\theta(t_0, r_0) \times D\left(0, \frac{\delta\delta_2}{2}\right)$. On the other hand, from the expansion (29), there exist two constants $m_1, m_2 > 0$ such that

$$m_1|t - t_0|^{-n_0/\mu} \leq |\phi(t)| \leq m_2|t - t_0|^{-n_0/\mu}, \tag{48}$$

for all $t \in D_\theta(t_0, r_0)$. In conclusion, we deduce from (47) and (48) that the estimates (46) hold, if $r_0 > 0$ is small enough.

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