RESEARCH ARTICLE

Decay of Correlations in a Topological Glass

Jean-Pierre Eckmann[a,b] and Maher Younan [a]
[a]*Département de Physique Théorique, Université de Genève;*
[b]*Section de Mathématiques, Université de Genève*

(*Received 00 Month 2011; final version received 00 Month 2011*)

Dedicated to David Sherrington, with admiration and best wishes

In this paper we continue the study of a topological glassy system. The state space of the model is given by all triangulations of a sphere with N nodes, half of which are red and half are blue. Red nodes want to have 5 neighbors while blue ones want 7. Energies of nodes with other numbers of neighbors are supposed to be positive. The dynamics is that of flipping the diagonal between two adjacent triangles, with a temperature dependent probability. We consider the system at very low temperatures.

We concentrate on several new aspects of this model: Starting from a detailed description of the stationary state, we conclude that pairs of defects (nodes with the "wrong" degree) move with very high mobility along 1-dimensional paths. As they wander around, they encounter single defects, which they then move "sideways" with a geometrically defined probability. This induces a diffusive motion of the single defects. If they meet, they annihilate, lowering the energy of the system. We both estimate the decay of energy to equilibrium, as well as the correlations. In particular, we find a decay like $t^{-0.4}$.

1. Introduction, the Model

This paper deals with a species of a class of models on topological studies of triangulations. Such models have been studied in several contexts 2-d gravitation, froth, [1, and references therein]. The variant we use here was introduced in [2], but it turned out that a very similar study was initiated earlier by Aste and Sherrington [3]. So, we hope that David will accept this paper as a small sign of recognition.

We reconsider here the model which was inspired by [4] and introduced in [2]. For completeness we repeat the definition of the model: We fix a (large) number N of nodes, half of which are red, and the other half blue. These nodes are the nodes of a topological triangulation T of the sphere S^2. The set of all possible such labelled triangulations will be denoted $\mathcal{T}_N$. We define a dynamics on $\mathcal{T}_N$ by the following Metropolis algorithm whose elementary steps are flips (T1 moves): A link is chosen uniformly at random (among the $3N-6$ links). In Fig. 2, if the link AB was chosen then the flip consists in replacing it by the link CD. This move is not admissible if the link CD already exists before the move. Otherwise it is admissible. Note that the number of nodes, N, does not change in this model. However, we will be interested in the behavior for $N \to \infty$.

The Metropolis algorithm is based on the energy function E on $\mathcal{T}_N$ which, for any triangulation $T \in \mathcal{T}_N$, is defined as

$$E(T) = \sum_{i \in \text{blue}} (d_i - 7)^2 + \sum_{i \in \text{red}} (d_i - 5)^2 \,,$$

where d_i is the degree (number of links) of the node i. Thus, this energy favors 7 links for the blue nodes and 5 for the red ones. *Mutatis mutandis*, the detailed definition of the energy is not important for the discussion of the model, and we will stick to this particular

form of the energy. Given an admissible flip, compute the energy of the triangulation before and after the flip; this defines

$$\mathrm{d}E = E_{\text{after}} - E_{\text{before}} \cdot$$

An admissible flip is performed if either $\mathrm{d}E \leq 0$ or, when $\mathrm{d}E > 0$, with probability $\exp(-\beta \mathrm{d}E)$, where β is the inverse temperature of the system.

Several properties of this model were discussed in [2], but here we study in more detail the dynamical properties of the model. In particular, we introduce a "charge" defined as follows:

Definition 1.1: The *charge* of a red node is defined by $d_i - 5$ and the charge of a blue node is defined by $d_i - 7$. We will say the charge is a *defect* + if it is $+1$ and - if it is -1. In general, the color of the charge will not matter and will not be mentioned.

In principle, all charges between -4 and $\mathcal{O}(N)$ can occur, since $d_i \geq 3$, but, obviously, at low temperatures mostly the charges +, 0, and - will come into play.

2. Equilibrium and the Approximation of the Dynamics

The dynamics of the model is given by the Metropolis algorithm. In it, a link is chosen uniformly at random among all possible links. The change of energy induced by the flipping of this link is called $\mathrm{d}E$. If $\mathrm{d}E \leq 0$ the flip is performed, if $\mathrm{d}E > 0$ the flip is performed with probability $p(\mathrm{d}E) = \exp(-\beta \mathrm{d}E)$. This process satisfies detailed balance, and most of the paper is dealing with the equilibrium properties of this process at low temperatures. Because of the detailed balance, the equilibrium measure μ has the property that the probability to see a given state whose energy is E is proportional to $\exp(-\beta E)$. We use this elementary observation to argue that at low temperature there are only few defects, by which we mean that there are few red nodes whose degree is not 5 and also few blue ones whose degree is not 7. Given that there are few of these "defects", we further assume that the "positions" of these defects are random in the sense that there are no strong conditional expectations: For example, having a defect +1 does not say that there is a defect -1 close-by. The upshot of this way of reasoning, which we corroborate by numerical studies, is that one can approximate the dynamics by just looking at defects.

Indeed, the full dynamics must be described by the evolution of correlation functions. It would have to take into account correlation functions between the charges (and the colors) of, say, the 4 nodes on a pair of triangles sharing an edge. Then, flipping that edge, the correlations of many neighboring triangles would be changed simultaneously, and this would necessitate considering a full hierarchy of correlations (like BBGKY). What we will see is that in this model, these higher order correlation functions do not influence our basic understanding of what is going on.

In contrast, the Euler relations play a small but not totally negligible role for the sizes of the systems we consider.

3. Description of the Stationary State

It will be useful to define throughout the paper the natural parameter

$$\varepsilon \equiv e^{-\beta} \, .$$

We are interested in a regime where the density c of charges (which equals E/N) is low but also, where the number $c \cdot N$ of charges is large, so that good statistics and a certain

independence of the Euler relations is attained. More precisely, we fix $\varrho \ll 1$ and $D_0 \gg 1$, and require $\varepsilon \leq \varrho$ and $N\varepsilon > D_0$. We furthermore consider the limit of large N.

The main result of this section is summarized in the following proposition:

Proposition 3.1: *Consider an equilibrium state at temperature $T \ll 1$ satisfying the above conditions on N and ε.*

(1) At first order in ε, the only charges present in the system are simple defects ± 1. Their density is $2\varepsilon + \mathcal{O}(\varepsilon^2)$.
(2) The distribution of the colors (red or blue) is independent in the limit $\varepsilon \to 0$.
(3) The distribution of the charges is independent in the limit $\varepsilon \to 0$.

Remark 1: *The meaning of $\varepsilon \to 0$ above is that the quantities become more and more decorrelated as $\varepsilon \to 0$ while still maintaining the inequalities $\varepsilon \leq \varrho$ and $N\varepsilon > D_0$.*

3.1. *Energy of the stationary state*

In this paragraph, we will calculate the energy of the stationary state in the limit specified above, as a function of the temperature.

Estimate 3.2: *Consider the region $\varepsilon N > D_0$ and $\varepsilon < \varrho$. For sufficiently large D_0 and sufficiently small ϱ the density of charges c is*

$$c \equiv E/N = 2\varepsilon + \mathcal{O}(\varepsilon^2) \, .$$

Proof: Assuming equilibrium, by detailed balance, the probability to see a defect of charge ± 1 is $\mathcal{O}(e^{-1\cdot\beta}) = \mathcal{O}(\varepsilon)$, while the probability to see higher charges is $\mathcal{O}(e^{-2^2\beta}) = \mathcal{O}(\varepsilon^4)$, by the assumption of equilibrium and the form of the Hamiltonian, since, if $(d_i - 5)^2 > 1$ then it is at least 4.

So it remains to estimate the coefficient in front of the factor ε. There are 4 cases to consider: The number of red nodes with degree 4 or 6, resp. the number of blue nodes with degree 6 or 8. All these cases cost energy 1 per instance, and thus these 4 numbers are equal by the virial theorem.

We also need to estimate the cases with 0 charge, *i.e.*, blue nodes with 7 neighbors and red nodes with 5 neighbors, which appear again equally often, by the virial theorem. Since there are $N/2$ nodes of each color, and each of the colors has 2 states of defect 1 (namely ± 1), we conclude that the expected total number of defects is

$$2 \cdot 2 \cdot \varepsilon \cdot (N/2) = 2\varepsilon N + \mathcal{O}(\varepsilon^2) \, . \tag{1}$$

□

3.2. *Distribution of the colors*

We next calculate the probabilities that a randomly chosen link connects 2 red (blue) nodes. We denote these probabilities by p_{rr} for red-red, p_{rb} for red-blue and so on. If there are no defects, *i.e.*, at order ε^0, all red nodes have 5 neighbors and all blue nodes have 7. This leads to the following relations:

$$2p_{\mathrm{rr}} + p_{\mathrm{rb}} = 5/6 \, ,$$
$$2p_{\mathrm{bb}} + p_{\mathrm{rb}} = 7/6 \, .$$

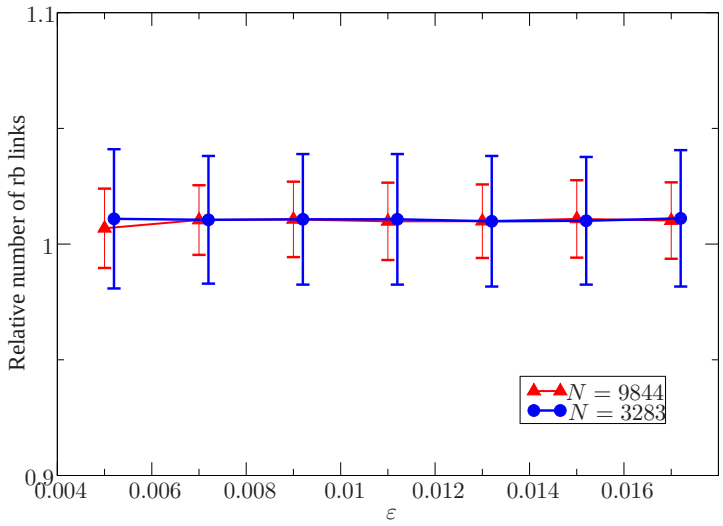

Figure 1. Numerical check of relation $p_{\rm rb} = 70/144$ by plotting $p_{\rm rb}/(70/144)$. The error bars are 3 σ and the data for $N = 3283$ are slightly shifted (in the x-direction) for better visibility.

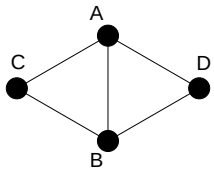

Figure 2. Labeling of the corners of 2 adjacent triangles

Assuming that the positions of the colors are uncorrelated, we find that the relative probabilities to find a red-red, resp. blue-blue pair are

$$p_{\rm rr}/p_{\rm bb} = 25/49 \ .$$

This leads to $p_{\rm rr} = 25/144$, $p_{\rm bb} = 49/144$, and $p_{\rm rb} = 70/144$. In Fig. 1 we show that numerical simulations confirm this simple approximation to a very high degree of fidelity.

3.3. *Energy cost of flips*

We adopt an approach similar to Sect. 3.2. We use the hypothesis that the charges are randomly distributed over the nodes to calculate the probability of finding a link with a given neighborhood of charges and compare it to simulation results. In this case however, given a link ℓ, the neighborhood we consider is the ordered set of all 4 nodes involved in its flipping. For example in Fig. 2, this set would be $(c(A), c(B), c(C), c(D))$ where $c(A)$ is the charge of the node A. This choice will be very useful for to study the dynamics later on since it determines the energy cost of flipping a given link:

$$\begin{aligned} {\rm d}E(\ell) &= \sum_{n\in\{A,B\}} (c(n)-1)^2 - (c(n))^2 + \sum_{n\in\{C,D\}} (c(n)+1)^2 - (c(n))^2 \\ &= 4 + 2\left(c(C) + c(D) - c(A) - c(B)\right) \ . \end{aligned} \tag{2}$$

It is easy to enumerate all the various cases and the energy cost associated with each of them. We restrict the discussion to those situations where the charges take values in $\{+1, 0, -1\}$. In principle, there are 3^4 configurations, which are reduced to 36, by symmetry. They are summarized in Table 1 (symmetrical cases omitted).

Table 1. The energy differences obtained by flipping the link between the first 2 values to a link between the second 2 values, as a function of the number of defects.

defects	initial state				$\mathrm{d}E$
0	0	0	0	0	4
1	+	0	0	0	2
1	0	0	0	-	2
1	0	-	0	0	6
1	0	0	+	0	6
2	+	+	0	0	0
2	+	0	0	-	0
2	0	0	-	-	0
2	+	-	0	0	4
2	+	0	+	0	4
2	0	-	0	-	4
2	0	0	+	-	4
2	-	-	0	0	8
2	0	-	+	0	8
2	0	0	+	+	8

defects	initial state				$\mathrm{d}E$
3	-	-	+	0	10
3	0	-	+	+	10
3	+	-	0	-	2
3	+	+	+	0	2
3	+	+	0	-	-2
3	+	0	-	-	-2
3	+	0	+	-	2
3	0	-	-	-	2
3	-	-	0	-	6
3	+	-	+	0	6
3	+	0	+	+	6
3	0	-	+	-	6

defects	initial state				$\mathrm{d}E$
4	+	-	-	-	0
4	+	+	+	-	0
4	-	-	+	+	12
4	-	-	-	-	4
4	+	-	+	-	4
4	+	+	-	-	-4
4	+	+	+	+	4
4	-	-	+	-	8
4	+	-	+	+	8

Note that if the defects of the original configuration are bounded by ± 1, then $\mathrm{d}E$ varies between -4 and 12.

3.4. *The number of local defect configurations*

We assume throughout that the number of red (blue) nodes is n_{r} (n_{b}) and that $\Delta \equiv n_{\mathrm{r}} - n_{\mathrm{b}} \in \{0, 1\}$. We denote by $p_{\pm}$ the probabilities to find charges ± 1, respectively. Assuming that there are no other charges (except 0), we can write

$$N \cdot (p_- + p_+) = E \,,$$
$$N \cdot (p_- - p_+) = 12 - \Delta \,,$$

where the second equation follows from the Euler formula. In equilibrium, $E = 2N\varepsilon$, by Eq. (1), and therefore we get

$$p_{\pm} = \varepsilon \mp 6/N \pm \Delta/(2N) + \mathcal{O}\left(\varepsilon^2\right) \,. \tag{3}$$

We will assume that $N\varepsilon \gg 6$ so that the second term in Eq. (3) can be neglected. In a similar way, one can show that

$$p_{\pm 2} = \varepsilon^4 + \mathcal{O}(\varepsilon^5) \,,$$

and combining these we find that the probability of nodes with charge 0 is

$$p_0 = 1 - 2\varepsilon + \mathcal{O}(\varepsilon^2) \,.$$

We next consider in more detail what happens in those pairs of triangles where a flip leads to $\mathrm{d}E = 0$. Looking again at Eq. (2) we see that the case $\mathrm{d}E = 0$ appears in 3 cases:
Case q_{+-} : One of A or B has charge + and C or D has charge - (and the others, charge 0).
Case q_{++}: A and B charge +, C and D charge 0.
Case q_{--} : A and B charge 0, C and D charge - .

Continuing with the independence assumption, we now look at the probability to find a configuration of type q_{++}, q_{+-} , and q_{--} . Note that there are $6N - 12$ half-links

emanating from the nodes, and we are to pair them up randomly. Note that if a site is red, it has 4, 5, 6 outgoing links, depending on whether its charge is $-$, 0, $+$, respectively. Similarly, the numbers for a blue node are 6, 7, 8. Therefore, given that there are on average $\varepsilon N/2$ defects of type red-4, red-6, blue-6, blue-8, there will be $4\varepsilon N/2$ links from the red-4, $6\varepsilon N/2$ from red-6 and blue-6, and $8\varepsilon N/2$ from blue-8. The blue-7 and red-5 occur with probability almost 1 and have therefore respectively $7N/2$ and $5N/2$ dangling edges (with a correction factor $1-\mathcal{O}(\varepsilon)$) which we omit throughout. The probabilities to see such dangling edges are the quantities above, divided by $6N-12$, the total number of dangling edges. We get, omitting higher order terms:

$$\begin{aligned} q_{++} &= (7p_+/6)^2 \cdot p_0^2 = 49\varepsilon^2/36 \,, \\ q_{--} &= (5p_-/6)^2 \cdot p_0^2 = 25\varepsilon^2/36 \,, \\ q_{+-} &= 4\,(5p_-/6)\,(7p_+/6) \cdot p_0^2 = 140\varepsilon^2/36 \,. \end{aligned} \tag{4}$$

We also get, by looking at Table 1:

$$\begin{aligned} p_{\mathrm{d}E=0} &= q_{++} + q_{--} + q_{+-} = 214\varepsilon^2/36 \,, \\ p_{\mathrm{d}E=2} &= 2\,(7p_+/6 + 5p_-/6) \cdot p_0^3 = 4\varepsilon \,, \\ p_{\mathrm{d}E=4} &= p_0^4 = 1-\mathcal{O}(\varepsilon) \,, \end{aligned} \tag{5}$$

The discussion of the other values of $\mathrm{d}E$ shows the limitations due to our closing assumptions: by the virial theorem, in total independence, we would simply have

$$p_{\mathrm{d}E=0} = p_{\mathrm{d}E=8} \text{ and } p_{\mathrm{d}E=2} = p_{\mathrm{d}E=6} \,. \tag{6}$$

But we could also have computed the probabilities as above, with the result:

$$\begin{aligned} p_{\mathrm{d}E=-2} &= 2\,(7p_+/6)^2 \cdot (5p_-/6) \cdot p_0 + 2\,(5p_-/6)^2 \cdot (7p_+/6) \cdot p_0 \\ &\approx 3.89\varepsilon^3 \end{aligned} \tag{7}$$

instead of $4\varepsilon^3 = p_{\mathrm{d}E=2} \cdot \varepsilon^2$ given by the stationarity assumption, which proves that the distribution of defects is not completely uncorrelated. We will say that the correlation is bounded by $0.1\varepsilon^3$, and can thus be neglected in the limit $\varepsilon \to 0$.

In Figs. 3 and 4 we show with 2 examples that the numerical simulations confirm these simple approximations to a very high degree of fidelity. Note that in [5], the *uniform* measure on $\mathcal{T}_N$ was considered, and even this leads to correlations of degrees of neighboring nodes.

4. Dynamics of the System (at Equilibrium)

We can use the results of the previous section to estimate the dynamics of the system under the Metropolis algorithm.

If a flip leads to an energy change $\mathrm{d}E$ then it is accepted in the Metropolis algorithm with probability

$$p_{\mathrm{acceptance}} = \varepsilon^{\max(0,\,\mathrm{d}E)} \,. \tag{8}$$

On the other hand, the probabilities to pick a link with fixed $\mathrm{d}E$ are given by Eq. (5) and Eq. (6). Multiplying these numbers with the probabilities of Eq. (8) leads to an estimate

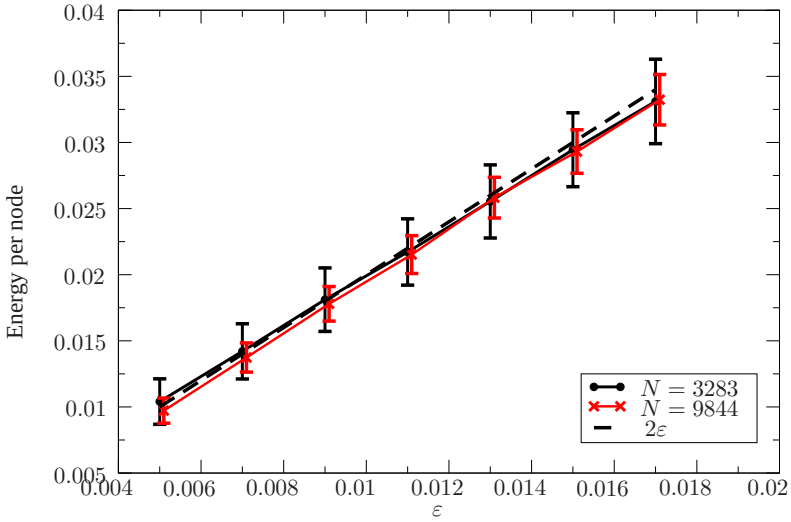

Figure 3. Numerical test of the mean energy per node (Estimate 3.2) for 950 realizations. The data for $N = 9844$ are slightly shifted for better visibility. Note the excellent fit with the theoretical curve, although the fluctuations are huge, getting better with larger system size (1 standard deviation shown).

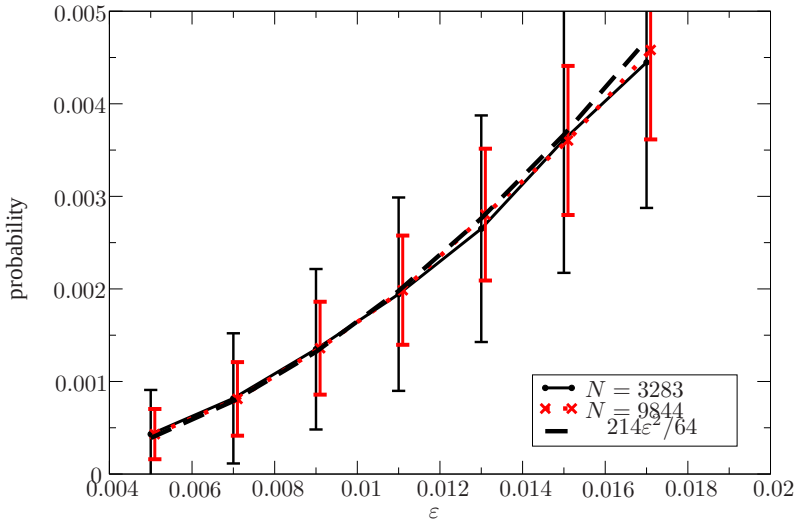

Figure 4. Numerical test of $p_{\mathrm{d}E=0}$ (Eq. (5)) from 950 realizations. The data for $N = 9844$ are slightly shifted for better visibility. Note the excellent fit with the theoretical curve, although the fluctuations are huge, getting better with larger system size (1 standard deviation shown).

of the probability that the flip in question actually happens. The results are summarized in Table 2 (calculated this time with the method of Eq. (7)).

Discussion: Inspection of Table 2 shows that the events with the highest transition rate are those which cost no energy, followed by those which have an energy cost of ± 2. Also note that the probability to find a link which will lead to a given $\mathrm{d}E$ is equal to the quantity in the table times $\varepsilon^{-\max(0,\mathrm{d}E)}$ since then we neglect the Metropolis factor. This leads to a table with the same prefactors, but with a power $\varepsilon^{|\mathrm{d}E-4|/2}$. In particular, *in the steady state, the local landscape is given by the 3rd column of Table 2: It is symmetric around* $\mathrm{d}E = 4$.

Henceforth, we will only consider the 3 most frequent types of flips (the others are an order ε less probable):

(1) Flips which change from 1 defect to 3 of them and which raise the energy by 2. These flips will be called *creation events*.

Table 2. The probabilities of transitions from the initial state. Data only shown to order ε^4. The third column shows the probabilities to pick a link leading to a given $\mathrm{d}E$. Higher order corrections are omitted.

$\mathrm{d}E$	transition rate	local landscape
-4	$1225/1296 \cdot \varepsilon^4$	$1225/1296 \cdot \varepsilon^4$
-2	$35/9 \cdot \varepsilon^3$	$35/9 \cdot \varepsilon^3$
0	$107/18 \cdot \varepsilon^2$	$107/18 \cdot \varepsilon^2$
2	$4 \cdot \varepsilon^3$	$4 \cdot \varepsilon^1$
4	$1 \cdot \varepsilon^4$	$1 \cdot \varepsilon^0$
6	$4 \cdot \varepsilon^7$	$4 \cdot \varepsilon^1$
8	$107/18 \cdot \varepsilon^{10}$	$107/18 \cdot \varepsilon^2$
10	$35/9 \cdot \varepsilon^{13}$	$35/9 \cdot \varepsilon^3$
12	$1225/1296 \cdot \varepsilon^{16}$	$1225/1296 \cdot \varepsilon^4$

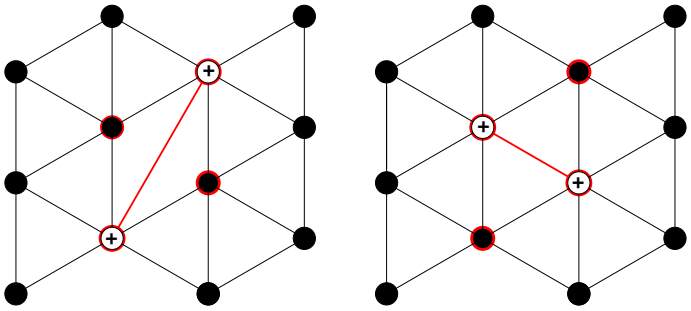

Figure 5. A flip from `++00` (on the left) and the resulting triangulation on the right. The affected nodes are supposed to be red, in this example. Note that the result is *again* of the type `++00`. Furthermore, *again* with $\mathrm{d}E = 0$ one can flip back. This is reminiscent of "blinkers" in the game of life [6, Chap25].

(2) Flips which start from 3 defects and end with 1 defect and which decrease the energy by 2. These flips will be called *annihilation events*. Creation and annihilation events are obviously dual to each other and equiprobable in the stationary state.
(3) Flips which do not change the energy, and in which a pair `++`, `+-`, or `--` is involved. These flips are by far the most probable. We will discuss below in more details the 3 configurations which lead to $\mathrm{d}E = 0$.

4.1. *The most probable flips*

As stated above, if $\varepsilon = 1\%$, then over 99% of the flips (which are accepted by the Metropolis algorithm) do not change the energy. It is clear that, in order to understand the dynamics of the system, one should start by studying these flips.

Looking at Table 1 we see that there are 3 candidates for $\mathrm{d}E = 0$ and they all involve only 2 defects. We will now show that the cases of `++00` and `00--` are quite different from that of `+00-` (and its 3 other variants `+0-0`, ...). In the first case, `++00`, which is similar to the case `00--`, the local neighborhood looks like in Fig. 5. In this case, what happens is a flipping back and forth between the 2 states, with probability $p = (3N-6)^{-1}$ (the probability to choose the colored link).

The case `+00-` is illustrated in Fig. 6. Here a new, and important phenomenon appears: The pattern, `+00-` which we will call a *pair*, is recreated, but *at a new position* a distance 1 away from the old one. We will also say that the pair `+-` *walks* one step.

The more important observation is that the pairs of defects must walk along a *predefined, 1-dimensional path* as shown in Fig. 7. This means that *the* $dE = 0$ *motion of* `+-` *pairs is a* **one***-dimensional random walk in the current triangulation* T*. This random walk (flipping back and forth on the predefined path) will continue until some other type of event happens.*

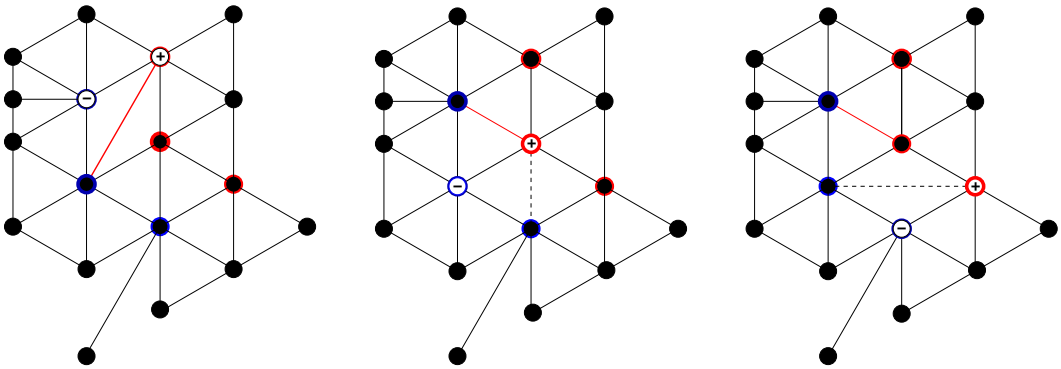

Figure 6. Change of pattern in the case `+00-`. In the sample only the relevant colors are as shown. Note that the effect of the flip is that the 2 defects move (in the picture) *down*. The reverse flip costs nothing $\mathrm{d}E = 0$. The second flip (dashed line) moves the defects one step further. Note that this motion must take place one a *predefined, 1-dimensional path*.

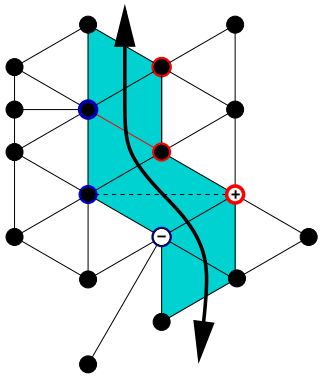

Figure 7. The same configuration as in Fig. 6 with the *1-dimensional* path of the pair of defects superimposed.

4.2. *Lifetime of pairs*

As we have seen, a pair of opposite charges `+-` can move through the system without energy cost. Its motion is a 1D random walk along a fixed 1-dimensional path. Edges are still chosen randomly and will be flipped if possible and if the Metropolis condition is met in case $\mathrm{d}E > 0$. Here we ask about the relative probabilities that a pair disappears, and we will show that *predominantly a pair will die when it collides with a defect.*

We need to compare 3 possibilities of which the first will be seen to be the most probable:

(1) The random walk reaches another defect.
(2) The pair is destroyed because a creation event involving 1 or 2 of its 2 defects occurs.
(3) Two independent random walks meet.

Our earlier discussion says that the concentration of pairs $\left(70\varepsilon^2/36\right)$ is much smaller than the concentration of defects (2ε), implying that the probability of 2 pairs meeting is insignificant when compared to the probability of a pair meeting a defect.

We next estimate the probability of destroying a pair as in case (2). On average, there are 7 links in the neighborhood of a given pair which increase E, and flipping such a link has an energy cost of 2. The probability of this to happen is $7\varepsilon^2/(3N)$. Since the pair moves every $\mathcal{O}(N)$ attempted flips, we conclude that, on average, a pair will do $\mathcal{O}\left(\varepsilon^{-2}\right)$ steps before it is destroyed as in case (2).

The number of steps needed for case (1) to happen depends obviously on the density of defects. We let ξ denote the average distance between defects (counted in number of links). Since the number of defects is $2\varepsilon N$ and the system is 2-dimensional, we conclude

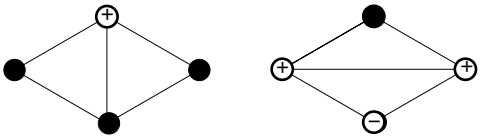

Figure 8. A creation event: a + - pair is created from a + defect, which is pushed one step.

that

$$\xi = \mathcal{O}(\varepsilon^{-1/2}) \ .$$

As long as the pair is not destroyed by the mechanism leading to case (2) it can thus do $\mathcal{O}(\varepsilon^{-2})$ steps by which time it can visit $\mathcal{O}(\varepsilon^{-1})$ defects.

This terminates the comparison of the 3 cases, and shows that a pair has the time to visit a very large number of defects before it is destroyed by the 2 other mechanisms.

5. The Geometry of Pair-Defect Collisions

In this section we consider the collisions between a pair and a defect. The discussion is really in two parts: On one hand, we must consider the probability that a collision between a pair and a defect is initiated. This depends on the density of the defects, and hence on ε. But, once a collision is initiated, we can ask what the effect of the collision is going to be. The next proposition shows that this effect is purely geometrical and independent of ε.

Proposition 5.1: *There are 9 topologically different possibilities $Q_i, i = 1, \ldots, 9$ for a collision to be initiated. For each of them, there are 2 purely geometrical constants $P_{\mathrm{m},i} > 0$ and $P_{\mathrm{d},i} > 0$ (depending on i) which tell us the probability that a collision leads to a move ($P_{\mathrm{m},i}$) of a defect (by 1 or 2 links) resp. the deletion of the pair ($P_{\mathrm{d},i}$).*

The remainder of the section deals with the proof of Proposition 5.1.

5.1. *Definition*

We will study in detail how collisions move defects. First of all, we will define what we mean by a collision. Assuming that the density of defects is very small, the only collisions we will consider are those involving 3 defects, namely, the pair + - which will collide with a defect + or -.

Definition 5.2: Consider some configuration T. Three defects $D_i, i = 1, 2, 3$ of T are said to be *in a collision* if there are $k \geq 2$ flips (k links of T) that do not increase the energy such that

(1) The only defects involved in these k flips are $D_i, i = 1, 2, 3$.
(2) All 3 defects are involved in these k flips.
(3) At least one of these k flips will move a pair (the others might be any of the 4 cases which do not increase the energy).

5.2. *Collision types*

In this section, we will describe all possible configurations of a collision and we will show that the probability of any such configuration depends solely on the topology (and not on the temperature).

The third condition of the definition of a collision states that we can always identify a pair; as a result, the set of all possible configurations of a collision can be obtained by

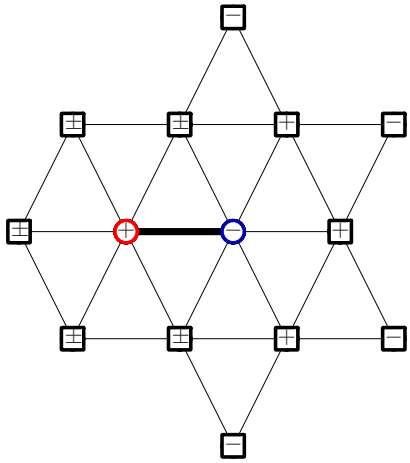

Figure 9. A figure showing all possible relative positions of a pair and a defect in collision. The pair is shown as a solid black line.

taking a pair and placing either a - defect or a + defect in any position where it can interact with one of the pair's 2 defects. As seen in the previous section, a + defect can interact with any defect if and only if both defects are at distance one. Two - defects can interact if and only if they are on opposite corners of 2 adjacent triangles. The last ingredient is that + defects can have a degree of 6 or 8 whereas - defects have a degree of 4 or 6. This yields a systematic method of constructing all possible configurations of a collision: consider a pair and let $\mathcal{U}_1$ be the set of all empty sites (charge 0) which are at distance 1 of any of the pair's 2 defects and $\mathcal{U}_2$ be the set of all empty sites which are opposite to the - defect of the pair. The set of all possible configurations of a collision is obtained by placing a + defect in any of $\mathcal{U}_1$'s sites or a - defect in any of $\mathcal{U}_2$'s sites, as shown in Fig. 9 in the case where the + defect is red and the - defect is blue. All in all, we get 9 different configurations of a pair and a defect (symmetrical case omitted).

Assuming that the defects are randomly distributed, it is clear from Fig. 9 that the probability of a collision to be of some type $Q_i \in \{1, \dots, 9\}$ is a temperature independent constant that can be calculated. To prove Proposition 5.1 one must study in detail each of the 9 cases. We will study in particular:

- What are the possible outcomes of each collision type and what is the (conditional on having initiated the collision Q_i) probability of each outcome?
- What is the probability (conditional on having initiated the collision Q_i) that a pair pushes a defect?

We can summarize the answers as follows:

- There will always be a defect left over at the end of the collision.
- Finding a pair and a defect at the end of the collision is possible in all 9 cases.
- An annihilation of the pair is possible in 2 of the 9 cases.
- It is possible that the defect is pushed in 8 cases. A defect can be pushed by more than 1 step.
- It is possible that the defect remains in its initial position in all 9 cases.

The relative probabilities of any of the above outcomes only depend on the local geometry. While all the cases have been worked out in detail, we illustrate the discussion for just 2 of them, and this will complete the proof of Proposition 5.1.

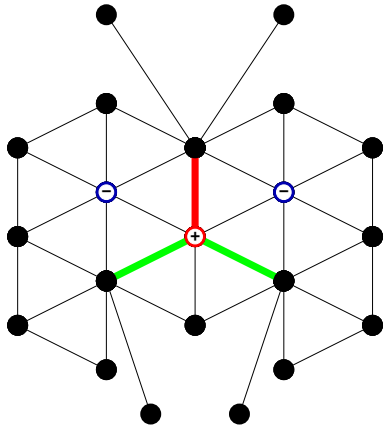

Figure 10. A collision where a an annihilation is possible. The green links show the way the pair enters (or exits) the collision. Flipping the red link will cause an annihilation event.

5.2.1. Example 1: A possible annihilation

There are 2 cases where an annihilation might occur. We consider here the case of Fig. 10. A +- pair collides with a - defect. For simplicity, assume that the +- pair came from the left. Once the pair and the defect are in collision, there are 3 links whose flipping leads to $\mathrm{d}E \leq 0$. Two of these links (the green ones) allow the pair to walk away from (or enter) the collision. Flipping the red link on the other hand causes an annihilation: the pair is destroyed and the defect is pushed by one step. We clearly see that there are 3 possible outcomes:

- The pair exits the collision through the same way it entered (in our case, on the left). The defect remains in its initial position.
- The pair exits the collision through the other green link. The defect moves 2 steps.
- An annihilation event occurs. The pair disappears and the defect moves 1 step.

The (conditional) probability of each outcome is 1/3 and the (conditional) probability that the defect will have moved at the end of the collision is 2/3.

5.2.2. Example 2: A bifurcation

Here, we look at the collision case of Fig. 11. No annihilation is possible here and the outcome of the collision is always one pair and one defect. The only relevant question is what is the probability that the defect will have moved at the end of the collision. But the combinatorics is more involved.

The pair enters and may exit the collision through a green link. Flipping a red link on the other hand will not end the collision. Notice that the fifth diagram contains 4 red links and no green ones. Moving a red link will visit the 6 figures sequentially. But moving the two lower red links in the lowest left figure will lead to another circle of five configurations, which is not shown in the figure. This collision case can be represented by a "state diagram" as in Fig. 12, where each node represents a state and each link represents the effect of flipping one of the colored links in Fig. 11. The pair enters the collision through a dangling link ℓ_1. It can wander around the vertices of the state diagram before exiting through a dangling link ℓ_2.

If $\ell_1 = \ell_2$, then it is as if the collision never occurred. In particular, the defect does not move. Furthermore, if ℓ_1 and ℓ_2 are of the same color, then the defect will remain in its initial position at the end of the collision. Using this remark and the diagrams of Fig. 11, one can explicitly compute the (conditional) probability that a pair pushes a defect if the collision is of the above type. This probability will be temperature independent.

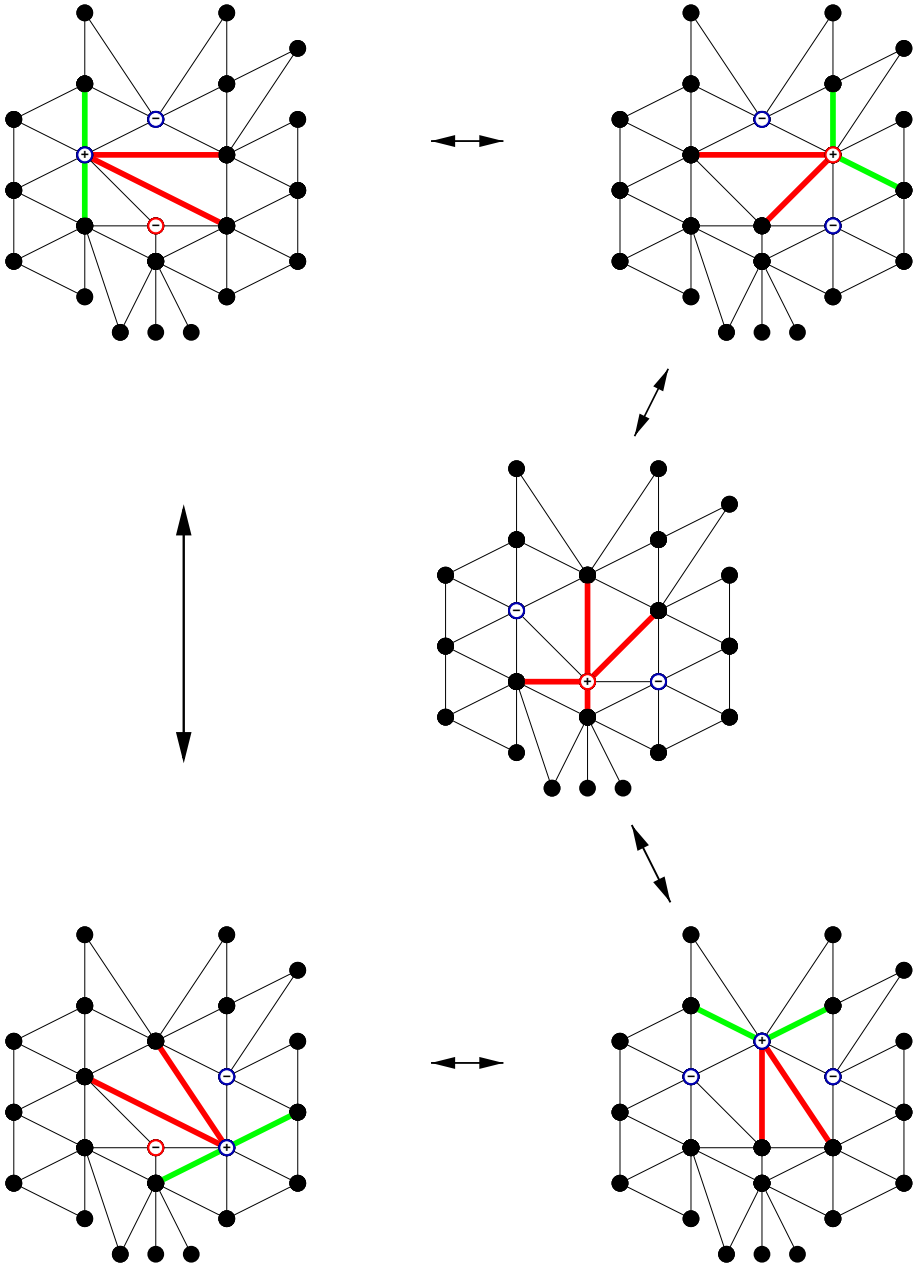

Figure 11. The central figure (with only red links) is symmetric along the axis -+-. If we flip the long vertical line, we arrive at the figure top-right. If we flip in it the red link which does not lead back to the center, we arrive at top-left. Flipping the red link which does not lead back to top-right, we arrive at bottom-left, then at bottom-right, and then back to the center. Since the same happens for the two lower links of the center, we see that the local state space is a figure "8" with 9 nodes of which 8 have two exits each. The state space can be symbolized as in Fig. 12.

The other 7 cases are treated similarly, and this completes the proof of Proposition 5.1.

Note that the proof means that collisions lead, on average to a *positive* probability of moving a defect. *This mechanism is the basic reason for the diffusive wandering of the defects in the triangulations. It is mediated by the collision of pairs with the defects. Clearly, if there are no pairs, the defects can not move by this mechanism, but only through much less probable events.*

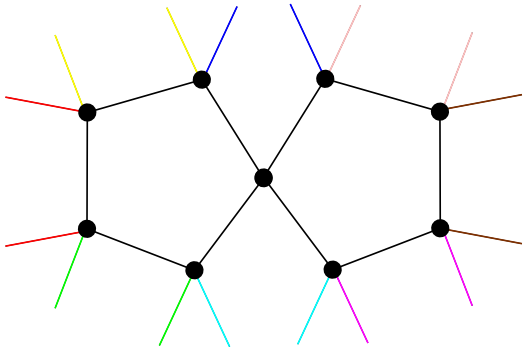

Figure 12. Each vertex represents one possible configuration during the collision of Fig. 11. Two vertices are linked if one can go from one configuration to the other by flipping a (red) link of Fig. 11. The pair enters and exits the collision through one of the 16 dangling links. If these 2 dangling links are the same or if they are of the same color, then the pair does not push the defect, otherwise, it is pushed by 1 step.

6. Relevant and Irrelevant Pairs

In Sect. 4.2, we have seen that a pair lives long enough to explore its 1D path, before being destroyed by other mechanisms. We now analyze in detail what can happen during this exploration phase.

When a pair is created, it is one step away from its birthplace. It will then perform a random walk on its predefined 1-dimensional path. Each time it comes back to its birthplace, it can die with probability $p_{\rm death} = 1/3$ as shown in Fig. 10. *If this happens, the triangulation will not have changed.* We will call this an ineffective pair. The probability $P_{\rm I} = P_{\rm I}(\xi)$ can be estimated as follows:

Assume that a defect X is at a distance ξ from the birthplace of the pair. Then, by extending slightly the gambler's ruin principle [7], *the probability* $P_{\rm R} = P_{\rm R}(\xi)$ *that the pair actually can reach* X *is* $(1 + (\xi - 1) \cdot p_{\rm death})^{-1} = \mathcal{O}(1/\xi)$. This implies that the probability for any event implying X when starting from the birthplace depends on ξ, and in the case of many defects, on their average distance (which we call again ξ). Thus,

$$P_{\rm I} = 1 - \mathcal{O}(1/\xi) \; , \quad P_{\rm R} = \mathcal{O}(1/\xi) \; . \tag{9}$$

7. Time Correlations at Equilibrium

Here, we estimate the rate of change of triangulations (as a function of time). Since our triangulations are purely topological, we need to define what we mean by the distance between 2 triangulations T_1 and T_2 in $\mathcal{T}_N$ (the space of triangulations of the sphere with N labeled nodes). There are many possible choices, see *e.g.*, [8] many of which lead to equivalent metrics. The one defined below is convenient for our purpose.

Let $\{T_1, T_2\} \subset \mathcal{T}_N$. Consider a node n of T_1. The flower $f\,(n, T_1)$ of n is defined as the ordered cyclic set of all neighbors of n in T_1. Two flowers are then said to be equal if one can be obtained from the other by a cyclic rotation. We can now define the following metric on $\mathcal{T}_N$:

$$d\,(T_1, T_2) = \sum_{n=1}^{N} d_n\,(T_1, T_2) \text{ and}$$

$$d_n\,(T_1, T_2) = \begin{cases} 0 & \text{if } f\,(n, T_1) = f\,(n, T_2)\;, \\ 1 & \text{otherwise}\;. \end{cases}$$

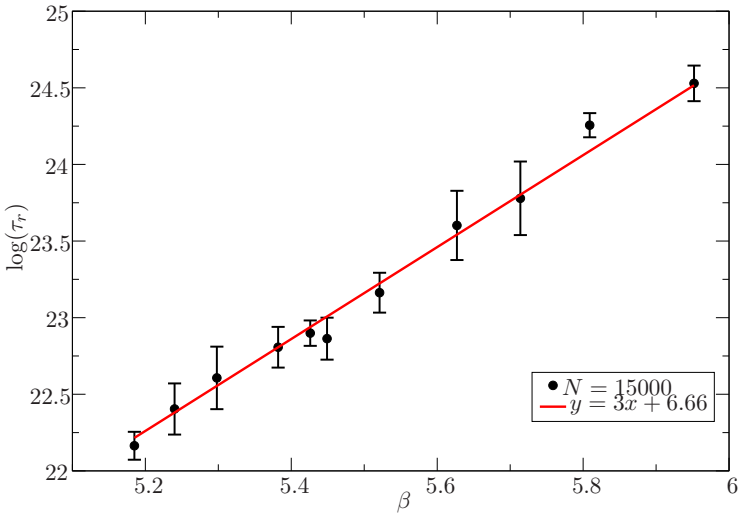

Figure 13. Decay rate of correlations at equilibrium. Numerical verification of Eqs. (10) and (11). The data are averages over 10 runs with $N = 15'000$. The error bars represent 1 standard deviation. The variable β is equal to $-\log(\varepsilon)$. The fits are for C between 0.5 and 0.001.

Using this metric, we define the following correlation function:

$$C(\vartheta) = 1 - \frac{d\left(T(t), T(t+\vartheta)\right)}{N} ,$$

where $T(t)$ is the system state at time t. Our result for the decay of this function at equilibrium, *i.e.*, when $t \to \infty$, is as follows:

Proposition 7.1: *The correlation function C decays like*

$$C(\vartheta) = e^{-\vartheta/\tau_r} , \tag{10}$$

with a relaxation time τ_r of the form

$$\tau_r = \mathcal{O}(e^{3\beta}) = \mathcal{O}(\varepsilon^{-3}) . \tag{11}$$

Proof: The correlation function $C(\vartheta)$ is nothing but the fraction of nodes whose flower is unchanged after ϑ time units. At equilibrium, the number of pairs p was established in Eq. (4) to be $p = 70/36 \cdot \varepsilon^2$. On the other hand, the density of defects in equilibrium is $\mathcal{O}(\varepsilon)$ and hence, their average distance ξ equals $\xi = \mathcal{O}(\varepsilon^{-1/2})$. By the estimates of Sect. 6 this means that the effective number of pairs which change the configuration in a permanent way is $\mathcal{O}(p \cdot \varepsilon^{-1/2})$. We further saw in Sect. 5 that the number of collisions a relevant pair will undergo is a temperature independent constant $\nu = \mathcal{O}(1)$. If ξ is the average distance between 2 defects, then, on average, this pair will change, on its way, the flowers of $2\nu\xi$ nodes. At time ϑ, each of these flowers is still unchanged with probability $C(\vartheta)$.

Since the pair makes a 1D random walk, all this happens within an average time interval $\delta\vartheta = \frac{1}{2}\nu^2\xi^2$. This in turn leads to

$$C(\vartheta + \delta\vartheta) = C(\vartheta) - 2pP_{\mathrm{R}}\nu\xi C(\vartheta) .$$

In the limit $\vartheta \gg \delta\vartheta$, we find

$$\dot{C}(\vartheta) = -4\frac{pP_{\mathrm{R}}}{\nu\xi}C(\vartheta) ,$$

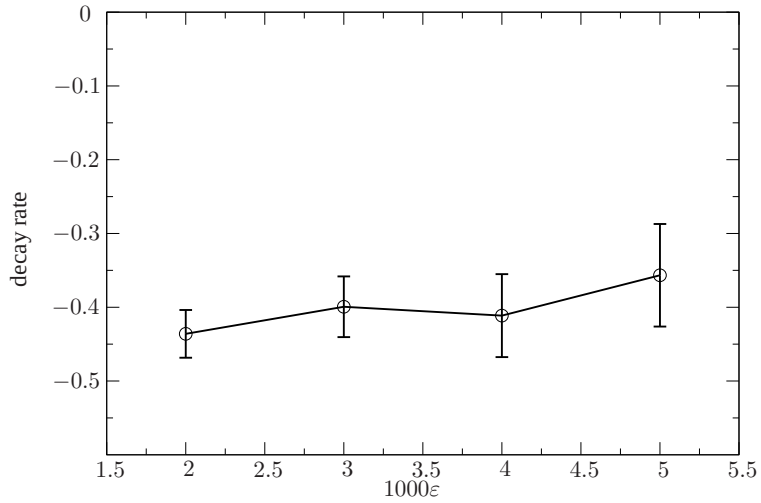

Figure 14. The decay rates of several simulations with $N = 15000$ and $\varepsilon = 0.002$ to 0.005.

and this leads to Eq. (10) with

$$\tau_r = \frac{\nu\xi(\varepsilon)}{4p(\varepsilon)P_{\mathrm{R}}(\xi)} \,. \tag{12}$$

Finally, using

$$P_{\mathrm{R}}(\xi) = \mathcal{O}(1/\xi) = \mathcal{O}\left(\varepsilon^{1/2}\right) \,, \tag{13}$$

Eq. (11) follows from Equations (12) and (13). □

8. The Aging Process

By the aging process, we mean the approach of the energy to its equilibrium value. Since the energy is by and large just the density $d(t)$ of defects we can formulate the result as

Estimate 8.1: *Under the assumptions $\varepsilon N > D_0$ and $\varepsilon < \varrho$ one has for the density d of defects:*

$$d(t) = \mathcal{O}(\left(\varepsilon^2 t\right)^{-2/5}) \,. \tag{14}$$

Note that this result differs from that proposed in [9], where the decay rate was given as $(\varepsilon^2 t)^{-1/2}$. This difference is caused by our observation that the diffusion constant of the defects actually depends on their density, because, if they are rarer, the pairs, which are the only ones able to move them around, need longer to find them.

Power decay rates are extremely hard to distinguish, but we have performed some tests which are illustrated in Fig. 14. They give a slight advantage to a decay of -0.4 as compared to -0.5.

Proof: We study the aging process by assuming that, in approach to equilibrium, the system is in a quasistationary state, with charge density $c = E/N$. Here, and in the sequel, time will be in units of $\tau = (3N - 6)/2$. Let $d(t)$ and $p(t)$ be the density of defects and pairs respectively. Then, up to terms of order $\mathcal{O}(\varepsilon^3)$ one has $c = d + 2p$.

As we will see in this section, the quasistationarity assumption simply means that the relaxation of the energy is a consequence of the *annihilation of colliding defects*. The

number of pairs is, up to fluctuations, essentially unchanged during the process we consider.

8.1. *Three timescales*

We saw that a fraction $1 - \mathcal{O}(\varepsilon)$ of all occurring flips in the system do not change the energy, and are either motions of pairs or blinkers. Of those, the only relevant ones are the wandering pairs, which induce diffusion of the defects as we have seen in Sect. 7. The discussion of the equilibrium probabilities apply also to states close to equilibrium, which is the regime we want to consider now.

The pair dynamics happens on the time scale $\tau_{\text{pair}} = \tau$ and it conserves both the number of pairs $p(t)$ and the number of defects $d(t)$.

The next slower time scale concerns creation and annihilation of pairs. Even though this changes $p(t)$, it conserves $d(t)$. Whenever one of these events happens, defects are pushed around by the pairs with some geometrically defined probability, and this leads to a diffusion, whose constant $D(t)$ measures this second time scale $\tau_{\text{diffusion}} = D^{-1}(t)$.

The third time scale τ_{meeting} is related to collision rate $\gamma(t)$ of defects; $\tau_{\text{meeting}} = \gamma^{-1}(t)$. They undergo a 2D random walk. Sooner or later, 2 defects of opposite charges will meet and will form a new pair which will run on timescale τ until it annihilates. In the regime we consider, *only* this sequence of events (collision and running pair) of the dynamics destroys 2 defects and, as a consequence, is responsible for the relaxation of the energy. Given the 3 time scales, the derivation of the decay rate is now rather straightforward.

8.2. *The quasistationarity assumption and the density of pairs*

By the previous discussion,

$$\tau_{\text{meeting}}(t) \gg \tau_{\text{diffusion}}(t) \gg \tau_{\text{pair}}(t) = \tau = 1 \, .$$

The orders of magnitude of these quantities near equilibrium are

$$\tau_{\text{meeting}}(t) = \mathcal{O}(\varepsilon^{-2} d^{-7/2}) \, , \quad \tau_{\text{diffusion}}(t) = \mathcal{O}(\varepsilon^{-2} d^{-1/2}), \quad \tau_{\text{pair}}(t) = 1 \, .$$

Consider a system for which, at time 0, $d(0) \gg 1$ and $p(0) \gg 1$. It is clear that the relaxation of pairs is much faster than that of defects. We will assume that pairs are always at equilibrium density, *i.e.*, that creation and destruction rates of pairs are equal and $p(t)$ is independent of t.

Remark 1 : The above discussion implies that $p(t)$ is constant over time intervals of order $\tau_{\text{meeting}}(t)$. In fact, both creation and annihilation events necessitate the presence of defects so that the creation and destruction rates of pairs will be linear in $d(t)$ at low density. This implies that p depends on t only through the value of $d(t)$. By abuse of notation, we will write $p(d)$ instead of $p\big(d(t)\big)$.

The creation rate of pairs is $12 d \varepsilon^2$ and the destruction rate is simply $p(d)/\tau_{\text{lifetime}}$. Therefore, by balancing the rates, we find:

$$p(d) = 12 d \, \tau_{\text{lifetime}} \, \varepsilon^2 \, . \tag{15}$$

Since a pair needs to diffuse from one defect to the other in order to annihilate, we estimate that $\tau_{\text{lifetime}} = \mathcal{O}\left(\xi^2\right) = \mathcal{O}(d^{-1})$. This implies that the density of pairs is $p(d) = \mathcal{O}(\varepsilon^2)$.

8.3. *The diffusion constant of single defects*

Repeating the arguments of Sect. 7 the average number of collisions ν and the average number of moved defects η are temperature independent constants. The diffusion constant of a defect is simply the probability that a given defect moves by one space unit during one time unit and it is given by

$$D \cdot d = \frac{2p(d)\, P_{\mathrm{R}}(\xi)\eta}{\nu^2 \xi^2} ,$$
$$D(d) = \mathcal{O}\big(p(d) \cdot P_{\mathrm{R}}(\mathcal{O}(d^{1/2}))\big) .$$

Using Equations (15) and (13), this leads to

$$D(d) = \mathcal{O}(\varepsilon^2 \cdot d^{1/2}) .$$

8.4. *Collision rate of single defects and relaxation coefficient*

The annihilation of 2 diffusive particles $A + B \to \emptyset$ has been studied in depth in [10–12]. Here, we use the mean field argument of [10], to deduce the collision rates. However, there will also be particle creation. On the other hand, *e.g.*, in [12] creation is indeed considered, but the study is for a fixed substrate, namely the lattice $\mathbb{Z}^2$, while our study is on a more floppy domain.

Given a 2D gas of 2 particles A and B of equal densities $d/2$ such that the diffusion constants $D_A = D_B = D$, it can be deduced from [10] that the collision rate γ is

$$\gamma(d) = \mathcal{O}(Dd^3) .$$

Extending this identity to a varying diffusion constant, we end up with

$$\dot{d} = -2\gamma(d) = -\mathcal{O}(\varepsilon^2 \cdot d^{7/2}) ,$$

where we assumed that we are far enough from equilibrium to neglect the creation rate of defects. □

Note that this result differs from that proposed in [9], where the decay rate was given as $(\varepsilon^2 t)^{-1/2}$. This difference is caused by our observation that the diffusion constant of the defects actually depends on their density, because, if they are rarer, the pairs, which are the only ones able to move them around need longer to find them.

Acknowledgements

This research was partially supported by the Fonds National Suisse.

References

[1] G. Schliecker, Adv. Physics 51 (2002) p.1319–1378.
[2] J.P. Eckmann, J. Stat. Phys. 129 (2007) p.289–309.
[3] T. Aste and D. Sherrington, J. Phys. A 32 (1999) p.7049–7056.
[4] E. Aharonov, E. Bouchbinder, H.G.E. Hentschel, V. Ilyin, N. Makedonska, I. Procaccia and N. Schupper, Europhysics Letters 77 (2007) p.56002.
[5] C. Godrèche, I. Kostov and I. Yekutieli, Phys. Rev. Lett. 69 (1992) p.2674–2677.
[6] E.R. Berlekamp, J.H. Conway and R.K. Guy *Winning ways for your mathematical plays. Vol. 2*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982 Games in particular.

[7] W. Feller *An introduction to probability theory and its applications. Vol. I*, John Wiley and Sons, Inc., New York, 1957 2nd ed.
[8] L. Davison and D. Sherrington, J. Phys. A 33 (2000) p.8615–8625.
[9] D. Sherrington, L. Davison, A. Buhot and J.P. Garraham, J. Physics: Condens, Matter 14 (2002) p.1673–1682.
[10] D. Toussaint and F. Wilczek, J. Chem. Phys. 78 (1983) p.2642–2647.
[11] B.P. Lee and J. Cardy, J. Stat. Phys. 80 (1995) p.971–1007.
[12] M. Sasada, ALEA Lat. Am. J. Probab. Math. Stat. 7 (2010) p.277–292.