A GENERAL APPROXIMATION OF QUANTUM GRAPH VERTEX COUPLINGS BY SCALED SCHRÖDINGER OPERATORS ON THIN BRANCHED MANIFOLDS

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Abstract. We demonstrate that any self-adjoint coupling in a quantum graph vertex can be approximated by a family of magnetic Schrödinger operators on a tubular network built over the graph. If such a manifold has a boundary, Neumann conditions are imposed at it. The procedure involves a local change of graph topology in the vicinity of the vertex; the approximation scheme constructed on the graph is subsequently 'lifted' to the manifold. For the corresponding operator a norm-resolvent convergence is proved, with the natural identification map, as the tube diameters tend to zero.

1. Introduction

The concept of quantum graph [EKK+08] serves as a laboratory to study quantum dynamics in situations when the configuration space has a complicated topology. At the same time, it is a useful tool in modelling numerous physical phenomena. To employ its full power, one should be able to understand the meaning of parameters associated with vertex coupling in such models, because one can typically associate many self-adjoint Hamiltonians with the same graph. An old and natural idea was to select plausible ones with the help of "fat-graph" approximations; the question to be answered, of course, is whether in this way one can obtain *all* the vertex couplings allowed by the sole requirement of probability current conservation at the graph vertices. The problem has attracted a lot of attention over the last decade; we refer to the monograph [P12] for an extensive bibliography.

As it is now well known the answer depends substantially on what boundary conditions one chooses for operators on the tube-like manifolds constructed over the graph "skeleton". The Dirichlet case is more difficult and to the date only some vertex couplings can be approximated in this way, cf. [P05, MV07, Gr08, ACF07, CE07, DC10] for more details. It requires an energy renormalisation; if one chooses the natural one which consists of subtracting the lowest transverse eigenvalue which blows up when the tube diameter ε tends to zero, a nontrivial limit is achieved when the fat graph from which one starts has a threshold resonance.

The situation is very different when the manifold boundary is either Neumann or absent. In this case no energy renormalisation is needed, of course. If the dynamics on the tubular network is described by the Laplace-Beltrami operator the limit is generically nontrivial and leads to the simplest coupling conditions conventionally labelled as Kirchhoff [FW93, Sa00, RS01, KuZ01, EP05, P06]. If one wants to get other vertex couplings, the approximation scheme has to be modified. The first step in this direction was undertaken in [EP09] when properly scaled potentials were added replacing the Laplacians by suitable Schrödinger operators. The strategy in this case was to proceed in two steps, first to construct an approximation on the graph itself and to "lift" subsequently the obtained procedure to the tubular manifold.

In this way we have been able in [EP09] to approximate two important coupling types usually referred to as δ and δ'_s. Referring to the graph approximation result obtained in [ET07] we conjectured existence of such approximation to any vertex coupling with real

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coefficients which covers all the couplings invariant with respect to the time reversal. The aim of the present paper is to show that one is not only able to prove the said conjecture but in fact can do better: following the "algebraic" work done in [CET10] we demonstrate here existence of a "fat-graph" approximation for *all* self-adjoint vertex couplings.

Let us recall briefly how the approximation constructed in [CET10] works, a detailed description will be given in Section 2 below. It has several steps:

(i) we change locally the graph topology disconnecting the edges and connecting the loose ends by addition finite edges the length of which tends to zero. Some of them may be missing, depending on the coupling we want to approximate
(ii) the additional edges will be coupled to the original ones by δ conditions of the strength dependent on the approximation parameter. We also add a parameter-dependent δ interaction to the centre of these finite edges
(iii) in order to accommodate the couplings with non-real coefficients we add magnetic fields described by (the tangent components of) appropriate vector potentials, also dependent on the approximation parameter

The main result of this paper consists of "lifting" this approximation to tubular networks and demonstrating that one can approximate in this way any self-adjoint vertex coupling. Since the approximation bears a local character we concentrate our attention on star graphs having a single vertex; an extension to general graphs satisfying suitable uniformity conditions can be performed in the same way as in [EP09] or [P12].

The paper is organised as follows: In the next section, we outline the approximation procedure on the graph level. In Section 3, we construct the graph-like manifold model. Moreover, we introduce the quadratic forms corresponding to our operators on the graph and the manifolds and relate them with the "free" operators, i.e the corresponding Laplacians. In Section 4 we briefly recall the convergence of operators and forms acting in different Hilbert spaces, apply the abstract conclusions to our situation here, and demonstrate our main result expressed in Theorem 4.7. In Section 5, we present some examples, including the case of a metric graph embedded in $\mathbb{R}^\nu$ when the manifold model is an ε-neighbourhood of the graph.

2. Approximation on the graph level

As we have indicated in the introduction the approximation is constructed in two steps. First we solve the problem on the graph level, and the obtained approximation is then "lifted" to network-type manifolds. The first part of this programme was realised in [CET10] and we summarise here the results as a necessary preliminary.

Any self-adjoint coupling in a vertex of degree n can be expressed through vertex conditions — one usually speaks about *admissible* conditions — which involve the boundary values $f(0), f'(0) \in \mathbb{C}^n$. They are conventionally written in the form

$$Af(0) + Bf'(0) = 0\,, \tag{2.1}$$

where A, B are $n \times n$ matrices such that the $n \times 2n$ matrix $(A|B)$ has maximum rank and AB^* is Hermitian, cf. [KS99]. A pair (A, B) describing a given coupling is naturally not unique and there are various ways how to remove the non-uniqueness, see e.g. [Ha00, Ku04]. The most suitable for our purpose is the one given by the following claim proved in [CET10]; it is simple but it requires an appropriate graph edge numbering.

Proposition 2.1. *For a quantum graph vertex of degree n, the following is valid:*
(a) If $S \in \mathbb{C}^{m\times m}$ with $m \le n$ is a Hermitian matrix and $T \in \mathbb{C}^{m\times(n-m)}$, then the equation

$$\begin{pmatrix} I^{(m)} & T \\ 0 & 0 \end{pmatrix} f'(0) = \begin{pmatrix} S & 0 \\ -T^* & I^{(n-m)} \end{pmatrix} f(0) \tag{2.2}$$

expresses admissible vertex conditions which make the graph Laplacian a self-adjoint operator.
(b) Conversely, for any self-adjoint vertex coupling there is a number $m \le n$ and a numbering of edges such that the coupling is described by the conditions (2.2) *with uniquely given matrices $T \in \mathbb{C}^{m\times(n-m)}$ and $S = S^* \in \mathbb{C}^{m\times m}$. If the edge numbering is given one can bring the coupling into the form* (2.2) *by a permutation $(1,\dots,n) \mapsto (\Pi(1),\dots,\Pi(n))$ of the edge indices with the matrices S, T uniquely determined by the permutation Π.*

Now can describe the approximation of such a general vertex coupling. For simplicity we consider a star graph of n semi-infinite edges; in view of the proposition we may suppose that the wave functions are coupled according to (2.2) renaming the edges if necessary. The construction has two main ingredients. First of all, we have to change locally the graph topology, adding vertices to the graph as well as new edges which would shrink to zero in the limit. In this way one is able to get (2.1) with *real* matrices A, B; to overcome this restriction we need to introduce also local magnetic fields, i.e. to place suitable vector potentials at the added edges.

The construction is sketched in Figure 1; we disconnect the edges of the star graph and connect their loose endpoints by line segments supporting appropriate operators according to the following rules:

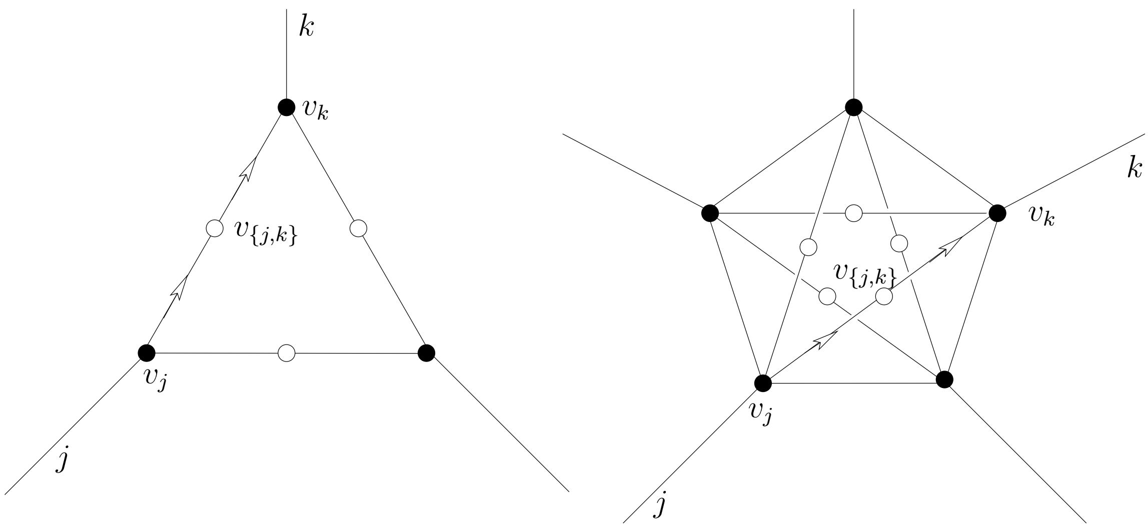

FIGURE 1. The approximation scheme for a vertex of degree $n = 3$ and $n = 5$. The inner edges are of length $2d$, some may be missing depending on the choice of the matrices S and T. The arrows symbolise the vector potential.

(i) As a convention, the rows of the matrix T are indexed from 1 to m, while the columns are indexed from $m+1$ to n. For the sake of brevity, we use in this section the symbol $\hat{n} := \{1,\dots,n\}$.
(ii) The external semi-infinite edges of the approximating graph, each parametrised by $s \in \mathbb{R}_+$ are at their endpoints v_j connected to the inner edges by δ coupling with the parameter $w_j(d)$ for each $j \in \hat{n}$ (see below).
(iii) Certain pairs v_j, v_k of external edge endpoints will be connected by segments (or *inner edges*, labelled by $\{j,k\}$) of length $2d$. This will be the case if one of the following conditions is satisfied, taking into account the convention (i):
 (a) $j \in \hat{m}$, $k \ge m+1$, and $T_{jk} \ne 0$ (or $j \ge m+1$, $k \in \hat{m}$, and $T_{kj} \ne 0$),
 (b) $j, k \in \hat{m}$ and $(\exists l \ge m+1)(T_{jl} \ne 0 \wedge T_{kl} \ne 0)$,
 (c) $j, k \in \hat{m}$, $S_{jk} \ne 0$, and the previous condition is not satisfied.

(iv) We denote the centre of such a connecting segment by $v_{\{j,k\}}$ and place there δ interaction with a parameter $w_{\{j,k\}}(d)$. We adopt another convention: the connecting edges will be regarded as union of two line segments of the length d, with the variable running from zero at $w_{\{j,k\}}$ to d at v_j or v_k.

(v) Finally, we put a vector potential on each connecting segment. What matters is its component tangential to the edge; we suppose it is constant along the edge and denote its value between the points $v_{\{j,k\}}$ and v_j as $A_{(j,k)}(d)$, and between the points $W_{\{j,k\}}$ and v_k as $A_{(k,j)}(d)$; recall that the two half-segments have opposite orientation, thus $A_{(k,j)}(d) = -A_{(j,k)}(d)$ holds for any pair $\{j,k\}$.

The choice of the dependence of $w_j(d)$, $w_{\{j,k\}}(d)$, and $A_{(j,k)}(d)$ on the length parameter d is naturally crucial; we will specify it below. We denote by $N_j \subset \hat{n}$ the set containing indices of all the external edges connected to the j-th one by an inner edge, i.e.

$$\begin{aligned} N_j &:= \{k\in\hat{m}:\, S_{jk}\neq 0\}\cup\{k\in\hat{m}:\, (\exists l\geq m+1)(T_{jl}\neq 0\wedge T_{kl}\neq 0)\}\\ &\quad\ \cup\{k\geq m+1:\, T_{jk}\neq 0\} \qquad\qquad \text{for } j\in\hat{m}\\ N_j &:= \{k\in\hat{m}:\, T_{kj}\neq 0\} \qquad\qquad\qquad \text{for } j\geq m+1 \end{aligned}$$

The definition of the set N_j has two simple consequences, namely

$$k\in N_j \Leftrightarrow j\in N_k \qquad \text{and} \qquad j\geq m+1\Rightarrow N_j\subset\hat{m}\,.$$

We employ the following symbols for wave function components on the edges: those on the j-th external one is denoted by f_j, while the wave function on the connecting segments is denoted $f_{(j,k)}$ on the interval between $v_{\{j,k\}}$ and v_j and $f_{(k,j)}$ on the other half of the segment; the conventions about parametrisation of the intervals have been specified above.

Next we shall write explicitly the coupling conditions involved in the above described scheme, first without the vector potentials; for simplicity we will often refrain from indicating the dependence of the parameters $w_j(d)$, $w_{\{j,k\}}(d)$ on the distance d. The δ interaction at the segment connecting the j-th and k-th outer edge (present for $j,k\in\hat{n}$ such that $k\in N_j$) is expressed through the conditions

$$f_{(j,k)}(0)=f_{(k,j)}(0)=:f_{\{j,k\}}(0)\,,\quad f'_{(j,k)}(0+)+f'_{(k,j)}(0+)=w_{\{j,k\}}f_{\{j,k\}}(0)\,,$$

while the δ coupling at the endpoint of the j-th external edge, $j\in\hat{n}$, means

$$f_j(0)=f_{(j,k)}(d)\quad\text{for all } k\in N_j,\quad f'_j(0)-\sum_{k\in N_j}f'_{(j,k)}(d-)=w_jf_j(0)\,.$$

It is not difficult to modify these conditions to include the vector potentials using a simple gauge transformation [CET10]: the continuity requirement is preserved, while the coupling parameter changes from $w_j(d)$ to $w_j(d)+\mathrm{i}\sum_{k\in N_j}A_{(j,k)}(d)$; in other words, the impact of the added potentials results into the phase shifts $dA_{(j,k)}(d)$ and $dA_{(k,j)}(d)$, respectively, on the appropriate parts of the connecting segments.

Using the above conditions one can find suitable candidates for $w_j(d)$, $w_{\{j,k\}}(d)$, and $A_{(j,k)}(d)$ by inserting the boundary values written as

$$\begin{gathered} f_{(j,k)}(d)=\mathrm{e}^{\mathrm{i}dA_{(j,k)}}\big(f_{(j,k)}(0)+df'_{(j,k)}(0)\big)+\mathscr{O}(d^2)\quad\text{and}\\ f'_{(j,k)}(d)=\mathrm{e}^{\mathrm{i}dA_{(j,k)}}f'_{(j,k)}(0)+\mathscr{O}(d) \end{gathered}$$

for any $j,k\in\hat{n}$ and fixing the d-dependence in such a way that the limit $d\to 0$ yields (2.2). The procedure is demanding and described in detail in [CET10], we will mention just its results. As for $A_{(j,k)}(d)$, we have the relations

$$A_{(j,k)}(d)=\begin{cases}\frac{1}{2d}\arg T_{jk} & \text{if } \operatorname{Re}T_{jk}\geq 0,\\ \frac{1}{2d}\big(\arg T_{jk}-\pi\big) & \text{if } \operatorname{Re}T_{jk}<0\end{cases} \tag{2.3a}$$

for all $j \in \hat{m}$, $k \in N_j \setminus \hat{m}$, while for $j \in \hat{m}$ and $k \in N_j \cap \hat{m}$ we put

$$A_{(j,k)}(d) = \begin{cases} \frac{1}{2d} \arg\Big(dS_{jk} + \sum_{l=m+1}^{n} T_{jl}\overline{T_{kl}}\Big) \\ \frac{1}{2d} \Big[\arg\Big(dS_{jk} + \sum_{l=m+1}^{n} T_{jl}\overline{T_{kl}}\Big) - \pi\Big] \end{cases} \tag{2.3b}$$

depending similarly on whether $\mathrm{Re}\,\big(dS_{jk} + \sum_{l=m+1}^{n} T_{jl}\overline{T_{kl}}\big)$ is nonnegative or not. Concerning $w_{\{j,k\}}(d)$, we require that

$$w_{\{j,k\}}(d) = \frac{1}{d}\left(-2 + \frac{1}{\langle T_{jk}\rangle}\right) \qquad \forall\, j \in \hat{m},\ k \in N_j \setminus \hat{m}\,. \tag{2.3c}$$

and

$$\frac{1}{2 + d \cdot w_{\{j,k\}}} = -\left\langle d \cdot S_{jk} + \sum_{l=m+1}^{n} T_{jl}\overline{T_{kl}} \right\rangle \qquad \forall\, j \in \hat{m},\ k \in N_j \cap \hat{m}\,, \tag{2.3d}$$

where we have employed the symbol $\langle c\rangle := \pm|c|$ for $\mathrm{Re}\, c \geq 0$ and $\mathrm{Re}\, c < 0$, respectively. Finally, the expressions for w_k are given by

$$w_k(d) = \frac{1 - |N_k| + \sum_{h=1}^{m} \langle T_{hk}\rangle}{d} \qquad \forall\, k \geq m+1\,, \tag{2.3e}$$

and

$$w_j(d) = S_{jj} - \frac{|N_j|}{d} - \sum_{\substack{k=1\\k\neq j}}^{m} \left\langle S_{jk} + \frac{1}{d}\sum_{l=m+1}^{n} T_{jl}\overline{T_{kl}} \right\rangle + \frac{1}{d}\sum_{l=m+1}^{n} (1 + \langle T_{jl}\rangle)\langle T_{jl}\rangle \tag{2.3f}$$

if $j \in \hat{m}$ and $k \in N_j \cap \hat{m}$.

Remark 2.2. For our later considerations it is crucial to know precisely the dependence of the magnetic and electric potentials $A_{(j,k)} = A_{(j,k)}(d)$ and $w_{\{j,k\}} = w_{\{j,k\}}(d)$ on the internal length d. We have $A_{(j,k)}(d) = \mathscr{O}(d^{-1})$ and $w_{\{j,k\}}(d) = \mathscr{O}(d^{-1})$ if $k \in N_j \setminus \hat{m}$. If $k \in N_j \cap \hat{m}$, then we have to distinguish two cases.

$$\sum_{l=m+1}^{n} T_{jl}\overline{T_{kl}} \neq 0, \tag{2.4}$$

then we again have $w_{\{j,k\}}(d) = \mathscr{O}(d^{-1})$. Otherwise, we collect another power of d^{-1} and obtain $w_{\{j,k\}}(d) = \mathscr{O}(d^{-2})$. We are not aware of any meaning of (2.4) in terms of the original vertex coupling or equivalent characterisations.

The choice of the parameters has been guided by formal considerations but it opens way to prove the convergence of the corresponding operators. Let us denote the Laplacian on the star graph $\Gamma(0)$ with the coupling (2.2) in the vertex as H^{star}, while H_d^{approx} will stand for the operators of the described approximating family; the symbols $R^{\mathrm{approx}}(z)$ and $R_d^{\mathrm{approx}}(z)$ will denote respectively the resolvents of those operators at the energy z outside the spectrum. We have to keep in mind that they act on different spaces: $R^{\mathrm{star}}(z)$ maps $\mathsf{L}_2(\Gamma(0))$ onto $\mathrm{dom}\, H^{\mathrm{star}}$, while the domain of $R_d^{\mathrm{approx}}(z)$ is $\mathsf{L}_2(\Gamma^{S,T}(d))$, where $\Gamma^{S,T}(d) = \Gamma(0) \sqcup \Gamma^{S,T}_{\mathrm{int}}(d)$ and where $\Gamma^{S,T}_{\mathrm{int}}(d)$ is the graph of connecting (inner) edges of length $2d$ described above. In order to compare the resolvents, we identify thus $R^{\mathrm{star}}(z)$ with the orthogonal sum

$$R_d^{\mathrm{star}}(z) := R^{\mathrm{star}}(z) \oplus 0 \tag{2.5}$$

adding the zero operator acting on $\mathsf{L}_2(\Gamma^{S,T}_{\mathrm{int}}(d))$. Then both operators act on the same space and one can estimate their difference; using explicit forms of the corresponding resolvent kernels one can check in a straightforward but rather tedious way the relation

$$\|R_d^{\mathrm{star}}(z) - R_d^{\mathrm{approx}}(z)\|_{\mathscr{B}_2} = \mathscr{O}(\sqrt{d}) \quad \text{as} \quad d \to 0+$$

for the Hilbert-Schmidt norm, see [CET10]. With the identification (2.5) in mind we can then state the indicated approximation result.

Theorem 2.3. *Let $w_j(d)$, $j \in \hat{n}$, $w_{\{j,k\}}(d)$, $j \in \hat{n}$, $k \in \mathbb{N}_j$ and $A^{(j,k)}(d)$ depend on the length d according to* (2.3a)–(2.3f). *Then the family H_d^{approx} converges to H^{star} in the norm-resolvent sense as $d \to 0+$.*

We present some examples of vertex coupling approximations in Section 5.2.

3. Approximation by Schrödinger operators on manifolds

Now we pass to the second step and show how the intermediate quantum graph constructed in Section 2 with δ couplings and vector potentials can be approximated by scaled magnetic Schrödinger operators on manifolds. For the sake of simplicity, we consider first an approximation using abstract manifolds without boundary, and discuss the case of a graph embedded in $\mathbb{R}^\nu$ subsequently in Section 5.1. To set up the approximation scheme, it is convenient to work with appropriate quadratic forms instead of the associated operators.

3.1. **The spaces and quadratic form on the graph level.** We start with the definition of the Hilbert space and quadratic form on the intermediate graph $\Gamma = \Gamma^{S,T}(d)$, where $d \in (0,1]$ denotes the approximation parameter of the previous section. It is convenient to modify slightly the convention (iv) concerning the internal edges $e = \{j,k\}$; from now on we shall consider each of them as a single edge with the δ interaction in the middle (i.e. at $v_{\{j,k\}}$) and identify this edge with the interval $[-d,d]$, oriented in such a way that the parameter increases from j to k if $j < k$. Concerning the vector potential, we set $A_e := A_{(j,i)} = -A_{(i,j)}$. For the sake of brevity, we use the symbols $A = (A_e)_e$, $w = (w_e, w_v)_{e,v}$ for the collections of magnetic potentials and δ interaction strengths, respectively. We will also often suppress in the sequel the dependence of the quantities on d, A, and w. With each outer edge $e \in \hat{n} = \{1,\dots,n\}$, we associate $I_e := [0,\infty)$, and for each inner edge $e \in \binom{\hat{n}}{2} = \{\,\{j,k\} \,|\, 1 \le j < k \le n\,\}$, we set $I_e = I_e(d) = [-d,d]$. As the Hilbert and Sobolev spaces on a fixed edge needed in our approximation we set

$$\mathscr{H}_e := \mathsf{L}_2(I_e) \qquad \text{and} \qquad \mathscr{H}_e^1 := \mathsf{H}^1(I_e),$$

where $\mathsf{L}_2(I)$ and $\mathsf{H}^1(I)$ denote as usual the space of square integrable functions and of once weakly differentiable and square integrable functions on the interval I, respectively.

For all the quadratic forms defined below, the domains consist of elements of $\mathscr{H}_e^1$. With the described parametrisation of an inner edge $e = \{j,k\}$ with $i < k$ the corresponding quadratic form is

$$\check{\mathfrak{h}}_e(f_e) := \int_{-d}^{d} \bigl|f_e'(s) + \mathrm{i} A_e f_e(s)\bigr|^2 \,\mathrm{d}s + w_e \bigl|f_e(0)\bigr|^2.$$

This form corresponds to the Laplacian on the edge with the magnetic potential A_e and the δ interaction at the point $s = 0$. It is convenient to introduce also a quadratic form which includes the effect of the δ interactions at the edge endpoints, namely

$$\mathfrak{h}_e(f_e) := \check{\mathfrak{h}}_e(f_e) + \frac{w_j}{|N_j|} \cdot \bigl|f_e(-d)\bigr|^2 + \frac{w_k}{|N_k|} \cdot \bigl|f_e(d)\bigr|^2.$$

On an outer edge, we simply set

$$\mathfrak{h}_e(f_e) := \int_0^\infty \bigl|f_e'(s)\bigr|^2 \,\mathrm{d}s.$$

The full Hilbert and Sobolev spaces are

$$\mathscr{H} := \bigoplus_e \mathscr{H}_e \qquad \text{and} \qquad \mathscr{H}^1 := \bigoplus_e \mathscr{H}_e^1 \cap \mathsf{C}(\Gamma),$$

where the sum runs over all the inner and outer edges. More explicitly, the Sobolev space $\mathscr{H}^1$ consists of all functions in $\mathsf{H}^1(I_e)$ on each edge, which are continuous on Γ, i.e. which have a common value

$$f(v) := f_e(v) := \begin{cases} f_e(0), & \text{if } e = j \text{ is an outer edge,} \\ f_e(-d), & \text{if } e = \{j,k\} \sim v = v_j \text{ is an inner edge, } j < k, \\ f_e(d), & \text{if } e = \{j,k\} \sim v = v_k \text{ is an inner edge,} \end{cases}$$

for all edges $e \sim v$, i.e. adjacent with v.

The quadratic form on the intermediate graph $\Gamma(d)$ is given by

$$\mathfrak{h}(f) := \sum_e \mathfrak{h}_e(f_e)$$

for $f = (f_e)_e \in \mathscr{H}^1$; the corresponding operator is the one described in Section 2 with δ interactions of strength w_j at vertex v_j and of strength w_e in the middle of the inner edge $e = \{j,k\}$, as well as vector potential $A_{(j,k)}$ supported by this edge.

For comparison reasons, we also need the *free* quadratic form, without both the magnetic potentials and the δ interactions, which is given by

$$\mathfrak{d}_e(f_e) := \int_{I_e} |f_e(s)|^2 \,\mathrm{d}s \qquad \text{and} \qquad \mathfrak{d}(f) := \sum_e \mathfrak{d}_e(f_e)$$

with the same domains as $\mathfrak{h}_e$ and $\mathfrak{h}$, respectively. It is easy to see that $\mathfrak{d}$ is a *closed* quadratic form, i.e. that $\operatorname{dom}\mathfrak{d} = \mathscr{H}^1$ with the norm given by $\|f\|^2_{\mathscr{H}^1} := \|f\|^2 + \mathfrak{d}(f)$ is complete, and therefore itself a Hilbert space. The operator corresponding to $\mathfrak{d}$ is the *free Laplacian* on $\Gamma(d)$, often also called *Kirchhoff Laplacian* on the graph.

Proposition 3.1.

(i) *The quadratic form $\mathfrak{h}$ is relatively form-bounded with respect to $\mathfrak{d}$ with relative bound zero. More precisely, for any $\eta > 0$ there is a constant $C_\eta > 0$ depending only on η, d, $\overline{A} := \max_e |A_e|$, and $\overline{w} := 3\max_{e,v}\{|w_e|, |w_v|\}$ such that*

$$\bigl|\mathfrak{h}(f) - \mathfrak{d}(f)\bigr| \le \eta\,\mathfrak{d}(f) + C_\eta \|f\|^2.$$

In particular, $\mathfrak{h}$ is also a closed form.

(ii) *We have $\mathfrak{d}(f) \le 2\bigl(\mathfrak{h}(f) + C_{1/2}\|f\|^2\bigr)$.*

Proof. (i) On the interval $[-d,d]$ we have the following standard estimate

$$\bigl|f(s)\bigr|^2 \le a\|f'\|^2 + \frac{2}{a}\|f\|^2 \tag{3.1}$$

for all $s \in [-d,d]$, $0 < a \le d$, and $f \in \mathsf{H}^1(-d,d)$. Moreover, for any $\eta > 0$ and $a, b \in \mathbb{R}$ we have

$$\frac{1}{1+\eta}\cdot a^2 - \frac{1}{\eta}\cdot b^2 \le (a+b)^2 \le (1+\eta)\cdot a^2 + \Bigl(1 + \frac{1}{\eta}\Bigr)\cdot b^2. \tag{3.2}$$

In particular, for an inner edge $e = \{j,k\}$ we have

$$\begin{aligned} \mathfrak{h}_e(f) - \mathfrak{d}_e(f) &= \|f' + \mathrm{i}A_e f\|^2 - \|f'\|^2 + w_e\bigl|f(0)\bigr|^2 + \frac{w_j}{|N_j|}\bigl|f(-d)\bigr|^2 + \frac{w_k}{|N_k|}\bigl|f(d)\bigr|^2 \\ &\le \Bigl(\frac{\eta}{2} + \overline{w}_e a\Bigr)\|f'\|^2 + \Bigl(\Bigl(1 + \frac{2}{\eta}\Bigr)|A_e|^2 + \frac{2\overline{w}_e}{a}\Bigr)\|f\|^2 \end{aligned}$$

on $[-d,d]$ using (3.1) with $s \in \{-d,0,d\}$ and the upper estimate in (3.2) with $\eta/2$ instead of η, where

$$\overline{w}_e := |w_e| + \frac{|w_j|}{|N_j|} + \frac{|w_k|}{|N_k|}\,.$$

Choosing

$$a := \min\Bigl\{\frac{\eta}{2\overline{w}_e}, d\Bigr\} \tag{3.3}$$

we can estimate the coefficient of $\mathfrak{d}_e(f) = \|f'\|^2$ by η.

For the opposite inequality, we have

$$\mathfrak{d}_e(f) - \mathfrak{h}_e(f) \le \Big(1 - \frac{1}{1+\eta/2} + \overline{w}_e a\Big)\|f'\|^2 + \Big(\frac{2|A_e|^2}{\eta} + \frac{2\overline{w}_e}{a}\Big)\|f\|^2$$

using now the lower estimate in (3.2) with $\eta/2$. In particular, with a as in (3.3) and with $1-(1+\eta/2)^{-1} \le \eta/2$ we can again estimate the coefficient of $\mathfrak{d}_e(f) = \|f'\|^2$ by η. As constant $C_{\eta,e}$ on each edge, we can therefore choose

$$C_{\eta,e} := \Big(1 + \frac{2}{\eta}\Big)|A_e|^2 + \max\Big\{\frac{4\overline{w}_e^2}{\eta}, \frac{2\overline{w}_e}{d}\Big\}. \tag{3.4}$$

Summing up all contributions for each edge, we can choose $C_\eta := \max_e C_{\eta,e}$, and this constant depends only on η, d, $\overline{A}$ and $\overline{w}$.

(ii) follows with $\eta = 1/2$. In particular,

$$C_{1/2} = C_{1/2}(d, A, w) = \mathscr{O}\big(\overline{A}^2\big) + \mathscr{O}\big(\overline{w}^2\big) + \mathscr{O}\Big(\frac{\overline{w}}{d}\Big). \tag{3.5}$$

□

3.2. **The spaces and quadratic form on the manifold level.** We now define the manifold model as in [EP09]. For a given $\varepsilon \in (0, d]$ we associate a connected $(m+1)$-dimensional manifold X_ε to the graph $\Gamma(d)$ as follows: To the edge e and the vertex v we associate the Riemannian manifolds

$$X_{\varepsilon,e} := I_e \times \varepsilon Y_e \quad \text{and} \quad X_{\varepsilon,v} := \varepsilon X_v, \tag{3.6}$$

respectively, where εY_e is a manifold Y_e of dimension $m > 0$ (called *transverse manifold)* equipped with the metric $h_{\varepsilon,e} := \varepsilon^2 h_e$. More precisely, the so-called *edge neighbourhood* $X_{\varepsilon,e}$ and the *vertex neighbourhood* $\varepsilon X_{\varepsilon,v}$ carry the metrics $g_{\varepsilon,e} = \mathrm{d}^2 s + \varepsilon^2 h_e$ and $g_{\varepsilon,v} = \varepsilon^2 g_v$, where h_e and g_v are ε-independent metrics on Y_e and X_v, respectively. We assume that for each edge e adjacent to v, the vertex neighbourhood $X_{\varepsilon,v}$ has a boundary component $\partial_e X_{\varepsilon,v} = \varepsilon \partial_e X_v$ *isometric* to the scaled transverse manifold εY_e. Fixing such an isometry and assuming that $X_{\varepsilon,v}$ has product structure near each of the boundary components $\partial_e X_{\varepsilon,v}$, we identify the boundary component $\partial_v X_{\varepsilon,e} = \{0\} \times \varepsilon Y_e$ of the edge neighbourhood $X_{\varepsilon,e}$ with $\partial_e X_{\varepsilon,v}$.

For simplicity, we assume here that the transversal manifold Y_e has no boundary and that its volume is normalised, i.e. $\operatorname{vol}_m Y_e = 1$.

On a Riemannian manifold X, we denote by $\mathsf{L}_2(X)$ the Hilbert space of square integrable functions on X with respect to the natural measure induced by the Riemannian metric. Moreover, we denote by $\mathsf{H}^1(X)$ the completion of the space of smooth functions with compact support (not necessarily vanishing on the boundary of X) with respect to the norm given by $\|u\|^2_{\mathsf{H}^1(X)} := \|u\|^2_{\mathsf{L}_2(X)} + \|\mathrm{d}u\|^2_{\mathsf{L}_2(X)}$, where $\mathrm{d}u$ denotes the exterior derivative of u on X.

We set

$$\mathscr{H}_{\varepsilon,e} := \mathsf{L}_2(I_e, \mathscr{K}_{\varepsilon,e}), \quad \mathscr{K}_{\varepsilon,e} := \mathsf{L}_2(\varepsilon Y_e) \quad \text{and} \quad \mathscr{H}_{\varepsilon,v} := \mathsf{L}_2(X_{\varepsilon,v}).$$

We will often identify an L_2-function u on $X_{\varepsilon,e}$ with the vector-valued function $I_e \to \mathscr{K}_{\varepsilon,e}$, $s \mapsto u(s) := u(s,\cdot)$.

For each inner edge, we set

$$\mathfrak{h}_{\varepsilon,e}(u_e) := \int_{-d}^{d} \Big(\big\|u_e'(s) + \mathrm{i}A_e u_e(s)\big\|^2 + \mathfrak{k}_{\varepsilon,e}(u_e(s))\Big)\,\mathrm{d}s + \frac{w_e}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \big\|u_e(s)\big\|^2\,\mathrm{d}s,$$

where u_e' denotes the derivative with respect to the longitudinal variable s and where

$$\mathfrak{k}_{\varepsilon,e}(\varphi) := \|\mathrm{d}_{Y_e}\varphi\|^2_{\mathsf{L}_2(\varepsilon Y_e)}.$$

Here, $\mathrm{d}_{Y_e}\varphi$ is the exterior derivative on the manifold Y_e. For each outer edge we set

$$\mathfrak{h}_{\varepsilon,e}(u_e) := \int_0^\infty \big(\|u_e'(s)\|^2_{\mathscr{K}_{\varepsilon,e}} + \mathfrak{k}_{\varepsilon,e}(u_e(s))\big)\,\mathrm{d}s$$

In both cases, $u_e \in \mathscr{H}^1_{\varepsilon,e} = \mathsf{H}^1(X_{\varepsilon,e})$. On a vertex neighbourhood, we set

$$\mathfrak{h}_{\varepsilon,v}(u_e) := \|\mathrm{d}_{X_v} u_v\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} + \frac{w_v}{\varepsilon \operatorname{vol} X_v}\|u_v\|^2_{\mathsf{L}_2(X_{\varepsilon,v})}.$$

The total Hilbert spaces here are

$$\mathscr{H}_\varepsilon := \bigoplus_e \mathscr{H}_{\varepsilon,e} \oplus \bigoplus_v \mathscr{H}_{\varepsilon,v} \qquad \text{and} \qquad \mathscr{H}^1_\varepsilon := \mathsf{H}^1(X_\varepsilon), \tag{3.7}$$

where the sum runs over all inner and outer edges. Now, the quadratic form on the manifold X_ε is given by

$$\mathfrak{h}_\varepsilon(u) := \sum_e \mathfrak{h}_{\varepsilon,e}(u_e) + \sum_v \mathfrak{h}_{\varepsilon,v}(u_e)$$

for $u \in \mathscr{H}^1$ with the obvious notation $u_e := u\restriction_{X_{\varepsilon,e}}$ and $u_v := u\restriction_{X_{\varepsilon,v}}$. The corresponding operator is a magnetic Schrödinger operator on X_ε with (constant) potential $w_e/(2\varepsilon)$ on $[-\varepsilon,\varepsilon] \times \varepsilon Y_e$ in the middle of an edge neighbourhood and $w_v/(\varepsilon \operatorname{vol} X_v)$ on each vertex neighbourhood. For the use of non-constant potentials we refer to [EP09].

For comparison reasons, we also need the *free* quadratic form (i.e. without magnetic and electric potentials), given by $\mathfrak{d}_{\varepsilon,e}(u_e) = \|\mathrm{d}u_e\|^2_{\mathsf{L}_2(X_{\varepsilon,e})}$, $\mathfrak{d}_{\varepsilon,v}(u_v) = \|\mathrm{d}u_v\|^2_{\mathsf{L}_2(X_{\varepsilon,v})}$ and

$$\mathfrak{d}_\varepsilon(u) := \|\mathrm{d}u\|^2_{\mathsf{L}_2(X_\varepsilon)} = \sum_e \mathfrak{d}_{\varepsilon,e}(u_e) + \sum_v \mathfrak{d}_{\varepsilon,v}(u_v)$$

with the same domains as for $\mathfrak{h}_{\varepsilon,e}$, $\mathfrak{h}_{\varepsilon,v}$ and $\mathfrak{h}_\varepsilon$. Since we define $\mathscr{H}^1 = \mathsf{H}^1(X_\varepsilon)$ as the completion of smooth functions with compact support with respect to the norm $\|u\|^2_{\mathscr{H}^1_\varepsilon} := \mathfrak{d}_\varepsilon(u) + \|u\|^2$, the quadratic form $\mathfrak{d}_\varepsilon$ is *closed*. The operator corresponding to $\mathfrak{d}$ is the *Laplacian* on X_ε.

Proposition 3.2.

(i) *The quadratic form $\mathfrak{h}_\varepsilon$ is relatively form-bounded with respect to $\mathfrak{d}_\varepsilon$ with relative bound zero. More precisely, for any $\eta > 0$ there is a constant $\widetilde{C}_\eta \ge C_\eta > 0$ depending only on η, d, $\overline{A} := \max_e |A_e|$, $\overline{w} := 3\max_{e,v}\{|w_e|, |w_v|\}$ and X_v such that*

$$\big|\mathfrak{h}_\varepsilon(u) - \mathfrak{d}_\varepsilon(u)\big| \le \eta\,\mathfrak{d}_\varepsilon(u) + \widetilde{C}_\eta \|u\|^2 \tag{3.8}$$

for all $0 < \varepsilon \le \varepsilon_0$, where $\varepsilon_0 := \eta c(v)/|w_v|$ and where $c(v)$ is a constant depending only on X_v. In particular, $\mathfrak{h}_\varepsilon$ is also a closed form.

(ii) *We have $\mathfrak{d}_\varepsilon(u) \le 2\big(\mathfrak{h}_\varepsilon(u) + \widetilde{C}_{1/2}\|u\|^2\big)$.*

Proof. The proof is very similar to the one of Proposition 3.1. For (i), we have the following vector-valued version of (3.1), namely,

$$\big\|u_e(s)\big\|^2_{\mathscr{K}_{\varepsilon,e}} \le a\|u_e'\|^2_{\mathscr{H}_{\varepsilon,e}} + \frac{2}{a}\|u_e\|^2_{\mathscr{H}_{\varepsilon,e}} \tag{3.9}$$

for all $s \in [-d,d]$, $0 < a \le d$ and $u \in \mathsf{H}^1(X_{\varepsilon,e})$. In particular, for an inner edge $e = \{j,k\}$ we have

$$\big|\mathfrak{h}_{\varepsilon,e}(u_e) - \mathfrak{d}_{\varepsilon,e}(u_e)\big| \le \eta\|u_e'\|^2_{\mathscr{H}_{\varepsilon,e}} + C_{\eta,e}\|u_e\|^2_{\mathscr{H}_{\varepsilon,e}}$$

with $C_{\eta,e}$ as in (3.4).

On a vertex neighbourhood, we have

$$\begin{aligned}\bigl|\mathfrak{h}_{\varepsilon,v}(u_v)-\mathfrak{d}_{\varepsilon,v}(u_v)\bigr| &= \frac{|w_v|}{\varepsilon\operatorname{vol}X_v}\|u_v\|^2_{\mathscr{H}_{\varepsilon,v}}\\ &\le \frac{|w_v|}{\varepsilon\operatorname{vol}X_v}\Bigl(\varepsilon^2 C(v)\|\mathrm{d}u_v\|^2_{\mathsf{L}_2(X_{\varepsilon,v})}+4\varepsilon c_{\mathrm{vol}}(v)\sum_{e\sim v}\Bigl(a\|u_e'\|^2_{\mathscr{H}_{\varepsilon,e}}+\frac{2}{a}\|u_e\|^2_{\mathscr{H}_{\varepsilon,e}}\Bigr)\Bigr)\end{aligned}$$

for $0<a\le d$ using [EP09, Lem. 2.9], where $c_{\mathrm{vol}}(v):=\operatorname{vol}X_v/\operatorname{vol}_m\partial X_v$ and $C(v)$ is another constant depending only on X_v, see [EP09] for details. Setting

$$a:=\min\{d,\eta\operatorname{vol}X_v/(4c_{\mathrm{vol}}(v)|w_v|\}\quad\text{and}\quad\varepsilon_0:=\min_v\frac{\operatorname{vol}X_v}{|w_v|C(v)},$$

and summing up all contributions, we can choose $\widetilde{C}_\eta>0$ such that (3.8) holds for all $0<\varepsilon\le\varepsilon_0$ with

$$\widetilde{C}_\eta=\widetilde{C}_\eta(d,A,w)=\mathscr{O}\Bigl(\overline{A}^2\Bigl(1+\frac{1}{\eta}\Bigr)\Bigr)+\mathscr{O}\Bigl(\frac{\overline{w}^2}{\eta}\Bigr)+\mathscr{O}\Bigl(\frac{\overline{w}}{d}\Bigr)\tag{3.10}$$

and the error term depend additionally only on X_v. The remaining assertion (ii) follows as before. □

4. Convergence of the operators

4.1. **Norm convergence of operators and forms acting in different Hilbert spaces.** Let us briefly review the concept of norm convergence of operators acting in different Hilbert spaces introduced first in [P06, App.]. A general spectral theory for quasi-unitary equivalent operators is developed in a more elaborated version in [P12, Ch. 4], see also [EP09].

Let $\mathscr{H}$ and $\mathscr{H}^1$ be Hilbert spaces such that $\mathscr{H}^1$ is a dense subspace of $\mathscr{H}$ with $\|f\|_{\mathscr{H}}\le\|f\|_{\mathscr{H}^1}$ and similarly for $\widetilde{\mathscr{H}}^1\subset\widetilde{\mathscr{H}}^1$. Let $\mathfrak{h}$ and $\widetilde{\mathfrak{h}}$ be closed, quadratic forms, semi-bounded from below with domain $\mathscr{H}^1$ and $\widetilde{\mathscr{H}}^1$, respectively.

Let $\delta>0$. We say that $\mathfrak{h}$ and $\widetilde{\mathfrak{h}}$ are *δ-quasi-unitarily equivalent*[1] if there are so-called *identification operators*

$$J\colon\mathscr{H}\longrightarrow\widetilde{\mathscr{H}},\quad J^1\colon\mathscr{H}^1\longrightarrow\widetilde{\mathscr{H}}^1\quad\text{and}\quad J'^1\colon\widetilde{\mathscr{H}}^1\longrightarrow\mathscr{H}^1,$$

such that these operators are *δ-quasi unitary*, i.e.

$$\|Jf-J^1f\|^2\le\delta^2\|f\|^2_{\mathscr{H}^1},\qquad\|J^*u-J'^1u\|^2\le\delta^2\|u\|^2_{\widetilde{\mathscr{H}}^1},\tag{4.1a}$$

$$\|J^*Jf-f\|^2\le\delta^2\|f\|^2_{\mathscr{H}^1},\qquad\|JJ^*u-u\|^2\le\delta^2\|u\|^2_{\widetilde{\mathscr{H}}^1},\tag{4.1b}$$

$$\bigl|\mathfrak{h}(J'^1u,f)-\widetilde{\mathfrak{h}}(u,J^1f)\bigr|\le\delta\|u\|_{\widetilde{\mathscr{H}}^1}\|f\|_{\mathscr{H}^1}\tag{4.1c}$$

for f and u in the appropriate spaces. The attribute *δ-quasi-unitary* refers to the fact that we have a quantitative generalisation of unitary operators. In particular, if $\delta=0$, then a δ-quasi-unitary operator is just unitary.

On the operator level, we have the following definition: Denote by H and $\widetilde{H}$ the (self-adjoint) operators associated to $\mathfrak{h}$ and $\widetilde{\mathfrak{h}}$. We say that H and $\widetilde{H}$ are *δ-quasi-unitarily equivalent* (see again Footnote 1) if there is an identification operator $J\colon\mathscr{H}\longrightarrow\widetilde{\mathscr{H}}$ such that

$$\bigl\|(\mathrm{id}-J^*J)R^\pm\bigr\|\le\delta,\quad\bigl\|(\mathrm{id}-JJ^*)\widetilde{R}^\pm\bigr\|\le\delta\quad\text{and}\quad\bigl\|JR^\pm-\widetilde{R}^\pm J\bigr\|\le\delta,\tag{4.2}$$

[1] We warn the reader that in [P12] the notion "δ-quasi-unitary equivalent" is defined in a slightly more general way (allowing e.g. a second identification operator $J'\colon\widetilde{\mathscr{H}}\longrightarrow\mathscr{H}$ such that $\|J^*-J'\|\le\delta$ to cover some more general situations). This should not cause any confusion here.

where $\|\cdot\|$ denotes the operator norm, and where $R^\pm := (H \mp \mathrm{i})^{-1}$ and $\widetilde{R}^\pm := (\widetilde{H} \mp \mathrm{i})^{-1}$ denote the resolvents, respectively. The resolvent estimates are supposed to hold for both signs.

We have the following relation between the quasi unitary equivalence for forms and operators. For convenience of the reader, we give a short proof of the first assertion here. The remaining assertions follow from abstract theory, see [P06, App. A] or [P12, Ch. 4].

Theorem 4.1. *Let $\delta > 0$ and $C \geq 1$. Assume that $\mathfrak{h}$ and $\widetilde{\mathfrak{h}}$ are δ-quasi-unitarily equivalent closed quadratic forms such that*

$$\|f\|^2_{\mathscr{H}^1} \leq 2(\mathfrak{h}(f) + C\|f\|^2) \quad \text{and} \quad \|u\|^2_{\widetilde{\mathscr{H}}^1} \leq 2(\widetilde{\mathfrak{h}}(u) + C\|u\|^2)$$

for all $f \in \mathscr{H}^1$ and $u \in \widetilde{\mathscr{H}^1}$. Then the following assertions hold:

(i) *The associated operators H and $\widetilde{H}$ are $(12C\delta)$-quasi-unitarily equivalent.*

(ii) *There is a universal constant $c(z) > 0$ depending only on z such that*

$$\|J(H-z)^{-1} - (\widetilde{H}-z)^{-1}J\| \leq c(z)C\delta, \tag{4.3a}$$

$$\|J(H-z)^{-1}J^* - (H_\varepsilon - z)^{-1}\| \leq c(z)C\delta \tag{4.3b}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, we can replace the function $\varphi(\lambda) = (\lambda - z)^{-1}$ in $\varphi(H) = (H-z)^{-1}$ etc. by any measurable, bounded function converging to a constant as $\lambda \to \infty$ and being continuous in a neighbourhood of $\sigma(H)$.

(iii) *Assume that $\widetilde{H} = H_\varepsilon$ is δ_ε-unitarily equivalent with H, where $\delta_\varepsilon \to 0$, then the spectrum of H_ε converges to the spectrum of H uniformly on any finite energy interval. The same is true for the essential spectrum.*

(iv) *Assume as before that $\widetilde{H} = H_\varepsilon$ is δ_ε-unitarily equivalent with H, where $\delta_\varepsilon \to 0$, then for any $\lambda \in \sigma_{\mathrm{disc}}(H)$ there exists a family $\{\lambda_\varepsilon\}_\varepsilon$ with $\lambda_\varepsilon \in \sigma_{\mathrm{disc}}(H_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$. Moreover, the multiplicity is preserved. If λ is a simple eigenvalue with normalised eigenfunction φ, then for ε small enough there exists a family of simple normalised eigenfunctions $\{\varphi_\varepsilon\}_\varepsilon$ of H_ε such that*

$$\|J\varphi - \varphi_\varepsilon\|_{\mathsf{L}_2(X_\varepsilon)} \to 0$$

holds as $\varepsilon \to 0$.

Proof. (i) From our assumption, we have

$$\|f\|^2_{\mathscr{H}^1} \leq 2(\mathfrak{h}(f) + C\|f\|^2) = 2\big|\mathfrak{h}(f) + \|f\|^2\big| + 2(C-1)\|f\|^2.$$

Moreover, the first term can be estimated as

$$\begin{aligned}
\big|\mathfrak{h}(f) + \|f\|^2\big|^2 &\leq 2\big(\mathfrak{h}(f)^2 + \|f\|^4\big) \\
&= 2\big|\mathfrak{h}(f) - \mathrm{i}\|f\|^2\big|\big|\mathfrak{h}(f) + \mathrm{i}\|f\|^2\big| \\
&= 2\big|\langle (H \mp \mathrm{i})f, f\rangle\big|\big|\langle f, (H \mp \mathrm{i})f\rangle\big| \\
&\leq 2\|f\|^2\|(H \mp \mathrm{i})f\|^2 \leq 2\|(H \mp \mathrm{i})f\|^4
\end{aligned}$$

using $\|(H \mp \mathrm{i})^{-1}\| \leq 1$ at the last step. In particular, we have

$$\|f\|^2_{\mathscr{H}^1} \leq (2\sqrt{2} + 2C - 2)\|(H \mp \mathrm{i})f\|^2 \leq 4C\|(H \mp \mathrm{i})f\|^2 \tag{4.4}$$

since $2\sqrt{2} - 2 \leq 2 \leq 2C$. Similarly, we can show the same estimate for u, and we have

$$\|f\|_{\mathscr{H}^1} \leq 2\sqrt{C}\|(H \mp \mathrm{i})f\| \quad \text{and} \quad \|u\|_{\widetilde{\mathscr{H}}^1} \leq 2\sqrt{C}\|(\widetilde{H} \mp \mathrm{i})u\|. \tag{4.5}$$

Therefore, we conclude

$$\|f - J^*Jf\| \leq \delta\|f\|_{\mathscr{H}^1} \leq 2\sqrt{C}\delta\|(H \mp \mathrm{i})f\|$$

by (4.1b), and in particular, $\|(\mathrm{id} - J^*J)R^\pm\| \leq 2\sqrt{C}\delta$. The second norm estimate in (4.2) follows similarly.

For the last norm estimate of the quasi-unitary equivalence of the operators in (4.2), set $f := R^{\pm} g \in \operatorname{dom} H$ and $u := \widetilde{R}^{\mp} v \in \operatorname{dom} \widetilde{H}$. Then we have

$$\begin{aligned}\langle (JR^{\pm} - \widetilde{R}^{\pm} J) g, v\rangle &= \langle Jf, v\rangle - \langle g, J^* u\rangle \\ &= \langle (J - J^1) f, v\rangle + \big(\langle J^1 f, (\widetilde{H} \pm \mathrm{i}) u\rangle - \langle (H \mp \mathrm{i}) f, J'^1 u\rangle\big) \\ &\qquad\qquad + \langle g, (J'^1 - J^*) u\rangle \\ &= \langle (J - J^1) f, v\rangle + \big(\widetilde{\mathfrak{h}}(J^1 f, u) - \mathfrak{h}(f, J'^1 u)\big) + \langle g, (J'^1 - J^*) u\rangle \\ &\qquad\qquad \mp \mathrm{i}\big(\langle (J^1 - J) f, u\rangle + \langle f, (J^* - J'^1) u\rangle\big),\end{aligned}$$

and therefore

$$\big|\langle (JR^{\pm} - \widetilde{R}^{\pm} J) g, v\rangle\big| \le (2\sqrt{C} + 4C + 3 \cdot 2\sqrt{C})\delta \|g\| \|v\| \le 12 C\delta \|g\| \|v\| \tag{4.6}$$

using (4.1) and (4.5).

Once we have the estimates of the quasi-unitary equivalence in (4.2), the remaining assertions follow as in [P06, App. A] or [P12, Ch. 4]. ☐

We remark that the convergence of higher-dimensional eigenspaces is also valid, however, it requires some technicalities which we skip here.

Remark 4.2. Note that we only obtain the quasi-unitary equivalence of the operators with a factor C and not $\sqrt{C}$. This is due to the fact that from (4.1c), we collect two factors $2\sqrt{C}$ for the estimates $\|R^{\mp} g\|_{\mathscr{H}^1} \le 2\sqrt{C}\|g\|_{\mathscr{H}}$ and $\|\widetilde{R}^{\mp} v\|_{\widetilde{\mathscr{H}}^1} \le 2\sqrt{C}\|v\|_{\widetilde{\mathscr{H}}}$ in (4.6).

4.2. **Quasi-unitary equivalence between the graph and manifold forms.** We now apply the abstract results of the previous section to our problem where

$$\mathscr{H} := \mathsf{L}_2(\Gamma^{S,T}(d)), \quad \mathscr{H}^1 := \mathsf{H}^1(\Gamma^{S,T}(d)), \quad \widetilde{\mathscr{H}} := \mathsf{L}_2(X_\varepsilon), \quad \widetilde{\mathscr{H}}^1 := \mathsf{H}^1(X_\varepsilon). \tag{4.7}$$

We start with the definition of the identification operator on an edge. Let

$$J_e \colon \mathscr{H}_e = \mathsf{L}_2(I_e) \longrightarrow \mathscr{H}_{\varepsilon,e} = \mathsf{L}_2(X_{\varepsilon,e}) \quad \text{be given by} \quad J_e f_e := f_e \otimes \mathbb{1}_{\varepsilon,e},$$

where $\mathbb{1}_{\varepsilon,e}$ is the (constant) eigenfunction of Y_e associated to the lowest (zero) eigenvalue equal to $\varepsilon^{-m/2}$. Since we assumed $\operatorname{vol} Y_e = 1$, the eigenfunction is normalised. Its adjoint acts as transverse averaging,

$$(J_e^* u_e)(s) = \langle u_e(s), \mathbb{1}_{\varepsilon,e}\rangle_{\mathscr{K}_{\varepsilon,e}} = \varepsilon^{m/2} \int_{Y_e} u_e(s, y_e)\, \mathrm{d}y_e.$$

Before defining the global identification operator, we need the following result:

Lemma 4.3. *For $0 < d \le 1$, $0 < \varepsilon \le 1$ and $f, g \in \mathsf{H}^1([-d, d])$ we have*

$$\Big|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(s)\overline{g}(s)\, \mathrm{d}s - f(0)\overline{g}(0)\Big| \le 2(\varepsilon/d)^{1/2}\|f\|_{\mathsf{H}^1}\|g\|_{\mathsf{H}^1}. \tag{4.8}$$

Proof. Note first that

$$|f(s)|^2 \le \frac{2}{d}\|f\|^2_{\mathsf{H}^1} \tag{4.9}$$

for $s \in [-d, d]$ by (3.1) since $d \in (0, 1]$ by assumption. From $f(s) - f(0) = \int_0^s f'(t)\, \mathrm{d}t$ we conclude

$$\big|f(s) - f(0)\big|^2 \le |s|\|f'\|^2. \tag{4.10}$$

Now, the left-hand side of (4.8) can be estimated by

$$\begin{aligned}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\big|f(s)-f(0)\big||g(s)|\,\mathrm{d}s+\frac{|f(0)|}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\big|g(s)-g(0)\big|\,\mathrm{d}s\\ \le\frac{1}{2\varepsilon}\Big(\int_{-\varepsilon}^{\varepsilon}|s|\,\mathrm{d}s\|f'\|^2\int_{-\varepsilon}^{\varepsilon}|g(s)|^2\,\mathrm{d}s\Big)^{1/2}+\frac{1}{2\varepsilon}\Big(\frac{2}{d}\|f\|^2_{\mathsf{H}^1}2\varepsilon\int_{-\varepsilon}^{\varepsilon}|s|\,\mathrm{d}s\|g'\|^2\Big)^{1/2}\\ \le\frac{1}{2}\|f'\|\sqrt{\frac{2}{d}}\|g\|_{\mathsf{H}^1}\sqrt{2\varepsilon}+\frac{1}{2}\sqrt{\frac{2}{d}}\|f\|_{\mathsf{H}^1}\sqrt{2\varepsilon}\|g'\|\end{aligned}$$

using (4.9)–(4.10) together with Cauchy-Schwarz inequality, from where the desired estimate follows. □

We can now compare the two contributions of the quadratic forms on an internal edge, including the potential in the middle of this edge. We could consider this inner point as a vertex, too, and use the arguments for vertex neighbourhoods as in [EP09]. Since this vertex has degree two only, we give a direct (and simpler) proof here:

Lemma 4.4. *We have*

$$\big|\mathfrak{h}_{\varepsilon,e}(J_ef_e,u_e)-\check{\mathfrak{h}}_e(f_e,J_e^*u_e)\big|\le 2|w_e|(\varepsilon/d)^{1/2}\|f\|_{\mathsf{H}^1(\Gamma)}\|u\|_{\mathsf{H}^1(X_\varepsilon)}$$

for all $f\in\mathscr{H}^1=\mathsf{H}^1(\Gamma)$, $u\in\widetilde{\mathscr{H}}^1=\mathsf{H}^1(X_\varepsilon)$, $0<\varepsilon\le 1$ *and* $0<d\le 1$.

Proof. We have

$$\begin{aligned}&\mathfrak{h}_{\varepsilon,e}(J_ef_e,u_e)-\check{\mathfrak{h}}_e(f_e,J_e^*u_e)\\ &=\int_{-d}^{d}\Big(\langle(f_e'\otimes\mathbb{1}_{\varepsilon,e}+\mathrm{i}A_ef_e\otimes\mathbb{1}_{\varepsilon,e})(s),u_e(s)\rangle_{\mathscr{K}_{\varepsilon,e}}-(f_e'(s)+\mathrm{i}A_ef_e(s))\overline{\langle u_e(s),\mathbb{1}_{\varepsilon,e}\rangle_{\mathscr{K}_{\varepsilon,e}}}\Big)\,\mathrm{d}s\\ &\qquad+w_e\Big(\int_{-\varepsilon}^{\varepsilon}\langle f_e(s)\mathbb{1}_{\varepsilon,e},u_e(s)\rangle_{\mathscr{K}_{\varepsilon,e}}\,\mathrm{d}s-f_e(0)\overline{\langle u_e(0),\mathbb{1}_{\varepsilon,e}\rangle_{\mathscr{K}_{\varepsilon,e}}}\Big).\end{aligned}$$

Note that in the first integral the term with the derivatives and the magnetic potential contributions respectively cancel. Moreover, the expression contains no contribution from the transversal (sesquilinear) form $\mathfrak{k}_{\varepsilon,e}$ since $\mathfrak{k}_{\varepsilon,e}(\mathbb{1}_{\varepsilon,e},\varphi)=0$ for any $\varphi\in\mathsf{L}_2(\varepsilon Y_e)$. The remaining (electric) potential term can be estimated by Lemma 4.3 with $f=f_e$ and $g(s)=\langle u_e(s),\mathbb{1}_{\varepsilon,e}\rangle_{\mathscr{K}_{\varepsilon,e}})$. □

As the global identification operator we define $J\colon\mathscr{H}\longrightarrow\widetilde{\mathscr{H}}$ by

$$Jf:=\bigoplus_e J_ef_e\oplus 0$$

with respect to the decomposition (3.7). In order to relate the Sobolev spaces of order one we correct the error made at the vertex neighbourhood by fixing the function to be constant there. Namely, we define $J^1\colon\mathscr{H}^1\longrightarrow\widetilde{\mathscr{H}}^1$ by

$$J^1f:=\bigoplus_e J_ef_e\oplus\varepsilon^{-m/2}\bigoplus_v f(v)\mathbb{1}_v,$$

where $\mathbb{1}_v$ is the constant function on X_v with value 1. Since f is continuous on the graph, Jf is continuous along the vertex and edge neighbourhood boundary, and therefore maps into the Sobolev space $\widetilde{\mathscr{H}}^1=\mathsf{H}^1(X_\varepsilon)$.

For the operator $J'^1\colon\widetilde{\mathscr{H}}^1\longrightarrow\mathscr{H}^1$, we have to modify J^* in such a way that the first order spaces are respected, namely we set

$$\begin{aligned}(J_e'^1u)(s):=(J_e^*u_e)(s)+\chi_-(s)\varepsilon^{m/2}\big(⨍_{v_j}u-(J_e^*u_e)(-d)\big)\\ +\chi_+(s)\varepsilon^{m/2}\big(⨍_{v_k}u-(J_e^*u_e)(d)\big)\end{aligned}$$

on an inner edge $e = \{j, k\}$, $j < k$, where $\chi_\pm$ are smooth functions with $\chi_\pm(\pm d) = 1$, $|\chi'_\pm| \le 2/d$ and $\chi_\pm(s) = 0$ for $\pm s \le 0$. Moreover,

$$\fint_v u := \frac{1}{\operatorname{vol} X_v} \langle u_v, \mathbb{1}_v \rangle := \frac{1}{\operatorname{vol} X_v} \int_{X_v} u_v \, \mathrm{d}x_v$$

is the average of a function u on the (unscaled) vertex neighbourhood X_v.

On an outer edge $e = j$ we set

$$(J'^1_e u)(s) := (J^*_e u_e)(s) + \chi(s) \varepsilon^{m/2} \big(\fint_{v_j} u - (J^*_e u_e)(0) \big)$$

where χ is a smooth function with $\chi(0) = 1$, $|\chi'| \le 2$ and $\chi(s) = 0$ for $s \ge 1$. Note that J'^1 differs from $J^* f$ only by a correction near the vertices. Since $(J'^1 u)_e(v) = \varepsilon^{m/2} \fint_v u$ independently of $e \sim v$, the function $J'^1 u$ is indeed continuous, and therefore an element of $\mathsf{H}^1(\Gamma)$.

Following [EP09, Prop. 3.2] and using additionally Lemma 4.4 together with the identification operators just defined, we can check the following claim:

Proposition 4.5. *Let $0 < d \le 1$, then the quadratic forms $\mathfrak{h}_\varepsilon$ and $\mathfrak{h}$ are δ_ε-quasi-unitary equivalent, where δ_ε depends on ε, d and $\overline{w} := 3 \max_{e,v}\{|w_e|, |w_v|\}$ as follows*

$$\delta_\varepsilon = \mathscr{O}\Big(\Big(\frac{\varepsilon}{d}\Big)^{1/2} (\overline{w} + 1) \Big) + \mathscr{O}\Big(\frac{\varepsilon^{1/2}}{d} \Big).$$

Moreover, the error depends additionally only on X_v and Y_e.

Corollary 4.6. *Assume that w_e, w_v and A_e are chosen as in (2.3), then $\overline{w} = \mathscr{O}(d^{-2})$ and $\overline{A} = \mathscr{O}(d^{-1})$. If in addition, $d = \varepsilon^\alpha$ with $0 < \alpha < 1/5$, then $\mathfrak{h}_\varepsilon$ and $\mathfrak{h}$ are δ_ε-quasi unitarily equivalent for all $0 < \varepsilon \le \varepsilon_1$, where $\delta_\varepsilon = \mathscr{O}(\varepsilon^{(1-5\alpha)/2})$, and where $\varepsilon_1 > 0$ is a constant.*

Finally, if $0 < \alpha < 1/13$, then the associated operators H_ε and H are $\widetilde{\delta}_\varepsilon$-quasi unitarily equivalent with $\widetilde{\delta}_\varepsilon = \mathscr{O}(\varepsilon^{(1-13\alpha)/2})$.

Proof. The quasi-unitary equivalence of the quadratic forms follows from Proposition 4.5, as well as the estimate on δ_ε. Moreover, $\varepsilon_0 = \varepsilon_0(\varepsilon)$ as given in Proposition 3.2 is generally of order $\mathscr{O}(1/\overline{w}) = \mathscr{O}(\varepsilon^{2\alpha})$, i.e. $\varepsilon_0 \le c\varepsilon^{2\alpha}$. In particular, we can choose, $\varepsilon_1 = c^{1/(1-2\alpha)}$.

For the last assertion, note that the constants $C_{1/2}$ and $\widetilde{C}_{1/2}$ of Propositions 3.1 and 3.2 fulfil $C_{1/2} = \mathscr{O}(\varepsilon^{-4\alpha})$ and $\widetilde{C}_{1/2} = \mathscr{O}(\varepsilon^{-4\alpha})$, cf. (3.5) and (3.10), since the term $\overline{w}^2 = \mathscr{O}(\varepsilon^{-4})$ is dominant. The result now follows from Theorem 4.1 (i) with $C := \max\{C_{1/2}, \widetilde{C}_{1/2}\}$, and therefore we have $\widetilde{\delta}_\varepsilon = 12 C \delta_\varepsilon = \mathscr{O}(\varepsilon^{-4\alpha + (1-5\alpha)/2}) = \mathscr{O}(\varepsilon^{(1-13\alpha)/2})$. □

Now we are in position to state and prove the main result of this article:

Theorem 4.7. *Assume that $\Gamma(0)$ is a star graph with vertex condition parametrised by matrices S and T as in Section 2 and let $0 < \alpha < 1/13$. Then there is a Schrödinger operator H_ε on an approximating manifold X_ε as constructed in Section 3.2 such that*

$$\| J R_d^{\mathrm{star}}(z) J^* - R_\varepsilon(z) \| = \mathscr{O}(\varepsilon^{\min\{1-13\alpha, \alpha\}/2})$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, where $R_\varepsilon(z) = (H_\varepsilon - z)^{-1}$.

Proof. The result is an immediate consequence of Corollary 4.6, Theorem 4.1 (ii) and Theorem 2.3. □

Remark 4.8. The error term in the theorem depends only on z and the building block manifolds X_v at the vertices and the transversal manifolds Y_e on the edges. If $\alpha = 1/14$, we obtain the error estimate $\mathscr{O}(\varepsilon^{1/28})$ which is the maximal value the function $\alpha \mapsto \min\{1 - 13\alpha, \alpha\}/2$ can achieve.

The error estimate we obtain here is of the same type that we obtained in [EP09, Sec. 4] when we approximated the δ's interaction despite the fact that the present approximation of this particular coupling is different, cf. Section 5.2 below.

If the condition (2.4) mentioned in Remark 2.2 is fulfilled for all j,k we obtain a slightly better estimate. In this case, we have $\overline{w} = \mathscr{O}(d^{-1})$ instead of $\mathscr{O}(d^{-2})$, and $\mathfrak{h}_\varepsilon$ and $\mathfrak{h}$ are δ_ε-quasi, where $\delta_\varepsilon = \mathscr{O}(\varepsilon^{(1-3\alpha)/2})$. Moreover, the associated operators H_ε and H are $\widetilde{\delta}_\varepsilon$-quasi unitarily equivalent with $\widetilde{\delta}_\varepsilon = \mathscr{O}(\varepsilon^{(1-7\alpha)/2})$. However, both assumptions made about α, namely $0 < \alpha < 1/13$ and $0 < \alpha < 1/7$, are for sure not optimal.

There is an obvious extension to the above convergence result for quantum graphs Γ_0 with more than one vertex. For quantum graphs with finitely many vertices, the convergence result holds without changes, and for infinitely many vertices, some uniformity conditions are needed. Such questions are discussed in detail in [P06] and [P12].

Remark 4.9. One may ask whether one can reformulate the "quasi-unitary equivalence" for the present situation using

$$\widetilde{J}\colon \mathsf{L}_2(\Gamma(0)) \longrightarrow \mathsf{L}_2(\Gamma^{S,T}(d)) = \mathsf{L}_2(\Gamma(0)) \oplus \mathsf{L}_2(\Gamma^{S,T}_{\mathrm{int}}(d)), \qquad \widetilde{J}f = f \oplus 0,$$

in which case $R_d^{\mathrm{star}}(z) = \widetilde{J}R^{\mathrm{star}}(z)\widetilde{J}^*$ by (2.5) and the resolvent convergence of Theorem 2.3 can be stated as

$$\|\widetilde{J}R^{\mathrm{star}}(z)\widetilde{J}^* - R_d^{\mathrm{approx}}(z)\|_{\mathscr{L}(\mathsf{L}_2(\Gamma^{S,T}(d)))} = \mathscr{O}(d^{1/2}) \tag{4.11}$$

for $d \to 0$. In fact, we are interested primarily in spectral consequences of such a reformulation which can be demonstrated in a more direct way. To this end, note that eq. (4.11) is just (4.3b) of Theorem 4.1 *without* the constant C. Moreover, from [P12, Thm. 4.2.9–10] one can conclude that (4.3a) is valid for more general φ than $\varphi(\lambda) = (\lambda - z)^{-1}$, see Theorem 4.1 (ii). Using arguments analogous to those in [P12, Sec. 4.2–4.3], we can deduce from (4.11) that (4.3b) also holds for such φ. Consequently, the spectral convergence stated in Theorem 4.1 (iii) and (iv) also holds in this situation.

5. Examples

5.1. **Embedded graphs and graph neighbourhoods.** Consider the situation when the graph is embedded in $\mathbb{R}^\nu$, $\nu \geq 2$. This may be a restriction to the vertex coupling if $\nu = 2$ and the vertex degree exceeds three; recall that the edges of the internal graph defined in (iii) of Section 1 are supposed to be non-intersecting. For $\nu \geq 3$ this difficulty can be avoided in the edges are properly curved. At the same time, irrespective of d the lengths of the edge parts of the manifold change as $\varepsilon \to 0$ by an amount given by the size of the vertex neighbourhoods. Let us point out briefly that for such embedded "fat graphs" curved and shortened edges lead to a small error in the approximation only.

Consider first the length change. In our case, the difference of the original edge length and the one of an embedded edge is of order $d - \varepsilon = d(1 - \varepsilon/d) = d(1 - \varepsilon^{1-\alpha})$. We have shown in [EP09, Lem. 2.7] that this leads to an additional error of order $\mathscr{O}(\varepsilon^{1-\alpha})$; expressed again in terms of quasi-unitary operators.

Furthermore, if we allow *curved* edges in the case of a graph embedded in $\mathbb{R}^\nu$, we still arrive at the same limit operator. The error is of order $\mathscr{O}(\varepsilon^{1-\alpha})$ (see [P12, Sec. 6.7 and Prop. 4.5.6] for details; the factor ε comes from the shrinking rate, the factor $\varepsilon^{-\alpha}$ from the curvature term of the embedded curve in dimension $\nu = 2$; the length shrinks by $d = \varepsilon^\alpha$ so its curvature is of order $\varepsilon^{-\alpha}$). Similar arguments apply for $\nu \geq 3$. In particular, combining the effect of shortening of edges and curved edges, and using the transitivity of quasi-unitary equivalence ([P12, Prop. 4.2.8]) we arrive at an error estimate which is not worse than the one in Theorem 4.7.

5.2. **Special vertex couplings and approximation by Schrödinger operators.** While the approximation described in Theorem 4.7 cover any self-adjoint coupling, for some of them we have better alternatives. This concerns, in particular, the δ coupling where a simple scaled potential does a better job as explained in [EP09]. On the other hand, for couplings with functions discontinuous at the vertex we do not have many alternatives.

It is illustrative to compare the approximation of the δ'_{s} coupling obtained from the graph-level approximations described in Section 2 with the one from [CE04] used in [EP09] for the approximation by Schrödinger operators. Recall that a δ'_{s} coupling of strength β in a vertex of degree n edges characterised by the condition

$$\frac{1}{\beta} J f(0) - f'(0) = 0,$$

where J is the $n \times n$ matrix with all entries one. In other words, the respective ST-parametrisation from Proposition 2.1 is given by $m = n$, $S = \beta^{-1} J$ and $T = 0$, and the strengths of the δ potentials required to approximate δ'_{s} according to Theorem 2.3 are

$$w_{\{jk\}} = -\frac{\beta}{d^2} - \frac{2}{d} \quad \text{and} \quad w_j = \frac{2-n}{\beta} - \frac{n-1}{d}.$$

In particular, all inner edges are present. If $n = 3$, for instance, we employ a small triangle graph of length scale $d = \varepsilon^{\alpha}$ attaching the “external” edges to its vertices (as sketched in Figure 1). The corresponding Schrödinger operator has a potential of order $-\varepsilon^{-\alpha-1}$ near v_j and of order $-\beta\varepsilon^{-2\alpha-1}$ at the midpoint of each edge $\{jk\}$; for simplicity the potentials can be chosen piecewise constant.

The approximation used in [EP09] is different. Here we keep the original star graph, but introduce additional δ-couplings on each edge at distance $d = \varepsilon^{\alpha}$ of the central vertex. The strength of the coupling at the central vertex is $-\beta/d^2$, hence the Schrödinger potential there is of order $-\beta\varepsilon^{-2\alpha-1}$. The strength of the coupling at the additional vertices is $-1/d$, hence the Schrödinger potential is of order $-\beta\varepsilon^{-\alpha-1}$. One sees that the approximation graph topology is different but the δ strengths in the two cases differ only in lower order terms[2] with respect to the length scale $d = \varepsilon^{\alpha}$.

Let us finally remark that δ'_{s} is not the only example of interest; our method makes it possible to approximate other couplings of potential importance such as the scale-invariant ones analysed recently in [CET11].

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