A CRITICAL KIRCHHOFF TYPE PROBLEM INVOLVING A NON–LOCAL OPERATOR

ALESSIO FISCELLA AND ENRICO VALDINOCI

Abstract. In this paper we show the existence of non-negative solutions for a Kirchhoff type problem driven by a non–local integrodifferential operator, that is

$$-M(\|u\|_Z^2)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2^*-2}u \quad \text{in } \Omega, \qquad u=0 \quad \text{in } \mathbb{R}^n\setminus\Omega.$$

where $\mathcal{L}_K$ is an integrodifferential operator with kernel K, Ω is a bounded subset of $\mathbb{R}^n$, M and f are continuous functions, $\|\cdot\|_Z$ is a functional norm and 2^* is a fractional Sobolev exponent.

1. Introduction

In this paper we deal with the following problem

$$\begin{cases} -M\left(\displaystyle\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy\right)\mathcal{L}_K u & \\ = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega, \\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega \end{cases} \tag{1}$$

where $n>2s$ with $s\in(0,1)$, $2^*=2n/(n-2s)$, λ is a positive parameter, $\Omega\subset\mathbb{R}^n$ is an open bounded set, M and f are two continuous functions whose properties will be introduced later and $\mathcal{L}_K$ is a non-local operator defined as follows:

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n}(u(x+y)+u(x-y)-2u(x))K(y)dy,$$

for all $x\in\mathbb{R}^n$, where $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ is a measurable function with the property that

$$\begin{aligned} &\text{there exists } \theta>0 \text{ and } s\in(0,1) \text{ such that} \\ &\theta\,|x|^{-(n+2s)} \le K(x) \le \theta^{-1}\,|x|^{-(n+2s)} \text{ for any } x\in\mathbb{R}^n\setminus\{0\}. \end{aligned} \tag{2}$$

It is immediate to observe that $mK\in L^1(\mathbb{R}^n)$ by setting $m(x)=\min\left\{|x|^2,1\right\}$. A typical example for K is given by $K(x)=|x|^{-(n+2s)}$. In this case problem (1) becomes

$$\begin{cases} M\left(\displaystyle\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dx\,dy\right)(-\Delta)^s u & \\ = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega, \\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{3}$$

Date: June 11, 2013.

where $-(-\Delta)^s$ is the fractional Laplace operator which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy$$

for $x \in \mathbb{R}^n$ (see [9] and references therein for further details on the fractional Laplacian and on the fractional Sobolev space $H^s(\mathbb{R}^n)$).

Problems (1) and (3) have a variational nature and the natural space where finding solutions for them is the homogeneous fractional Sobolev space $H^s_0(\Omega)$ (see [9]). In order to study (1) and (3) it is important to encode the 'boundary condition' $u = 0$ in $\mathbb{R}^n \setminus \Omega$ (which is different from the classical case of the Laplacian) in the weak formulation, by considering also that in the norm $\|u\|_{H^s(\mathbb{R}^n)}$ the interaction between Ω and $\mathbb{R}^n \setminus \Omega$ gives positive contribution. The functional space that takes into account these boundary condition will be denoted by Z and it was introduced in [11] in the following way.

First, we denote by X the linear space of Lebesgue measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\text{the map } (x, y) \mapsto (u(x) - u(y))^2 K(x - y) \text{ is in } L^1\big(Q,\, dxdy\big),$$

where $Q := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. The space X is endowed with the norm

$$\|u\|_X = \Big(\|u\|_{L^2(\Omega)} + \int_Q |u(x) - u(y)|^2 K(x - y)dx\,dy\Big)^{1/2}. \tag{4}$$

It is immediate to observe that bounded and Lipschitz functions belong to X, thus X is not reduced to $\{0\}$ (see [18, 19] for further details on space X). Now, the functional space Z denotes the closure of $C_0^\infty(\Omega)$ in X. By [11, Lemma 4], the space Z is an Hilbert space which can be endowed with the norm defined as

$$\|u\|_Z = \Big(\int_Q |u(x) - u(y)|^2 K(x - y)dx\,dy\Big)^{1/2}. \tag{5}$$

Note that in (4) and (5) the integrals can be extended to all $\mathbb{R}^{2n}$, since $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

In view of our problem, we suppose that $M : \mathbb{R}^+ \to \mathbb{R}^+$ verifies the following conditions:

$$M \text{ is an increasing and continuous function;} \tag{6}$$

$$\text{there exists } m_0 > 0 \text{ such that } M(t) \geq m_0 = M(0) \text{ for any } t \in \mathbb{R}^+. \tag{7}$$

A typical example for M is given by $M(t) = m_0 + tb$ with $b \geq 0$.

Also, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies:

$$\lim_{|t|\to 0} \frac{f(x,t)}{t} = 0, \text{ uniformly in } x \in \Omega; \tag{8}$$

$$\text{there exists } q \in (2, 2^*) \text{ such that } \lim_{|t|\to\infty} \frac{f(x,t)}{t^{q-1}} = 0 \text{ uniformly in } x \in \Omega; \tag{9}$$

there exists $\sigma \in (2, 2^*)$ such that for any $x \in \Omega$ and $t > 0$

$$0 < \sigma F(x,t) = \sigma \int_0^t f(x,s)ds \le t f(x,t)\,. \tag{10}$$

Moreover, since we intend to find non-negative solution, we assume this further condition for f

$$f(x,t) = 0 \quad \text{for any } x \in \Omega \text{ and } t \le 0. \tag{11}$$

An example of a function satisfying the conditions (8)–(11) is given by

$$f(x,t) = \begin{cases} 0 & \text{if } t < 0, \\ a(x)t^{q-1} & \text{if } 0 < t < 1, \\ a(x)t^{q_1-1} & \text{if } t \ge 1, \end{cases}$$

with $2 < q_1 < q$, $a \in L^\infty(\Omega)$ and $a(x) > 0$ for any $x \in \Omega$.

The weak formulation of (1) is given by the following problem

$$\begin{cases} M(\|u\|_Z^2)\displaystyle\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy \\ \qquad = \lambda \displaystyle\int_\Omega f(x,u(x))\varphi(x)\,dx + \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx \quad \forall \varphi \in Z \\ u \in Z. \end{cases} \tag{12}$$

Thanks to our assumptions on Ω, M, f and K, all the integrals in (12) are well defined if u, $\varphi \in Z$. We also point out that the odd part of function K gives no contribution to the integral of the left-hand side of (12). Therefore, it would be not restrictive to assume that K is even.

Recently, some studies have been performed for critical problems in a non-local setting; we refer the interested readers to [3, 6, 17, 18, 19, 20]. Inspired by the above articles, in this paper we would like to investigate the existence of a nontrivial solution for problem (12), by extending the result in classical Laplacian case dealt with in [10].

Theorem 1. *Let $s \in (0,1)$, $n > 2s$ and Ω be a bounded open subset of $\mathbb{R}^n$. Assume that the functions K, M and f satisfy conditions* (2) *and* (6)–(10)*. Then there exists $\lambda^* > 0$ such that problem* (12) *has a nontrivial solution u_λ for all $\lambda \ge \lambda^*$. Such solution also verifies*

$$\lim_{\lambda\to\infty} \|u_\lambda\|_Z = 0.$$

The paper is organized as follows. In Section 2 we introduce a truncated problem whose weak solution will be a weak solution of the original problem (1). In Section 3 we prove some technical lemmas. In Section 4 we prove the existence of a solution for the truncated problem and our main result. Finally, in Section 5 we study the sign of the weak solutions of problem (1).

The paper ends with an appendix which presents some detailed motivation for our nonlocal equation, starting from some classical models for vibrating strings.

2. The auxiliary problem

In order to prove Theorem 1 we first study an auxiliary truncated problem, by assuming that M is unbounded (otherwise the truncation on M is not necessary).

Given σ as in (10) and $a \in \mathbb{R}$ such that $m_0 < a < \frac{\sigma}{2} m_0$, by (6) there exists $t_0 > 0$ such that $M(t_0) = a$. Now, by setting

$$M_a(t) := \begin{cases} M(t) & \text{if } 0 \le t \le t_0, \\ a & \text{if } t \ge t_0, \end{cases}$$

we can introduce the following auxiliary problem

$$\begin{cases} -M_a(\|u\|_Z^2)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2^*-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{13}$$

with f and λ defined as in Problem (1). By (6) we note also that

$$M_a(t) \le a \quad \text{for any } t \ge 0. \tag{14}$$

We obtain the following result.

Theorem 2. *Let $s \in (0,1)$, $n > 2s$ and Ω be a bounded open subset of $\mathbb{R}^n$. Assume that conditions* (2) *and* (6)–(10) *hold true. Then there exists $\lambda_0 > 0$ such that problem* (13) *has a nontrivial weak solution, for all $\lambda \ge \lambda_0$ and for all $a \in (m_0, \frac{\sigma}{2} m_0)$.*

3. Variational formulation and technical lemmas

For the proof of Theorem 2, we observe that problem (13) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J}_{a,\lambda} : Z \to \mathbb{R}$ defined as follows

$$\mathcal{J}_{a,\lambda}(u) = \frac{1}{2}\widehat{M_a}(\|u\|_Z^2) - \lambda \int_\Omega F(x,u(x))dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*} dx\,.$$

where

$$\widehat{M_a}(t) = \int_0^t M_a(s)ds.$$

Note that the functional $\mathcal{J}_{a,\lambda}$ is Fréchet differentiable in $u \in Z$ and for any $\varphi \in Z$

$$\begin{aligned} \mathcal{J}'_{a,\lambda}(u)(\varphi) = {} & M_a(\|u\|_Z^2)\int_Q \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \\ & - \lambda \int_\Omega f(x,u(x))\varphi(x)\,dx - \int_\Omega |u(x)|^{2^*-2} u(x)\varphi(x)dx\,. \end{aligned} \tag{15}$$

Now we prove that the functional $\mathcal{J}_{a,\lambda}$ has the geometric features required by the Mountain Pass Theorem.

Lemma 3. *Let K, M and f be three functions satisfying* (2) *and* (6)–(10). *Then there exist two positive constants ρ and α such that*

$$\mathcal{J}_{a,\lambda}(u) \ge \alpha > 0, \tag{16}$$

for any $u \in Z$ with $\|u\|_Z = \rho$.

Proof. By (8) and (9) it follows that, for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|F(x,t)| \le \epsilon\,|t|^2 + \delta\,|t|^q\,. \tag{17}$$

By (7) and (17) we get

$$\mathcal{J}_{a,\lambda}(u) \ge \frac{m_0}{2}\|u\|_Z^2 - \epsilon\lambda\int_\Omega |u(x)|^2\,dx - \delta\lambda\int_\Omega |u(x)|^q\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx.$$

So, by using a fractional Sobolev inequality (see [9, Theorem 6.5]), there is a positive constant $C = C(\Omega)$ such that

$$\mathcal{J}_{a,\,\lambda}(u) \geq \Big(\frac{m_0}{2} - \epsilon\lambda C\Big) \|u\|_Z^2 - \delta\lambda C \|u\|_Z^q - C \|u\|_Z^{2^*}.$$

Therefore, by fixing ϵ such that $k := \dfrac{m_0}{2} - \epsilon\lambda C > 0$, since $2 < q < 2^*$, the result follows by choosing ρ sufficiently small. □

Lemma 4. *Let K, M and f be three functions satisfying* (2) *and* (6)–(10). *Then there exists an $e \in Z$ with $\mathcal{J}_{a,\,\lambda}(e) < 0$ and $\|e\|_Z > \rho$.*

Proof. We fix $u_0 \in Z$ such that $\|u_0\|_Z = 1$ and $u_0 \geq 0$ a.e. in $\mathbb{R}^n$. Now, let $t > 0$. By using (10) and (14), we get

$$\mathcal{J}_{a,\,\lambda}(tu_0) \leq a\frac{t^2}{2} - c_1 t^\sigma \lambda \int_\Omega |u_0(x)|^\sigma \, dx + c_2 |\Omega| - \frac{t^{2^*}}{2^*} \int_\Omega |u_0(x)|^{2^*} \, dx.$$

Since $\sigma > 2$, passing to the limit as $t \to +\infty$, we get that $\mathcal{J}_{a,\,\lambda}(tu_0) \to -\infty$, so that the assertion follows taking $e = t_* u_0$, with $t_* > 0$ large enough. □

Now, in order to prove the boundedness of the a Palais–Smale sequence we set

$$c_{a,\,\lambda} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{J}_{a,\,\lambda}(\gamma(t)) > 0$$

where

$$\Gamma := \{\gamma \in C([0,1],\, Z) : \gamma(0) = 0,\, \mathcal{J}_{a,\,\lambda}(\gamma(1)) < 0\}.$$

Lemma 5. *Let K, M and f be three functions satisfying* (2) *and* (6)–(10). *Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z such that*

$$\mathcal{J}_{a,\,\lambda}(u_j) \to c_{a,\,\lambda}, \tag{18}$$

and

$$\sup\left\{ \left|\mathcal{J}'_{a,\,\lambda}(u_j)(\phi)\right| : \phi \in Z,\, \|\phi\|_Z = 1 \right\} \to 0 \quad \forall \phi \in Z, \tag{19}$$

as $j \to +\infty$. Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z.

Proof. By (18) and (19) there exists $C > 0$ such that

$$|\mathcal{J}_{a,\,\lambda}(u_j)| \leq C \;\text{ and }\; \left| \mathcal{J}'_{a,\,\lambda}(u_j)\left(\frac{u_j}{\|u_j\|_Z}\right)\right| \leq C, \tag{20}$$

for any $j \in \mathbb{N}$. Moreover, by (7), (10), and (14) it follows that

$$\begin{aligned} &\mathcal{J}_{a,\,\lambda}(u_j) - \frac{1}{\sigma}\mathcal{J}'_{a,\,\lambda}(u_j)(u_j) \\ &\quad \geq \frac{1}{2}\widehat{M_a}(\|u_j\|_Z^2) - \frac{1}{\sigma} M_a(\|u_j\|_Z^2)\, \|u_j\|_Z^2 \geq \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right) \|u_j\|_Z^2 \end{aligned} \tag{21}$$

So, by combining (20) with (21) and by remembering that $m_0 < a < \dfrac{\sigma}{2} m_0$, we can conclude the proof. □

The following result is needed to study the asymptotic behaviour of the solution of problem (12).

Lemma 6. *Let K, M and f be three functions satisfying* (2) *and* (6)–(10)*. Then*

$$\lim_{\lambda\to+\infty} c_{a,\lambda} = 0.$$

Proof. Let $e \in Z$ be the function given by Lemma 4 and let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence sucht that $\lambda_j \to +\infty$. Since $\mathcal{J}_{a,\lambda}$ satisfies the Mountain Pass geometry, it follows that there exists $t_\lambda > 0$ verifying $\mathcal{J}_{a,\lambda}(t_\lambda e) = \max_{t\geq 0} J_{a,\lambda}(te)$. Hence, $\mathcal{J}'_{a,\lambda}(t_\lambda e)(e) = 0$ and by (15) we get

$$t_\lambda \|e\|_Z^2 M_a(t_\lambda^2 \|e\|_Z^2) = \lambda \int_\Omega f(x, t_\lambda e(x))e(x)\,dx + t_\lambda^{2^*-1} \int_\Omega |e(x)|^{2^*}\,dx\,. \tag{22}$$

Now, by construction $e \geq 0$ a.e. in $\mathbb{R}^n$. So, by (10), (14) and (22) it follows

$$a\|e\|_Z^2 \geq t_\lambda^{2^*-2} \int_\Omega |e(x)|^{2^*}\,dx,$$

which implies that t_λ is bounded for any $\lambda > 0$. Thus, there exists $\beta \geq 0$ such that $t_{\lambda_j} \to \beta$ as $j \to +\infty$. So, by using also (14) and (22) there exists $D > 0$ such that

$$\lambda_j \int_\Omega f(x, t_{\lambda_j} e(x))e(x)\,dx + t_{\lambda_j}^{2^*-1} \int_\Omega |e(x)|^{2^*}\,dx = t_{\lambda_j} M_a(t_{\lambda_j}^2 \|e\|_Z^2) \leq D \tag{23}$$

for any $j \in \mathbb{N}$. We claim that $\beta = 0$. Indeed, if $\beta > 0$ then by (8), (9) for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(x,t)| \leq 2\epsilon\,|t| + q\delta\,|t|^{q-1}\ .$$

and so, by the Dominated Convergence Theorem,

$$\int_\Omega f(x, t_{\lambda_j} e(x))e(x)\,dx \to \int_\Omega f(x, \beta e(x))e(x)\,dx \quad \text{as } j \to +\infty.$$

By remembering that $\lambda_j \to +\infty$, we get

$$\lim_{j\to+\infty} \lambda_j \int_\Omega f(x, t_{\lambda_j} e(x))e(x)\,dx + t_{\lambda_j}^{2^*-1} \int_\Omega |e(x)|^{2^*}\,dx = +\infty$$

which contradicts (23). Thus, we have that $\beta = 0$. Now, we consider the following path $\gamma_*(t) = te$ for $t \in [0,1]$ which belongs to Γ. By using (10) we get

$$0 < c_{a,\lambda} \leq \max_{t\in[0,1]} \mathcal{J}_{a,\lambda}(\gamma_*(t)) \leq \mathcal{J}_{a,\lambda}(t_\lambda e) \leq \frac{1}{2}\widehat{M_a}(t_\lambda^2 \|e\|_Z^2). \tag{24}$$

By (6) and by remembering that $\beta = 0$ we have

$$\lim_{\lambda\to+\infty} \widehat{M_a}(t_\lambda^2 \|e\|_Z^2) = 0,$$

and so by using also (24) we can conclude the proof. □

We conclude this section by proving the following proposition. This technical result will be useful in applying the concentration-compacteness principle (see [15, Theorem 2]) to prove Theorem 2.

Proposition 7. *Let $p \in \mathbb{R}^n$, $\delta \in (0,1)$, $u \in L^{2^*}(\mathbb{R}^n)$.*

Let either $U \times V = B_\delta(p) \times \mathbb{R}^n$ or $U \times V = \mathbb{R}^n \times B_\delta(p)$. Then

$$\lim_{\delta\to 0} \delta^{-2} \int_U \int_{V\cap\{|x-y|\leq\delta\}} |u(x)|^2 |x-y|^{2-n-2s}\,dx\,dy = 0 \tag{25}$$

and

$$\lim_{\delta\to 0}\int_U\int_{V\cap\{|x-y|>\delta\}}|u(x)|^2|x-y|^{-n-2s}\,dx\,dy=0. \tag{26}$$

Proof. We set

$$\xi_\delta := \left(\int_{B_\delta(p)}|u(x)|^{2^*}\,dx\right)^{2/2^*}$$

and we remark that

$$\lim_{\delta\to 0}\xi_\delta=0. \tag{27}$$

Also we observe that, using the Hölder inequality with exponents $2^*/2=n/(n-2s)$ and $n/2s$, we have

$$\int_{B_\delta(p)}|u(x)|^2\,dx\le\left(\int_{B_\delta(p)}|u(x)|^{2^*}\,dx\right)^{2/2^*}\left(\int_{B_\delta(p)}1\,dx\right)^{2s/n}\le C\xi_\delta\delta^{2s}, \tag{28}$$

for some $C>0$ independent of δ (in what follows we will possibly change C from line to line). Moreover

$$(U\times V)\cap\{|x-y|\le\delta\}\subseteq B_{2\delta}(p)\times B_{2\delta}(p). \tag{29}$$

Indeed, if $(x,y)\in U\times V=B_\delta(p)\times\mathbb{R}^n$, with $|x-y|\le\delta$, we have that

$$|p-y|\le|p-x|+|x-y|\le\delta+\delta,$$

and so we get (29). On the other hand, if $(x,y)\in U\times V=\mathbb{R}^n\times B_\delta(p)$ with $|x-y|\le\delta$, we obtain

$$|p-x|\le|p-y|+|y-x|\le\delta+\delta,$$

and this completes the proof of (29).

Now we use (29), we change variable $z:=x-y$ and we conclude that

$$\begin{aligned}&\int_{x\in U}\int_{y\in V\cap\{|x-y|\le\delta\}}|u(x)|^2|x-y|^{2-n-2s}\,dx\,dy\\&\quad\le\int_{x\in B_{2\delta}(p)}\int_{y\in B_{2\delta}(p)\cap\{|x-y|\le\delta\}}|u(x)|^2|x-y|^{2-n-2s}\,dx\,dy\\&\quad\le\int_{x\in B_{2\delta}(p)}\int_{z\in B_\delta}|u(x)|^2|z|^{2-n-2s}\,dx\,dz\\&\quad\le C\delta^{2-2s}\int_{x\in B_{2\delta}(p)}|u(x)|^2\,dx.\end{aligned}$$

Using this and (28) we obtain

$$\begin{aligned}&\delta^{-2}\int_U\int_{V\cap\{|x-y|\le\delta\}}|u(x)|^2|x-y|^{2-n-2s}\,dx\,dy\\&\quad\le C\delta^{-2s}\int_{x\in B_{2\delta}(p)}|u(x)|^2\,dx\le C\xi_\delta.\end{aligned}$$

This and (27) imply (25).

Now we prove (26). For this, we fix an auxiliary parameter $K > 2$ (such parameter will be taken arbitrarily large at the end, after sending $\delta \to 0$). We observe that

$$U \times V \subseteq \big(B_{K\delta}(p) \times \mathbb{R}^n\big) \cup \big((\mathbb{R}^n \setminus B_{K\delta}(p)) \times B_\delta(p)\big). \tag{30}$$

Indeed, if $U \times V = B_\delta(p) \times \mathbb{R}^n$, then of course $U \times V \subseteq B_{K\delta}(p) \times \mathbb{R}^n$, hence (30) is obviuous. If instead $(x, y) \in U \times V = \mathbb{R}^n \times B_\delta(p)$, we distinguish two cases: if $x \in B_{K\delta}(p)$ then $(x, y) \in B_{K\delta}(p) \times \mathbb{R}^n$; if $x \in \mathbb{R}^n \setminus B_{K\delta}(p)$, then

$$(x, y) \in (\mathbb{R}^n \setminus B_{K\delta}(p)) \times V = (\mathbb{R}^n \setminus B_{K\delta}(p)) \times B_\delta(p).$$

This completes the proof of (30).

Now we compute

$$\begin{aligned}
&\int_{x\in B_{K\delta}(p)} \int_{y\in\mathbb{R}^n\cap\{|x-y|>\delta\}} |u(x)|^2 |x-y|^{-n-2s}\, dx\, dy \\
&\quad = \int_{x\in B_{K\delta}(p)} \int_{z\in\mathbb{R}^n\setminus B_\delta} |u(x)|^2 |z|^{-n-2s}\, dx\, dz \\
&\quad = C\delta^{-2s} \int_{x\in B_{K\delta}(p)} |u(x)|^2\, dx \\
&\quad \le C\xi_{K\delta},
\end{aligned} \tag{31}$$

where (28) has been used again in the last step.

Now we observe that if $x \in \mathbb{R}^n \setminus B_{K\delta}(p)$ and $y \in B_\delta(p)$ then

$$\begin{aligned}
|x-y| \ge |x-p| - |y-p| &= \frac{|x-p|}{2} + \frac{|x-p|}{2} - |y-p| \\
&\ge \frac{|x-p|}{2} + \frac{K\delta}{2} - \delta \ge \frac{|x-p|}{2}.
\end{aligned}$$

As a consequence, using the Hölder inequality with exponents $2^*/2 = n/(n-2s)$ and $n/2s$, we infer that

$$\begin{aligned}
&\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(p)} \int_{y\in B_\delta(p)} |u(x)|^2 |x-y|^{-n-2s}\, dx\, dy \\
&\le C \int_{x\in\mathbb{R}^n\setminus B_{K\delta}(p)} \int_{y\in B_\delta(p)} |u(x)|^2 |x-p|^{-n-2s}\, dx\, dy \\
&= C\delta^n \int_{x\in\mathbb{R}^n\setminus B_{K\delta}(p)} |u(x)|^2 |x-p|^{-n-2s}\, dx \\
&\le C\delta^n \left(\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(p)} |u(x)|^{2^*}\, dx\right)^{2/2^*} \left(\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(p)} |x-p|^{-(n+2s)n/2s}\, dx\right)^{2s/n} \\
&\le C\delta^n \|u\|^2_{L^{2^*}(\mathbb{R}^n)} \left(\int_{K\delta}^{+\infty} \rho^{-((n+2s)n/2s)+(n-1)} d\rho\right)^{2s/n} \\
&= C\delta^n \|u\|^2_{L^{2^*}(\mathbb{R}^n)} \left((K\delta)^{-n^2/2s}\right)^{2s/n} \\
&= CK^{-n} \|u\|^2_{L^{2^*}(\mathbb{R}^n)}.
\end{aligned} \tag{32}$$

By collecting the results in (30), (31) and (32), we obtain that

$$\begin{aligned}
&\int_U \int_{V\cap\{|x-y|>\delta\}} |u(x)|^2|x-y|^{-n-2s}\,dx\,dy \\
\le\ & \int_{x\in B_{K\delta}(p)} \int_{y\in\mathbb{R}^n\cap\{|x-y|>\delta\}} |u(x)|^2|x-y|^{-n-2s}\,dx\,dy \\
&\quad + \int_{x\in\mathbb{R}^n\setminus B_{K\delta}(p)} \int_{y\in B_\delta(p)} |u(x)|^2|x-y|^{-n-2s}\,dx\,dy \\
\le\ & C\xi_{K\delta} + CK^{-n}\|u\|^2_{L^{2^*}(\mathbb{R}^n)}.
\end{aligned}$$

From this, we send first $\delta\to 0$ and then $K\to+\infty$ and we readily obtain (26) (recall again (27)). $\square$

4. Proof of Theorems 1 and 2

Proof of Theorem 2. By Lemmas 3 and 4 the functional $\mathcal{J}_{a,\lambda}$ satisfies the geometric structure required by the Mountain Pass Theorem (see [16, Theorem 2.2]) Now, it remains to check the validity of the Palais-Smale condition. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z verifying (18) and (19). Since by Lemma 5 $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z, by applying also [11, Lemma 4] and [2, Theorem IV.9], up to a subsequence, there exists $u\in Z$ such that u_j converges to u weakly in Z, strongly in $L^q(\Omega)$ with $q\in[1,2^*)$ and a.e. in Ω. Also, in particular there exists $h\in L^2(\Omega)$ such that

$$|u_j(x)|\le h(x) \quad \text{for any } j\in\mathbb{N} \text{ and a.e. } x\in\Omega. \tag{33}$$

We point out the above inequality and convergences are also verified in all $\mathbb{R}^n$, since $u_j=0=u$ a.e. in $\mathbb{R}^n\setminus\Omega$; in particular we shall assume that $h(x)=0$ for a.e. $x\in\mathbb{R}^n\setminus\Omega$. Moreover, up to a subsequence, there is $\alpha\ge 0$ such that $\|u_j\|_Z\to\alpha$, so by using (6) it follows that $M_a(\|u_j\|_Z^2)\to M_a(\alpha^2)$ as $j\to+\infty$.

Now, we claim that

$$\|u_j\|_Z^2 \to \|u\|_Z^2 \qquad \text{as } j\to+\infty, \tag{34}$$

which clearly implies that $u_j\to u$ in Z as $j\to+\infty$. By [11, Lemma 4] we know that $\{u_j\}_{j\in\mathbb{N}}$ is also bounded in $H_0^s(\Omega)$. So, by Phrokorov's Theorem we may suppose that there exist two positive measures μ and ν on $\mathbb{R}^n$ such that

$$|(-\Delta)^s u_j|^2\,dx \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |u_j|^{2^*} \rightharpoonup \nu \tag{35}$$

in the sense of measures. Moreover, by [15, Theorem 2] we obtain an at most countable set of distinct points $\{x_i\}_{i\in J}$, positive numbers $\{\nu_i\}_{i\in J}$, $\{\mu_i\}_{i\in J}$ and a positive measure $\widetilde{\mu}$ with $Supp\,\widetilde{\mu}\subset\overline{\Omega}$ such that

$$\nu = |u|^{2^*}\,dx + \sum_{i\in J}\nu_i\delta_{x_i}, \tag{36}$$

and

$$\mu = |(-\Delta)^s u|^2\,dx + \widetilde{\mu} + \sum_{i\in J}\mu_i\delta_{x_i}, \qquad \nu_i\le S\mu_i^{2^*/2}, \tag{37}$$

with S the best constant of the Sobolev embedding.

Our goal is to show that the set J is empty. We argue by contradiction and suppose $J\ne\emptyset$. Then we fix $i\in J$ and for any $\delta>0$ we set $\psi_\delta(x):=\psi((x-x_i)/\delta)$ where

$\psi \in C_0^\infty(\mathbb{R}^n,[0,1])$ is such that $\psi \equiv 1$ in $B(0,1)$ and $\psi \equiv 0$ in $\mathbb{R}^n \setminus B(0,2)$. Since for a fixed $\delta > 0$ $\{\psi_\delta u_j\}_{j\in\mathbb{N}}$ is bounded in Z uniformly in j, by (19) it follows that $\mathcal{J}'_{a,\lambda}(u_j)(\psi_\delta u_j) \to 0$ as $j \to +\infty$, that is

$$
\begin{aligned}
&M_a(\|u_j\|_Z^2)\int_{\mathbb{R}^{2n}} u_j(x)\big(u_j(x)-u_j(y)\big)\big(\psi_\delta(x)-\psi_\delta(y)\big)K(x-y)\,dx\,dy \\
&= -M_a(\|u_j\|_Z^2)\int_{\mathbb{R}^{2n}} \psi_\delta(y)\,|u_j(x)-u_j(y)|^2\,K(x-y)\,dx\,dy \qquad (38)\\
&\quad + \lambda\int_\Omega f(x,u_j(x))\psi_\delta(x)u_j(x)dx + \int_\Omega |u_j(x)|^{2^*}\,\psi_\delta(x)dx\, + o_j(1),
\end{aligned}
$$

as $j \to +\infty$.

By the Cauchy-Schwartz inequality we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{2n}} u_j(x)\big(u_j(x)-u_j(y)\big)\big(\psi_\delta(x)-\psi_\delta(y)\big)K(x-y)\,dx\,dy \\
&\le \left(\int_{\mathbb{R}^{2n}} |u_j(x)|^2\,|\psi_\delta(x)-\psi_\delta(y)|^2\,K(x-y)\,dx\,dy\right)^{1/2} \qquad (39)\\
&\quad \left(\int_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2\,K(x-y)\,dx\,dy\right)^{1/2},
\end{aligned}
$$

where the last term in the right-hand side is finite uniformly in j.

Now, we claim that

$$
\lim_{\delta\to 0}\left[\lim_{j\to+\infty}\int_{\mathbb{R}^{2n}} |u_j(x)|^2\,|\psi_\delta(x)-\psi_\delta(y)|^2\,K(x-y)\,dx\,dy\right] = 0. \qquad (40)
$$

For this, we first fix $\delta > 0$ and we observe that $u_j(x) \to u(x)$ a.e. $x \in \Omega$ as $j \to +\infty$. Since $u_j = 0 = u$ a.e. in $\mathbb{R}^n \setminus \Omega$, $u_j(x) \to u(x)$ a.e. $x \in \mathbb{R}^n$ as $j \to +\infty$. On the other hand, by (2), (33), the boundedness and Lipschitz regularity of ψ_δ we get, for some $L > 0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^{2n}} |u_j(x)|^2\,|\psi_\delta(x)-\psi_\delta(y)|^2\,K(x-y)\,dx\,dy \\
&\le \frac{1}{\theta}\int_{\mathbb{R}^{2n}} |u_j(x)|^2\,|\psi_\delta(x)-\psi_\delta(y)|^2\,|x-y|^{-n-2s}\,dx\,dy \\
&\le \frac{L^2\delta^{-2}}{\theta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\cap\{|x-y|\le\delta\}} |u_j(x)|^2\,|x-y|^{2-n-2s}\,dx\,dy \\
&\quad + \frac{4}{\theta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\cap\{|x-y|>\delta\}} |u_j(x)|^2\,|x-y|^{-n-2s}\,dx\,dy \\
&\le C\frac{(L^2\delta^{-2}+4)}{\theta}\int_{\mathbb{R}^n} |u_j(x)|^2\,dx\,dy \le C\frac{(L^2\delta^{-2}+4)}{\theta}\int_{\mathbb{R}^n} |h(x)|^2\,dx\,dy < +\infty
\end{aligned}
$$

with $C = C(n,s,\delta) > 0$. Thus, by the Dominated Convergence Theorem it follows that

$$
\begin{aligned}
&\lim_{j\to+\infty}\int_{\mathbb{R}^{2n}} |u_j(x)|^2\,|\psi_\delta(x)-\psi_\delta(y)|^2\,K(x-y)\,dx\,dy \qquad (41)\\
&= \int_{\mathbb{R}^{2n}} |u(x)|^2\,|\psi_\delta(x)-\psi_\delta(y)|^2\,K(x-y)\,dx\,dy
\end{aligned}
$$

with $\delta > 0$ fixed.

By arguing as above we have

$$
\begin{aligned}
&\int_{U\times V} |u(x)|^2\, |\psi_\delta(x)-\psi_\delta(y)|^2\, K(x-y)\, dx\, dy \\
&\qquad \le \frac{1}{\theta}\int_{\mathbb{R}^{2n}} |u(x)|^2\, |\psi_\delta(x)-\psi_\delta(y)|^2\, |x-y|^{-n-2s}\, dx\, dy \\
&\qquad \le \frac{L^2}{\theta}\delta^{-2}\int_U\int_{V\cap\{|x-y|\le\delta\}} |u(x)|^2\, |x-y|^{2-n-2}\, dx\, dy \\
&\qquad\quad + \frac{4}{\theta}\int_U\int_{V\cap\{|x-y|>\delta\}} |u(x)|^2\, |x-y|^{-n-2}\, dx\, dy
\end{aligned}
\tag{42}
$$

where U and V are two generic subsets of $\mathbb{R}^n$. Now, we will prove that the term on the right-hand in (41) goes to 0 as $\delta\to 0$, by using (42) case by case. First, we observe that when $U=V=\mathbb{R}^n\setminus B(x_i,\delta)$ all the integrals in (42) are equal to 0. When $U\times V=B(x_i,\delta)\times\mathbb{R}^n$ and $U\times V=\mathbb{R}^n\times B(x_i,\delta)$, we can use Proposition 7 together with (42) to get

$$
\lim_{\delta\to 0}\int_{U\times V} |u(x)|^2\, |\psi_\delta(x)-\psi_\delta(y)|^2\, K(x-y)\, dx\, dy = 0.
$$

Thus, by using these last results we get

$$
\lim_{\delta\to 0}\int_{\mathbb{R}^{2n}} |u(x)|^2\, |\psi_\delta(x)-\psi_\delta(y)|^2\, K(x-y)\, dx\, dy = 0,
$$

and by combining this formula with (41) we prove (40); from this, by using also (39) it follows that

$$
\lim_{\delta\to 0}\left[\lim_{j\to+\infty} M_a(\|u_j\|_Z^2)\int_{\mathbb{R}^{2n}} u_j(x)\big(u_j(x)-u_j(y)\big)\big(\psi_\delta(x)-\psi_\delta(y)\big)K(x-y)\, dx\, dy = 0\right].
\tag{43}
$$

Now, by Hölder inequality and (2) we observe that, for any $x\in\mathbb{R}^n$,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n} \frac{u_j(x)-u_j(y)}{|x-y|^{n+2s}}dx\right|^2 \\
&\le 2\,|u_j(y)|^2\left|\int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+2s}}dx\right|^2 + 2\left|\int_\Omega \frac{u_j(x)-u_j(y)}{|x-y|^{n+2s}}dx\right|^2 \\
&\le C\,|u_j(y)|^2 + 2\frac{|\Omega|}{\theta}\int_\Omega |u_j(x)-u_j(y)|^2\, K(x-y)dx,
\end{aligned}
\tag{44}
$$

with $C=C(\Omega)>0$. So, by (35) and (44) we get

$$
\begin{aligned}
&\liminf_{j\to+\infty}\int_{\mathbb{R}^n}\psi_\delta(y)\int_\Omega |u_j(x)-u_j(y)|^2\, K(x-y)\, dx\, dy \\
&\quad\ge \frac{\theta}{2\,|\Omega|}\frac{1}{c(n,s)}\liminf_{j\to+\infty}\int_{\mathbb{R}^n}\psi_\delta(y)\, c(n,s)\left|\int_{\mathbb{R}^n}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\, dx\right|^2 dy \\
&\qquad - C\liminf_{j\to+\infty}\int_{\mathbb{R}^n}\psi_\delta(y)\, |u_j(y)|^2\, dy \\
&\quad\ge \frac{\theta}{2\,|\Omega|}\frac{1}{c(n,s)}\int_{\mathbb{R}^n}\psi_\delta(y)d\mu - C\int_{B(x_i,\delta)} |u(y)|^2\, dy.
\end{aligned}
\tag{45}
$$

Moreover, by (8), (9) for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(x,t)| \leq 2\epsilon\,|t| + q\delta\,|t|^{q-1}\ . \tag{46}$$

and so, by the Dominated Convergence Theorem we get

$$\int_{B(x_i,\delta)} f(x,u_j(x))u_j(x)\psi_\delta(x)dx \to \int_{B(x_i,\delta)} f(x,u(x))u(x)\psi_\delta(x)dx \tag{47}$$

as $j \to +\infty$; we also observe that the resulting integral goes to 0 as $\delta \to 0$. So, by (35) it follows that

$$\int_\Omega |u_j(x)|^{2^*}\,\psi_\delta(x)dx \to \int_\Omega \psi_\delta(x)d\nu \quad \text{as } j \to +\infty$$

and by combining this last formula with (38), (43), (45) and (47) we get

$$\begin{aligned}\int_\Omega \psi_\delta(x)d\nu &+ \int_{B(x_i,\delta)} f(x,u(x))u(x)\psi_\delta(x)dx\\ &\geq M_a(\alpha^2)C\left(\int_\Omega \psi_\delta(y)d\mu - \int_{B(x_i,\delta)} |u(y)|^2\,dy\right) + o_\delta(1),\end{aligned}$$

recalling that $M_a(\|u_j\|_Z^2) \to M_a(\alpha^2)$ as $j \to +\infty$. By sending $\delta \to 0$ and by using (7) we conclude that $\nu_i \geq M_a(\alpha^2)\mu_i \geq m_0 C\mu_i$ and by using also the inequality in (37) we get

$$\nu_i \geq \frac{(m_0C)^{n/2s}}{S^{(n-2s)/2s}}, \tag{48}$$

for any $i \in J$. Now we shall prove that the above expression cannot occur, and so the set J is empty. By (18) and (19) we get

$$\lim_{j\to+\infty}\left(\mathcal{J}_{a,\,\lambda}(u_j) - \frac{1}{\sigma}\mathcal{J}'_{a,\,\lambda}(u_j)(u_j)\right) = c_{a,\,\lambda}. \tag{49}$$

Moreover, by (7), (10) and remembering that $m_0 < a < \dfrac{\sigma}{2}m_0$ we have

$$\begin{aligned}J_{a,\,\lambda}(u_j) &- \frac{1}{\sigma}\mathcal{J}'_{a,\,\lambda}(u_j)(u_j)\\ &\geq \frac{1}{2}\widehat{M_a}(\|u_j\|_Z^2) - \frac{1}{\sigma}M_a(\|u_j\|_Z^2)\,\|u_j\|_Z^2 + \left(\frac{1}{\sigma} - \frac{1}{2^*}\right)\int_\Omega |u_j(x)|^{2^*}\,dx\\ &\geq \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right)\|u_j\|_Z^2 + \left(\frac{1}{\sigma} - \frac{1}{2^*}\right)\int_\Omega |u_j(x)|^{2^*}\,dx\\ &\geq \left(\frac{1}{\sigma} - \frac{1}{2^*}\right)\int_\Omega \psi_\delta(x)\,|u_j(x)|^{2^*}\,dx.\end{aligned} \tag{50}$$

By combining (49) and (50) we get

$$c_{a,\,\lambda} \geq \left(\frac{1}{\sigma} - \frac{1}{2^*}\right)\int_\Omega \psi_\delta(x)d\nu,$$

from which, by sending $\delta \to 0$ and by using (48), it follows that

$$c_{a,\,\lambda} \geq \left(\frac{1}{\sigma} - \frac{1}{2^*}\right)\frac{(m_0C)^{n/2s}}{S^{(n-2s)/2s}},$$

which leads to an absurd by Lemma 6. Thus, J is empty and by (35) and (36) it follows that u_j converges to u in $L^{2^*}(\Omega)$. So, by (19) with $\phi = u_j$, (46) and the Dominated Convergence Theorem we have

$$\lim_{j\to+\infty} M_a(\|u_j\|_Z^2)\,\|u_j\|_Z^2 = \lambda\int_\Omega f(x,u(x))u(x)dx + \int_\Omega |u(x)|^{2^*}\,dx. \tag{51}$$

Moreover, by remembering that $u_j \rightharpoonup u$ in Z, $M_a(\|u_j\|_Z^2) \to M_a(\alpha^2)$ and by using (19), (46) and the Dominated Convergence Theorem we have

$$M_a(\alpha^2)\,\langle u,\,\varphi\rangle_Z = \lambda\int_\Omega f(x,u(x))\varphi(x)\,dx - \int_\Omega |u(x)|^{2^*-2}\,u(x)\varphi(x)dx\,, \tag{52}$$

for any $\varphi \in Z$. So, by combining (51) and (52) it follows that

$$M_a(\|u_j\|_Z^2)\,\|u_j\|_Z^2 \to M_a(\alpha^2)\,\|u\|_Z^2 \qquad \text{as } j\to+\infty,$$

from which we conclude the proof of claim (34).

Therefore, we have proved the Palais-Smale condition and by the Mountain Pass Theorem we obtain a critical point $u \in Z$ for the functional $\mathcal{J}_{a,\,\lambda}$ at level $c_{a,\,\lambda}$. Since $J_{a,\,\lambda}(u) = c_{a,\,\lambda} > 0 = J_{a,\,\lambda}(0)$ we conclude that $u \not\equiv 0$. □

Proof of Theorem 1. By Theorem 2, for any $\lambda \geq \lambda_0$ let u_λ be a solution of problem (13). Now, we claim that

$$\text{there exists } \lambda^* \geq \lambda_0 \text{ such that } \|u_\lambda\|_Z \leq t_0 \text{ for any } \lambda \geq \lambda^*\,. \tag{53}$$

where t_0 is given as at the beginning of Section 2. We argue by contradiction and suppose that there is a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ such that $\|u_{\lambda_j}\|_Z \geq t_0$. Since u_{λ_j} is a critical point of the functional $\mathcal{J}_{a,\,\lambda_j}$, by using also (7) and (10) it follows that

$$\begin{aligned} c_{a,\,\lambda_j} &\geq \frac{1}{2}\widehat{M_a}(\|u_{\lambda_j}\|_Z^2) - \frac{1}{\sigma}M_a(\|u_{\lambda_j}\|_Z^2)\,\|u_{\lambda_j}\|_Z^2 \\ &\geq \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right)\|u_{\lambda_j}\|_Z^2 \geq \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right)t_0^2, \end{aligned}$$

which contradicts Lemma 6 since $m_0 < a < \dfrac{\sigma}{2}m_0$. So, by (53) we get $M_a(\|u_\lambda\|_Z^2) = M(\|u_\lambda\|_Z^2)$ which inplies that u_λ is a solution of problem (1) for any $\lambda \geq \lambda_0$.

Moreover, arguing as above we have

$$c_{a,\,\lambda} \geq \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right)\|u_\lambda\|_Z^2\,,$$

and so, since $m_0 < a < \dfrac{\sigma}{2}m_0$ and by Lemma 6, it follows that $\lim\limits_{\lambda\to+\infty}\|u_\lambda\|_Z = 0$. □

5. Existence of non-negative solutions

In this section we study the sign of solutions of problem (12). For this, we first introduce the following technical lemma.

Lemma 8. *Let $u \in Z$. Then the absolute value of u, denoted by $|u|$, is in Z.*

Proof. We fix $a > 0$. Since $u \in Z$, by costruction there exists $w \in C_0^\infty(\Omega)$ such that

$$\|u - w\|_X < \frac{a}{2}. \tag{54}$$

Now, for any $\epsilon > 0$ and $x \in \mathbb{R}^n$, we set $v_\epsilon(x) := \big(\epsilon^2 + w^2(x)\big)^{1/2} - \epsilon$. We observe that $v_\epsilon = 0 = w$ in $\mathbb{R}^n \setminus \Omega$ and it is a smooth function by construction. Hence, $v_\epsilon \in C_0^\infty(\Omega)$. Also, we have $v_\epsilon(x) \to |w(x)|$ a.e. $x \in \mathbb{R}^n$ as $\epsilon \to 0$. Since $|v_\epsilon| \le |w|$ for any $\epsilon > 0$, by the Dominated Convergence Theorem, $v_\epsilon \to |w|$ in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$.

On the other hand,

$$|\nabla v_\epsilon| = \frac{|w|\,|\nabla w|}{(\epsilon^2 + w^2)^{1/2}} \le |\nabla w|\,,$$

uniformly in ϵ. Therefore, by the boundedness and Lipschitz regularity of w it follows that

$$\begin{aligned}
&|v_\epsilon(x) - |w(x)| - v_\epsilon(y) + |w(y)|\,|^2\, K(x-y) \\
&\qquad \le 2\left(|v_\epsilon(x) - v_\epsilon(y)|^2 + |\,|w(x)| - |w(y)|\,|^2\right) K(x-y) \\
&\qquad \le C\, \min\left\{1,\, |x-y|^2\right\} K(x-y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n).
\end{aligned}$$

which is clearly finite thanks to (2). Thus, by the Dominated Convergence Theorem we get $v_\epsilon \to |w|$ in X as $\epsilon \to 0$, in particular

$$\|v_\epsilon - |w|\|_X < \frac{a}{2} \tag{55}$$

for ϵ sufficiently small, say $\epsilon \le \bar\epsilon$, with $\bar\epsilon = \bar\epsilon(a) > 0$.

By (54) and (55) it is easy to see that

$$\||u| - v_{\bar\epsilon}\|_X \le \||u| - |w|\|_X + \||w| - v_{\bar\epsilon}\|_X \le \|u - w\|_X + \||w| - v_{\bar\epsilon}\|_X < a.$$

This concludes the proof. ∎

Corollary 9. *Let all the assumptions of Theorem 1 be satisfied and assume* (11) *in addition. Then problem* (12) *has a non-negative solution* u_λ *for all* $\lambda \ge \lambda^*$*, where* λ^* *is the parameter given in Theorem 1.*

Proof. We fix $\lambda \ge \lambda^*$. Let $u_\lambda \in Z$ be a solution of problem (12), given by Theorem 1. By Lemma 8 we have $u_\lambda^- \in Z$. So, by (12) with $\varphi = u_\lambda^-$ we get

$$\begin{aligned}
&M(\|u_\lambda\|_Z^2) \int_{\mathbb{R}^{2n}} (u_\lambda(x) - u_\lambda(y))(u_\lambda^-(x) - u_\lambda^-(y)) K(x-y)\,dx\,dy \\
&\qquad = \lambda \int_\Omega f(x, u_\lambda(x)) u_\lambda^-(x)\,dx + \int_\Omega \left|u_\lambda^-(x)\right|^{2^*} dx.
\end{aligned} \tag{56}$$

Now, we observe that

$$\begin{aligned}
&(u_\lambda(x) - u_\lambda(y))(u_\lambda^-(x) - u_\lambda^-(y)) \\
&\qquad = -u_\lambda^+(x) u_\lambda^-(y) - u_\lambda^-(x) u_\lambda^+(y) - (u_\lambda^-(x) - u_\lambda^-(y))^2 \le -\left|u_\lambda^-(x) - u_\lambda^-(y)\right|^2,
\end{aligned}$$

for a.e. $x,\, y \in \mathbb{R}^n$. Moreover, by (11) we get $f(x, u_\lambda(x))u_\lambda^-(x) = 0$ for a.e. $x \in \mathbb{R}^n$. Thus, by (56) it follows that

$$0 \le -\int_{\mathbb{R}^{2n}} \left|u_\lambda^-(x) - u_\lambda^-(y)\right|^2 K(x-y)dx\,dy - \int_\Omega \left|u_\lambda^-(x)\right|^{2^*} dx \le -\left\|u_\lambda^-\right\|_Z^2$$

which implies $u_\lambda^- \equiv 0$. □

Appendix A. Some motivation for a fractional Kirchhoff equation

The goal of these last pages is to give some motivation for the problem studied in this paper. For this, we would like first to recall some basic facts on the classical Kirchhoff equation: our explanations will be oversimplified, and even crude in some parts, and we will not attempt a rigorous mathematical justification of all the asymptotics that we are going to discuss heuristically.

We will consider the one-dimensional case for simplicity. For this we take the physical model of an elastic string constrained at the extrema. For concreteness, the string will be represented by the graph of a function $u : [-1, 1] \times [0, +\infty) \to \mathbb{R}$, and the end-point constraint reads $u(-1, t) = u(1, t) = 0$ for any $t \ge 0$. As usual we will write $u = u(x, t)$, where x is the space variable and t is the time.

For further use, we can indeed identify this finite string with an infinite string, that is constrained outside $(-1, 1)$, i.e. consider the function $u : [-1, 1] \times [0, +\infty) \to \mathbb{R}$, with $u(x, t) = 0$ for any $x \in \mathbb{R} \setminus (-1, 1)$ and any $t \ge 0$.

Then, the acceleration u_{tt} of the vertical displacement u of the vibrating string (that from now on will be assumed suitably small with respect to the length of the string) must be compensated, by Newton's law, by the elastic force of the string and by the external force field f: so we obtain the classical equation for the vibrating string:

$$u_{tt} = Mu_{xx} + f.$$

If we look for stationary solutions, i.e. solutions $u(x)$ that do not depend on time, the equation boils down to

$$Mu_{xx} + f = 0. \tag{57}$$

To a first approximation, for homogeneous strings, the elastic tension term M is simply a positive constant m_0. Several corrections to the model were proposed in order to take into account some discrepancies between the theory and the experimental data, since *"it is well known that the classical linearized analysis of the vibrating string can lead to results which are reasonably accurate only when the minimum (rest position) tension and the displacements are of such magnitude that the relative change in tension during the motion is small"*, see [7].

A classical modification of the above model is then to suppose that the tension increases if so does the length of the string. This ansatz is coherent with the common experience that a taut string reacts more strongly than a slack one. It is conceivable then to make the above ansatz quantitaive and suppose, for simplicity, that the tension, for small deformations of the string, takes (at least for small elongations of the string) the linear form

$$M(\ell) = m_0 + 2b\ell, \tag{58}$$

where $b > 0$ is constant and ℓ is the increment in the length of the string with respect to its rest position (in which the string has length 2), i.e.

$$\ell = \int_{-1}^{1} \sqrt{1 + u_x^2}\, dx - 2. \tag{59}$$

For small deformations, $\sqrt{1 + u_x^2} = 1 + \frac{u_x^2}{2}$ up to higher order terms, and so

$$\ell = \frac{1}{2} \int_{-1}^{1} u_x^2\, dx.$$

By plugging this into (58) we obtain

$$M = m_0 + b \int_{-1}^{1} u_x^2\, dx = m_0 + b \int_{\mathbb{R}} u_x^2\, dx,$$

where we used the notation for which u is defined to vanish outside $(-1, 1)$. By inserting this into (57), one obtains the classical version of the Kirchhoff equation

$$M\left(\int_{-\infty}^{+\infty} u_x^2\, dx\right) u_{xx} + f = 0, \tag{60}$$

with $M(t) = m_0 + bt$. As a historical remark, we mention that the equation was first introduced in [12, 13] and then, probably independently, proposed in [7, 8]; see also [14] for a comparison between the theory and the experimental data.

We observe that the first term in (60) can be interpreted in a variational way, as arising from an energy of the form

$$\frac{1}{2} \widehat{M}\left(\int_{-\infty}^{+\infty} u_x^2\, dx\right), \tag{61}$$

where $\widehat{M}$ is a primitive of M.

With this respect, the Kirchhoff equation of nonlocal type that we studied originates from the idea that the energy in (61) does not depend on the H^1 norm of the function that parameterizes the graph of the string, but rather on its H^s norm, namely we replaced (61) with

$$\frac{1}{2} \widehat{M}\left(\int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^{1+2s}}\, dx\, dy\right),$$

or even with more general kinds of fractional norms. In this sense, while the "nonlocal" feature of the tension in the classical Kirchhoff equation surfaces from the average of a "local" object (namely u_x^2), in the equation we took into account the "nonlocal" aspect of the tension arises from an object which is "nonlocal" as well. In general, we think it could be interesting to study even more general models in which the tension of the string is related to "nonlocal" measurments of the modification of the string from its rest position. Some of these models may be variational in nature (as the one considered here), some others may be not.

Another way of obtaining the model we study from the classical Kirchhoff equation goes as follows. Following [4], for $\sigma \in (0, 1)$, we consider the σ-length of the string as follows. Let $E := \{(x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } x_2 < u(x_1)\}$ be the subgraph of u. We assume that the oscillation of the string does not exceed a size of order ϵ, i.e. $|u| < \epsilon$

and so $\partial E \subset \{(x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } |x_2| < \epsilon\}$. Then we define the length of the string in the set $Q := [-1, 1] \times [-\epsilon, \epsilon]$ as

$$\ell_\sigma(u) := I(E \cap Q, \mathbb{R}^2 \setminus E) + I(Q \setminus E, E \setminus Q),$$

where, for any couple of disjoint measurable sets $X, Y \subset \mathbb{R}^2$ we set

$$I(X, Y) := \int_{X\times Y} \frac{dx\, dy}{|x-y|^{2+\sigma}}.$$

It is known that (up to a suitable rescaling) ℓ_σ tends to the classical length of the string as $\sigma \to 1$ (see [1, 5]). Of course, the fractional length of the string at rest here is simply $\ell_\sigma(0)$, and so the difference between the fractional length of the string and its original value is

$$\ell_\sigma := \ell_\sigma(u) - \ell_\sigma(0).$$

So it is conceivable to replace in the model the dependence from the classical length with the dependence of this "nonlocal" version of length, i.e. to substitute (58) with

$$M(\ell_\sigma) = m_0 + 2b\ell_\sigma. \tag{62}$$

Moreover, ℓ_σ may be computed in terms of u thanks to the following geometric observation. Let

$$\begin{aligned}
&E^+ := \{(x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } 0 < x_2 < u(x_1)\},\\
&E^- := \{(x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } u(x_1) < x_2 < 0\},\\
&W^+ := \mathbb{R} \times (0, +\infty),\\
&W^- := \mathbb{R} \times (-\infty, 0)\\
\text{and}\qquad &Q^\pm := Q \cap W^\pm.
\end{aligned}$$

Then

$$\begin{aligned}
&\ell_\sigma(u)\\
=\ &I((Q^- \setminus E^-) \cup E^+, (W^+ \setminus E^+) \cup E^-) + I((Q^+ \setminus E^+) \cup E^-, W^- \setminus Q)\\
=\ &I(Q^- \setminus E^-, W^+ \setminus E^+) + I(Q^- \setminus E^-, E^-)\\
&+I(E^+, W^+ \setminus E^+) + I(E^+, E^-)\\
&+I(Q^+ \setminus E^+, W^- \setminus Q) + I(E^-, W^- \setminus Q)
\end{aligned}$$

and

$$\ell_\sigma(0) = I(Q^-, W^+) + I(Q^+, W^- \setminus Q).$$

Moreover

$$\begin{aligned}
&I(Q^-, W^+) - I(Q^- \setminus E^-, W^+ \setminus E^+)\\
=\ &I(Q^- \setminus E^-, E^+) + I(E^-, W^+ \setminus E^+) + I(E^-, E^+)
\end{aligned}$$

and

$$I(Q^+, W^- \setminus Q) - I(Q^+ \setminus E^+, W^- \setminus Q) = I(E^+, W^- \setminus Q).$$

As a consequence

$$\begin{aligned}
\ell_\sigma \ = \ &I(Q^- \setminus E^-, E^-) + I(E^+, W^+ \setminus E^+)\\
&+I(E^-, W^- \setminus Q) - I(Q^- \setminus E^-, E^+)\\
&-I(E^-, W^+ \setminus E^+) - I(E^+, W^- \setminus Q).
\end{aligned}$$

By collecting all the terms involving E^+ and E^- and using that $I(X,Y)=I(Y,X)$ we obtain

$$\begin{aligned} \ell_\sigma &= I(E^+,W^+\setminus E^+)-I(E^+,W^-\setminus E^-) \\ &\quad +I(E^-,W^-\setminus E^-)-I(E^-,W^+\setminus E^+). \end{aligned} \tag{63}$$

We now write separately the first two terms. For typographical convenience we use the notation of writing the integrating variables next to their integral sign. Also, we set $u^+ := \max\{u,0\}$ and $u^- := \max\{-u,0\}$: notice that $u^\pm \geq 0$ and $u = u^+ - u^-$. In this way, $E^+ = \{(x_1,x_2)\in\mathbb{R}^2 \text{ s.t. } 0<x_2<u^+(x_1)\}$, $E^- := \{(x_1,x_2)\in\mathbb{R}^2 \text{ s.t. } -u^-(x_1)<x_2<0\}$,

$$\begin{aligned} & I(E^+,W^+\setminus E^+) \\ = & \int_{\mathbb{R}} dx_1 \int_0^{u^+(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \int_{u^+(y_1)}^{+\infty} dy_2 \big(|x_1-y_1|^2+|x_2-y_2|^2\big)^{-(2+\sigma)/2} \end{aligned}$$

and

$$\begin{aligned} & I(E^+,W^-\setminus E^-) \\ = & \int_{\mathbb{R}} dx_1 \int_0^{u^+(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \int_{-\infty}^{-u^-(y_1)} dy_2 \big(|x_1-y_1|^2+|x_2-y_2|^2\big)^{-(2+\sigma)/2}. \end{aligned}$$

Thus, we set $\psi=\psi(x_1,y_1,z_2) := \big(|x_1-y_1|^2+|z_2|^2\big)^{-(2+\sigma)/2}$, we make the substitution $z_2 := y_2-x_2$ and we get

$$\begin{aligned} & I(E^+,W^+\setminus E^+)-I(E^+,W^-\setminus E^-) \\ & = \int_{\mathbb{R}} dx_1 \int_0^{u^+(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \left[\int_{u^+(y_1)-x_2}^{+\infty} dz_2 - \int_{-\infty}^{-u^-(y_1)-x_2} dz_2 \right] \psi. \end{aligned} \tag{64}$$

Now we observe that

$$\int_{-\infty}^0 dz_2\ \psi = \int_0^{+\infty} dz_2\ \psi,$$

since ψ is even in z_2. Therefore

$$\begin{aligned} & \left[\int_{u^+(y_1)-x_2}^{+\infty} dz_2 - \int_{-\infty}^{-u^-(y_1)-x_2} dz_2 \right] \psi \\ = & \left[\int_{u^+(y_1)-x_2}^{0} dz_2 + \int_0^{+\infty} dz_2 - \int_{-\infty}^0 dz_2 - \int_0^{-u^-(y_1)-x_2} dz_2 \right] \psi \\ = & \left[\int_{u^+(y_1)-x_2}^{0} dz_2 - \int_0^{-u^-(y_1)-x_2} dz_2 \right] \psi, \end{aligned}$$

hence (64) becomes

$$\begin{aligned} & I(E^+,W^+\setminus E^+)-I(E^+,W^-\setminus E^-) \\ & = -\int_{\mathbb{R}} dx_1 \int_0^{u^+(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \left[\int_0^{u^+(y_1)-x_2} dz_2 + \int_0^{-u^-(y_1)-x_2} dz_2 \right] \psi. \end{aligned} \tag{65}$$

At this point, we make the crude approximation

$$\int_0^{\epsilon'} dz_2\ \psi \simeq \psi\Big|_{z_2=0} \epsilon' = |x_1-y_1|^{-(2+\sigma)}\epsilon', \tag{66}$$

when ϵ' is of the order of ϵ. As a matter of fact, such approximation is not fully justified when x_1 and y_1 are in a neighborhood of size much smaller than ϵ, due to the singularity of the kernel: since this appendix is mainly motivational, and should not be interpreted in a strictly rigorous mathematical language, we neglect this subtle point and just take the ansatz that (66) is reasonable for most of the points of integration x_1 and y_1 and see what happens. Similarly, we observe that, for $s := \frac{\sigma+1}{2}$, at least formally and in the principal value sense

$$\begin{aligned}\|u\|^2_{H^s(\mathbb{R})} &= \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u(x_1)-u(y_1)|^2}{|x_1-y_1|^{1+2s}} \\ &= \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u(x_1)|^2 - u(x_1)u(y_1) + |u(y_1)|^2 - u(y_1)u(x_1)}{|x_1-y_1|^{2+\sigma}} \\ &= 2\int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u(x_1)|^2 - u(x_1)u(y_1)}{|x_1-y_1|^{2+\sigma}},\end{aligned} \tag{67}$$

thanks to the symmetric role played by x_1 and y_1.

From (66) we obtain the approximation

$$\begin{aligned}&\left[\int_0^{u^+(y_1)-x_2} dz_2 + \int_0^{-u^-(y_1)-x_2} dz_2\right]\psi \\ &\qquad \simeq |x_1-y_1|^{-(2+\sigma)}\big(u^+(y_1)-u^-(y_1)-2x_2\big) \\ &\qquad = |x_1-y_1|^{-(2+\sigma)}\big(u(y_1)-2x_2\big).\end{aligned}$$

Therefore, up to terms that we neglected,

$$\begin{aligned}&\int_0^{u^+(x_1)} dx_2 \left[\int_0^{u^+(y_1)-x_2} dz_2 + \int_0^{-u^-(y_1)-x_2} dz_2\right]\psi \\ = {}& \int_0^{u^+(x_1)} dx_2 |x_1-y_1|^{-(2+\sigma)}\big(u(y_1)-2x_2\big) \\ = {}& -|x_1-y_1|^{-(2+\sigma)}\big(|u^+(x_1)|^2 - u^+(x_1)u(y_1)\big).\end{aligned}$$

Consequently, (65) becomes

$$\begin{aligned}&I(E^+, W^+\setminus E^+) - I(E^+, W^-\setminus E^-) \\ &\quad = \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u^+(x_1)|^2 - u^+(x_1)u(y_1)}{|x_1-y_1|^{2+\sigma}}.\end{aligned} \tag{68}$$

Notice also that a reflection of the vertical variable transforms the set E^+ of the function u into the set E^- for the function $-u$, and also $(-u)^+ = u^-$. Hence the symmetric version of (68) reads

$$\begin{aligned}&I(E^-, W^-\setminus E^-) - I(E^-, W^+\setminus E^+) \\ &\quad = \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u^-(x_1)|^2 + u^-(x_1)u(y_1)}{|x_1-y_1|^{2+\sigma}}.\end{aligned} \tag{69}$$

Moreover, since, at any point x_1 either $u^+(x_1)=0$ or $u^-(x_1)=0$, we see that

$$|u(x_1)|^2 = |u^+(x_1)|^2 + |u^-(x_1)|^2.$$

Accordingly, by plugging (68) and (69) into (63) and we obtain the approximation

$$\ell_\sigma = \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u(x_1)|^2 - u(x_1)u(y_1)}{|x_1 - y_1|^{2+\sigma}} = \frac{1}{2}\|u\|^2_{H^s(\mathbb{R})},$$

where in the last step we used (67). By inserting this expression into (62) we obtain the approximated tension

$$M = m_0 + 2b\|u\|^2_{H^s(\mathbb{R})}.$$

Hence, a nonlocal model for the vibrating string may be obtained from (57), by considering the above tension and by replacing the local spatial second derivative with the nonlocal operator $-(-\Delta)^s$: in this way we obtain the nonlocal equation

$$-M\left(\|u\|^2_{H^s(\mathbb{R})}\right)(-\Delta)^s u + f = 0.$$

Acknowledgments

We thank Filippo Gazzola and Clelia Marchionna for a very nice conversation.

References

[1] L. Ambrosio, G. De Philippis and L. Martinazzi, *Gamma-convergence of nonlocal perimeter functionals*, Manuscripta Math., 134, no. 3-4, 377–403 (2011).
[2] H. Brézis, Analyse fonctionelle. Théorie et applications, *Masson*, Paris (1983).
[3] B. Barrios, E. Colorado, A. De Pablo and U. Sanchez, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations, 252, 6133–6162 (2012).
[4] L. Caffarelli, J. M. Roquejoffre and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math., 63, no. 9, 1111–1144 (2010).
[5] L. Caffarelli and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations, 41, no. 1-2, 203–240 (2011).
[6] A. Capella, *Solutions of a pure critical exponent problem involving the half-Laplacian in annular-shaped domains*, Commun. Pure Appl. Anal., 10, no. 6, 1645–1662 (2011).
[7] G. F. Carrier, *On the nonlinear vibration problem of the elastic string*, Quart. Appl. Math., 3, 157–165 (1945).
[8] G. F. Carrier, *A note on the vibrating string*, Quart. Appl. Math., 7, 97–101 (1949).
[9] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, B. Sci. Math., in press, available on line at `http://arxiv.org/abs/1104.4345` .
[10] G. M. Figueiredo, *Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument*, J. Math. Anal. Appl., 401, no. 2, 706–713 (2013).
[11] A. Fiscella, *Saddle point solutions for non-local elliptic operators*, submitted paper, available on line at `http://arxiv.org/abs/1210.8401` .
[12] G. R. Kirchhoff, Vorlesungen über mathematische Physik: Mechanik, *Leipzig*, B. G. Teubner (1876).
[13] G. R. Kirchhoff, Vorlesungen über mathematische Physik. Band 1: Mechanik, *Leipzig*, B. G. Teubner (1883).
[14] D. W. Oplinger, *Frequency response of a nonlinear stretched string*, J. Acoust. Soc. Am., 32, no. 12, 1529–1538 (1960).
[15] G. Palatucci and A. Pisante, *Sobolev embeddings and concentration-compactness alternative for fractional Sobolev spaces*, submitted paper , available on line at `http://mipa.unimes.fr/preprints.html` .
[16] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Reg. Conf. Ser. Math.*, 65, *American Mathematical Society*, Providence, RI (1986).
[17] R. Servadei and E. Valdinoci, *A Brezis–Nirenberg result for non–local critical equations in low dimension*, Commun. Pure Appl. Anal., 12, no. 6 (2013).
[18] R. Servadei and E. Valdinoci, *Fractional Laplacian equations with critical Sobolev exponent*, preprint available at `http://www.math.utexas.edu/mp_arc-bin/mpa?yn=12-58` .

[19] R. Servadei and E. Valdinoci, *The Brezis–Nirenberg result for the fractional Laplacian*, preprint available at http://www.math.utexas.edu/mp_arc-bin/mpa?yn=11-196 .

[20] J. Tan, *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations, 36, no. 1-2, 21–41 (2011).

Dipartimento di Matematica "Federigo Enriques", Università di Milano, Via Cesare Saldini, 50 20133 Milano ITALY

E-mail address: alessio.fiscella@unimi.it

Weierstrass Institut für Angewandte Analysis und Stochastik, Hausvogteiplatz 11A, D-10117, Berlin, Germany

E-mail address: enrico.valdinoci@wias-berlin.de