On the connections between symmetries and conservation rules of dynamical systems

Giampaolo Cicogna*

Dipartimento di Fisica "E.Fermi" dell'Università di Pisa
and Istituto Nazionale di Fisica Nucleare, Sez. di Pisa

Largo B. Pontecorvo 3, Ed. B-C, I-56127, Pisa, Italy

Abstract

The strict connection between Lie point-symmetries of a dynamical system and its constants of motion is discussed and emphasized, through old and new results. It is shown in particular how the knowledge of a symmetry of a dynamical system can allow to obtain conserved quantities which are invariant under the symmetry. In the case of Hamiltonian dynamical systems it is shown that, if the system admits a symmetry of "weaker" type (specifically, a λ or a Λ-symmetry), then the generating function of the symmetry is not a conserved quantity, but the deviation from the exact conservation is "controlled" in a well defined way. Several examples illustrate the various aspects.

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1 Introduction

The role and the relevance of methods based on the analysis of symmetry properties of differential equations (both ordinary and partial) are well known, for what concerns not only the problem of finding explicit solutions, but also of examining "structural" properties (a typical and relevant feature is, e.g., the presence of conservation rules).

*Email: cicogna@df.unipi.it

There is an enormous literature on this subject: see e.g. [1, 2, 3, 4] for some classical texts where general procedures and standard applications can be found. However, there is a particular context where symmetry methods meet some intrinsic difficulty: this is the case of dynamical systems (DS), i.e. systems of first-order time-evolution differential equations of the form

$$\dot{u}_a \;=\; f_a(u,t) \qquad u_a = u_a(t) \quad (a = 1, \ldots, n) \tag{1}$$

with $\dot{u} = \mathrm{d}u/\mathrm{d}t$ and (sufficiently smooth) given functions $f_a = f_a(u,t)$. The present paper is devoted to investigate precisely this case. I am referring more specifically to Lie point-symmetries, i.e. to continuous transformations generated by infinitesimal vector fields X which can be written in the form

$$X \;=\; \varphi_a(u,t)\frac{\partial}{\partial u_a} + \tau(u,t)\frac{\partial}{\partial t} \equiv \varphi \cdot \nabla_u + \tau \partial_t \ . \tag{2}$$

The problem of finding all symmetries admitted by a DS is quite difficult. On the other hand, expectedly, the presence of some symmetry is strictly related to the determination of first integrals of the differential problem, also called, in this context, constants of motion of the dynamical flow. The determination of constants of motion for a time-evolution process is clearly a basic result, not only in view of obtaining full solutions of the problem, but also for their physical interpretation as "conserved quantities" along the time evolution.

Sect.2 is devoted to present the equations providing the conditions for the existence of Lie point-symmetries of a given DS, and to discuss the close connection existing between the problem of solving these equations and of detecting constants of motion. It can be noticed that this connection, in the present context, is not *directly* related to the celebrated Noether theorem. It will be useful, instead, in view of our discussion, to recall two classical, perhaps less known, old results. In few words, these results are very interesting as they illustrate the strict relationship existing, also in the context of DS, between the notions of symmetries and of constants of motion, but on the other hand they are not very useful in practice because they require the knowledge of "many" constants of motion in order to find (possibly all) symmetries of the DS.

In Sect.3, the point of view is partly reversed: I will assume that *just only one* symmetry is known. Indeed, although finding all symmetries can be a difficult problem, it often happens that one is able to detect a single symmetry, and often this symmetry has a rather simple expression. Sect.4 is devoted to show two quite general new classes of examples where this situation occurs. The first one includes Lorenz-like DS, the second one deals with systems related to higher order ODE's.

I will then show that this symmetry can be used to introduce some suitable "symmetry-adapted coordinates" where the problem of determining the constants of motion becomes easier and provides conserved quantities which are also invariant under the symmetry. This is completely confirmed by the examples considered in Sect.4.

In addition, it is shown that a "weaker" notion of symmetry can be introduced to the same purpose. I am referring to the notion of λ-symmetry [5, 6], which will be briefly recalled especially for what concerns its application to DS [7, 8]. In particular, the presence of a λ-symmetry allows the introduction of suitable coordinates exactly as standard symmetries.

In Sect.5, the particular case of *Hamiltonian* DS will be considered. A classical result ensures that if such a DS exhibits a symmetry admitting a generating function G, this function is automatically a conserved quantity invariant under the symmetry. If instead the DS admits a λ-symmetry, the generating function is no longer a conserved quantity, but the "breaking" of the conservation is "controlled" in a well defined way. In this context, also a generalization of the notion of λ-symmetry will be usefully introduced [8, 9]. Two examples illustrate the results, with the explicit calculation of the "deviation" from the exact conservation rule.

This is a full paper presented within ICNAAM 2011; a very short and preliminary sketch of part of these results can be found in the enlarged Abstracts of the Conference Proceedings [10].

2 Symmetries and constants of motion: some classical facts

Following the standard procedure, see e.g. [2, 4], a vector field X (2) is a Lie point-symmetry for the DS (1) (according to an usually accepted abuse of language, I will denote by X both the symmetry and its infinitesimal generator) if the following condition is satisfied (sum over repeated indices unless otherwise stated)

$$[\,f,\varphi\,]_a \;=\; -\frac{\partial}{\partial t}(\varphi_a-\tau\, f_a)+\frac{\partial \tau}{\partial u_b}f_af_b \qquad (a,b=1,\ldots,n) \tag{3}$$

where $[\,f,\varphi\,]_a$ is defined by

$$[\,f,\varphi\,]_a \;=\; f_b\frac{\partial}{\partial u_b}\varphi_a-\varphi_b\frac{\partial}{\partial u_b}f_a\equiv(f\cdot\nabla)\varphi_a-(\varphi\cdot\nabla)f_a\;.$$

It is not restrictive to put $\tau=0$, possibly introducing "evolutionary" vector field

$$X_{\rm ev}:=(\varphi-\tau\dot u)\cdot\nabla \;=\; (\varphi-\tau f)\cdot\nabla\equiv\widetilde\varphi\cdot\nabla$$

so the symmetry condition becomes

$$[\,f,\varphi\,]_{\,a}+\frac{\partial}{\partial t}\varphi_a \;=\; 0\;. \tag{4}$$

Despite this apparently simple form, it is in general very difficult to obtain a complete solution to this set of determining equations. *In principle* there are n (functionally independent, locally defined) solutions $\varphi^{(a)}$; denoting by $\kappa=\kappa(u,t)$ any constant of motion of the DS, i.e. any function such that

$$D_t\kappa\equiv\partial_t\kappa+f\cdot\nabla\kappa \;=\; 0$$

the most general symmetry of the DS can be written as

$$X = \sum_{a=1}^{n} \kappa^{(a)} \varphi^{(a)} \cdot \nabla \equiv \kappa^{(a)} X^{(a)} .$$

Apart from this very general result, the relationship between symmetries and constants of motion is actually much closer. To illustrate this point, and also in view of our discussion, let me recall the two following, perhaps less known, classical results.

a. This result is due to Ovsjannikov [11] and shows how symmetries of a DS can be deduced from the knowledge of its constants of motion.

Proposition 1 *Assume that n functionally independent constants of motion $\kappa^{(a)}$ of the given DS are known; then the linear system of n^2 equations*

$$\sum_{a=1}^{n} p_{ab} \frac{\partial \kappa^{(a)}}{\partial u_c} = \delta_{bc}$$

can be solved for the n^2 quantities p_{ab}. Then

$$X^{(a)} = \sum_{b=1}^{n} p_{ab} \frac{\partial}{\partial u_b}$$

are n independent symmetries for the DS.

b. The following result, based on the notion of Liouville vector field, has been restated by G. Ünal [12], and also used by J. Zhang and Y. Li [13]. Let me recall the main fact in the following form.

A DS is said to admit a *Liouville vector field* $Y = \psi(u,t) \cdot \nabla$ if

$$\partial_t \psi_a + [f, \psi]_a + (\mathtt{Div}\, f) \psi_a = 0$$

Clearly, if $\mathtt{Div}\, f = 0$ then Y is a standard symmetry; if instead $\mathtt{Div}\, f \neq 0$, putting

$$Y = q\, X$$

then X is a standard symmetry for the DS if q is a scalar function solving

$$\partial_t q + f \cdot \nabla q + (\mathtt{Div}\, f) q = 0 .$$

Then one has:

Proposition 2 *If the Liouville vector field Y satisfies $\mathtt{Div}\, \psi = 0$, then there are $n-1$ constants of motion $\widehat{\kappa}^{(a)}$ such that*

$$Y = \psi_a \frac{\partial}{\partial u_a} = \varepsilon_{abc\ldots l} \widehat{\kappa}^{(1)}_{,b}\, \widehat{\kappa}^{(2)}_{,c} \ldots \widehat{\kappa}^{(n-1)}_{,l} \frac{\partial}{\partial u_a} \tag{5}$$

where $\widehat{\kappa}^{(a)}_{,b} = \partial \widehat{\kappa}^{(a)} / \partial u_b$. In addition, the above constants of motion $\widehat{\kappa}^{(a)}$ are invariant under both the vector fields Y and X:

$$Y \widehat{\kappa}^{(a)} = X \widehat{\kappa}^{(a)} = 0 .$$

The last sentence says that the quantities $\widehat{\kappa}^{(a)}$ are simultaneously invariant under the dynamical flow and under the symmetry[1].

Both the above results are conceptually greatly relevant, but clearly of little practical use if one wants to explicitly find symmetries (or constants of motion as well) of a given DS. In the following, I try to partly reverse the approach: I will assume that just *only one* symmetry is known, and then try to deduce any possible information from it.

3 Symmetry adapted coordinates

Often, given a DS, one symmetry of it is easily seen, either by direct inspection or by simple calculations, as we shall see in the following section. Then (remembering also Proposition 2) the idea is to use invariants under this symmetry to construct (one or more) constants of motion.

To this purpose, the presence of a λ-symmetry (instead of a standard one) may equally well help in the calculations. Let me briefly recall the basic definitions of λ-symmetry for what concerns the application in this context.

The notion of λ-symmetry has been originally introduced in 2001 by C. Muriel & J.L.Romero in the context of ODE's [5, 6]. Since then, this notion has received many very important applications and extensions, which cannot be recalled here (for a fairly complete list of references, see e.g. [14, 15]). In our case, a DS admits a λ-symmetry $X = \varphi \cdot \nabla$ if there is a C^∞ function $\lambda = \lambda(u, \dot{u}, t)$ such that the following condition holds

$$[\, f, \varphi\,]_{\,a} + \frac{\partial}{\partial t}\varphi_a \; = \; -\lambda\, \varphi_a \tag{6}$$

to be compared with the standard condition (4).

λ-symmetries are not properly symmetries, indeed, e.g., they do not transform solutions into other solutions, nevertheless they share with standard symmetries many useful properties; in particular they indicate, as well as standard symmetries, a convenient choice of variables in view of our procedure.

Let us assume then that the given DS admits either a standard or a λ-symmetry $X = \varphi \cdot \nabla$. Introduce then n functionally independent quantities which are left fixed by this symmetry: choose the time t as one of these, and the remaining $n-1$, denoted by $w_j = w_j(u)$, independent of t:

$$X\, w_j \; = \; X\, t \; = \; 0 \qquad (j = 1, \ldots, n-1)\ .$$

Let ζ be the "rectifying" coordinate along the action of X, i.e.

$$X\, \zeta \; = \; 1 \qquad \text{or} \qquad X \; = \; \frac{\partial}{\partial \zeta}\ .$$

[1] In [12] this result is stated saying that the quantities $\widehat{\kappa}^{(a)}$ appearing in (5) are 'the' first integrals of the DS. Clearly, not all the first integrals satisfy (5) nor are symmetry-invariant.

Notice that, even in the case of λ-symmetry, all these coordinates depend only on X and not on λ. Choose now w_j, ζ as new dependent variables (with t still as independent one), and rewrite the DS in terms of these, i.e. in the form $\dot{w}_j = W_j$, $\dot{\zeta} = Z$: one immediately has that the r.h.s. W_j, Z of the new DS turn out to be independent of ζ if the symmetry is standard [2], and that only Z may depend on ζ if the symmetry is a λ-symmetry [7, 8]: this can be summarized writing

$$\dot{w}_j \;=\; W_j(w,t) \quad , \quad \dot{\zeta} \;=\; Z(w,[\zeta],t) \; . \tag{7}$$

If now we look for constants of motion of the DS *expressed as functions of* w_j, ζ, t, i.e. $\kappa = \kappa(w,\zeta,t)$, we can conclude with the following

Proposition 3 *The constants of motion $\kappa^{(a)}(w,\zeta,t)$ solve the characteristic equation*

$$\frac{\mathrm{d}w_1}{W_1} \;=\; \ldots \;=\; \frac{\mathrm{d}w_{n-1}}{W_{n-1}} \;=\; \frac{\mathrm{d}\zeta}{Z} \;=\; \mathrm{d}t$$

where W_j, Z are defined in (7).

The advantage of this procedure is clear: we have a *reduction* of the initial problem to a system of $n-1$ equations involving $n-1$ variables w_j; for the same reason, also the search for the constants of motion through the above characteristic equation is easier. One obtains in this way, by construction, conserved quantities which are also symmetry-invariant; this agrees of course with the classical Frobenious theorem [2]. Notice that this is a special case of a more general problem of finding suitable reduction procedures of DS; this and other related aspects will be discussed in a paper by G. Gaeta, S. Walcher and the present author (in preparation).

4 Two classes of examples

The two following propositions provide two quite general classes of DS where the presence of one symmetry is guaranteed and constants of motion can be successfully deduced.

Proposition 4 *A DS of the form*

$$\dot{u}_a \;=\; \sigma_{(a)} u_a + g_a(u) \qquad (a=1,\ldots,n;\ \text{no sum over } a)$$

where $\sigma_{(a)}$=const ($\neq 0$) admits the symmetry

$$X \;=\; \exp(\lambda t)\, g\cdot\nabla$$

($\lambda =$ *const* (*possibly zero*)) *if $g_a(u)$ have the form*

$$g_a(u) \;=\; u_a^{\left(1-(\lambda/\sigma_{(a)})\right)} P_a(u) \qquad (a,b,c=1,\ldots,n)$$

where P_a are any smooth functions of the ratios $u_b^{\sigma_{(c)}}/u_c^{\sigma_{(b)}}$ with "exchanged" exponents.

Several examples of this situation are known: it includes e.g. generalized Lorenz systems [13], etc.

Remark. If $\lambda \neq 0$, one can equivalently say that $X' := g \cdot \nabla$ is a λ-*symmetry* with $\lambda(u, \dot{u}, t) = \lambda = \text{const}$. This follows from the general property that if X_λ is a λ-symmetry with some $\lambda(u, \dot{u}, t)$ then

$$X := \exp\left(\int \lambda(u, \dot{u}, t)\, \mathrm{d}t\right) X_\lambda$$

is a (possibly *nonlocal*) standard symmetry[2].

Here an explicit example for Proposition 4.

Example 1. With $n = 3$ and $u \equiv (x, y, z)$ consider the DS:

$$\dot{x} = -x + x^2 P_1 \ , \ \dot{y} = -y + y^2 P_2 \ , \ \dot{z} = -2z + z^{(3/2)} P_3$$

where P_a are functions of $x/y, x^2/z$, and which can be cast in analytic form

$$\dot{x} = -x + Q_1(xy, y^2, z) \ , \ \dot{y} = -y + Q_2(x^2, xy, z) \ , \ \dot{z} = -2z + Q_3(x^3, yz, xz) \ .$$

This DS admits the symmetry

$$X \ = \ e^t \left(Q_1 \frac{\partial}{\partial x} + Q_2 \frac{\partial}{\partial y} + Q_3 \frac{\partial}{\partial z} \right) \ .$$

Choose e.g. (this is a variant of an example given in [12])

$$Q_1 = z - 2y^2 \quad , \quad Q_2 = 2xy \quad , \quad Q_3 = 4xz$$

then, with the notations of the above section,

$$w_1 = z/y^2 \quad , \quad w_2 \ = \ x^2 + y^2 - z/2 \ ,$$

$$\zeta \ = \ \frac{e^{-t}}{4v} \log \left| \frac{x - v}{x + v} \right| \qquad \left(\text{here} \quad v = |w_2|^{1/2}\right) \ .$$

Following the above procedure, the DS becomes

$$\dot{w}_1 = 0 \quad , \quad \dot{w}_2 = -2w_2 \quad , \quad \dot{\zeta} = e^{-t}$$

and exactly *three* functionally independent constants of motion can be found

$$\kappa_1 = w_1 = z/y^2 \quad , \quad \kappa_2 = e^{2t}(x^2 + y^2 - z/2)$$

$$\kappa_3 = e^{-t} \left(1 + \frac{1}{4v} \log \left| \frac{x - v}{x + v} \right| \right) \ .$$

[2] According to this remark, all examples given in [13] are actually equivalent to standard (not properly λ) symmetries.

Proposition 5 *Let $n = 2$, $u \equiv (x, y)$; the DS*

$$\dot{x} = y \qquad \dot{y} = y^2\gamma^{-1}\gamma_x + \gamma\, F(\gamma^{-1}y)$$

where $\gamma = \gamma(x) \neq 0$ and F are any given smooth functions, admits the symmetry

$$X = \gamma\frac{\partial}{\partial x} + y\gamma_x\frac{\partial}{\partial y} \ .$$

Here one has

$$w = \gamma^{-1}y \qquad , \qquad \zeta = \int \gamma^{-1}\mathrm{d}x$$

and the DS becomes

$$\dot{w} = F(w) \qquad , \qquad \dot{\zeta} = w \ .$$

Two constants of motion are easily obtained:

$$\kappa_1 = \zeta - \int w\, F^{-1}(w)\,\mathrm{d}w \qquad , \qquad \kappa_2 = t - \int F^{-1}(w)\,\mathrm{d}w \ .$$

This type of DS is specially interesting because the DS is equivalent to the ODE

$$\ddot{x} = \dot{x}^2\gamma^{-1}\gamma_x + \gamma\, F(\gamma^{-1}\dot{x})$$

and the above symmetry of the DS is in this case automatically extended to become a symmetry for the ODE:

$$X_{ODE} = \gamma\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} \ .$$

Notice that the ODE becomes just $\ddot{\zeta} = F(\dot{\zeta}) = F(w)$. Similarly, constants of motion for the DS become first integrals for the ODE simply replacing y with $\dot{x}$.

It is clearly possible to extend in a suitable way this example to DS and to the corresponding ODE to the case $n > 2$.

An explicit example follows.

Example 2. Choosing $\gamma = e^x$, $F = -y^2\, e^{-2x} = -w^2$, the DS is

$$\dot{x} = y \quad , \quad \dot{y} = y^2\left(1 - e^{-x}\right)$$

with symmetry $X = \exp(x)(\partial_x + y\partial_y)$, and the ODE is

$$\ddot{x} = \dot{x}^2 - e^{-x}\,\dot{x}^2 \ .$$

Thanks to the new variables, the general solution is easily get and two constants of motion for the DS (and for the ODE, replacing y with $\dot{x}$) are

$$\kappa_1 = \log|y| - x - e^{-x} \quad , \quad \kappa_2 = t - e^x/y \ .$$

5 Hamiltonian DS, Λ-symmetries and "controlled failure" of conservation rules

Let me now consider the specially interesting case in which the DS is a *Hamiltonian* DS, i.e. the DS is obtained from a given Hamiltonian function H. Changing accordingly the notations, with $n = 2m$, the n variables $u = u_a(t)$ are replaced by the canonical variables $q_\alpha(t), p_\alpha(t)$ ($\alpha = 1, \ldots, m$):

$$u \equiv (q_1, \ldots, q_n, p_1, \ldots, p_n) \equiv (q, p) \in \mathbf{R}^{2m}$$

and the DS is the system of the Hamilton equations of motion for the given Hamiltonian $H = H(q, p, t)$:

$$\dot{u} \;=\; J\nabla H \;=\; F(u, t) \qquad ; \qquad \nabla \equiv (\nabla_q, \nabla_p)$$

where

$$J \;=\; \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

is the standard symplectic matrix and I_m the $m \times m$ identity matrix. In the same way, vector fields X will be written

$$X \;=\; \varphi_\alpha(q, p, t)\frac{\partial}{\partial q_\alpha} + \psi_\alpha(q, p, t)\frac{\partial}{\partial p_\alpha} \equiv \Phi \cdot \nabla \qquad ; \qquad \Phi \equiv (\varphi_\alpha, \psi_\alpha).$$

I will now restrict the attention on vector fields admitting a *generating function* $G = G(q, p, t)$, i.e. vector fields X satisfying

$$\Phi \;=\; J\nabla G \quad \text{or} \quad \varphi = \nabla_p G,\ \psi = -\nabla_q G\ .$$

The connection between symmetries and constants of motion in the Hamiltonian context is even more stringent. It is well known indeed, since the end of XIX century [16] (see also, e.g., [2]), that if X is a Lie point-symmetry for a Hamiltonian DS, i.e. $[F, \Phi] + \partial_t \Phi = 0$, then

$$\nabla(D_t G) \;=\; 0 \qquad \text{or} \qquad D_t G \;=\; g(t)\ . \tag{8}$$

This follows from the identity

$$\nabla_a(D_t G) = \nabla_a(\{G, H\} + \partial_t G) = -J_{ab}([F, \Phi]_b + \partial_t \Phi_b) \qquad (a, b = 1, \ldots, n) \tag{9}$$

where $\{\cdot, \cdot\}$ is the standard Poisson bracket.

Then G is a constant of motion apart from an additional time dependent function g. Let me now consider for simplicity generating functions $G(q, p)$ not depending explicitly on t; combining this with the obvious property $X(G) = 0$ if $X = J\nabla G \cdot \nabla$, the following standard result can be stated for convenience

Proposition 6 *Let the Hamiltonian DS $\dot{u} = J\nabla H$ admit a symmetry $X = \Phi \cdot \nabla$ where $\Phi = J\nabla G$. Then the generating function $G(q, p)$ is a conserved quantity invariant under X.*

In terms of our previous arguments, G can then be chosen as one of symmetry-invariant variables w, and in this case it is automatically (and trivially) also a constant of motion.

A less trivial and more interesting situation occurs if the Hamiltonian DS does admit some X as a λ-symmetry: in this case the identity (9), thanks to (6) becomes (let me now write $\dot{G}$ instead of $D_t G$)

$$\nabla_a(\dot{G}) \ = \ \lambda(J\Phi)_a \ = \ -\lambda \, \nabla_a \, G \ . \tag{10}$$

In addition, in this context, it may be useful to introduce an extension of the notion of λ-symmetry, replacing the scalar function λ with a $n \times n$ matrix Λ (depending in general on $q, p, \dot{q}, \dot{p}, t$): the λ-symmetry condition (6) for the DS $\dot{u} = F(u) = J\nabla H$ is replaced by (see [8, 9])

$$[F, \Phi]_a \ = \ -(\Lambda\Phi)_a \tag{11}$$

and the above identity (9) must be modified accordingly, giving the following

Proposition 7 *If a DS admits $X = J\nabla G \cdot \nabla$ as a Λ-symmetry, then $\dot{G}$ obeys the equation*

$$\nabla_a(\dot{G}) \ = \ (J \, \Lambda \, \Phi)_a \ = \ (J \, \Lambda \, J \, \nabla)_a \, G \ . \tag{12}$$

The two equations (10,12) clearly point out an interesting property of λ (or Λ) symmetries: they can be viewed as "perturbations" of the "exact" symmetry. Equations (10,12) indeed express the "deviation" from the exact conservation rule $\dot{G} = 0$ produced by the presence of a nonzero λ (or Λ).

Some examples will clarify this point. The first one deals with the case of a λ-symmetry (i.e. with a scalar function λ); it is quite simple and can be useful to illustrate the idea. The second one deals with a Toda-like Hamiltonian and a Λ-symmetry with the introduction of a Λ matrix. In both cases, the deviation from the exact conservation of G will be explicitly evaluated and the "controlled" failure of the conservation rule clearly described.

Example 3. Consider the Hamiltonian in $m = 2$ degrees of freedom

$$H \ = \ \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}q_1 p_1^3 + \frac{1}{2}q_2^2 p_1^2 \ .$$

The vector field $X = \partial/\partial q_1$ is a λ-symmetry for the Hamilton equations of motion $\dot{u} = F(u) = J\nabla H$ (which can be easily written), with λ given by the scalar function $\lambda = 3p_1^2/2$, namely

$$[F, \Phi] \ = \ -\frac{3}{2}p_1^2\Phi \qquad , \qquad \Phi = (1, 0, 0, 0)^t \ .$$

Expectedly, the generating function $G = p_1$ is not conserved, indeed one has

$$\dot{G} \ = \ -\frac{1}{2}G^3 \ = \ -\frac{1}{3}\lambda \, G \ .$$

Elementary integration gives $G(t) = G_0\big(1 + t\,G_0^2\big)^{-1/2}$ with $G_0 = G(0)$, and

$$\dot G \;=\; -\frac{1}{2}G_0^3\big(1 + t\,G_0^2\big)^{-3/2}$$

which precisely expresses "how much" G is not conserved and indicates in particular that G is "almost conserved" for great values of t.

Example 4. Consider now the following 2 degrees of freedom Toda Hamiltonian

$$H \;=\; \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + e^{q_1+q_2} + e^{q_1-q_2} \ .$$

It is easy to write the corresponding Hamilton equations of motion and to verify that the vector field

$$X \;=\; \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}$$

with generating function $G = p_1 + p_2$ is a Λ-symmetry for this system with Λ given by the 4×4 matrix

$$\Lambda \;=\; -2\,e^{q_1+q_2}\begin{pmatrix} 0 & 0 \\ I_2 & 0 \end{pmatrix}$$

namely

$$[F, \Phi] \;=\; -\Lambda\Phi \qquad , \qquad \Phi = (1, 1, 0, 0)^t \ .$$

In agreement with the above discussion and Proposition 7, one obtains

$$\dot G = \dot p_1 + \dot p_2 = -2\,e^{q_1+q_2} \qquad , \qquad \nabla(\dot G) = -2\,e^{q_1+q_2}\,\Phi \ .$$

Introducing the variables

$$w_1 = q_1 - q_2 \quad , \quad w_2 = p_1 - p_2 \quad , \quad w_3 = G = p_1 + p_2 \quad , \quad \zeta = q_1 + q_2$$

the DS becomes

$$\dot w_1 = w_2 \quad , \quad \dot w_2 = -2e^{w_1} \quad , \quad \dot w_3 = -2e^{\zeta} \quad , \quad \dot\zeta = w_3 \ .$$

It can be noted that this DS has not the "reduced" form as said in Proposition 3 and eq. (7), where ζ is present only in the r.h.s. of $\dot\zeta$: indeed the reduced form (7) is granted only if Λ is a scalar, $\Lambda = \lambda I$. Anyway, the system is easily solvable; in particular one has

$$\zeta \;=\; 2\log\Big(\frac{|c_1|}{\cosh(c_1 t + c_2)}\Big) \quad , \quad \dot G \;=\; -2e^{\zeta}$$

$$G \;=\; -2c_1\tanh(c_1 t + c_2) \quad , \quad |\dot G(t)| \le 2|c_1|$$

where c_1, c_2 are arbitrary constants, which shows that G is not a conserved quantity, as expected, however – for any choice of c_1, c_2 – both $\dot G(t)$ and $G(t)$ are determined and bound quantities.

6 Conclusion

Finding Lie point-symmetries of a dynamical system is in general a quite difficult task, and it is strictly connected with the searching for its constants of motion. In this paper I have discussed and emphasized this close connection, using old and new results. I have shown in particular that the knowledge of a symmetry of the DS can allow to directly obtain quantities which are conserved and invariant under the symmetry.

The particular case of Hamiltonian DS is specially interesting: a classical result ensures that if a symmetry of the DS admits a generating function G, then G is automatically a constant of motion and a symmetry-invariant. If instead the Hamiltonian DS admits a symmetry of "weaker" type (specifically, a λ or a Λ-symmetry) then the generating function is no longer a conserved quantity, but we have seen that the deviation from the exact conservation rule $\dot{G} = 0$ is "controlled" in a well defined way.

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