Macroscopic Conductivity of Free Fermions in Disordered Media

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December 8, 2013

Abstract

We conclude our analysis of the linear response of charge transport in lattice systems of free fermions subjected to a random potential by deriving general mathematical properties of its conductivity at the macroscopic scale. The present paper belongs to a succession of studies on Ohm and Joule's laws from a thermodynamic viewpoint starting with [BPK1, BPK2, BPK3]. We show, in particular, the existence and finiteness of the conductivity measure $\mu_{\mathbf{\Sigma}}$ for macroscopic scales. Then we prove that, similar to the conductivity measure associated to Drude's model, $\mu_{\mathbf{\Sigma}}$ converges in the weak*–topology to the trivial measure in the case of perfect insulators (strong disorder, complete localization), whereas in the limit of perfect conductors (absence of disorder) it converges to an atomic measure concentrated at frequency $\nu = 0$. However, the AC–conductivity $\mu_{\mathbf{\Sigma}}|_{\mathbb{R}\backslash\{0\}}$ does not vanish in general: We show that $\mu_{\mathbf{\Sigma}}(\mathbb{R}\backslash\{0\}) > 0$, at least for large temperatures and a certain regime of small disorder.

1 Introduction

We define in [BPK3] AC–conductivity measures for free fermions on the lattice subjected to a random potential by using the second principle of thermodynamics, which corresponds to the positivity of the heat production for cyclic processes on equilibrium states. Such measures were introduced for the first time in [KLM, KM] by using a different approach.

In [BPK3] we prove moreover Ohm and Joule's laws from first principles of thermodynamics and quantum mechanics for electric fields that is time– and space–dependent. The microscopic theory usually explaining these laws is based on Drude's model (1900) combined with quantum corrections. [Cf. the Landau theory of fermi liquids.] Indeed, although the motion of electrons and ions is treated classically and the interaction between these two species is modeled by perfectly elastic random collisions, this quite elementary model provides a qualitatively good description of DC– and AC–conductivities in metals. Recall that well–known computations using Drude's model predict that the conductivity $\Sigma_{\mathrm{Drude}}(t)$ behaves like

$$\Sigma_{\mathrm{Drude}}(t) = D \exp(-\mathrm{T}^{-1} t) \ , \qquad t \in \mathbb{R}_0^+ \ , \tag{1}$$

where $\mathrm{T} > 0$ is related to the mean time interval between two collisions of a charged carrier with defects in the crystal, whereas $D \in \mathbb{R}^+$ is some strictly positive constant. In particular, for any electromagnetic potential $\mathbf{A} \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3; (\mathbb{R}^3)^*)$ with corresponding electric field (in the Weyl gauge)

$$E_{\mathbf{A}}(t,x) := -\partial_t \mathbf{A}(t,x) \ , \quad t \in \mathbb{R}, \ x \in \mathbb{R}^3 \ ,$$

the heat production at large times is in this case equal to

$$\int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2 \Sigma_{\mathrm{Drude}}(s_1 - s_2) \int_{\mathbb{R}^3} \mathrm{d}^3 x \langle E_{\mathbf{A}}(s_2,x), E_{\mathbf{A}}(s_1,x)\rangle$$

for any $t \geq t_0$, where t_0 is the time when the electromagnetic potential is turned on, i.e., $\mathbf{A}(t,\cdot) = 0$ for all $t \leq t_0$. Then, since $s \mapsto E_{\mathbf{A}}(s,x)$ is smooth and compactly supported for all $x \in \mathbb{R}^3$, we deduce from Fubini's theorem and (1) that

$$\begin{aligned} & \int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2 \Sigma_{\mathrm{Drude}}(s_1 - s_2) \int_{\mathbb{R}^3} \mathrm{d}^3 x \, \langle E_{\mathbf{A}}(s_2,x), E_{\mathbf{A}}(s_1,x)\rangle \\ = \ & \frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}^3 x \int_{\mathbb{R}} \mathrm{d}\nu |\hat{E}_{\mathbf{A}}(\nu,x)|^2 \vartheta_{\mathrm{T}}(\nu) \ , \end{aligned}$$

where $\nu \mapsto \hat{E}_{\mathbf{A}}(\nu,x)$ and

$$\nu \mapsto \vartheta_{\mathrm{T}}(\nu) \sim \frac{\mathrm{T}}{1 + \mathrm{T}^2 \nu^2}$$

are the Fourier transforms of the maps

$$s \mapsto E_{\mathbf{A}}(s,x) \qquad \text{and} \qquad s \mapsto \exp\left(-\mathrm{T}^{-1} |s|\right) \ ,$$

respectively, at any fixed $x \in \mathbb{R}^3$. In particular,

$$|\hat{E}_{\mathbf{A}}(\nu, x)|^2 \vartheta_{\mathrm{T}}(\nu)\, \mathrm{d}\nu$$

is the heat production due to the component of frequency ν of the electric field, in accordance with Joule's law in the AC–regime.

Thus, the (positive) measure $\vartheta_{\mathrm{T}}(\nu)\mathrm{d}\nu$ is the in–phase conductivity measure of Drude's model. Its restriction to $\mathbb{R}\backslash\{0\}$ can be interpreted as an (in–phase) *AC*–conductivity measure. In the limit of the perfect insulator ($\mathrm{T} \to 0$) the in–phase conductivity measure $\vartheta_{\mathrm{T}}(\nu)\mathrm{d}\nu$ converges in the weak*–topology to the trivial measure $(0 \cdot \mathrm{d}\nu)$. On the other hand, in the limit of the perfect conductor ($\mathrm{T} \to \infty$), only the in–phase AC–conductivity measure of Drude's model, as defined above, converges in the weak*–topology to the trivial measure $(0 \cdot \mathrm{d}\nu)$ on $\mathbb{R}\backslash\{0\}$. Indeed, as $\mathrm{T} \to \infty$, the in–phase conductivity measure $\vartheta_{\mathrm{T}}(\nu)\mathrm{d}\nu$ converges in the weak*–topology to the atomic measure $D\delta_0$ concentrated at $\nu = 0$ with $D \in \mathbb{R}^+$ being some strictly positive constant. Here, $\delta_0(B) := \mathbf{1}[0 \in B]$ for any Borel set $B \subset \mathbb{R}$.

One aim of this paper is to verify this phenomenology for our many–body quantum system. To this end, we represent the conductivity measure – up to some explicit atomic correction at zero frequency ($\nu = 0$) – as the spectral measure of some self adjoint operator with respect to (w.r.t.) a fixed vector. This proof uses analyticity properties of correlation functions of KMS states. It involves the so–called Duhamel two–point function as explained in [BPK2, Section A] and requires the construction of a Hilbert space of (here called) "current *Duhamel* fluctuations". Using these objects we derive various mathematical properties of the conductivity $\mathbf{\Sigma}$ of the fermion system. In particular, $\mathbf{\Sigma}$ is shown to be a time–correlation function of some unitary evolution. This yield the existence of the conductivity measure $\mu_{\mathbf{\Sigma}}$ as a spectral measure (up to an explicit atomic correction).

Another important outcome of this approach is the finiteness of $\mu_{\mathbf{\Sigma}}$, i.e., $\mu_{\mathbf{\Sigma}}(\mathbb{R}) < \infty$. Moreover, the conductivity measure is not anymore restricted to $\mathbb{R}\backslash\{0\}$. It also includes DC–conductivities, in contrast with [BPK3].

Similar to Drude's model, we also show that the AC–conductivity measure $\mu_{\mathbf{\Sigma}}|_{\mathbb{R}\backslash\{0\}}$ converges in the weak*–topology to the trivial measure in the case of perfect conductors, i.e., the absence of disorder, as well as in the case of perfect insulators, i.e., in the case of strong disorder. Note that the fact that the AC–conductivity measure becomes zero does not imply, in general, that there are no currents in presence of electric fields. It only implies that the so–called in–phase current, which is the component of the total current producing heat, also called active current, is zero. Furthermore, the AC–conductivity $\mu_{\mathbf{\Sigma}}|_{\mathbb{R}\backslash\{0\}}$ is in general non–vanishing: We show in Theorem 4.7 that $\mu_{\mathbf{\Sigma}}(\mathbb{R}\backslash\{0\}) > 0$, for large temperatures and a certain regime of small disorder.

More precisely, we show that, for any cyclic process driven by the external electric filed, the heat production vanishes in both limits of perfect conductors and perfect insulators, but the full conductivity does not vanish in the case of perfect conductors (cf. Theorem 4.6). In this last case, *exactly* like in Drude's model, the conductivity measure $\mu_{\mathbf{\Sigma}}$ converges in the weak*–topology to the atomic measure $\tilde{D}\delta_0$ with $\tilde{D} \in \mathbb{R}^+$ being the *explicit* strictly positive constant (30) and $\delta_0(B) := \mathbf{1}[0 \in B]$ for any Borel set $B \subset \mathbb{R}$.

To conclude, our main assertions are Theorems 3.1 (current Duhamel fluctuations), 4.1 (mathematical properties of the paramagnetic conductivity), 4.6 (asymptotic behavior of the conductivity), and 4.7 (strict positivity of the heat production). This paper is organized as follows:

- The random fermion system is defined in Section 2. The mathematical framework of this study is the one of [BPK1, BPK2, BPK3].
- In Section 3 we define the Hilbert space of "current Duhamel fluctuations".
- In Section 4 we derive important mathematical properties of the conductivity of the fermion system.
- Section 5 gathers technical proofs related to the asymptotic behavior of the conductivity and the strict positivity of the heat production. Both studies use explicit computations based on results of [BPK2, BPK3].

Notation 1.1 (Generic constants)
To simplify notation, we denote by D any generic positive and finite constant. These constants do not need to be the same from one statement to another.

2 Setup of the Problem

Let $d \in \mathbb{N}$, $\mathfrak{L} := \mathbb{Z}^d$ and $(\Omega, \mathfrak{A}_\Omega, \mathfrak{a}_\Omega)$ be the probability space defined as follows: Set $\Omega := [-1,1]^{\mathfrak{L}}$ and let Ω_x, $x \in \mathfrak{L}$, be an arbitrary element of the Borel σ–algebra of the interval $[-1,1]$ w.r.t. the usual metric topology. Then, $\mathfrak{A}_\Omega$ is the

σ–algebra generated by cylinder sets $\prod_{x\in\mathfrak{L}}\Omega_x$, where $\Omega_x = [-1,1]$ for all but finitely many $x \in \mathfrak{L}$. The measure $\mathfrak{a}_\Omega$ is the product measure

$$\mathfrak{a}_\Omega\Big(\prod_{x\in\mathfrak{L}}\Omega_x\Big) := \prod_{x\in\mathfrak{L}}\mathfrak{a}_{\mathbf{0}}(\Omega_x)\ , \tag{2}$$

where $\mathfrak{a}_{\mathbf{0}}$ is any fixed probability measure on the interval $[-1,1]$. We denote by $\mathbb{E}[\,\cdot\,]$ the expectation value associated with $\mathfrak{a}_\Omega$.

For simplicity and without loss of generality (w.l.o.g.), we assume that the expectation of the random variable at any single site is zero:

$$\mathbb{E}\left[\omega(0)\right] = \int_\Omega \omega(0)\mathrm{d}\mathfrak{a}_{\mathbf{0}}(\omega) = 0\ . \tag{3}$$

We can easily remove this condition by replacing ω by $\omega - \mathbb{E}[\omega(0)]$ and adding $\mathbb{E}[\omega(0)]$ to the discrete Laplacian defined below.

Note that the i.i.d. property of the potential is not essential for our results. We could take any ergodic ensemble instead. However, this assumption and (3) extremely simplify the proof of the asymptotic behavior of the conductivity (Theorem 4.6) and of the strict positivity of the heat production (Theorem 4.7).

For any realization $\omega \in \Omega$, $V_\omega \in \mathcal{B}(\ell^2(\mathfrak{L}))$ is the self–adjoint multiplication operator with the function $\omega : \mathfrak{L} \to [-1,1]$. Then we consider the *Anderson tight–binding model* $(\Delta_\mathrm{d} + \lambda V_\omega)$ acting on the Hilbert space $\ell^2(\mathfrak{L})$, where $\Delta_\mathrm{d} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ is (up to a minus sign) the usual d–dimensional discrete Laplacian given by

$$[\Delta_\mathrm{d}(\psi)](x) := 2d\psi(x) - \sum_{z\in\mathfrak{L},\ |z|=1}\psi(x+z)\ , \qquad x\in\mathfrak{L},\ \psi\in\ell^2(\mathfrak{L})\ . \tag{4}$$

To define the one–particle dynamics like in [KLM], we use the unitary group $\{\mathrm{U}_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ generated by the random Hamiltonian $(\Delta_\mathrm{d} + \lambda V_\omega)$ for $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$:

$$\mathrm{U}_t^{(\omega,\lambda)} := \exp(-it(\Delta_\mathrm{d} + \lambda V_\omega)) \in \mathcal{B}(\ell^2(\mathfrak{L}))\ , \qquad t\in\mathbb{R}\ . \tag{5}$$

Denote by $\mathcal{U}$ the CAR C^*–algebra associated to the infinite system. Annihilation and creation operators of (spinless) fermions with wave functions $\psi \in \ell^2(\mathfrak{L})$ are defined by

$$a(\psi) := \sum_{x\in\mathfrak{L}}\overline{\psi(x)}a_x \in \mathcal{U}\ ,\quad a^*(\psi) := \sum_{x\in\mathfrak{L}}\psi(x)a_x^* \in \mathcal{U}\ .$$

Here, $\{a_x, a_x^*\}_{x\in\mathfrak{L}} \subset \mathcal{U}$ and the identity $\mathbf{1} \in \mathcal{U}$ are generators of $\mathcal{U}$ and satisfy the canonical anti–commutation relations. For all $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, the condition

$$\tau_t^{(\omega,\lambda)}(a(\psi)) = a((\mathrm{U}_t^{(\omega,\lambda)})^*(\psi)) \ , \qquad t \in \mathbb{R},\ \psi \in \ell^2(\mathfrak{L}) \ , \tag{6}$$

uniquely defines a family $\tau^{(\omega,\lambda)} := \{\tau_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ of (Bogoliubov) automorphisms of $\mathcal{U}$, see [BR2, Theorem 5.2.5]. The one–parameter group $\tau^{(\omega,\lambda)}$ is strongly continuous and defines (free) dynamics on the C^*–algebra $\mathcal{U}$. For any realization $\omega \in \Omega$ and strength $\lambda \in \mathbb{R}_0^+$ of disorder, the thermal equilibrium state of the system at inverse temperature $\beta \in \mathbb{R}^+$ (i.e., $\beta > 0$) is by definition the unique $(\tau^{(\omega,\lambda)}, \beta)$–KMS state $\varrho^{(\beta,\omega,\lambda)}$, see [BR2, Example 5.3.2.] or [P, Theorem 5.9]. It is a gauge–invariant quasi–free state which is uniquely characterized by its symbol

$$\mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)} := \frac{1}{1 + \mathrm{e}^{\beta(\Delta_{\mathrm{d}} + \lambda V_\omega)}} \in \mathcal{B}(\ell^2(\mathfrak{L})) \tag{7}$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$.

3 Hilbert Space of Current Duhamel Fluctuations

We study in [BPK3, Theorem 4.1] the rate at which resistance in the fermion system converts electric energy into heat energy. This thermal effect results from short range bond *current fluctuations*.

Short range bond currents are the elements of the linear subspace

$$\mathcal{I} := \mathrm{lin}\left\{\mathrm{Im}(a^*(\psi_1)\, a(\psi_2)) : \psi_1, \psi_2 \in \ell^1(\mathfrak{L}) \subset \ell^2(\mathfrak{L})\right\} \subset \mathcal{U} \ . \tag{8}$$

As usual, $\mathrm{lin}\{\mathcal{M}\}$ denotes the linear hull of the subset $\mathcal{M}$ of a vector space. For all $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, the one–parameter (Bogoliubov) group $\tau^{(\omega,\lambda)} = \{\tau_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ preserves the space $\mathcal{I}$. Indeed, the unitary group $\{\mathrm{U}_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ (see (5) and (6)) defines a strongly continuous group on $(\ell^1(\mathfrak{L}) \subset \ell^2(\mathfrak{L}), \|\cdot\|_1)$.

For any $l \in \mathbb{R}^+$ we define the box

$$\Lambda_l := \{(x_1, \ldots, x_d) \in \mathfrak{L} \, : \, |x_1|, \ldots, |x_d| \leq l\} \ . \tag{9}$$

The *fluctuation observable* of the current $I \in \mathcal{I}$ is defined by

$$\mathbb{F}^{(l)}(I) = \frac{1}{|\Lambda_l|^{1/2}} \sum_{x\in\Lambda_l} \left\{\chi_x(I) - \varrho^{(\beta,\omega,\lambda)}(\chi_x(I))\,\mathbf{1}\right\} \ , \qquad I \in \mathcal{I} \ , \tag{10}$$

where χ_x, $x \in \mathfrak{L}$, are (space) translations, i.e., the $*$–automorphisms of $\mathcal{U}$ uniquely defined by

$$\chi_x(a_y) = a_{y+x} \ , \quad y \in \mathbb{Z}^d \ .$$

We showed in [BPK3, Eq. (40)] that the paramagnetic conductivity, which is responsible for heat production, can be written in terms of Green–Kubo relations involving time–correlations of *bosonic* fields coming from current fluctuations in the system. In [BPK3, Section 3.3] we introduced the Hilbert space of current fluctuations from $\mathbb{F}^{(l)}$ and the sesquilinear form on $\mathcal{U}$ naturally defined by teh state $\varrho^{(\beta,\omega,\lambda)}$. This is related to the usual construction of a GNS representation of the $(\tau^{(\omega,\lambda)},\beta)$–KMS state $\varrho^{(\beta,\omega,\lambda)}$.

As showed in [BPK2, Section A], another natural GNS representation of $\varrho^{(\beta,\omega,\lambda)}$ can be constructed from the Duhamel two–point function defined by

$$(B_1,B_2)_{\sim}^{(\omega)} \equiv (B_1,B_2)_{\sim}^{(\beta,\omega,\lambda)} := \int_0^\beta \varrho^{(\beta,\omega,\lambda)} \left(B_1^* \tau_{i\alpha}^{(\omega,\lambda)}(B_2) \right) \mathrm{d}\alpha \qquad (11)$$

for any $B_1, B_2 \in \mathcal{U}$. This *positive definite* sesquilinear form has appeared in different contexts like in linear response theory and we recommend [BPK2, Section A] for more details. We name this GNS representation the *Duhamel GNS representation* of the $(\tau^{(\omega,\lambda)},\beta)$–KMS state $\varrho^{(\beta,\omega,\lambda)}$, see [BPK2, Definition A.6]. It turns out that a Hilbert space of current fluctuations constructed from the scalar product of Duhamel GNS representation is easier to handle and in some sense more natural.

Indeed, define the bond current observable

$$I_{\mathbf{x}} := -2\,\mathrm{Im}(a_{x^{(2)}}^* a_{x^{(1)}}) \in \mathcal{I} \ ,$$

for any pair $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$, where $\{\mathfrak{e}_x\}_{x\in\mathfrak{L}}$ is the canonical orthonormal basis $\mathfrak{e}_x(y) \equiv \delta_{x,y}$ of $\ell^2(\mathfrak{L})$. Then we introduce a (random) *positive definite* sesquilinear form on $\mathcal{I}$ by

$$(I,I')_{\mathcal{I},l}^{(\omega)} \equiv (I,I')_{\mathcal{I},l}^{(\beta,\omega,\lambda)} := (\mathbb{F}^{(l)}(I), \mathbb{F}^{(l)}(I'))_{\sim}^{(\omega)} \ , \qquad I,I' \in \mathcal{I} \ , \qquad (12)$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$.

Using [BPK2, Eqs. (24), (103)], the space–averaged paramagnetic transport coefficient

$$t \mapsto \Xi_{\mathrm{p},l}^{(\omega)}(t) \equiv \Xi_{\mathrm{p},l}^{(\beta,\omega,\lambda)}(t) \in \mathcal{B}(\mathbb{R}^d)$$

satisfies, w.r.t. the canonical orthonormal basis $\{e_k\}_{k=1}^d$ of $\mathbb{R}^d$, the equality

$$\left\{ \Xi_{\mathrm{p},l}^{(\omega)}(t) \right\}_{k,q} \equiv \left\{ \Xi_{\mathrm{p},l}^{(\beta,\omega,\lambda)}(t) \right\}_{k,q} = \left(I_{0,e_k}, \tau_t^{(\omega,\lambda)}(I_{0,e_q}) \right)_{\mathcal{I},l}^{(\omega)} - \left(I_{0,e_k}, I_{0,e_q} \right)_{\mathcal{I},l}^{(\omega)} \tag{13}$$

for any $l,\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k,q \in \{1,\ldots,d\}$ and $t \in \mathbb{R}$. For the basic definition of the space–averaged paramagnetic transport coefficient $\Xi_{\mathrm{p},l}^{(\omega)}$ we refer to [BPK2, Eq. (33)]. One may take in this paper Equation (13) as its definition. The above expression was indeed crucial to study the mathematical properties of $\Xi_{\mathrm{p},l}^{(\omega)}$, see [BPK2, Theorem 3.1, Corollary 3.2].

Furthermore, the deterministic paramagnetic transport coefficient

$$t \mapsto \mathbf{\Xi}_{\mathrm{p}}(t) \equiv \mathbf{\Xi}_{\mathrm{p}}^{(\beta,\lambda)}(t) \in \mathcal{B}(\mathbb{R}^d)$$

is defined by

$$\mathbf{\Xi}_{\mathrm{p}}(t) := \lim_{l\to\infty} \mathbb{E}\left[\Xi_{\mathrm{p},l}^{(\omega)}(t) \right] \tag{14}$$

for any $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$, see [BPK3, Eq. (32)]. We define the limiting positive sesquilinear form in $\mathcal{I}$ by

$$(I,I')_{\mathcal{I}} \equiv (I,I')_{\mathcal{I}}^{(\beta,\lambda)} := \lim_{l\to\infty} \mathbb{E}\left[(I,I')_{\mathcal{I},l}^{(\omega)} \right], \qquad I,I' \in \mathcal{I}, \tag{15}$$

via the following theorem:

Theorem 3.1 (Sesquilinear form from current Duhamel fluctuations)
Let $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}_0^+$. Then, one has:
(i) *The positive sesquilinear form $(\cdot,\cdot)_{\mathcal{I}}$ is well–defined, i.e., the limit exists:*

$$\lim_{l\to\infty} \mathbb{E}\left[(I,I')_{\mathcal{I},l}^{(\omega)} \right] \in \mathbb{R}, \qquad I,I' \in \mathcal{I}.$$

(ii) *There is a measurable subset $\tilde{\Omega} \equiv \tilde{\Omega}^{(\beta,\lambda)} \subset \Omega$ of full measure such that, for any $\omega \in \tilde{\Omega}$,*

$$(I,I')_{\mathcal{I}} = \lim_{l\to\infty} (I,I')_{\mathcal{I},l}^{(\omega)}, \qquad I,I' \in \mathcal{I}.$$

Proof: The proof is very similar to the one of [BPK3, Theorem 5.26], which concerns the (well–defined) limit

$$\langle I,I' \rangle_{\mathcal{I}} \equiv \langle I,I' \rangle_{\mathcal{I}}^{(\beta,\lambda)} := \lim_{l\to\infty} \mathbb{E}\left[\langle I,I' \rangle_{\mathcal{I},l}^{(\omega)} \right] \in \bar{\mathbb{R}}, \qquad I,I' \in \mathcal{I}. \tag{16}$$

Here, for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$,

$$\langle I, I' \rangle_{\mathcal{I},l}^{(\omega)} \equiv \langle I, I' \rangle_{\mathcal{I},l}^{(\beta,\omega,\lambda)} := \varrho^{(\beta,\omega,\lambda)} \left(\mathbb{F}^{(l)} (I)^* \, \mathbb{F}^{(l)} (I') \right) , \qquad I, I' \in \mathcal{I} .$$

Here, $\mathbb{F}^{(l)}$ is the fluctuation observable defined by (10). In particular, one has the inequality

$$(I, I')_{\mathcal{I},l}^{(\omega)} \leq \langle I, I' \rangle_{\mathcal{I},l}^{(\omega)} , \qquad I, I' \in \mathcal{I} , \tag{17}$$

which results from [BPK2, Theorem A.4] for $\mathcal{X} = \mathcal{U}$ and $\varrho = \varrho^{(\beta,\omega,\lambda)}$. By [BPK2, Lemma 5.10], this implies the existence of a constant $D \in \mathbb{R}^+$ such that, for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and all $\psi_1, \psi_2, \psi_1', \psi_2' \in \ell^1(\mathfrak{L})$,

$$\left| \Big(\mathrm{Im}(a^* (\psi_1) \, a (\psi_2)), \mathrm{Im}(a^* (\psi_1') \, a (\psi_2')) \Big)_{\mathcal{I},l}^{(\omega)} \right| \leq D \, \|\psi_1\|_1 \, \|\psi_2\|_1 \, \|\psi_1'\|_1 \, \|\psi_2'\|_1 \ . \tag{18}$$

Then, an analogue of [BPK3, Lemma 5.25] for $(\cdot, \cdot)_{\mathcal{I}}$ is proven by using the Akcoglu–Krengel ergodic theorem, see [BPK3, Sections 5.2, 5.4]. We omit the details since one uses very similar arguments to those proving [BPK3, Theorem 5.17] and the proof is even simpler. ■

Remark 3.2 (Auto–correlation upper bounds)
The positive sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{I}}$ defines the Hilbert space $\mathcal{H}_{\mathrm{fl}}$ of current fluctuations as explained in [BPK3, Section 3.3]. By (17), $(\cdot, \cdot)_{\mathcal{I}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{I}}$ are related to each other via the auto–correlation upper bounds:

$$(I, I')_{\mathcal{I}} \leq \langle I, I' \rangle_{\mathcal{I}} , \qquad I, I' \in \mathcal{I} .$$

Hence, we define the kernel

$$\tilde{\mathcal{I}}_0 := \{ I \in \mathcal{I} : (I, I)_{\mathcal{I}} = 0 \}$$

of the positive sesquilinear form $(\cdot, \cdot)_{\mathcal{I}}$. The quotient $\mathcal{I}/\tilde{\mathcal{I}}_0$ is a pre–Hilbert space and its completion w.r.t. the scalar product

$$([I], [I'])_{\mathcal{I}/\tilde{\mathcal{I}}_0} := \ (I, I')_{\mathcal{I}} , \qquad [I], [I'] \in \mathcal{I}/\tilde{\mathcal{I}}_0 , \tag{19}$$

is the Hilbert space

$$\left(\tilde{\mathcal{H}}_{\mathrm{fl}}, (\cdot, \cdot)_{\tilde{\mathcal{H}}_{\mathrm{fl}}} \right) \tag{20}$$

of current *Duhamel* fluctuations. The dynamics defined by $\tau^{(\omega,\lambda)} = \{\tau_t^{(\omega,\lambda)}\}_{t \in \mathbb{R}}$ on $\mathcal{U}$ induces a unitary time evolution on $\tilde{\mathcal{H}}_{\mathrm{fl}}$:

Theorem 3.3 (Dynamics of current Duhamel fluctuations)
Let $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}_0^+$. Then, there is a measurable subset $\tilde{\Omega} \equiv \tilde{\Omega}^{(\beta,\lambda)} \subset \Omega$ of full measure such that, for any $\omega \in \tilde{\Omega}$, there is a unique, strongly continuous one–parameter unitary group $\{\tilde{\mathrm{V}}_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ on the Hilbert space $\tilde{\mathcal{H}}_{\mathrm{fl}}$ obeying, for any $t \in \mathbb{R}$,

$$\tilde{\mathrm{V}}_t^{(\omega,\lambda)}([I]) = [\tau_t^{(\omega,\lambda)}(I)] \ , \qquad [I] \in \mathcal{I}/\tilde{\mathcal{I}}_0 \ .$$

Proof: The proof is essentially the same as the one of [BPK3, Theorem 5.27]. We omit the details. Note that one uses (17)–(18) combined with [BPK2, Corollary A.8]. ■

Remark 3.4 (Deterministic unitary group)
As in [BPK3, Section 5.5.3], by using the Duhamel representation [BPK2, Definition A.6] one can construct a unique, strongly continuous one–parameter deterministic unitary group $\{\hat{\mathrm{V}}_t^{(\lambda)}\}_{t\in\mathbb{R}}$ on a direct integral Hilbert space.

By using the Hilbert space $\tilde{\mathcal{H}}_{\mathrm{fl}}$ (20) of current Duhamel fluctuations, we infer from Equations (13) and (14)–(15) that

$$\{\Xi_{\mathrm{p}}(t)\}_{k,q} = \Big([I_{0,e_k}], \tilde{\mathrm{V}}_t^{(\omega,\lambda)}([I_{0,e_q}])\Big)_{\tilde{\mathcal{H}}_{\mathrm{fl}}} - \Big([I_{0,e_k}], [I_{0,e_q}]\Big)_{\tilde{\mathcal{H}}_{\mathrm{fl}}} \tag{21}$$

for any $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$, $k, q \in \{1, \ldots, d\}$ and all $t \in \mathbb{R}$. Here, ω belongs to some measurable subset of full measure defined such that the strongly continuous one–parameter unitary group $\{\tilde{\mathrm{V}}_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ exists, see Theorem 3.3. Equation (21) is the analogue of (13) for $\Xi_{\mathrm{p},l}^{(\omega)}$. As a consequence, we can now follow the same strategy as in [BPK2, Section 5.1.2]. This is performed in the next section.

4 Macroscopic Conductivity of Fermion Systems

As in [BPK3, Definition 3.2], for any $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}_0^+$, the macroscopic conductivity is the map

$$t \mapsto \mathbf{\Sigma}(t) \equiv \mathbf{\Sigma}^{(\beta,\lambda)}(t) := \begin{cases} 0 & , \qquad t \leq 0 \ . \\ \Xi_{\mathrm{d}} + \Xi_{\mathrm{p}}(t) & , \qquad t \geq 0 \ . \end{cases} \tag{22}$$

Here, Ξ_{p} is the deterministic paramagnetic transport coefficient defined by (14), whereas the time–independent operator $\Xi_{\mathrm{d}} \in \mathcal{B}(\mathbb{R}^d)$ is the diamagnetic transport coefficient, which equals

$$\{\Xi_{\mathrm{d}}\}_{k,q} = 2\delta_{k,q} \operatorname{Re}\left\{\mathbb{E}\left[\langle \mathfrak{e}_{e_k}, \mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)} \mathfrak{e}_0 \rangle\right]\right\} \tag{23}$$

for any $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$ and $k, q \in \{1, \ldots, d\}$, see [BPK2, Eq. (37)]. $\langle \cdot, \cdot \rangle$ is the scalar product in $\ell^2(\mathfrak{L})$ and recall that the positive bounded operator $\mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)}$ is defined by (7).

Since we assume the random potential to be i.i.d. the paramagnetic and diamagnetic transport coefficients turn out to be both a multiple of the identity, see [BPK3, Eqs. (68)–69)]. In particular, there is a function

$$\boldsymbol{\sigma}_{\mathrm{p}} \equiv \boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,\lambda)} \in C(\mathbb{R}; \mathbb{R}_0^-)$$

and a constant $\boldsymbol{\sigma}_{\mathrm{d}} \equiv \boldsymbol{\sigma}_{\mathrm{d}}^{(\beta,\lambda)}$ such that

$$\Xi_{\mathrm{p}}(t) = \boldsymbol{\sigma}_{\mathrm{p}}(t)\ \mathrm{Id}_{\mathbb{R}^d}\ , \qquad \Xi_{\mathrm{d}} = \boldsymbol{\sigma}_{\mathrm{d}}\ \mathrm{Id}_{\mathbb{R}^d}\ , \tag{24}$$

for any $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$. Note additionally that, for all $t \in \mathbb{R}$, $\boldsymbol{\sigma}_{\mathrm{p}}(t) = \boldsymbol{\sigma}_{\mathrm{p}}(|t|)$ with $\boldsymbol{\sigma}_{\mathrm{p}}(0) = 0$ and

$$\boldsymbol{\sigma}_{\mathrm{p}}(t) \in [-2\|[I_{0,e_1}]\|_{\tilde{\mathcal{H}}_{\mathrm{fl}}}^2, 0]\ ,$$

see (21). Thus the *in–phase* conductivity of the fermion system equals

$$\boldsymbol{\sigma}(t) \equiv \boldsymbol{\sigma}^{(\beta,\lambda)}(t) := \boldsymbol{\sigma}_{\mathrm{p}}(t) + \boldsymbol{\sigma}_{\mathrm{d}}\ , \qquad t \in \mathbb{R}\ . \tag{25}$$

Clearly, $\boldsymbol{\sigma} \in C(\mathbb{R}; \mathbb{R})$ satisfies $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(-t)$ with $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_{\mathrm{d}}$. Since the diamagnetic conductivity $\boldsymbol{\sigma}_{\mathrm{d}}$ is an explicit constant, that is,

$$\boldsymbol{\sigma}_{\mathrm{d}} = 2 \operatorname{Re}\left\{\mathbb{E}\left[\langle \mathfrak{e}_{e_1}, \mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)} \mathfrak{e}_0 \rangle\right]\right\}\ , \tag{26}$$

the study of the in–phase conductivity $\boldsymbol{\sigma}$ corresponds to the analysis of the properties of $\boldsymbol{\sigma}_{\mathrm{p}}$. We follow the same strategy as in [BPK2, Section 5.1.2].

First, we denote by $i\tilde{\mathcal{L}}_{\mathrm{fl}}^{(\omega)}$ the anti–self–adjoint operator acting on $\tilde{\mathcal{H}}_{\mathrm{fl}}$ generating the unitary group $\{\tilde{\mathrm{V}}_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ of Theorem 3.3. Then, one deduces from Equation (21) and the spectral theorem the existence of the paramagnetic conductivity measure μ_{p}, like in [BPK2, Theorem 5.4]:

Theorem 4.1 (Paramagnetic conductivity measures)
Let $\beta \in \mathbb{R}^+$and $\lambda \in \mathbb{R}_0^+$. Then, there is a positive symmetric measure $\mu_{\mathrm{p}} \equiv \mu_{\mathrm{p}}^{(\beta,\lambda)}$ on $\mathbb{R}$ such that $\mu_{\mathrm{p}}(\mathbb{R}) < \infty$ uniformly w.r.t. $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$, while

$$\boldsymbol{\sigma}_{\mathrm{p}}(t) = \int_{\mathbb{R}} (\cos(t\nu) - 1)\, \mu_{\mathrm{p}}(\mathrm{d}\nu) , \qquad t \in \mathbb{R} . \tag{27}$$

Proof: As explained above, the existence of the finite positive symmetric measure μ_{p} on $\mathbb{R}$ satisfying (27) is a consequence of the spectral theorem applied to $i\tilde{\mathcal{L}}_{\mathrm{fl}}^{(\omega)}$ together with $\boldsymbol{\sigma}_{\mathrm{p}}(t) = \boldsymbol{\sigma}_{\mathrm{p}}(|t|)$ and $\boldsymbol{\sigma}_{\mathrm{p}}(0) = 0$. See Equations (21) and (24). Observe also that μ_{p} is clearly a deterministic measure. Moreover,

$$\mu_{\mathrm{p}}(\mathbb{R}) = ([I_{0,e_1}], [I_{0,e_1}])_{\tilde{\mathcal{H}}_{\mathrm{fl}}}$$

and we thus deduce from (18) that this quantity is uniformly bounded w.r.t. $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}_0^+$. ■

Remark 4.2 (On the strict negativity of the paramagnetic conductivity)
In contrast to the standard Liouvillian $\tilde{\mathcal{L}}$ in [BPK2, Eq. (105)], it is a priori not clear whether the kernel of $\tilde{\mathcal{L}}_{\mathrm{fl}}^{(\omega)}$ is empty or not. Thus, we define $\mathrm{P}_{\mathrm{fl}}^{(\omega)}$ to be the orthogonal projection on the kernel of $\tilde{\mathcal{L}}_{\mathrm{fl}}^{(\omega)}$. By (21) and (24) combined with the stationarity of KMS states, one can prove that $\boldsymbol{\sigma}_{\mathrm{p}}(t) = 0$ for $t \neq 0$ iff $\mathrm{P}_{\mathrm{fl}}^{(\omega)}[I_{0,e_1}] = [I_{0,e_1}]$. In particular, if $\boldsymbol{\sigma}_{\mathrm{p}}(t) = 0$ for some $t \in \mathbb{R}\backslash\{0\}$ then $\boldsymbol{\sigma}_{\mathrm{p}}$ is the zero function on $\mathbb{R}$ .In the same way, if there is $t \in \mathbb{R}$ where $\boldsymbol{\sigma}_{\mathrm{p}}(t) \neq 0$ then $\boldsymbol{\sigma}_{\mathrm{p}}(t) < 0$ for all $t \in \mathbb{R}\backslash\{0\}$.

Note that Theorem 4.1 is a reminiscent of [BPK2, Theorem 3.1 (v)] where we show the existence of a local paramagnetic conductivity measure $\mu_{\mathrm{p},l}^{(\omega)} \equiv \mu_{\mathrm{p},l}^{(\beta,\omega,\lambda)}$. It is a positive operator valued measure that satisfies

$$\int_{\mathbb{R}} (1 + |\nu|)\, \|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}(\mathrm{d}\nu) < \infty ,$$

uniformly w.r.t. $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and

$$\Xi_{\mathrm{p},l}^{(\omega)}(t) = \int_{\mathbb{R}} (\cos(t\nu) - 1)\, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu) , \qquad t \in \mathbb{R} .$$

Recall that $\Xi_{\mathrm{p},l}^{(\omega)}$ is the space–averaged paramagnetic transport coefficient, see (13). For all $l \in \mathbb{R}^+$, the map $\omega \mapsto \mu_{\mathrm{p},l}^{(\omega)}$ is measurable w.r.t. the σ–algebra $\mathfrak{A}_\Omega$ and the weak* topology for ($\mathcal{B}(\mathbb{R}^d)$–valued) measures on $\mathbb{R}$. $\mathbb{E}[\mu_{\mathrm{p},l}^{(\omega)}]$, seen as a weak integral, is a finite positive measure. Indeed, as $l \to \infty$, it converges to the positive measure $\mu_{\mathrm{p}} \,\mathrm{Id}_{\mathbb{R}^d}$, with μ_{p} as in Theorem 4.1:

Theorem 4.3 (From microscopic to macroscopic conductivity measures)
Let $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}_0^+$. Then there is a measurable set $\tilde{\Omega} \equiv \tilde{\Omega}(\beta, \lambda) \subset \Omega$ of full measure such that, for all $\omega \in \tilde{\Omega}$, $\mu_{\mathrm{p},l}^{(\omega)}$ converges in the weak*–topology to $\mu_{\mathrm{p}} \,\mathrm{Id}_{\mathbb{R}^d}$, as $l \to \infty$. In particular, $\mathbb{E}[\mu_{\mathrm{p},l}^{(\omega)}]$ converges in the weak*–topology to $\mu_{\mathrm{p}} \,\mathrm{Id}_{\mathbb{R}^d}$, as $l \to \infty$.

Proof: The limit in [BPK3, Theorem 3.1 (p)] is uniform w.r.t. times t in compact sets. This implies the weak*–convergence of $\mu_{\mathrm{p},l}^{(\omega)}$ towards $\mu_{\mathrm{p}} \,\mathrm{Id}_{\mathbb{R}^d}$ for ω in a measurable set $\tilde{\Omega}(\beta, \lambda) \subset \Omega$ of full measure. ■

Corollary 4.4 (First moment of the paramagnetic conductivity measure)
Let $\beta \in \mathbb{R}^+$and $\lambda \in \mathbb{R}_0^+$. Then,

$$\int_{\mathbb{R}} (1 + |\nu|) \, \mu_{\mathrm{p}}(\mathrm{d}\nu) < \infty \ ,$$

uniformly w.r.t. $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}_0^+$. In particular, the family $\{\boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,\lambda)}\}_{\beta \in \mathbb{R}^+, \lambda \in \mathbb{R}_0^+}$ of maps from $\mathbb{R}$ to $\mathbb{R}_0^-$ is equicontinuous.

Proof: By Theorem 4.1, it suffices to prove that

$$\int_0^\infty \nu \ \mu_{\mathrm{p}}(\mathrm{d}\nu) < \infty \ ,$$

uniformly w.r.t. $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}_0^+$. By using Theorem 4.3 and [BPK2, Theorem 5.5], we arrive at

$$\lim_{\nu_0 \to \infty} \int_0^{\nu_0} \nu \ \mu_{\mathrm{p}}(\mathrm{d}\nu) \leq 2\langle [I_{0,e_1}], [I_{0,e_1}] \rangle_{\mathcal{I}} \ .$$

Combined with [BPK2, Lemma 5.10], this implies the existence of a constant $D \in \mathbb{R}^+$ not depending on $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}_0^+$ such that

$$\lim_{\nu_0 \to \infty} \int_0^{\nu_0} \nu \ \mu_{\mathrm{p}}(\mathrm{d}\nu) \leq D < \infty \ .$$

Since μ_{p} is a positive measure, the above limit exists and the equicontinuity of the paramagnetic conductivity is deduced like in the proof of [BPK2, Corollary 3.2 (iv)]. ∎

Note that the diamagnetic conductivity $\boldsymbol{\sigma}_{\mathrm{d}}$ is constant in time and its Fourier transform is the atomic measure $\boldsymbol{\sigma}_{\mathrm{d}}\delta_0$, see (23). Since the conductivity Σ (22) is the sum of the paramagnetic and diamagnetic conductivities, we define the *in–phase* conductivity measure $\mu_{\boldsymbol{\Sigma}} \equiv \mu_{\boldsymbol{\Sigma}^{(\beta,\lambda)}}$ by

$$\mu_{\boldsymbol{\Sigma}} := \mu_{\mathrm{p}} + \left(\boldsymbol{\sigma}_{\mathrm{d}} - \mu_{\mathrm{p}}\left(\mathbb{R}\right)\right)\delta_0 \tag{28}$$

for any $\beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}_0^+$. By Theorem 4.1, the in–phase conductivity $\boldsymbol{\sigma}$ given in (25), equals

$$\boldsymbol{\sigma}(t) = \int_{\mathbb{R}} \cos(t\nu)\mu_{\boldsymbol{\Sigma}}(\mathrm{d}\nu) = \int_{\mathbb{R}} \left(\cos(t\nu) - 1\right)\mu_{\mathrm{p}}(\mathrm{d}\nu) + \boldsymbol{\sigma}_{\mathrm{d}}\ , \qquad t \in \mathbb{R}\ .$$

The restricted measure $\mu_{\mathrm{AC}} := \mu_{\mathrm{p}}|_{\mathbb{R}\backslash\{0\}}$ is the (in–phase) *AC*–conductivity measure described in [BPK3, Theorem 4.4], which was deduced from the second principle of thermodynamics. The additional information we obtain here is the finiteness of μ_{p}, i.e., $\mu_{\mathrm{p}}(\mathbb{R}) < \infty$. The (in–phase) *DC*–conductivity measure is the atomic measure

$$\mu_{\mathrm{DC}} := \left(\boldsymbol{\sigma}_{\mathrm{d}} - \mu_{\mathrm{p}}\left(\mathbb{R}\backslash\{0\}\right)\right)\delta_0\ .$$

Remark 4.5 (On the strict negativity of the paramagnetic conductivity)
Similar to [BPK2, Theorem 5.9], the conductivity measure μ_{AC} can be reconstructed from some macroscopic quantum current viscosity. We refrain from doing it here.

Note that the case $\lambda = 0$ can be interpreted as the perfect conductor. Indeed, by explicit computations using the dispersion relation

$$E(p) := 2\left[d - (\cos(p_1) + \cdots + \cos(p_d))\right]\ , \qquad p \in [-\pi,\pi]^d\ , \tag{29}$$

of the (up to a minus sign) discrete Laplacian Δ_{d},

$$\left\langle \mathfrak{e}_x, \mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,0)}\mathfrak{e}_y \right\rangle = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{1}{1+\mathrm{e}^{\beta E(p)}}\mathrm{e}^{-ip\cdot(x-y)}\mathrm{d}^d p\ ,$$

we obtain

$$\boldsymbol{\sigma}_{\mathrm{d}}^{(\beta,0)} = \frac{2}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{\cos(p_1)}{1+\mathrm{e}^{\beta E(p)}}\mathrm{d}^d p \neq 0 \tag{30}$$

for any $\beta \in \mathbb{R}^+$, whereas $\mu_{\mathrm{p}}^{(\beta,0)}(\mathbb{R}) = 0$ (cf. Lemma 5.4). Hence, the heat production vanishes in this special case. Similarly, the limit $\lambda \to \infty$ corresponds to the perfect insulator and also gives a vanishing heat production for any cyclic processes involving the external electromagnetic field:

Theorem 4.6 (Conductivity – Asymptotics)
Let $\beta \in \mathbb{R}^+$ and assume that $\mathfrak{a}_{\mathbf{0}}$ is absolutely continuous w.r.t. the Lebesgue measure when we perform the limit $\lambda \to \infty$.
(p) *Paramagnetic conductivity: $\boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,\lambda)}(t)$ converges uniformly on compact sets to zero, as $\lambda \to 0^+$ or $\lambda \to \infty$. In particular, μ_{p} converges in the weak*–topology to the trivial measure in these two cases.*
(d) *Diamagnetic conductivity: $\boldsymbol{\sigma}_{\mathrm{d}}^{(\beta,\lambda)}$ converges to $\boldsymbol{\sigma}_{\mathrm{d}}^{(\beta,0)}$, as $\lambda \to 0^+$, and to zero, as $\lambda \to \infty$.*

Proof: (p) The assertions follow from Proposition 5.3 and Lemma 5.4.
(d) The corresponding assertions for $\boldsymbol{\sigma}_{\mathrm{d}}$ can be shown by using the same kind of (explicit) computation as for $\boldsymbol{\sigma}_{\mathrm{p}}$ and are even much simpler to prove than for the paramagnetic case. Indeed, they follow from (26) and direct estimates: To study the limit $\lambda \to 0^+$, use (49) to get that, for any $\beta, \lambda \in \mathbb{R}^+$,

$$\left\| \mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)} - \mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,0)} \right\|_{\mathrm{op}} \leq \left\| \mathrm{e}^{\beta\Delta_{\mathrm{d}}} - \mathrm{e}^{\beta(\Delta_{\mathrm{d}}+\lambda V_\omega)} \right\|_{\mathrm{op}} \leq \beta \mathrm{e}^{2d\beta} \left|\lambda\right| \ .$$

Under the condition that $\mathfrak{a}_{\mathbf{0}}$ is absolutely continuous w.r.t. the Lebesgue measure, by a similar but easier computation using Duhamel expansions as done in Section 5.1, one verifies that

$$\lim_{\lambda\to\infty} \mathbb{E}\left[\langle \mathfrak{e}_{e_1}, \mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)} \mathfrak{e}_0 \rangle \right] = 0 \ .$$

This shows the case $\lambda \to \infty$, by Equation (26) ■

By the second principle of thermodynamics, the fermion system cannot transfer any energy to the electromagnetic field. In fact, the fermion system even absorbs, in general, some non–vanishing amount of electromagnetic energy in form of heat. To explain this, let $\mathcal{S}(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d)$ be the Fréchet space of Schwartz functions $\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ endowed with the usual locally convex topology. The electromagnetic potential is here an element $\mathbf{A} \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d) \subset \mathcal{S}(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d)$ and the electric field equals

$$E_{\mathbf{A}}(t,x) := -\partial_t \mathbf{A}(t,x) \ , \quad t \in \mathbb{R},\ x \in \mathbb{R}^d \ . \tag{31}$$

Then one gets:

Theorem 4.7 (Absorption of electromagnetic energy)
Let $\lambda_0 \in \mathbb{R}^+$. Then there is $\beta_0 \in \mathbb{R}^+$ such that, for any $\beta \in (0, \beta_0)$ and $\lambda \in (\lambda_0/2, \lambda_0)$,

$$\mu_{\mathrm{AC}}\left(\mathbb{R}\backslash\{0\}\right) > 0 \ .$$

Equivalently, there is a meager set $\mathcal{Z} \subset C_0^\infty(\mathbb{R}\times\mathbb{R}^d;\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}\times\mathbb{R}^d;\mathbb{R}^d)$ such that, for all $\mathbf{A} \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d;\mathbb{R}^d)\backslash\mathcal{Z}$,

$$\int_{\mathbb{R}} \mathrm{d}s_1 \int_{\mathbb{R}} \mathrm{d}s_2\ \mathbf{\Sigma}(s_1 - s_2) \int_{\mathbb{R}^d} \mathrm{d}^d x \left\langle E_{\mathbf{A}}(s_2, x), E_{\mathbf{A}}(s_1, x)\right\rangle > 0 \ .$$

Proof: Use Lemmata 5.5 and 5.6. ■

It means that the paramagnetic conductivity $\boldsymbol{\sigma}_{\mathrm{p}}$ is generally non–zero and thus causes a strictly positive heat production for non–vanishing electric fields. This is the case of usual conductors.

5 Technical Proofs

We gather here some technical assertions used to prove Theorems 4.6–4.7. We divide the section in two parts. The first subsection is a study of asymptotic properties of the paramagnetic conductivity, whereas the second one is a proof that the fermion system generally absorbs a non–vanishing amount of electromagnetic work in form of heat.

Before starting our proofs, we recall some definitions used in [BPK2, BPK3]: First, $C_{t+i\alpha}^{(\omega)}$ is the complex–time two–point correlation function, see [BPK3, Section 5.1] for more details. For all $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and $\alpha \in [0, \beta]$, it equals

$$C_{t+i\alpha}^{(\omega)}(\mathbf{x}) = \langle \mathfrak{e}_{x^{(2)}}, \mathrm{e}^{-it(\Delta_{\mathrm{d}}+\lambda V_\omega)} F_\alpha^\beta\left(\Delta_{\mathrm{d}} + \lambda V_\omega\right) \mathfrak{e}_{x^{(1)}} \rangle \ , \quad \mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2 \ , \tag{32}$$

where the real function F_α^β is defined, for any $\beta \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, by

$$F_\alpha^\beta\left(\varkappa\right) := \frac{\mathrm{e}^{\alpha\varkappa}}{1 + \mathrm{e}^{\beta\varkappa}} \ , \qquad \varkappa \in \mathbb{R} \ . \tag{33}$$

Then we set for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$, $\alpha \in [0, \beta]$, $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$ and $\mathbf{y} := (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$,

$$\mathfrak{C}_{t+i\alpha}^{(\omega)}(\mathbf{x}, \mathbf{y}) = \sum_{\pi,\pi' \in S_2} \varepsilon_\pi \varepsilon_{\pi'} C_{t+i\alpha}^{(\omega)}(y^{\pi'(1)}, x^{\pi(1)}) C_{-t+i(\beta-\alpha)}^{(\omega)}(x^{\pi(2)}, y^{\pi'(2)}) \ , \tag{34}$$

compare with [BPK2, Eq. (93)]. Here, $\pi, \pi' \in S_2$ are by definition permutations of $\{1,2\}$ with signatures $\varepsilon_\pi, \varepsilon_{\pi'} \in \{-1,1\}$. In [BPK3, Eq. (141)] we define the function

$$\Gamma_{1,1}(t) := \lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x,y\in\Lambda_l} \mathbb{E}\left[\int_0^\beta \mathfrak{C}_{t+i\alpha}^{(\omega)}(x, x-e_1, y, y-e_1)\mathrm{d}\alpha\right] \tag{35}$$

and, by [BPK3, Eq. (147)], observe that

$$\boldsymbol{\sigma}_{\mathrm{p}}(t) = \Gamma_{1,1}(t) - \Gamma_{1,1}(0) \tag{36}$$

for any $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$. Now we are ready to prove Theorems 4.6 and 4.7.

5.1 Asymptotics of Paramagnetic Conductivity

Here we study the asymptotic properties of the paramagnetic conductivity $\boldsymbol{\sigma}_{\mathrm{p}}$, as $\lambda \to 0^+$ and $\lambda \to \infty$. In other words, we prove Theorem 4.6 (p). We break this proof in several lemmata and one proposition.

By (36) and [BPK3, Lemma 5.16], for any $\varepsilon, \beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$ and $\upsilon \in (0, \beta/2)$,

$$\boldsymbol{\sigma}_{\mathrm{p}}(t) = 4d\left(\tilde{\Gamma}_{\upsilon,\varepsilon,1,1}(t) - \tilde{\Gamma}_{\upsilon,\varepsilon,1,1}(0)\right) + \mathcal{O}(\upsilon) + \mathcal{O}_\upsilon(\varepsilon) , \tag{37}$$

uniformly for times t in compact sets. The term of order $\mathcal{O}_\upsilon(\varepsilon)$ vanishes when $\varepsilon \to 0^+$ for any fixed $\upsilon \in (0, \beta/2)$. By [BPK3, Eqs. (139) and (142)],

$$\tilde{\Gamma}_{\upsilon,\varepsilon,1,1}(t) = \lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x,y\in\Lambda_l} \mathbb{E}\left[\int_\upsilon^{\beta-\upsilon} \mathfrak{B}_{t+i\alpha,\upsilon,\varepsilon}^{(\omega)}(x, x-e_1, y, y-e_1)\mathrm{d}\alpha\right] < \infty \tag{38}$$

for all $\varepsilon, \beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and $\upsilon \in (0, \beta/2)$, with

$$\mathfrak{B}_{t+i\alpha,\upsilon,\varepsilon}^{(\omega)}(\mathbf{x}, \mathbf{y}) := \sum_{\pi,\pi'\in S_2} \varepsilon_\pi \varepsilon_{\pi'} B_{t+i\alpha,\upsilon,\varepsilon}^{(\omega)}(y^{\pi'(1)}, x^{\pi(1)}) \times B_{-t+i(\beta-\alpha),\upsilon,\varepsilon}^{(\omega)}(x^{\pi(2)}, y^{\pi'(2)})$$

and

$$B_{t+i\alpha,\upsilon,\varepsilon}^{(\omega)}(\mathbf{x}) = \int_{|\nu|<M_{\beta,\upsilon,\varepsilon}} \hat{F}_\alpha^\beta(\nu) \langle \mathfrak{e}_{x^{(2)}}, \mathrm{e}^{-i(t-\nu)(\Delta_{\mathrm{d}}+\lambda V_\omega)} \mathfrak{e}_{x^{(1)}} \rangle \mathrm{d}\nu \tag{39}$$

for any $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$ and $\mathbf{y} := (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$. Here, $M_{\beta,\upsilon,\varepsilon}$ is a constant only depending on $\beta, \upsilon, \varepsilon$ and $\hat{F}_{\alpha}^{\beta}$ is the Fourier transform of the function F_{α}^{β} (33). See [BPK3, Eq. (87)].

Thus, by (37), it suffices to obtain the asymptotics $\lambda \to 0^+$ and $\lambda \to \infty$ of the function $\tilde{\Gamma}_{\upsilon,\varepsilon,1,1}$. To this end we use the finite sum approximation

$$\xi_{\nu,t,N}^{(\omega,\lambda)} := \mathrm{e}^{-i(t-\nu)\lambda V_{\omega}} + \sum_{n=1}^{N-1} (-i)^n \int_{\nu}^{t} \mathrm{d}\nu_1 \int_{\nu}^{\nu_1} \mathrm{d}\nu_2 \cdots \int_{\nu}^{\nu_{n-1}} \mathrm{d}\nu_n \, \mathrm{e}^{-i(t-\nu_1)\lambda V_{\omega}} \Delta_{\mathrm{d}} \\ \times \mathrm{e}^{-i(\nu_1-\nu_2)\lambda V_{\omega}} \Delta_{\mathrm{d}} \mathrm{e}^{-i(\nu_2-\nu_3)\lambda V_{\omega}} \cdots \mathrm{e}^{-i(\nu_{n-1}-\nu_n)\lambda V_{\omega}} \Delta_{\mathrm{d}} \mathrm{e}^{-i(\nu_n-\nu)\lambda V_{\omega}}$$

of the unitary operator $\mathrm{e}^{-i(t-\nu)(\Delta_{\mathrm{d}}+\lambda V_{\omega})}$ for any $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $N \in \mathbb{N}$ and $\nu, t \in \mathbb{R}$. Indeed, using Duhamel's formula one gets that

$$\lim_{N\to\infty} \left\| \xi_{\nu,t,N}^{(\omega,\lambda)} - \mathrm{e}^{-i(t-\nu)(\Delta_{\mathrm{d}}+\lambda V_{\omega})} \right\|_{\mathrm{op}} = 0 \tag{40}$$

uniformly for $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\nu \in [-M_{\beta,\upsilon,\varepsilon}, M_{\beta,\upsilon,\varepsilon}]$ and times t in compact sets. Hence, we replace $\mathrm{e}^{-i(t-\nu)(\Delta_{\mathrm{d}}+\lambda V_{\omega})}$ in (39) by its approximation $\xi_{\nu,t,N}^{(\omega,\lambda)}$ and define

$$\tilde{B}_{t+i\alpha,\upsilon,\varepsilon,N}^{(\omega,\lambda)}(\mathbf{x}) := \int_{|\nu|<M_{\beta,\upsilon,\varepsilon}} \hat{F}_{\alpha}^{\beta}(\nu) \langle \mathfrak{e}_{x^{(2)}}, \xi_{\nu,t,N}^{(\omega,\lambda)} \mathfrak{e}_{x^{(1)}} \rangle \mathrm{d}\nu \tag{41}$$

as well as

$$\tilde{\mathfrak{B}}_{t+i\alpha,\upsilon,\varepsilon,N}^{(\omega,\lambda)}(\mathbf{x},\mathbf{y}) := \sum_{\pi,\pi'\in S_2} \varepsilon_{\pi}\varepsilon_{\pi'} \tilde{B}_{t+i\alpha,\upsilon,\varepsilon,N}^{(\omega)}(y^{\pi'(1)}, x^{\pi(1)}) \\ \times \tilde{B}_{-t+i(\beta-\alpha),\upsilon,\varepsilon,N}^{(\omega)}(x^{\pi(2)}, y^{\pi'(2)})$$

for any $\varepsilon, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$ and $\mathbf{y} := (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$. Indeed, one has:

Lemma 5.1 (Finite sum approximation)
Let $\varepsilon, \beta \in \mathbb{R}^+$, $t \in \mathbb{R}$ and $\upsilon \in (0, \beta/2)$. Then,

$$\lim_{N\to\infty} \frac{1}{|\Lambda_l|} \sum_{x,y\in\Lambda_l} \int_{\upsilon}^{\beta-\upsilon} \Big| \mathfrak{B}_{t+i\alpha,\upsilon,\varepsilon}^{(\omega)}(x, x-e_1, y, y-e_1) \\ - \tilde{\mathfrak{B}}_{t+i\alpha,\upsilon,\varepsilon,N}^{(\omega,\lambda)}(x, x-e_1, y, y-e_1) \Big| \mathrm{d}\alpha = 0$$

uniformly for $l \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$.

Proof: The map $(\alpha,\nu)\mapsto \hat{F}_{\alpha}^{\beta}(\nu)$ is absolutely integrable in

$$(\alpha,\nu)\in[\upsilon,\beta-\upsilon]\times[-M_{\beta,\upsilon,\varepsilon},M_{\beta,\upsilon,\varepsilon}]$$

for any $\varepsilon,\beta\in\mathbb{R}^{+}$ and $\upsilon\in(0,\beta/2)$. Therefore, the assertion is directly proven by using (40) to compute the difference between (39) and (41). We omit the details. See similar arguments to the proof of [BPK3, Lemma 5.11]. ■

As a consequence, we only need to bound, for any $\varepsilon,\beta\in\mathbb{R}^{+}$, $\upsilon\in(0,\beta/2)$, and $l,N\in\mathbb{N}$, the function

$$\mathfrak{q}_{\upsilon,\varepsilon,N,l}^{(\beta,\omega,\lambda)}(t):=\frac{1}{|\Lambda_{l}|}\sum_{x,y\in\Lambda_{l}}\mathbb{E}\left[\int_{\upsilon}^{\beta-\upsilon}\tilde{\mathfrak{B}}_{t+i\alpha,\upsilon,\varepsilon,N}^{(\omega,\lambda)}(x,x-e_{1},y,y-e_{1})\mathrm{d}\alpha\right],$$

as $\lambda\rightarrow0^{+}$ and $\lambda\rightarrow\infty$, uniformly for all $l\in\mathbb{R}^{+}$.

Lemma 5.2 (Asymptotics of the finite sum approximation)
Let $\varepsilon,\beta\in\mathbb{R}^{+}$, $\lambda\in\mathbb{R}_{0}^{+}$, $t\in\mathbb{R}$, $\upsilon\in(0,\beta/2)$, and $N\in\mathbb{N}$. Then,

$$\lim_{\lambda\rightarrow0}\mathbb{E}\left[\mathfrak{q}_{\upsilon,\varepsilon,N,l}^{(\beta,\omega,\lambda)}(t)\right]=\mathbb{E}\left[\mathfrak{q}_{\upsilon,\varepsilon,N,l}^{(\beta,\omega,0)}(t)\right]$$

uniformly for $l\in\mathbb{R}^{+}$. If the probability measure $\mathfrak{a}_{0}$ is in addition absolutely continuous w.r.t. the Lebesgue measure then

$$\lim_{\lambda\rightarrow\infty}\mathbb{E}\left[\mathfrak{q}_{\upsilon,\varepsilon,N,l}^{(\beta,\omega,\lambda)}(t)\right]=0$$

uniformly for $l\in\mathbb{R}^{+}$.

Proof: The function $\mathfrak{q}_{\upsilon,\varepsilon,N,l}^{(\beta,\omega,\lambda)}(t)$ is a finite sum of terms of the form

$$\begin{aligned}&\frac{(-i)^{n_{1}+n_{2}}}{|\Lambda_{l}|}\sum_{x,y\in\Lambda_{l}}\sum_{\pi,\pi'\in S_{2}}\varepsilon_{\pi}\varepsilon_{\pi'}\int_{\upsilon}^{\beta-\upsilon}\mathrm{d}\alpha\int_{|\nu|<M_{\beta,\upsilon,\varepsilon}}\mathrm{d}\nu\int_{|u|<M_{\beta,\upsilon,\varepsilon}}\mathrm{d}u\\&\int_{\nu}^{t}\mathrm{d}\nu_{1}\cdots\int_{\nu}^{\nu_{n_{1}-1}}\mathrm{d}\nu_{n_{1}}\int_{u}^{-t}\mathrm{d}u_{1}\cdots\int_{u}^{u_{n_{2}-1}}\mathrm{d}u_{n_{2}}\ \hat{F}_{\alpha}^{\beta}(\nu)\,\hat{F}_{\beta-\alpha}^{\beta}(u)\\&\times\langle\mathfrak{e}_{x_{\pi(1)}},\mathrm{e}^{-i(t-\nu_{1})\lambda V_{\omega}}\Delta_{\mathrm{d}}\mathrm{e}^{-i(\nu_{1}-\nu_{2})\lambda V_{\omega}}\Delta_{\mathrm{d}}\cdots\\&\qquad\cdots\mathrm{e}^{-i\left(\nu_{n_{1}-1}-\nu_{n_{1}}\right)\lambda V_{\omega}}\Delta_{\mathrm{d}}\mathrm{e}^{-i(\nu_{n_{1}}-\nu)\lambda V_{\omega}}\mathfrak{e}_{y_{\pi'(1)}}\rangle\\&\times\langle\mathfrak{e}_{y_{\pi'(2)}},\mathrm{e}^{-i(-t-u_{1})\lambda V_{\omega}}\Delta_{\mathrm{d}}\mathrm{e}^{-i(u_{1}-u_{2})\lambda V_{\omega}}\Delta_{\mathrm{d}}\cdots\\&\qquad\cdots\mathrm{e}^{i\left(u_{n_{2}-1}-u_{n_{2}}\right)\lambda V_{\omega}}\Delta_{\mathrm{d}}\mathrm{e}^{-i(u_{n_{2}}-u)\lambda V_{\omega}}\mathfrak{e}_{x_{\pi(2)}}\rangle\end{aligned}$$

for $n_1, n_2 \in \mathbb{N}_0 \cap [0, N]$. Here, $(x_1, x_2) := (x, x - e_1)$, $(y_1, y_2) := (y, y - e_1)$. [By abuse of notation, the case $n_1 = 0$ or $n_2 = 0$ means that there is no integral but a term $\mathrm{e}^{-i(t-\nu)\lambda V_\omega}$ inside the corresponding scalar product.] From this and the translation invariance of the probability measure $\mathfrak{a}_\Omega$, we get that $\mathbb{E}[\mathfrak{q}_{\upsilon,\varepsilon,N,l}^{(\beta,\omega,\lambda)}(t)]$ is a finite sum of terms of the form

$$
\begin{aligned}
&(-i)^{n_1+n_2} \sum_{x \in \mathfrak{L}} \sum_{\pi,\pi' \in S_2} \varepsilon_\pi \varepsilon_{\pi'} \int_\upsilon^{\beta-\upsilon} \mathrm{d}\alpha \int_{|\nu|<M_{\beta,\upsilon,\varepsilon}} \mathrm{d}\nu \int_{|u|<M_{\beta,\upsilon,\varepsilon}} \mathrm{d}u \qquad (42)\\
&\int_\nu^t \mathrm{d}\nu_1 \cdots \int_\nu^{\nu_{n_1-1}} \mathrm{d}\nu_{n_1} \int_u^{-t} \mathrm{d}u_1 \cdots \int_u^{u_{n_2-1}} \mathrm{d}u_{n_2} \hat{F}_\alpha^\beta(\nu)\, \hat{F}_{\beta-\alpha}^\beta(u)\\
&\sum_{z \in \Lambda_l} \frac{\mathbf{1}[x+z \in \Lambda_l]}{|\Lambda_l|} \mathbb{E}\Big[\langle \mathfrak{e}_{x_{\pi(1)}}, \mathrm{e}^{-i(t-\nu_1)\lambda V_\omega} \Delta_{\mathrm{d}} \mathrm{e}^{-i(\nu_1-\nu_2)\lambda V_\omega} \Delta_{\mathrm{d}} \cdots\\
&\qquad\qquad \cdots \mathrm{e}^{-i\left(\nu_{n_1-1}-\nu_{n_1}\right)\lambda V_\omega} \Delta_{\mathrm{d}} \mathrm{e}^{-i(\nu_{n_1}-\nu)\lambda V_\omega} \mathfrak{e}_{y_{\pi'(1)}} \rangle\\
&\times \langle \mathfrak{e}_{y_{\pi'(2)}}, \mathrm{e}^{-i(-t-u_1)\lambda V_\omega} \Delta_{\mathrm{d}} \mathrm{e}^{-i(u_1-u_2)\lambda V_\omega} \Delta_{\mathrm{d}} \cdots\\
&\qquad\qquad \cdots \mathrm{e}^{-i\left(u_{n_2-1}-u_{n_2}\right)\lambda V_\omega} \Delta_{\mathrm{d}} \mathrm{e}^{-i(u_{n_2}-u)\lambda V_\omega} \mathfrak{e}_{x_{\pi(2)}} \rangle \Big] ,
\end{aligned}
$$

where $(x_1, x_2) := (x, x - e_1)$, $(y_1, y_2) := (0, -e_1)$. Note that

$$
\int_\upsilon^{\beta-\upsilon} \mathrm{d}\alpha \int_{|\nu|<M_{\beta,\upsilon,\varepsilon}} \mathrm{d}\nu \int_{|u|<M_{\beta,\upsilon,\varepsilon}} \mathrm{d}u \left| \hat{F}_\alpha^\beta(\nu)\, \hat{F}_{\beta-\alpha}^\beta(u) \right| < \infty
$$

and the volume of integration in (42) of the ν_a– and u_b–integrals, $a = 1, \ldots, n_1$, $b = 1, \ldots, n_2$, gives a factor

$$
\frac{|t-\nu|^{n_1}|t+u|^{n_2}}{n_1! n_2!}.
$$

By developing the Laplacians Δ_{d}, note that, whenever $t \neq \nu$, $t \neq -u$,

$$
\begin{aligned}
&\sum_{z \in \Lambda_l} \frac{\mathbf{1}[x+z \in \Lambda_l]}{|\Lambda_l|} \mathbb{E}\Big[\langle \mathfrak{e}_{x_{\pi(1)}}, \mathrm{e}^{-i(t-\nu_1)\lambda V_\omega} \Delta_{\mathrm{d}} \mathrm{e}^{-i(\nu_1-\nu_2)\lambda V_\omega} \Delta_{\mathrm{d}}\\
&\qquad\qquad \cdots \mathrm{e}^{-i\left(\nu_{n_1-1}-\nu_{n_1}\right)\lambda V_\omega} \Delta_{\mathrm{d}} \mathrm{e}^{-i(\nu_{n_1}-\nu)\lambda V_\omega} \mathfrak{e}_{y_{\pi'(1)}} \rangle\\
&\times \langle \mathfrak{e}_{y_{\pi'(2)}}, \mathrm{e}^{-i(-t-u_1)\lambda V_\omega} \Delta_{\mathrm{d}} \mathrm{e}^{-i(u_1-u_2)\lambda V_\omega} \Delta_{\mathrm{d}}\\
&\qquad\qquad \cdots \mathrm{e}^{-i\left(u_{n_2-1}-u_{n_2}\right)\lambda V_\omega} \Delta_{\mathrm{d}} \mathrm{e}^{-i(u_{n_2}-u)\lambda V_\omega} \mathfrak{e}_{x_{\pi(2)}} \rangle \Big]
\end{aligned}
$$

is a sum of $(2d+1)^{n_1+n_2}$ terms of the form, up to constants bounded in absolute value by $(2d)^{n_1+n_2}$,

$$\sum_{z\in\Lambda_l} \frac{\mathbf{1}[x+z\in\Lambda_l]}{|\Lambda_l|}\mathbf{1}[x\in\Lambda_{2N+1}]\mathbb{E}\left[\mathrm{e}^{\pm i\mathfrak{t}_1\lambda V_\omega(x_1)}\cdots\mathrm{e}^{\pm i\mathfrak{t}_n\lambda V_\omega(x_n)}\right] \tag{43}$$

where $n\in\mathbb{N}$, $n\leq n_1+n_2\leq 2N$, $\mathfrak{t}_1,\ldots,\mathfrak{t}_n\in\mathbb{R}^+$ and $x_1\in\{x,x-e_1\}$, $x_2\ldots,x_{n-1}\in\mathfrak{L}$, $x_n\in\{0,-e_1\}$ with $x_j\neq x_p$ for $j\neq p$. By Lebesgue's dominated convergence theorem, it suffices to analyze (43) either in the limit $\lambda\to\infty$ or $\lambda\to 0^+$. By (2),

$$\mathbb{E}\left[\mathrm{e}^{\pm i\mathfrak{t}_1\lambda V_\omega(x_1)}\cdots\mathrm{e}^{\pm i\mathfrak{t}_n\lambda V_\omega(x_n)}\right]=\mathbb{E}\left[\mathrm{e}^{\pm i\mathfrak{t}_1\lambda V_\omega(x_1)}\right]\cdots\mathbb{E}\left[\mathrm{e}^{\pm i\mathfrak{t}_n\lambda V_\omega(x_n)}\right] \tag{44}$$

for any $n\in\mathbb{N}$, $\mathfrak{t}_1,\ldots,\mathfrak{t}_n\in\mathbb{R}^+$ and $x_1,\ldots,x_n\in\mathfrak{L}$ with $x_j\neq x_p$ for $j\neq p$. Since

$$\lim_{\lambda\to 0}\mathbb{E}\left[\mathrm{e}^{\pm i\mathfrak{t}\lambda V_\omega(x)}\right]=1$$

for all $x\in\mathfrak{L}$ and $\mathfrak{t}\in\mathbb{R}^+$, we deduce from (44) that

$$\lim_{\lambda\to 0}\mathbb{E}\left[\mathrm{e}^{\pm i\mathfrak{t}_1\lambda V_\omega(x_1)}\cdots\mathrm{e}^{\pm i\mathfrak{t}_n\lambda V_\omega(x_n)}\right]=1$$

and one gets the first assertion of the lemma by Lebesgue's dominated convergence theorem.

If, additionally, the probability measure $\mathfrak{a}_0$ is a absolutely continuous w.r.t. the Lebesgue measure, then from the Riemann–Lebesgue lemma we have the limit

$$\lim_{\lambda\to\infty}\mathbb{E}\left[\mathrm{e}^{\pm i\mathfrak{t}\lambda V_\omega(x)}\right]=0$$

for all $x\in\mathfrak{L}$ and $\mathfrak{t}\in\mathbb{R}^+$. From (44), we then obtain that

$$\lim_{\lambda\to\infty}\mathbb{E}\left[\mathrm{e}^{\pm i\mathfrak{t}_1\lambda V_\omega(x_1)}\cdots\mathrm{e}^{\pm i\mathfrak{t}_n\lambda V_\omega(x_n)}\right]=0\ .$$

Using this and Lebesgue's dominated convergence theorem, one thus gets the second assertion. ∎

We are now in position to compute the asymptotics, as $\lambda\to 0^+$ and $\lambda\to\infty$, of the paramagnetic conductivity $\boldsymbol{\sigma}_{\mathrm{p}}$, which equals (37).

Proposition 5.3 (Asymptotics of the paramagnetic conductivity)
Let $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$. Then,

$$\lim_{\lambda \to 0} \boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,\lambda)}(t) = \boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,0)}(t) \ .$$

If the probability measure $\mathfrak{a}_0$ is in addition absolutely continuous w.r.t. the Lebesgue measure then

$$\lim_{\lambda \to \infty} \boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,\lambda)}(t) = 0 \ .$$

Proof: Let $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$. By Lemmata 5.1–5.2,

$$\begin{aligned} & \lim_{\lambda \to 0} \frac{1}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} \mathbb{E} \left[\int_{\upsilon}^{\beta-\upsilon} \mathfrak{B}_{t+i\alpha,\upsilon,\varepsilon}^{(\beta,\omega,\lambda)}(x, x-e_1, y, y-e_1) \mathrm{d}\alpha \right] \\ = \ & \frac{1}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} \mathbb{E} \left[\int_{\upsilon}^{\beta-\upsilon} \mathfrak{B}_{t+i\alpha,\upsilon,\varepsilon}^{(\beta,\omega,0)}(x, x-e_1, y, y-e_1) \mathrm{d}\alpha \right] \end{aligned}$$

uniformly for all $l \in \mathbb{R}^+$, whereas

$$\lim_{\lambda \to \infty} \frac{1}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} \mathbb{E} \left[\int_{\upsilon}^{\beta-\upsilon} \mathfrak{B}_{t+i\alpha,\upsilon,\varepsilon}^{(\beta,\omega,\lambda)}(x, x-e_1, y, y-e_1) \mathrm{d}\alpha \right] = 0$$

uniformly for all $l \in \mathbb{R}^+$, provided the probability measure $\mathfrak{a}_0$ is absolutely continuous w.r.t. the Lebesgue measure. Thus, by using these limits together with (37)–(38) we arrive at the assertions. ■

Finally, to get Theorem 4.6, we need to compute explicitly the paramagnetic conductivity $\boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,\lambda)}$ at $\lambda = 0$. This is done in the next lemma:

Lemma 5.4 (Paramagnetic conductivity at constant potential)
For any $\beta \in \mathbb{R}^+$ and $t \in \mathbb{R}$, $\boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,0)}(t) = 0$.

Proof: Let $\beta \in \mathbb{R}^+$. By (14) and [BPK2, Lemma 5.2], note that

$$\boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,0)}(t) = \lim_{l \to \infty} \frac{1}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} \int_0^{\beta} \left(\mathfrak{D}_{t+i\alpha}(x,y) - \mathfrak{D}_{i\alpha}(x,y) \right) \mathrm{d}\alpha \ , \tag{45}$$

where, for any $x, y \in \mathfrak{L}$,

$$\mathfrak{D}_{t+i\alpha}(x,y) := \mathfrak{C}_{t+i\alpha}^{(\beta,\omega,0)}(x, x-e_1, y, y-e_1) \ .$$

Observe also that $\mathfrak{C}_{t+i\alpha}^{(\beta,\omega,0)}$, which is defined by (34), does not depend on $\omega \in \Omega$. Explicit computations show that $\mathfrak{D}_{t+i\alpha}(x,y)$ equals

$$\begin{aligned}\mathfrak{D}_{t+i\alpha}(x,y) &= \frac{2}{(2\pi)^{2d}} \int_{[-\pi,\pi]^d} \mathrm{d}^d p \int_{[-\pi,\pi]^d} \mathrm{d}^d p' \, \frac{\mathrm{e}^{\beta E(p')} \mathrm{e}^{(\alpha - it)(E(p)-E(p'))}}{\left(1+\mathrm{e}^{\beta E(p)}\right)\left(1+\mathrm{e}^{\beta E(p')}\right)} \\ &\quad \times \left(1 - \cos\left(p_1 - p_1'\right)\right) \mathrm{e}^{i(p+p')\cdot(x-y)}\end{aligned}$$

for any $t \in \mathbb{R}$, $\alpha \in [0,\beta]$ and $x,y \in \mathfrak{L}$, with $E(p) = E(-p)$ being the dispersion relation (29) of Δ_{d}. By performing the transformation $p \to p - p'$ and then $p' \to p' + p/2$ together with $E(p) = E(-p)$ we deduce that

$$\int_0^\beta \mathfrak{D}_{t+i\alpha}(x,y)\mathrm{d}\alpha = \int_{[-\pi,\pi]^d} \mathfrak{d}_t(p)\, \mathrm{e}^{ip\cdot(x-y)} \mathrm{d}^d p \tag{46}$$

for all $t \in \mathbb{R}$ and $x,y \in \mathfrak{L}$, with $\mathfrak{d}_t$ being the function defined on $[-\pi,\pi]^d$ by

$$\begin{aligned}\mathfrak{d}_t(p) &:= \frac{2}{(2\pi)^{2d}} \int_{[-\pi,\pi]^d} \mathrm{d}^d p' \, \frac{\mathrm{e}^{\beta E(p'+p/2)} \mathrm{e}^{-it(E(p'-p/2)-E(p'+p/2))}}{\left(1+\mathrm{e}^{\beta E(p'-p/2)}\right)\left(1+\mathrm{e}^{\beta E(p'+p/2)}\right)} \\ &\quad \times \frac{\left(\mathrm{e}^{\beta(E(p'-p/2)-E(p'+p/2))} - 1\right)}{\left(E(p'-p/2) - E(p'+p/2)\right)} \left(1 - \cos\left(2p_1'\right)\right) \ .\end{aligned}$$

Consequently, using (46) one gets, for any $l \in \mathbb{R}^+$ and $t \in \mathbb{R}$, the equality

$$\frac{1}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} \int_0^\beta \mathfrak{D}_{t+i\alpha}(x,y)\mathrm{d}\alpha = \int_{[-\pi,\pi]^d} \gamma_l(p)\, \mathfrak{d}_t(p)\, \mathrm{d}^d p \ , \tag{47}$$

where the function γ_l is defined on $[-\pi,\pi]^d$ by

$$\gamma_l(p) := \left| \frac{1}{|\Lambda_l|^{1/2}} \sum_{x \in \Lambda_l} \mathrm{e}^{ip\cdot x} \right|^2 = \frac{1}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} \mathrm{e}^{ip\cdot(x-y)} \ .$$

Observe that, for any $l \in \mathbb{R}^+$ and all $\varepsilon \in \mathbb{R}^+$,

$$\int_{[-\pi,\pi]^d} \gamma_l(p)\, \mathrm{d}^d p = (2\pi)^{2d} \qquad \text{and} \qquad \lim_{l\to\infty} \int_{[-\pi,\pi]^d \setminus \mathcal{B}(0,\varepsilon)} \gamma_l(p)\, \mathrm{d}^d p = 0 \ ,$$

where $\mathcal{B}(0,\varepsilon) \subset \mathbb{R}^d$ is the ball of radius ε centered at 0. From this we infer that

$$\lim_{l\to\infty} \left| \int_{[-\pi,\pi]^d} \gamma_l(p)\, \mathfrak{d}_t(p)\, \mathrm{d}^d p - \int_{\mathcal{B}(0,\varepsilon)} \gamma_l(p)\, \mathfrak{d}_t(p)\, \mathrm{d}^d p \right| = 0 \tag{48}$$

for all $\varepsilon \in \mathbb{R}^+$ and any $t \in \mathbb{R}$. Meanwhile, remark that

$$\mathfrak{d}_t(p) - \mathfrak{d}_0(p) = \mathcal{O}(|tp|) \ .$$

Then, using the continuity of the function $\mathfrak{d}_0(\cdot)$ together with (45), (47) and (48), it follows that $\boldsymbol{\sigma}_{\mathrm{p}}^{(\beta,0)}(t) = 0$ for all $t \in \mathbb{R}$. ■

Therefore, Theorem 4.6 (p) follows from Proposition 5.3 and Lemma 5.4.

5.2 On the Strict Positivity of the Heat Production

In this subsection we aim to prove Theorem 4.7: First, we study the asymptotics of the paramagnetic conductivity $\boldsymbol{\sigma}_{\mathrm{p}}$ at $\beta, \lambda, t = 0$. Then, we show that the behavior of $\boldsymbol{\sigma}_{\mathrm{p}}$ near this point implies strict positivity of the heat production, at least for short pulses of the electric field and small $\beta, \lambda > 0$. This result corresponds to Lemma 5.5. The latter can be extended at small $\beta, \lambda > 0$ by an analyticity argument to all electric fields outside a meager set, see Lemma 5.6.

Lemma 5.5 (Non–vanishing AC–conductivity measure – I)
Let $\mathbf{A} \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d) \backslash \{0\}$ *be such that, for some* $k \in \{1, \ldots, d\}$*,*

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} s \{E_{\mathbf{A}}(s,x)\}_k \mathrm{d}s \right)^2 \mathrm{d}^d x > 0$$

and define, for all $T \in \mathbb{R}^+$*, the time–rescaled potential*

$$\mathbf{A}^{(T)}(t,x) := \mathbf{A}(T^{-1}t, x) \ , \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d \ .$$

For any $\lambda_0 \in \mathbb{R}^+$*, there are* $\beta_0, T_0 \in \mathbb{R}^+$ *such that, for* $\beta \in (0, \beta_0)$*,* $\lambda \in (\lambda_0/2, \lambda_0)$ *and* $T \in (T_0/2, T_0)$*,*

$$\int_{\mathbb{R}} \mathrm{d}s_1 \int_{\mathbb{R}} \mathrm{d}s_2 \, \boldsymbol{\sigma}_{\mathrm{p}}(s_1 - s_2) \int_{\mathbb{R}^d} \mathrm{d}^d x \, \langle E_{\mathbf{A}^{(T)}}(s_2, x), E_{\mathbf{A}^{(T)}}(s_1, x) \rangle > 0 \ .$$

Proof: Let $\lambda_0 \in \mathbb{R}^+$. Using Duhamel's formula note first that

$$\mathrm{e}^{(\alpha - it)(\Delta_{\mathrm{d}} + \lambda V_\omega)} = \mathrm{e}^{(\alpha - it)\Delta_{\mathrm{d}}} + \int_0^1 \mathrm{e}^{(\alpha - it)(1-\gamma)\Delta_{\mathrm{d}}} \, (\alpha - it) \, \lambda V_\omega \mathrm{e}^{(\alpha - it)\gamma(\Delta_{\mathrm{d}} + \lambda V_\omega)} \mathrm{d}\gamma \tag{49}$$

for any $\alpha \in [0,\beta]$ and $t \in \mathbb{R}$. Since all operators in this last equation are bounded, it follows that, if $\lambda \in [0,\lambda_0]$ and $\beta \in \mathbb{R}^+$ is sufficiently small, the Neumann series for $\left(1+\mathrm{e}^{\beta(\Delta_{\mathrm{d}}+\lambda V_\omega)}\right)^{-1}$ absolutely converges:

$$\begin{aligned}
&\left(1+\mathrm{e}^{\beta(\Delta_{\mathrm{d}}+\lambda V_\omega)}\right)^{-1} \qquad (50)\\
=\ & \sum_{n=0}^{\infty}\left\{-\beta\lambda\left(1+\mathrm{e}^{\beta\Delta_{\mathrm{d}}}\right)^{-1}\int_0^1 \mathrm{e}^{\beta(1-\gamma)\Delta_{\mathrm{d}}}V_\omega \mathrm{e}^{\beta\gamma(\Delta_{\mathrm{d}}+\lambda V_\omega)}\mathrm{d}\gamma\right\}^n\left(1+\mathrm{e}^{\beta\Delta_{\mathrm{d}}}\right)^{-1}\ .
\end{aligned}$$

By (49)–(50), one gets the existence of a constant $D \in \mathbb{R}^+$ such that, for $\lambda \in [0,\lambda_0]$ and any sufficiently small $\beta \in (0,1)$, $\alpha \in [0,\beta]$ and $\omega \in \Omega$,

$$\left\|F_\alpha^\beta\left(\Delta_{\mathrm{d}}+\lambda V_\omega\right)-F_\alpha^\beta\left(\Delta_{\mathrm{d}}\right)\right\|_{\mathrm{op}} \leq D\beta\lambda \qquad (51)$$

with F_α^β defined by (33).

We define the approximated complex–time two–point correlation function $\tilde{C}_{t+i\alpha}^{(\omega)}$, for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and $\alpha \in [0,\beta]$, by

$$\tilde{C}_{t+i\alpha}^{(\omega)}(\mathbf{x}) := \langle \mathfrak{e}_{x^{(2)}}, \mathrm{e}^{-it(\Delta_{\mathrm{d}}+\lambda V_\omega)}F_\alpha^\beta\left(\Delta_{\mathrm{d}}\right)\mathfrak{e}_{x^{(1)}}\rangle\ , \quad \mathbf{x} := (x^{(1)},x^{(2)}) \in \mathfrak{L}^2\ , \quad (52)$$

compare with (32), the original form of $C_{t+i\alpha}^{(\omega)}$. For any $\mathbf{x} := (x^{(1)},x^{(2)}) \in \mathfrak{L}^2$ and $\mathbf{y} := (y^{(1)},y^{(2)}) \in \mathfrak{L}^2$, let us define

$$\tilde{\mathfrak{C}}_{t+i\alpha}^{(\omega)}(\mathbf{x},\mathbf{y}) := \sum_{\pi,\pi' \in S_2} \varepsilon_\pi \varepsilon_{\pi'} \tilde{C}_{t+i\alpha}^{(\omega)}(y^{\pi'(1)},x^{\pi(1)})\tilde{C}_{-t+i(\beta-\alpha)}^{(\omega)}(x^{\pi(2)},y^{\pi'(2)})\ .$$

From (34)–(35) and (51) we thus deduce that

$$\Gamma_{1,1}(t) = \lim_{l\to\infty}\frac{1}{|\Lambda_l|}\sum_{x,y\in\Lambda_l}\mathbb{E}\left[\int_0^\beta \tilde{\mathfrak{C}}_{t+i\alpha}^{(\omega)}(x,x-e_1,y,y-e_1)\mathrm{d}\alpha\right]+\mathcal{O}(\beta^2\lambda) \quad (53)$$

uniformly for $t \in \mathbb{R}$.

Next, we define an approximation of $\tilde{C}_{t+i\alpha}^{(\omega)}$ by

$$\begin{aligned}
\hat{C}_{t+i\alpha}^{(\omega)}(\mathbf{x}) \ :=\ & \left\langle \mathfrak{e}_{x^{(2)}}, \mathrm{e}^{-it\Delta_{\mathrm{d}}}F_\alpha^\beta\left(\Delta_{\mathrm{d}}\right)\mathfrak{e}_{x^{(1)}}\right\rangle \qquad (54)\\
& -\frac{\lambda}{2}\left\langle \mathfrak{e}_{x^{(2)}}, \left(itV_\omega+\frac{t^2}{2}(V_\omega\Delta_{\mathrm{d}}+\Delta_{\mathrm{d}}V_\omega+\lambda V_\omega^2)\right)\mathfrak{e}_{x^{(1)}}\right\rangle
\end{aligned}$$

for all $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$, $\alpha \in [0,\beta]$ and $\mathbf{x} := (x^{(1)},x^{(2)}) \in \mathfrak{L}^2$. Indeed, by (49) and a power expansion of $F_\alpha^\beta\left(\Delta_{\mathrm{d}}\right)$ at $\alpha,\beta = 0$, there is a constant

$D \in \mathbb{R}^+$ such that, for any $\lambda \in [0, \lambda_0]$, sufficiently small $\beta \in (0,1)$, $\alpha \in [0, \beta]$, $\omega \in \Omega$ and $t \in \mathbb{R}$,

$$\begin{aligned}
&\left\| \left(\mathrm{e}^{-it(\Delta_{\mathrm{d}}+\lambda V_\omega)} - \mathrm{e}^{-it\Delta_{\mathrm{d}}}\right) F_\alpha^\beta \left(\Delta_{\mathrm{d}}\right) + \frac{1}{2}\int_0^1 \mathrm{e}^{-it(1-\gamma)\Delta_{\mathrm{d}}} it\lambda V_\omega \mathrm{e}^{-it\gamma(\Delta_{\mathrm{d}}+\lambda V_\omega)} \mathrm{d}\gamma \right\|_{\mathrm{op}} \\
\leq \;& D\beta\lambda \left|t\right| \;. \qquad (55)
\end{aligned}$$

Meanwhile, note that

$$\begin{aligned}
&\int_0^1 \mathrm{e}^{-it(1-\gamma)\Delta_{\mathrm{d}}} it V_\omega \mathrm{e}^{-it\gamma(\Delta_{\mathrm{d}}+\lambda V_\omega)} \mathrm{d}\gamma \qquad (56) \\
= \;& itV_\omega + \frac{t^2}{2}\left(V_\omega \Delta_{\mathrm{d}} + \Delta_{\mathrm{d}} V_\omega + \lambda V_\omega^2\right) + \mathcal{O}(|t|^3)
\end{aligned}$$

uniformly for $\lambda \in [0, \lambda_0]$ and $\omega \in \Omega$. Thus, by combining (52)–(56), for $\lambda \in [0, \lambda_0]$, we arrive at the equality

$$\begin{aligned}
\Gamma_{1,1}(t) \;=\;& \lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x,y\in\Lambda_l} \mathbb{E}\left[\int_0^\beta \widehat{\mathfrak{C}}_{t+i\alpha}^{(\omega)}(x, x-e_1, y, y-e_1)\mathrm{d}\alpha\right] \\
&+\mathcal{O}(\beta^2\lambda) + \mathcal{O}(\beta\lambda\left|t\right|^3) \qquad (57)
\end{aligned}$$

for sufficiently small β and $|t|$, where

$$\widehat{\mathfrak{C}}_{t+i\alpha}^{(\omega)}(\mathbf{x}, \mathbf{y}) := \sum_{\pi,\pi' \in S_2} \varepsilon_\pi \varepsilon_{\pi'} \hat{C}_{t+i\alpha}^{(\omega)}(y^{\pi'(1)}, x^{\pi(1)}) \hat{C}_{-t+i(\beta-\alpha)}^{(\omega)}(x^{\pi(2)}, y^{\pi'(2)})$$

for all $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$ and $\mathbf{y} := (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$.

We now use that V_ω is an i.i.d. potential satisfying $\mathbb{E}[V_\omega(x)] = 0$ for all $x \in \mathfrak{L}$ to compute that, for any $\mathbf{x} := (x^{(1)}, x^{(2)})$ and $\mathbf{y} := (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$, $x^{(1)} \neq x^{(2)}$, $y^{(1)} \neq y^{(2)}$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\beta \widehat{\mathfrak{C}}_{t+i\alpha}^{(\omega)}(\mathbf{x}, \mathbf{y})\mathrm{d}\alpha\right] - \int_0^\beta \mathfrak{C}_{t+i\alpha}^{(0)}(\mathbf{x}, \mathbf{y})\mathrm{d}\alpha \qquad (58) \\
= \;& -\frac{\lambda^2 t^2}{4}\mathbb{E}\left[V_\omega^2\right] \sum_{\pi,\pi' \in S_2} \varepsilon_\pi \varepsilon_{\pi'} \left\{ \left(\left(\int_0^\beta \langle \mathfrak{e}_{x^{\pi(1)}}, \mathrm{e}^{-it\Delta_{\mathrm{d}}} F_\alpha^\beta \left(\Delta_{\mathrm{d}}\right) \mathfrak{e}_{y^{\pi'(1)}} \rangle \mathrm{d}\alpha\right) \delta_{x^{\pi(2)}, y^{\pi'(2)}} \right.\right. \\
&\left. + \left(\int_0^\beta \langle \mathfrak{e}_{y^{\pi'(2)}}, \mathrm{e}^{it\Delta_{\mathrm{d}}} F_{\beta-\alpha}^\beta \left(\Delta_{\mathrm{d}}\right) \mathfrak{e}_{x^{\pi(2)}} \rangle \mathrm{d}\alpha\right) \delta_{y^{\pi'(1)}, x^{\pi(1)}} \right\} + \frac{\beta\lambda^2 t^4}{16}\mathbf{D}\left(\mathbf{x}, \mathbf{y}\right) \;,
\end{aligned}$$

where, for any $\mathbf{x} = (x^{(1)}, x^{(2)}), \mathbf{y} = (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$, $x^{(1)} \neq x^{(2)}$, $y^{(1)} \neq y^{(2)}$,

$$\begin{aligned}\mathbf{D}(\mathbf{x}, \mathbf{y}) \quad := \quad & \sum_{\pi, \pi' \in S_2} \varepsilon_\pi \varepsilon_{\pi'} \left\{ \lambda^2 \left(\mathbb{E}\left[V_\omega^2 \right] \right)^2 \delta_{y^{\pi'(1)}, x^{\pi(1)}} \delta_{x^{\pi(2)}, y^{\pi'(2)}} \right. \\ & \left. + \mathbb{E}\left[\langle \mathfrak{e}_{x^{\pi(1)}}, (V_\omega \Delta_{\mathrm{d}} + \Delta_{\mathrm{d}} V_\omega) \, \mathfrak{e}_{y^{\pi'(1)}} \rangle \langle \mathfrak{e}_{y^{\pi'(2)}}, (V_\omega \Delta_{\mathrm{d}} + \Delta_{\mathrm{d}} V_\omega) \, \mathfrak{e}_{x^{\pi(2)}} \rangle \right] \right\} \ .\end{aligned}$$

Note that, for each $\lambda \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$, $\mathbf{D} \equiv \mathbf{D}^{(\lambda)}$ can be seen as the kernel (w.r.t. the canonical basis $\{\mathfrak{e}_x \otimes \mathfrak{e}_{x'}\}_{x, x' \in \mathfrak{L}}$) of a bounded operator on $\ell^2(\mathfrak{L}) \otimes \ell^2(\mathfrak{L})$ with operator norm uniformly bounded w.r.t. λ on compact sets. Therefore, it is straightforward to deduce that

$$\lim_{l \to \infty} \frac{1}{|\Lambda_l|} \sum_{x, y \in \Lambda_l} \mathbf{D}(x, x - e_1, y, y - e_1) = \mathcal{O}(1) \tag{59}$$

uniformly for λ in compact sets. For more details on the last equation, see for instance the proofs of [BPK2, Lemma 5.3] and [BPK3, Lemma 5.10].

Because of Lemma 5.4 and (35)–(36), note that

$$\begin{aligned} & \lim_{l \to \infty} \frac{1}{|\Lambda_l|} \sum_{x, y \in \Lambda_l} \int_0^\beta \mathfrak{C}_{t + i\alpha}^{(0)}(x, x - e_1, y, y - e_1) \mathrm{d}\alpha \\ = \quad & \lim_{l \to \infty} \frac{1}{|\Lambda_l|} \sum_{x, y \in \Lambda_l} \int_0^\beta \mathfrak{C}_{i\alpha}^{(0)}(x, x - e_1, y, y - e_1) \mathrm{d}\alpha \end{aligned}$$

does not depend on $t \in \mathbb{R}$. Using this, for $\lambda \in [0, \lambda_0]$, we infer from (36) and (57)–(59) the existence of a constant $D \in \mathbb{R}^+$ such that the paramagnetic conductivity σ_{p} is of the form

$$\sigma_{\mathrm{p}}(t) = -D\lambda^2 \beta t^2 + \mathcal{O}(\beta^2 \lambda) + \mathcal{O}(\beta \lambda |t|^3) \tag{60}$$

for $\lambda \in [0, \lambda_0]$ and sufficiently small $\beta, |t|$.

Now we choose sufficiently small $\beta_0, T_0 > 0$ and estimate the energy increment caused by the time–rescaled potential $\mathbf{A}^{(T)} \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d) \backslash \{0\}$ for $T \in (T_0/2, T_0)$, $\lambda \in (\lambda_0/2, \lambda_0)$, $\beta \in (0, \beta_0)$. We assume w.l.o.g. that $E_{\mathbf{A}}$ is zero in all but the first component which equals a function $\mathcal{E}_t \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$ for any $t \in \mathbb{R}$. Then, by (60) and Fubini's theorem, we have

$$\begin{aligned} & \int_{\mathbb{R}} \mathrm{d}s_1 \int_{\mathbb{R}} \mathrm{d}s_2 \boldsymbol{\sigma}_{\mathrm{p}}(s_1 - s_2) \int_{\mathbb{R}^d} \mathrm{d}^d x \, \langle E_{\mathbf{A}^{(T)}}(s_2, x), E_{\mathbf{A}^{(T)}}(s_1, x) \rangle \\ = \quad & -D\lambda^2 \beta T^2 \int_{\mathbb{R}^d} \mathrm{d}^d x \int_{\mathbb{R}} \mathrm{d}s_1 \int_{\mathbb{R}} \mathrm{d}s_2 (s_1 - s_2)^2 \mathcal{E}_{s_2}(x) \mathcal{E}_{s_1}(x) \\ & + \mathcal{O}(\beta^2 \lambda) + \mathcal{O}(\beta \lambda T^3) \ . \end{aligned} \tag{61}$$

Because $\mathbf{A} \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d) \backslash \{0\}$, we infer from (31) that

$$\int_{\mathbb{R}} \mathcal{E}_s(x) \mathrm{d}s = 0$$

and, for all $x \in \mathbb{R}^d$,

$$-\int_{\mathbb{R}} \mathrm{d}s_1 \int_{\mathbb{R}} \mathrm{d}s_2 (s_1 - s_2)^2 \mathcal{E}_{s_2}(x) \mathcal{E}_{s_1}(x) = 2 \left(\int_{\mathbb{R}} s \mathcal{E}_s(x) \mathrm{d}s \right)^2 . \tag{62}$$

As a consequence, if

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} s \mathcal{E}_s(x) \mathrm{d}s \right)^2 \mathrm{d}^d x > 0 ,$$

then (61)–(62) yield the lemma, provided $\lambda_0 T_0^2 \gg \beta_0, T_0^3$. ■

Note that Lemma 5.5 implies that, for any $\lambda \in \mathbb{R}^+$ and sufficiently small $\beta \in \mathbb{R}^+$, the AC–conductivity measure is non–zero, i.e.,

$$\mu_{\mathrm{AC}} (\mathbb{R} \backslash \{0\}) = \mu_{\mathrm{p}} (\mathbb{R} \backslash \{0\}) > 0 . \tag{63}$$

This property implies the following result:

Lemma 5.6 (Non–vanishing AC–conductivity measure – II)
If (63) holds then the set

$$\mathcal{Z} := \left\{ \varphi \in \mathcal{S}(\mathbb{R}; \mathbb{R}) : \int_{\mathbb{R}} \mathrm{d}s_1 \int_{\mathbb{R}} \mathrm{d}s_2 \, \boldsymbol{\sigma}_{\mathrm{p}}(s_2 - s_1) \varphi(s_1) \varphi(s_2) = 0 \right\}$$

is meager in the Fréchet space $\mathcal{S}(\mathbb{R}; \mathbb{R})$ of Schwartz functions equipped with the usual locally convex topology.

Proof: By (63), there is at least one point $\nu_0 \in \mathbb{R} \backslash \{0\}$ such that $\mu_{\mathbf{\Sigma}} (\mathcal{V}) \neq 0$ for all open neighborhoods $\mathcal{V}$ of ν_0. To see this, observe that

$$\mathbb{R} \backslash \{0\} = \bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, n \right] \cup \left[-n, -\frac{1}{n} \right] ,$$

and thus there is $n \in \mathbb{N}$ such that

$$\mu_{\mathrm{AC}} \left(\left[\frac{1}{n}, n \right] \cup \left[-n, -\frac{1}{n} \right] \right) > 0 .$$

Then, by compactness, there is $\nu_0 \in \left[\frac{1}{n}, n\right] \cup \left[-n, -\frac{1}{n}\right]$ such that

$$\mu_{\mathrm{AC}}\left(\mathcal{V} \cap \left(\left[\frac{1}{n}, n\right] \cup \left[-n, -\frac{1}{n}\right]\right)\right) \neq 0$$

for all open neighborhoods $\mathcal{V}$ of ν_0.

Take now any non–zero function $\varphi \in C_0^\infty(\mathbb{R};\mathbb{R}) \subset \mathcal{S}(\mathbb{R};\mathbb{R})$. Its Fourier transform $\hat{\varphi}$ obeys

$$\left|\frac{d^n \hat{\varphi}}{d\nu^n}(\nu)\right| \leq D_1 D_2^n \,, \qquad n \in \mathbb{N} \,,\ \nu \in \mathbb{R} \,,$$

for some constants $D_1, D_2 \in \mathbb{R}^+$. In particular, there is a unique continuation of $\hat{\varphi} : \mathbb{R} \to \mathbb{C}$ to an entire function, again denoted by $\hat{\varphi} : \mathbb{C} \to \mathbb{C}$. Hence, the set of zeros of $\hat{\varphi}$ has no accumulation points.

If $\hat{\varphi}(\nu_0) \neq 0$ then, by continuity of $\hat{\varphi}$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \boldsymbol{\sigma}_{\mathrm{p}}(s_1 - s_2)\varphi(s_2)\varphi(s_1)\mathrm{d}s_2\mathrm{d}s_1 = \int_{\mathbb{R}\backslash\{0\}} |\hat{\varphi}(\nu)|^2 \mu_{\mathrm{AC}}(\mathrm{d}\nu) > 0 \,. \qquad (64)$$

If $\hat{\varphi}(\nu_0) = 0$ then, for all $\alpha \in (0,1)$, we define the rescaled function $\hat{\varphi}_\alpha(\nu)$ by $\hat{\varphi}(\alpha\nu)$, which is the Fourier transform of $\alpha^{-1}\varphi(\alpha^{-1}x)$. For sufficiently small $\varepsilon \in \mathbb{R}^+$ and all $\alpha \in (1-\varepsilon, 1)$,

$$\int_{\mathbb{R}\backslash\{0\}} |\hat{\varphi}_\alpha(\nu)|^2 \mu_{\mathrm{AC}}(\mathrm{d}\nu) > 0 \,,$$

because the set of zeros of $\hat{\varphi}$ has no accumulation points. On the other hand, $\alpha^{-1}\varphi(\alpha^{-1}x)$ converges in $\mathcal{S}(\mathbb{R};\mathbb{R})$ to $\varphi(x)$, as $\alpha \to 1$. Thus, the complement of $\mathcal{Z}$ is dense in $\mathcal{S}(\mathbb{R};\mathbb{R})$, by density of the set $C_0^\infty(\mathbb{R};\mathbb{R})$ in $\mathcal{S}(\mathbb{R};\mathbb{R})$. Since $\mu_{\mathrm{AC}} := \mu_{\mathrm{p}}|_{\mathbb{R}\backslash\{0\}}$ with $\mu_{\mathrm{p}}(\mathbb{R}) < \infty$ (Theorem 4.1), note that the map

$$\hat{\varphi} \mapsto \int_{\mathbb{R}\backslash\{0\}} |\hat{\varphi}(\nu)|^2 \mu_{\mathrm{AC}}(\mathrm{d}\nu)$$

is continuous on $\mathcal{S}(\mathbb{R};\mathbb{R})$. Because the Fourier transform is a homeomorphism of $\mathcal{S}(\mathbb{R};\mathbb{R})$, by the first equation in (64), the map

$$\varphi \mapsto \int_{\mathbb{R}}\int_{\mathbb{R}} \boldsymbol{\sigma}_{\mathrm{p}}(s_1 - s_2)\varphi(s_2)\varphi(s_1)\mathrm{d}s_2\mathrm{d}s_1$$

is also continuous on $\mathcal{S}(\mathbb{R};\mathbb{R})$ and the complement of $\mathcal{Z}$ is hence an open set. ■

Acknowledgments: We would like to thank Volker Bach, Horia Cornean, Abel Klein and Peter Müller for relevant references and interesting discussions as well as important hints. JBB and WdSP are also very grateful to the organizers of the Hausdorff Trimester Program entitled "*Mathematical challenges of materials science and condensed matter physics*" for the opportunity to work together on this project at the Hausdorff Research Institute for Mathematics in Bonn. This work has also been supported by the grant MTM2010-16843 of the Spanish "Ministerio de Ciencia e Innovación"as well as the FAPESP grant 2013/13215–5.

References

[BR2] O. BRATTELI AND D.W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics, Vol. II, 2nd ed.* Springer-Verlag, New York, 1996.

[BPK1] J.-B. BRU, W. DE SIQUEIRA PEDRA AND C. KURIG, Heat Production of Non–Interacting Fermions Subjected to Electric Fields, to appear in *Comm. Pure Appl. Math.* (2014), 48 pages.

[BPK2] J.-B. BRU, W. DE SIQUEIRA PEDRA AND C. KURIG, Microscopic Conductivity Distributions of Non–Interacting Fermions, *Preprint (2013)*.

[BPK3] J.-B. BRU, W. DE SIQUEIRA PEDRA AND C. KURIG, AC–Conductivity Measure from Heat Production of Free Fermions in Disordered Media, *Preprint (2013)*.

[KLM] A. KLEIN, O. LENOBLE, AND P. MÜLLER, On Mott's formula for the ac-conductivity in the Anderson model, *Annals of Mathematics* **166** (2007) 549–577.

[KM] A. KLEIN AND P. MÜLLER, The Conductivity Measure for the Anderson Model, *Journal of Mathematical Physics, Analysis, Geometry* **4** (2008) 128–150.

[P] C.-A. PILLET, Quantum Dynamical Systems, in *Open Quantum Systems I: The Hamiltonian Approach,* Vol. 1880 of Lecture Notes in Mathematics, editors: S. Attal, A. Joye, C.-A. Pillet. Springer–Verlag, 2006.