# Tensor product of Hilbert spaces



## §1. Introduction

Let $\mathscr{H} = \{x, x', \dots\}$ be a Hilbert space, with scalar product $(x|x')$, and $\mathscr{K} = \{y, y', \dots\}$ a Hilbert space with scalar product $(y|y')$. We do not rule out finite-dimensional or inseparable Hilbert spaces.

If $\mathrm{E}, \mathrm{F}, \mathrm{G}$ are complex vector spaces, a mapping $\varphi : \mathrm{E} \times \mathrm{F} \to \mathrm{G}$ is *bilinear* if

$$\begin{aligned} \varphi(x_1 + x_2, y) &= \varphi(x_1, y) + \varphi(x_2, y) \\ \varphi(\lambda x, y) &= \lambda\varphi(x, y) \\ \varphi(x, y_1 + y_2) &= \varphi(x, y_1) + \varphi(x, y_2) \\ \varphi(x, \lambda y) &= \lambda\varphi(x, y) \end{aligned}$$

for all vectors $x \in \mathrm{E}$, $y \in \mathrm{F}$ and (complex) scalars $\lambda$; $\varphi$ is called *sesquilinear* if the last condition is replaced by $\varphi(x, \lambda y) = \overline{\lambda}\varphi(x, y)$, where $\overline{\lambda}$ denotes the conjugate of $\lambda$ ("linear in $x$, semilinear in $y$").

Definition 1. — A *tensor product* of $\mathscr{H}$ with $\mathscr{K}$ is a Hilbert space $\mathscr{P}$, together with a bilinear mapping $\varphi : \mathscr{H} \times \mathscr{K} \to \mathscr{P}$, such that

(1) the set of all vectors $\varphi(x, y)$ $(x \in \mathscr{H}, y \in \mathscr{K})$ forms a *total* subset of $\mathscr{P}$, that is, its closed linear span is equal to $\mathscr{P}$;

(2) $\big(\varphi(x_1, y_1)|\varphi(x_2, y_2)\big) = (x_1|x_2)(y_1|y_2)$ for $x_1, x_2 \in \mathscr{H}$, $y_1, y_2 \in \mathscr{K}$.

We refer to the pair $(\mathscr{P}, \varphi)$ as the tensor product.

If $(\mathscr{P}, \varphi)$ is a tensor product of $\mathscr{H}$ with $\mathscr{K}$, it is customary to write $x \otimes y$ in place of $\varphi(x, y)$, and $\mathscr{H} \otimes \mathscr{K}$ in place of $\mathscr{P}$. Thus a tensor product of $\mathscr{H}$ with $\mathscr{K}$ is a Hilbert space $\mathscr{H} \otimes \mathscr{K}$ and a mapping $(x, y) \mapsto x \otimes y$ of $\mathscr{H} \times \mathscr{K}$ into $\mathscr{H} \otimes \mathscr{K}$ such that

$$\begin{aligned} (x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y \\ (\lambda x) \otimes y &= \lambda(x \otimes y) \\ x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2 \\ x \otimes (\lambda y) &= \lambda(x \otimes y) \end{aligned} \tag{0}$$

(1) the vectors $x \otimes y$ form a total subset of $\mathscr{H} \otimes \mathscr{K}$ ,
(2) $(x_1 \otimes y_1 | x_2 \otimes y_2) = (x_1|x_2)(y_1|y_2)$ .

We shall denote by $\mathscr{H} \odot \mathscr{K}$ the linear subspace of $\mathscr{H} \otimes \mathscr{K}$ generated by the vectors $x \otimes y$ . This consists of all finite linear combinations

$$\sum_1^n \lambda_k (x_k \otimes y_k) = \sum_1^n (\lambda_k x_k) \otimes y_k \,,$$

thus $\mathscr{H} \odot \mathscr{K}$ is the set of all finite sums $\sum_1^n x_k \otimes y_k$ . Condition (1) is equivalent to the assertion that $\mathscr{H} \odot \mathscr{K}$ is a *dense* linear subspace of $\mathscr{H} \otimes \mathscr{K}$ ; equivalently, the only vector orthogonal to every $x \otimes y$ is the zero vector.

In condition (2), putting $x_1 = x_2 = x$ and $y_1 = y_2 = y$ , we see that $\|x \otimes y\| = \|x\| \, \|y\|$ .

In this section we shall prove several important properties of tensor products, leading up to the theorem that the tensor product of $\mathscr{H}$ with $\mathscr{K}$ is "essentially unique". Of course we may be working in a vacuum—it is conceivable that the axioms for a tensor product are inconsistent or only sometimes consistent. The purpose of §2 is to show that a tensor product always does exist.

In Propositions 1,2,3, we assume given a tensor product $(\mathscr{H} \otimes \mathscr{K}, \otimes)$ , where the symbol $\otimes$ in the second coordinate abbreviates the associated bilinear mapping.

Proposition 1. — *If* $\sum_1^n x_k \otimes y_k = 0$ , *and the* $y_k$ *are linearly independent, then* $x_1 = \ldots = x_n = 0$ .

*Proof.* Let $z_1, \ldots, z_m \in \mathscr{H}$ be orthonormal vectors spanning the same linear subspace as $x_1, \ldots, x_n$ . Say

$$x_k = \sum_{r=1}^m \lambda_{kr} z_r \qquad (k = 1, \ldots, n) \,.$$

Then

$$\begin{aligned} 0 = \sum_k x_k \otimes y_k &= \sum_k \Big( \sum_r \lambda_{kr} z_r \Big) \otimes y_k \\ &= \sum_k \sum_r \lambda_{kr} \, z_r \otimes y_k = \sum_r z_r \otimes \Big( \sum_k \lambda_{kr} y_k \Big). \end{aligned}$$

Writing $y_r' = \sum_k \lambda_{kr} y_k$ $(r = 1, \ldots, m)$ , we have $\sum_r z_r \otimes y_r' = 0$ . The terms in this last sum are orthogonal: if $r \neq s$ , then

$$(z_r \otimes y_r' | z_s \otimes y_s') = (z_r|z_s)(y_r'|y_s') = 0 \cdot (y_r'|y_s') = 0 \,.$$

Also, clearly $\|z_r \otimes y'_r\| = \|y'_r\|$. By the "Pythagorean relation",

$$0 = \Big\| \sum_r z_r \otimes y'_r \Big\|^2 = \sum_r \|z_r \otimes y'_r\|^2 = \sum_r \|y'_r\|^2 \,,$$

hence $y'_1 = \ldots = y'_m = 0$. Then $0 = \sum_k \lambda_{kr} y_k$ for $r = 1, \ldots, m$; since the $y_k$ are linearly independent, $\lambda_{kr} = 0$ for all $k, r$. But then $x_k = \sum_r \lambda_{kr} z_r = 0$ for all $k$. ♢

PROPOSITION 2. — *If $\mathscr{E}$ is a complex vector space, and $\alpha : \mathscr{H} \odot \mathscr{K} \to \mathscr{E}$ is any linear mapping, then $\psi(x,y) = \alpha(x \otimes y)$ defines a bilinear mapping $\psi : \mathscr{H} \times \mathscr{K} \to \mathscr{E}$.*

*Proof.* For example,

$$\psi(x_1 + x_2, y) = \alpha\big((x_1 + x_2) \otimes y\big) = \alpha(x_1 \otimes y + x_2 \otimes y) = \\ \alpha(x_1 \otimes y) + \alpha(x_2 \otimes y) = \psi(x_1, y) + \psi(x_2, y)\,. \diamondsuit$$

*Lemma. — If $\psi : \mathscr{H} \times \mathscr{K} \to \mathscr{E}$ is a bilinear mapping, then every relation $\sum_1^n x_k \otimes y_k = 0$ implies $\sum_1^n \psi(x_k, y_k) = 0$.*

*Proof.* Let $y'_1, \ldots, y'_m$ be linearly independent vectors spanning the same linear subspace as $y_1, \ldots, y_n$. Say

$$y_k = \sum_{r=1}^m \lambda_{kr} y'_r \qquad (k = 1, \ldots, n)\,.$$

Then

$$0 = \sum_k x_k \otimes y_k = \sum_k \sum_r \lambda_{kr}\, x_k \otimes y'_r = \sum_r \Big( \sum_k \lambda_{kr} x_k \Big) \otimes y'_r\,.$$

By Prop. 1, $\sum_k \lambda_{kr} x_k = 0$ for all $r$, hence

$$0 = \sum_r \psi(0, y'_r) = \sum_r \psi\Big( \sum_k \lambda_{kr} x_k, y'_r \Big) = \sum_r \sum_k \lambda_{kr}\, \psi(x_k, y'_r) \\ = \sum_k \psi\Big( x_k, \sum_r \lambda_{kr} y'_r \Big) = \sum_k \psi(x_k, y_k). \diamondsuit$$

PROPOSITION 3. — *If $\psi : \mathscr{H} \times \mathscr{K} \to \mathscr{E}$ is any bilinear mapping, there exists a unique linear mapping $\alpha : \mathscr{H} \odot \mathscr{K} \to \mathscr{E}$ such that $\psi(x,y) = \alpha(x \otimes y)$.*

*Proof.* Given $u \in \mathscr{H} \odot \mathscr{K}$, let us define $\alpha(u) \in \mathscr{E}$. Say $u = \sum_1^n x_k \otimes y_k$; we set $\alpha(u) = \sum_1^n \psi(x_k, y_k)$. If also $u = \sum_1^m x'_r \otimes y'_r$, so that

$$\sum_1^n x_k \otimes y_k + \sum_1^m (-x'_r) \otimes y'_r = 0\,,$$

then by the Lemma,

$$\sum_1^n \psi(x_k, y_k) + \sum_1^m \psi(-x'_r, y'_r) = 0$$

and so $\sum_1^n \psi(x_k, y_k) = \sum_1^m \psi(x'_r, y'_r)$. Thus $\alpha : \mathscr{H} \odot \mathscr{K} \to \mathscr{E}$ is unambiguously defined.

Clearly $\alpha$ is linear. If $\beta : \mathscr{H} \odot \mathscr{K} \to \mathscr{E}$ is another linear mapping satisfying $\psi(x,y) = \beta(x \otimes y)$, then for any $u = \sum_1^n x_k \otimes y_k \in \mathscr{H} \odot \mathscr{K}$,

$$\beta(u) = \beta\Big(\sum_1^n x_k \otimes y_k\Big) = \sum_1^n \beta(x_k \otimes y_k) = \sum_1^n \psi(x_k, y_k) = \alpha(u)\,,$$

thus $\alpha = \beta$. ♢

Theorem 1. — *If* $(\mathscr{H} \otimes \mathscr{K}, \otimes)$ *and* $(\mathscr{H} \mathbin{\overline{\otimes}} \mathscr{K}, \overline{\otimes})$ *are any two tensor products of* $\mathscr{H}$ *with* $\mathscr{K}$, *then there exists a unique bounded linear mapping* $U : \mathscr{H} \otimes \mathscr{K} \to \mathscr{H} \mathbin{\overline{\otimes}} \mathscr{K}$ *such that* $U(x \otimes y) = x \mathbin{\overline{\otimes}} y$. *This mapping is* **unitary**, *i.e., it is one-one onto and* $(Uu|Uv) = (u|v)$ *for all* $u, v \in \mathscr{H} \otimes \mathscr{K}$.

*Proof.* We denote by $\mathscr{H} \mathbin{\overline{\odot}} \mathscr{K}$ the linear subspace of $\mathscr{H} \mathbin{\overline{\otimes}} \mathscr{K}$ spanned by the vectors $x \mathbin{\overline{\otimes}} y$.

Taking $\mathscr{E} = \mathscr{H} \mathbin{\overline{\odot}} \mathscr{K}$, and $\psi(x,y) = x \mathbin{\overline{\otimes}} y$, Prop. 3 provides a linear mapping $U : \mathscr{H} \odot \mathscr{K} \to \mathscr{H} \mathbin{\overline{\odot}} \mathscr{K}$ such that $x \mathbin{\overline{\otimes}} y = U(x \otimes y)$. Clearly $U$ is 'onto' (i.e., 'surjective'), by the definition of $\mathscr{H} \mathbin{\overline{\odot}} \mathscr{K}$.

The mapping $U$ preserves scalar products. For, if $u = \sum_k x_k \otimes y_k$ and $v = \sum_r x'_r \otimes y'_r$,

$$\begin{aligned}
(Uu|Uv) &= \big(\sum_k x_k \mathbin{\overline{\otimes}} y_k \big| \sum_r x'_r \mathbin{\overline{\otimes}} y'_r\big) = \sum_k \sum_r (x_k \mathbin{\overline{\otimes}} y_k | x'_r \mathbin{\overline{\otimes}} y'_r) \\
&= \sum_k \sum_r (x_k|x'_r)(y_k|y'_r) = \sum_k \sum_r (x_k \otimes y_k | x'_r \otimes y'_r) \\
&= \big(\sum_k x_k \otimes y_k \big| \sum_r x'_r \otimes y'_r\big) = (u|v)\,.
\end{aligned}$$

Since $\mathscr{H} \odot \mathscr{K}$ is dense in $\mathscr{H} \otimes \mathscr{K}$, and $\mathscr{H} \mathbin{\overline{\otimes}} \mathscr{K}$ is complete, there is a unique continuous linear extension $U : \mathscr{H} \otimes \mathscr{K} \to \mathscr{H} \mathbin{\overline{\otimes}} \mathscr{K}$. Clearly $U$ still preserves scalar products, hence norm; its range is therefore complete, hence closed. But the range contains $\mathscr{H} \mathbin{\overline{\odot}} \mathscr{K}$, and so must be dense. We conclude that $U(\mathscr{H} \otimes \mathscr{K}) = \mathscr{H} \mathbin{\overline{\otimes}} \mathscr{K}$. ♢

## §2. Construction of $\mathscr{H} \otimes \mathscr{K}$ .

As in §1, $\mathscr{H} = \{x, \ldots\}$ and $\mathscr{K} = \{y, \ldots\}$ are fixed Hilbert spaces.

*Lemma* 1. — *If* $T : \mathscr{K} \to \mathscr{H}$ *is conjugate-linear* (briefly, c-linear) *and bounded, there exists a unique c-linear bounded* $T^{\#} : \mathscr{H} \to \mathscr{K}$ *such that*

$$(T^{\#}x|y) = (Ty|x) \qquad (x \in \mathscr{H},\, y \in \mathscr{K})\,.$$

*Proof.* For fixed $x \in \mathscr{H}$ , $y \mapsto (x|Ty)$ is a bounded linear form on $\mathscr{K}$ ; let $T^{\#}x$ denote the unique vector in $\mathscr{K}$ such that $(x|Ty) = (y|T^{\#}x)$ for all $y \in \mathscr{K}$ .

The conjugate-linearity of $T^{\#}$ results from the calculations

$$\begin{aligned}\big(T^{\#}(x_1 + x_2)|y\big) &= (Ty|x_1 + x_2) = (Ty|x_1) + (Ty|x_2)\\ &= (T^{\#}x_1|y) + (T^{\#}x_2|y) = (T^{\#}x_1 + T^{\#}x_2|y)\,,\end{aligned}$$

and

$$\big(T^{\#}(\lambda x)\big|y\big) = (Ty|\lambda x) = \overline{\lambda}\,(Ty|x) = \overline{\lambda}\,(T^{\#}x|y) = (\overline{\lambda}\,T^{\#}x|y)\,.$$

The norm of $T^{\#}x$ is the same as the norm of the bounded linear form it defines on $\mathscr{K}$ . Since

$$|(y|T^{\#}x)| = |(x|Ty)| \leqslant \|x\|\,\|Ty\| \leqslant \|x\|\,\|T\|\,\|y\|\,,$$

one has $\|T^{\#}x\| \leqslant \|x\|\,\|T\|$ . Thus $T^{\#}$ is bounded, indeed $\|T^{\#}\| \leqslant \|T\|$ . ♢

Definition 1. — We denote by $\mathscr{L}_c$ the set of all bounded c-linear mappings $T : \mathscr{K} \to \mathscr{H}$ ; more precisely, $\mathscr{L}_c(\mathscr{K}, \mathscr{H})$ . Clearly $\mathscr{L}_c$ is a complex vector space, relative to the natural operations (e.g., $(\lambda T)x = \lambda(Tx)$).

If $T \in \mathscr{L}_c(\mathscr{K}, \mathscr{H})$ , then $T^{\#} \in \mathscr{L}_c(\mathscr{H}, \mathscr{K})$ . Some further properties of the correspondence $T \mapsto T^{\#}$ :

*Lemma* 2. — *If* $S, T \in \mathscr{L}_c(\mathscr{K}, \mathscr{H})$ , *and* $\lambda$ *is complex, then*

(1) $T^{\#\#} = T$,

(2) $\|T^{\#}\| = \|T\|$ ,

(3) $(S + T)^{\#} = S^{\#} + T^{\#}$ ,

(4) $(\lambda T)^{\#} = \lambda T^{\#}$ .

(5) *If* $A \in \mathscr{L}(\mathscr{H})$ *and* $B \in \mathscr{L}(\mathscr{K})$ *are bounded linear operators then* $(ATB)^{\#} = B^*T^{\#}A^*$ .

*Proof.* (1) $(T^{\#\#}y|x) = (T^{\#}x|y) = (Ty|x)$ .

(2) It was shown in the proof of Lemma 1 that $\|T^{\#}\| \leqslant \|T\|$ . Hence $\|T\| = \|(T^{\#})^{\#}\| \leqslant \|T^{\#}\| \leqslant \|T\|$ .

(3), (4) Routine.

(5) It is clear that $ATB \in \mathscr{L}_c(\mathscr{K}, \mathscr{H})$. Moreover,

$$\begin{aligned}\big((ATB)^{\#}x|y\big) &= (ATBy|x) \\ &= (TBy|A^*x) = (T^{\#}A^*x|By) = (B^*T^{\#}A^*x|y)\,. \diamondsuit\end{aligned}$$

Corollary. — *$T \mapsto T^{\#}$ is a linear norm-preserving mapping of $\mathscr{L}_c(\mathscr{K}, \mathscr{H})$ onto $\mathscr{L}_c(\mathscr{H}, \mathscr{K})$.*

Let $\{e_j\}_{j\in \mathrm{J}}$ be an orthonormal basis for $\mathscr{H}$, and $\{f_k\}_{k\in \mathrm{K}}$ an orthonormal basis for $\mathscr{K}$.

*Lemma* 3. — *If $T \in \mathscr{L}_c$, then*

$$\sum_k \|Tf_k\|^2 = \sum_j \|T^{\#}e_j\|^2 \qquad (\text{perhaps } = +\infty)\,.$$

*In particular, each expression is independent of the specific basis chosen.*

*Proof.* By Parseval's identity,

$$\begin{aligned}\|Tf_k\|^2 &= \sum_j \big|(Tf_k|e_j)\big|^2, \\ \|T^{\#}e_j\|^2 &= \sum_k \big|(T^{\#}e_j|f_k)\big|^2\,.\end{aligned}$$

Since $(Tf_k|e_j) = (T^{\#}e_j|f_k)$, the lemma results from $\sum_j \sum_k = \sum_k \sum_j$ (certainly valid for positive-term sums). $\diamondsuit$

Definition 2. — We denote by $\mathscr{T}$ the set of all $T \in \mathscr{L}_c(\mathscr{K}, \mathscr{H})$ for which $\sum_{k\in \mathrm{K}} \|Tf_k\|^2 < +\infty$. By Lemma 3, $\mathscr{T}$ does not depend on the specific basis $\{f_k\}$. More precisely: $\mathscr{T} = \mathscr{T}(\mathscr{K}, \mathscr{H})$. It will be shown in this section that $\mathscr{T}$ is a Hilbert space and that it has the properties required of the Hilbert space $\mathscr{P}$ in the definition of the tensor product (Def. 1 of §1).

*Remark.* — By $\sum_{k\in \mathrm{K}} \|Tf_k\|^2 < +\infty$ is meant that the finite sums are bounded; the value of $\sum_k$ is the LUB (least upper bound, or supremum) of these finite sums. In particular, $Tf_k = 0$ except for countably many $k$.

*Lemma* 4. — *$\mathscr{T}$ is a linear subspace of $\mathscr{L}_c$.*

*Proof.* Let $S,T \in \mathscr{T}$, $\lambda$ complex. Since $\{\|Sf_k\|\}$ and $\{\|Tf_k\|\}$ are square-summable, so is the sequence $\{\|Sf_k\| + \|Tf_k\|\}$ (recall the Hilbert space of square-summable numerical sequences). Thus $\big\{\big(\|Sf_k\| + \|Tf_k\|\big)^2\big\}$ is summable. Since $\|(S+T)f_k\| \leqslant \|Sf_k\| + \|Tf_k\|$, $\{\|(S+T)f_k\|^2\}$ is summable by the "comparison test". Thus $S + T \in \mathscr{T}$.

Also $\{\|(\lambda T)f_k\|\} = \{|\lambda|\, \|Tf_k\|\}$ is square-summable, so $\lambda T \in \mathscr{T}$. $\diamondsuit$

*Remark.* — It is clear from Lemma 3 that $T \mapsto T^{\#}$ effects a (norm-preserving) linear isomorphism of $\mathscr{T}(\mathscr{K},\mathscr{H})$ onto $\mathscr{T}(\mathscr{H},\mathscr{K})$.

*Lemma* 5. — *If* $S,\, T \in \mathscr{T}$, *then* $\sum_{k\in \mathrm{K}}(Sf_k|Tf_k) = \sum_{j\in \mathrm{J}}(S^{\#}e_j|T^{\#}e_j)$, *and both series are absolutely convergent. In particular, each expression is independent of the specific basis.*

*Proof.* For each $k \in \mathrm{K}$, the form $(S,T) \to (Sf_k|Tf_k)$ on $\mathscr{T} \times \mathscr{T}$ is linear in $S$ and conjugate-linear in $T$ (i.e., it is sesquilinear), hence, by the "polarization identity" (see, e.g., Th. 3 on p. 29 of *Introduction to Hilbert space* [Oxford UP, 1961; 2nd. edn., Chelsea, 1976; AMS-Chelsea, 1999),

$$\begin{aligned}(Sf_k|Tf_k) =& \frac{1}{4}\big\{\|(S+T)f_k\|^2 - \|(S-T)f_k\|^2 \\ &+ i\,\|(S+iT)f_k\|^2 - i\,\|(S-iT)f_k\|^2\big\}.\end{aligned}$$

Similarly, for each $j \in \mathrm{J}$ the form $\psi(S,T) = (S^{\#}e_j|T^{\#}e_j)$ being sesquilinear (linear in $S$, conjugate-linear in $T$),

$$\begin{aligned}\psi(S,T) =& \frac{1}{4}\big\{\psi(S+T,S+T) - \psi(S-T,S-T) \\ &+ i\,\psi(S+iT,S+iT) - i\,\psi(S-iT,S-iT)\big\}.\end{aligned}$$

that is,

$$\begin{aligned}(S^{\#}e_j|T^{\#}e_j) =& \frac{1}{4}\big\{\|(S+T)^{\#}e_j\|^2 - \|(S-T)^{\#}e_j\|^2 \\ &+ i\,\|(S+iT)^{\#}e_j\|^2 - i\,\|(S-iT)^{\#}e_j\|^2\big\}.\end{aligned}$$

The first and third displayed formulas exhibit the asserted absolute convergence; summing the first formula over $k$ and the third over $j$, the assertion of the lemma follows from Lemma 3. $\diamondsuit$

Definition 3. — If $S,T \in \mathscr{T}$, we set

$$(S|T) = \sum_{k\in \mathrm{K}}(Sf_k|Tf_k) = \sum_{j\in \mathrm{J}}(S^{\#}e_j|T^{\#}e_j)$$

(the sums being absolutely convergent, and independent of specific orthonormal basis).

*Remark.* — Now that the possible visual conflict with the complex number $i$ is past, we will write $\{f_i\}_{i\in \mathrm{I}}$ instead of $\{f_k\}_{k\in \mathrm{K}}$ for the given orthonormal basis of $\mathscr{K}$.

*Remark.* — If $S,T \in \mathscr{T}(\mathscr{K},\mathscr{H})$, then $(S|T) = (S^{\#}|T^{\#})$ (the right hand side is given by Def. 3 applied in $\mathscr{T}(\mathscr{H},\mathscr{K})$).

*Lemma* 6. — *$\mathscr{T}$ is a pre-Hilbert space, with $(S|T)$ as scalar product.*

*Proof.* Clearly $(S,T) \mapsto (S|T)$ is sesquilinear. Moreover, $(T|T) = \sum_i \|Tf_i\|^2 \geqslant 0$, and $(T|T) = 0$ if and only if $Tf_i = 0$ for all $i$, that is, $T = 0$. ♢

It will be shown later in this section that $\mathscr{T}$ is a Hilbert space, i.e. is complete relative to the scalar product $(S|T)$.

Definition 4. — If $T \in \mathscr{T}$, we write $\|T\|_2 = (T|T)^{1/2}$. The notation $\|T\|$ is reserved for the bound of $T$ as an operator on $\mathscr{K}$.

*Remark.* — $T \mapsto T^{\#}$ effects a scalar-product preserving linear isomorphism of $\mathscr{T}(\mathscr{K},\mathscr{H})$ onto $\mathscr{T}(\mathscr{H},\mathscr{K})$. In particular, $\|T^{\#}\|_2 = \|T\|_2$.

In order to show that $\mathscr{T}$ is complete with respect to the norm $\|T\|_2$, it will be convenient to discuss *matrices.*

If $T \in \mathscr{L}_c(\mathscr{K},\mathscr{H})$, then $Tf_i = \sum_j \lambda_{ij} e_j$ for suitable unique scalars $\lambda_{ij}$. The array $(\lambda_{ij})$ is called the **matrix** of $T$ (relative to the given orthonormal bases). By Parseval's identity, $\|Tf_i\|^2 = \sum_j |\lambda_{ij}|^2$, hence

$$\sum_i \|Tf_i\|^2 = \sum_{i,j} |\lambda_{ij}|^2 .$$

Thus $T \in \mathscr{T} = \mathscr{T}(\mathscr{K},\mathscr{H})$ if and only if $\sum_{i,j} |\lambda_{ij}|^2 < +\infty$, i.e., its matrix is "square-summable".

Denote by $\mathscr{M}$ the set of all matrices $(\lambda_{ij})$ for which $\sum_{i,j} |\lambda_{ij}|^2 < +\infty$. As is well-known, $\mathscr{M}$ is a Hilbert space relative to the operations

$$\begin{aligned}(\lambda_{ij}) + (\mu_{ij}) &= (\lambda_{ij} + \mu_{ij}) \\ \lambda\,(\lambda_{ij}) &= (\lambda\,\lambda_{ij}) \\ \big((\lambda_{ij})\big|(\mu_{ij})\big) &= \sum_{i,j} \lambda_{ij}\overline{\mu_{ij}}\end{aligned}$$

Thus we have a mapping $\theta : \mathscr{T} \to \mathscr{M}$, $\theta(T)$ being the matrix of $T$ relative to the given orthonormal bases. If $S$ has matrix $(\mu_{ij})$ and $T$ has matrix $(\lambda_{ij})$, clearly $S+T$ has matrix $(\mu_{ij} + \lambda_{ij})$; in other words $\theta(S+T) = \theta(S) + \theta(T)$. Similarly $\theta(\lambda T) = \lambda\,\theta(T)$. Also clearly $S = T$ if and only if $\mu_{ij} = \lambda_{ij}$ for all $i,j$, that is, if and only if $\theta(S) = \theta(T)$. Thus $\theta$ is a one-one (i.e., "injective") linear mapping of $\mathscr{T}$ into $\mathscr{M}$.

Actually $\theta$ is *onto* (i.e., "surjective"). For, suppose $(\lambda_{ij}) \in \mathscr{M}$ is given; we seek $T \in \mathscr{T}$ with $Tf_i = \sum_j \lambda_{ij} e_j$ for all $i$. If $y = \sum_i \alpha_i f_i \in \mathscr{K}$, where $\sum_i |\alpha_i|^2 < +\infty$, we must have

$$\begin{aligned} Ty &= \sum_i \overline{\alpha_i}\, Tf_i \\ &= \sum_i \overline{\alpha_i} \sum_j \lambda_{ij} e_j \\ &= \sum_j \big( \sum_i \overline{\alpha_i}\, \lambda_{ij} \big) e_j \end{aligned} \tag{1}$$

Thus we wish to define $Ty$ by the formula (1). To do this, we must verify that

$$\sum_j \big| \sum_i \overline{\alpha_i}\, \lambda_{ij} \big|^2 < +\infty\,. \tag{2}$$

Now, for each $j$, $\{\lambda_{ij}\}$ is square-summable with respect to $i$; so is $\{\alpha_i\}$; working in the sequence space $\ell^{(2)}(\mathrm{I})$, we have, by the Schwarz inequality,

$$\big| \sum_i \overline{\alpha_i}\, \lambda_{ij} \big|^2 \leqslant \big( \sum_i |\alpha_i|^2 \big) \big( \sum_i |\lambda_{ij}|^2 \big) = \|y\|^2 \sum_i |\lambda_{ij}|^2\,;$$

summing over $j$,

$$\sum_j \big| \sum_i \overline{\alpha_i}\, \lambda_{ij} \big|^2 \leqslant \|y\|^2 \sum_{j,i} |\lambda_{ij}|^2 = \|y\|^2\, \|(\lambda_{ij})\|^2\,.$$

Hence it is permissible to define $Ty$ by (1), and we have

$$\|Ty\| \leqslant \|(\lambda_{ij})\| \cdot \|y\|\,. \tag{3}$$

It is readily verified that $T : \mathscr{K} \to \mathscr{H}$ is bounded and conjugate-linear.

*Lemma 7. — The correspondence* $T \mapsto (\lambda_{ij})$ ($T \in \mathscr{T}$), *where* $(\lambda_{ij})$ *is the matrix of* $T$ *relative to the given orthonormal bases, is a linear isomorphism of* $\mathscr{T}$ *onto* $\mathscr{M}$, *preserving scalar product. In particular* $\mathscr{T}$ *is a* **Hilbert space** *relative to* $(S|T)$.

*Proof.* By the preceding remarks, we need only show that $(T|S) = \big(\theta(T)\big|\theta(S)\big)$. Let $T$ have matrix $(\lambda_{ij})$, and $S$ have matrix $(\mu_{ij})$. Then

$$\begin{aligned} (T|S) &= \sum_{i\in \mathrm{I}} (Tf_i|Sf_i) \\ &= \sum_{i\in \mathrm{I}} \big( \sum_{j\in \mathrm{J}} \lambda_{ij} e_j \big| \sum_{k\in \mathrm{J}} \mu_{ik} e_k \big) = \sum_{i\in \mathrm{I}} \sum_{j\in \mathrm{J}} \sum_{k\in \mathrm{J}} \lambda_{ij}\, \overline{\mu_{ik}}\, \delta_{jk} \\ &= \sum_{i\in \mathrm{I}} \sum_{j\in \mathrm{J}} \lambda_{ij}\, \overline{\mu_{ij}} = \big( (\lambda_{ij}) \big| (\mu_{ij}) \big) = \big(\theta(T)\big|\theta(S)\big).\ \diamondsuit \end{aligned}$$

COROLLARY. — *If* $T \in \mathscr{T}$ , *then* $\|T\| \leqslant \|T\|_2$ .

*Proof.* Relation (3) preceding Lemma 7. ♢

*Lemma* 8. — *If* $x \in \mathscr{H}$ , $y \in \mathscr{K}$ *are fixed, the mapping*

$$y' \mapsto (y|y') \cdot x \qquad (y' \in \mathscr{K})$$

*defines an element of* $\mathscr{T}$ , *to be denoted* $x \otimes y$ $\big($*thus* $(x \otimes y)y' = (y|y')x\big)$ .

*Proof.* Set $Ty' = (y|y') \cdot x$ $(y' \in \mathscr{K})$ . Clearly $T : \mathscr{K} \to \mathscr{H}$ is $c$-linear, and $T$ is bounded because

$$\|Ty'\| = |(y|y')| \cdot \|x\| \leqslant \big(\|y\| \cdot \|x\|\big)\|y'\| .$$

Thus $T \in \mathscr{L}_c$ . It remains to show that $T \in \mathscr{T}$ . This is clear if $y = 0$ . Otherwise, since $Ty' = \big(\frac{1}{\|y\|} \cdot y\big|y'\big) \cdot \|y\|\, x$ , we may suppose that $\|y\| = 1$ . Expand $\{y\}$ to an orthonormal basis $\{y, z_k\}$ of $\mathscr{K}$ . Since $Tz_k = (y|z_k)x = 0 \cdot x = 0$ for all $k$ , $\|Ty\|^2 + \sum_k \|Tz_k\|^2 = \|Ty\|^2 < +\infty$ , hence $T \in \mathscr{T}$ by Def. 2. ♢

COROLLARY. — *If* $x \in \mathscr{H}$, $y \in \mathscr{K}$ , $T \in \mathscr{T}$, *then* $(x \otimes y|T) = (x|Ty)$ .

*Proof.* Recall that if $\{f_i\}_{i\in \mathrm{I}}$ is the given orthonormal basis of $\mathscr{K}$ and if $y \in \mathscr{K}$ , then $y = \sum_i (y|f_i)f_i$ and the formula $\sigma(y) = \{(y|f_i)\}$ defines a Hilbert space isomorphism $\sigma : \mathscr{K} \to \ell^{(2)}(\mathrm{I})$ (*cf.* P.R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*, §14, p. 27, Th. 1). Thus,

$$\begin{aligned}
(x \otimes y|T) &= \sum_i \big((x \otimes y)f_i\big|Tf_i\big) && \text{(Def. 3)}\\
&= \sum_i \big((y|f_i)x\big|Tf_i\big) && \text{(see Lemma 8)}\\
&= \sum_i (y|f_i)(x|Tf_i) = \sum_i (y|f_i)\,\overline{(Tf_i|x)}\\
&= \sum_i (y|f_i)\,\overline{(T^{\#}x|f_i)} && \text{(see Lemma 1)}\\
&= \big(\{(y|f_i)\}\big|\{T^{\#}x|f_i\}\big) && (\text{in } \ell^{(2)}(\mathrm{I}))\\
&= \big(\sigma(y)\big|\sigma(T^{\#}x)\big) = (y|T^{\#}x)\\
&= (x|Ty) \,.\ \diamondsuit && \text{(Lemma 1)}
\end{aligned}$$

*Lemma* 9. — *For* $x$, $x_1$, $x_2 \in \mathscr{H}$ *and* $y$, $y_1$, $y_2 \in \mathscr{K}$ ,

(1) $$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y \,,\ (\lambda x) \otimes y = \lambda(x \otimes y)$$
$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2 \,,\ x \otimes (\lambda y) = \lambda(x \otimes y)$$

(2) $$(x_1 \otimes y_1|x_2 \otimes y_2) = (x_1|x_2)(y_1|y_2) \ \big(\textit{hence } \|x \otimes y\|_2 = \|x\|\,\|y\|\,\big)$$

(3) $$(x \otimes y)^{\#} = y \otimes x$$

*Proof.* (1) The expression $(x \otimes y)y' = (y|y')x$ is linear in $x$ and linear in $y$.

(2) Here, $\|x \otimes y\|_2$ denotes norm in the Hilbert space $\mathscr{H} \otimes \mathscr{K}$. By the preceding Corollary,

$$\begin{aligned}(x_1 \otimes y_1|x_2 \otimes y_2) &= \big(x_1\big|(x_2 \otimes y_2)y_1\big) = \big(x_1\big|(y_2|y_1)x_2\big) \\ &= (x_1|x_2)\overline{(y_2|y_1)} = (x_1|x_2)(y_1|y_2)\,.\end{aligned}$$

(3) Citing Lemma 1 and 8,

$$\begin{aligned}\big((x \otimes y)^{\#}x'\big|y'\big) &= \big((x \otimes y)y'\big|x'\big) = \big((y|y')x\big|x'\big) \\ &= (y|y')(x|x') \\ &= \big((x|x')y\big|y'\big) = \big((y \otimes x)x'\big|y'\big).\ \diamondsuit\end{aligned}$$

Definition 5. — We denote by $\mathscr{T}_0 = \mathscr{T}_0(\mathscr{K}, \mathscr{H})$ the set of all finite sums of elements $x \otimes y$ of $\mathscr{H} \otimes \mathscr{K}$ ($x \in \mathscr{H}$, $y \in \mathscr{K}$).

*Remarks.* — Since $\lambda(x \otimes y) = (\lambda x) \otimes y$, $\mathscr{T}_0$ is a linear subspace of $\mathscr{T}$. By part (3) of Lemma 9, $T \mapsto T^{\#}$ maps $\mathscr{T}_0(\mathscr{K}, \mathscr{H})$ onto $\mathscr{T}_0(\mathscr{H}, \mathscr{K})$.

*Lemma* 10. — *$\{e_j \otimes f_i\}$ is an orthonormal basis for $\mathscr{T}$. In particular, $\mathscr{T}_0$ is a* **dense** *linear subspace of the Hilbert space $\mathscr{T}$.*

*Proof.* If $j \neq j'$ or $i \neq i'$, clearly $e_j \otimes f_i \perp e_{j'} \otimes f_{i'}$ by part (2) of Lemma 9. Also $\|e_j \otimes f_i\|_2 = \|e_j\|\,\|f_i\| = 1$. Thus the vectors $e_j \otimes f_i$ form an orthonormal set. If $T \in \mathscr{T}$ is orthogonal to every $e_j \otimes f_i$ then, for all $i, j$,

$$0 = (e_j \otimes f_i|T) = (e_j|Tf_i)$$

by the corollary to Lemma 8, hence $Tf_i = 0$ for all $i$, whence $T = 0$. Thus the $e_j \otimes f_i$ form a “complete orthonormal system”, i.e., an orthonormal basis. $\diamondsuit$

Summarizing, and fulfilling the promise of §1,

Theorem 2. — *There exists a tensor product of $\mathscr{H}$ with $\mathscr{K}$.*

*Proof.* In Def. 1 of §1, take $\mathscr{P} = \mathscr{T}$ and $\varphi(x, y) = x \otimes y$. $\diamondsuit$

*Remark.* — The mapping $x \otimes y \mapsto y \otimes x$ extends to a unitary mapping of $\mathscr{H} \otimes \mathscr{K}$ onto $\mathscr{K} \otimes \mathscr{H}$. For, $(x \otimes y)^{\#} = y \otimes x$, and $T \mapsto T^{\#}$ is a unitary mapping of $\mathscr{T}(\mathscr{K}, \mathscr{H})$ onto $\mathscr{T}(\mathscr{H}, \mathscr{K})$; for instance, $\|T^{\#}\|_2 = \|T\|_2$ was remarked following Def. 4.

Proposition 4. — *If $T \in \mathscr{L}_c(\mathscr{K}, \mathscr{H})$, then $T \in \mathscr{T}_0$ if and only if $T$ has finite-dimensional range.*

*Proof.* “only if”: $x \otimes y$ has range of dimension $\leqslant 1$, hence $\sum_1^n x_k \otimes y_k$ has range of dimension $\leqslant n$ (spanned by $x_1, \ldots, x_n$).

"if": Suppose $T \in \mathscr{L}_c(\mathscr{K}, \mathscr{H})$ has finite-dimensional range, spanned say by the orthonormal vectors $x_1, \ldots, x_n$. Then

$$Ty = \sum_1^n \alpha_k(y) x_k \qquad (y \in \mathscr{K}),$$

for suitable unique conplex coefficients $\alpha_k(y)$ (depending on $y$). Since $T$ is c-linear, so are the $\alpha_k$. Moreover,

$$\|Ty\|^2 = \sum_1^n |\alpha_k(y)|^2$$

shows that the $\alpha_k$ are *bounded* c-linear forms. Hence there exist vectors $y_1, \ldots, y_n \in \mathscr{K}$ such that

$$\alpha_k(y) = (y_k|y) \qquad (y \in \mathscr{K})$$

(consider the bounded *linear* forms $y \mapsto \overline{\alpha_k(y)}$). Then $Ty = \sum_1^n (y_k|y) x_k = \left(\sum_1^n x_k \otimes y_k\right) y$, so $T = \sum_1^n x_k \otimes y_k \in \mathscr{T}_0$. ♢

Corollary. — *Every $T \in \mathscr{T}$ is the uniform limit of finite-dimensional c-linear bounded operators* (*hence is "completely continuous"*).

*Proof.* Since $\mathscr{T}_0$ is dense in $\mathscr{T}$, there is a sequence $T_n \in \mathscr{T}_0$ such that $\|T_n - T\|_2 \to 0$ (Lemma 10). Then $\|T_n - T\| \leqslant \|T_n - T\|_2 \to 0$ (see the Corollary of Lemma 7). ♢

## §3. Tensor product of operators

*Lemma* 11. — *If $T \in \mathscr{T}(\mathscr{K}, \mathscr{H})$, $A \in \mathscr{L}(\mathscr{H})$, $B \in \mathscr{L}(\mathscr{K})$, then $ATB^* \in \mathscr{T}(\mathscr{K}, \mathscr{H})$, and $\|ATB^*\|_2 \leqslant \|A\|\,\|B\|\,\|T\|_2$.*

*Proof.* Obviously $ATB^* \in \mathscr{L}_c(\mathscr{K}, \mathscr{H})$. Moreover,

$$\begin{aligned}\sum_{i \in \mathrm{I}} \|ATB^* f_i\|^2 \leqslant \|A\|^2 \sum_{i \in \mathrm{I}} \|TB^* f_i\|^2 &= \|A\|^2 \sum_{j \in \mathrm{J}} (TB^*)^{\#} e_j\|^2 \\ &= \|A\|^2 \sum_{j \in \mathrm{J}} \|BT^{\#} e_j\|^2 \leqslant \|A\|^2 \|B\|^2 \sum_{j \in \mathrm{J}} \|T^{\#} e_j\|^2 \\ &= \|A\|^2 \|B\|^2 \sum_{i \in \mathrm{I}} \|T f_i\|^2\end{aligned}$$

(for the first and third equalities, cite Lemma 3; for the second equality, in each term cite (5) of Lemma 2 with $A = I$ and $B$ replaced by $B^*$). ♢

Definition 7. — For fixed $A \in \mathscr{L}(\mathscr{H})$, $B \in \mathscr{L}(\mathscr{K})$, by Lemma 11 the mapping $T \mapsto ATB^*$ $(T \in \mathscr{T})$ is a bounded linear mapping of the Hilbert space $\mathscr{T}$ into $\mathscr{T}$. We denote it by $A \otimes B$. Thus $A \otimes B \in \mathscr{L}(\mathscr{T})$.

THEOREM 3. — *If* $A \in \mathscr{L}(\mathscr{H})$, $B \in \mathscr{L}(\mathscr{K})$, *there exists a unique* $C \in \mathscr{L}(\mathscr{H} \otimes \mathscr{K})$ *such that*

$$C(x \otimes y) = Ax \otimes By \qquad \text{for all } x \in \mathscr{H}, y \in \mathscr{K} .$$

*Namely:* $C = A \otimes B$.

*Proof.* Uniqueness: Obvious from the density of $\mathscr{H} \odot \mathscr{K}$ in $\mathscr{H} \otimes \mathscr{K}$ (Lemma 10).

Existence: Let $A \in \mathscr{L}(\mathscr{H})$, $B \in \mathscr{L}(\mathscr{K})$, $x \in \mathscr{H}$, $y \in \mathscr{K}$, and let us write $C = A \otimes B \in \mathscr{L}(\mathscr{H} \otimes \mathscr{K})$ and $T = x \otimes y \in \mathscr{T} = \mathscr{L}_c(\mathscr{K}, \mathscr{H})$.

By definition, $C$ sends $T$ to $ATB^*$ (Def. 7), so we are to show that $ATB^* = Ax \otimes By$. For every $y' \in \mathscr{K}$,

$$\begin{aligned} TB^*y' &= (x \otimes y)(B^*y') = (y|B^*y')x \qquad \text{(see Lemma 8)} \\ &= (By|y')x \end{aligned}$$

therefore $ATB^*y' = (By|y')Ax = (Ax \otimes By)y'$ (Lemma 8 again), that is, $ATB^* = Ax \otimes By$. Thus $C$ sends $T = x \otimes y$ to $ATB^* = Ax \otimes By$. ♢

THEOREM 4. — *With the obvious notations,*

(1)
$$\begin{aligned} (A_1 + A_2) \otimes B &= A_1 \otimes B + A_2 \otimes B \\ (\lambda \mathrm{A}) \otimes B &= \lambda(A \otimes \mathrm{B}) \\ A \otimes (B_1 + B_2) &= A \otimes B_1 + A \otimes B_2 \\ A \otimes (\lambda B) &= \lambda(A \otimes B) \end{aligned}$$

(2) $I \otimes I = I$ ( $I$ *the identity mapping on* $\mathscr{H}$, $\mathscr{K}$, $\mathscr{H} \otimes \mathscr{K}$ *).*

(3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

(4) $(A \otimes B)^* = A^* \otimes B^*$

(5) $\|A \otimes B\| = \|A\| \, \|B\|$.

(6) $A \otimes B$ *is invertible if and only if* $A$ *and* $B$ *are both invertible, in which case* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

(7) *If* $\sigma$ *denotes spectrum, then*

$$\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B) = \{\lambda\mu : \ \lambda \in \sigma(A) , \mu \in \sigma(B)\} .$$

*Proof.* (1) For example,

$$\begin{aligned} [(A_1 + A_2) \otimes B]x \otimes y &= [(A_1 + A_2)x] \otimes By \\ &= (A_1x + A_2x) \otimes By \\ &= A_1x \otimes By + A_2x \otimes By \\ &= (A_1 \otimes B + A_2 \otimes B)x \otimes y . \end{aligned}$$

Quote Lemma 10 (or Theorem 3).

(2) $(I \otimes I)x \otimes y = Ix \otimes Iy = x \otimes y = I(x \otimes y)$.

(3)
$$\begin{aligned}[(A \otimes B)(C \otimes D)]x \otimes y &= (A \otimes B)(Cx \otimes Dy) = \mathrm{A}(Cx) \otimes B(Dy) \\ &= [(AC)x] \otimes [(BD)y] \\ &= (AC \otimes BD)(x \otimes y)\end{aligned}$$

(4)
$$\begin{aligned}\big((A \otimes B)^* x \otimes y \big| u \otimes v\big) &= \big(x \otimes y \big| (A \otimes B)u \otimes v\big) = (x \otimes y | Au \otimes Bv) \\ &= (x|Au)(y|Bv) = (A^*x|u)(B^*y|v) \\ &= (A^*x \otimes B^*y | u \otimes v)\end{aligned}$$

(5) The inequality $\|A \otimes B\| \leqslant \|A\| \, \|B\|$ is shown by Lemma 11 and the definition of $A \otimes B$ (Def. 7). To prove the reverse inequality, choose sequences $x_n \in \mathscr{H}$, $y_n \in \mathscr{K}$ such that $\|x_n\| = \|y_n\| = 1$ and $\|Ax_n\| \to \|A\|$, $\|By_n\| \to \|B\|$. Then

$$\|Ax_n \otimes By_n\| = \|Ax_n\| \, \|By_n\| \to \|A\| \, \|B\| .$$

Since $\|x_n \otimes y_n\| = \|x_n\| \, \|y_n\| = 1$, we have $\|(A \otimes B)(x_n \otimes y_n)\| \leqslant \|A \otimes B\|$, whence $\|A\| \, \|B\| \leqslant \|A \otimes B\|$ on passing to the limit.

(6) If $A$ and $B$ are invertible, then

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1}) \otimes (BB^{-1}) = I \otimes I = I ,$$

and similarly $(A^{-1} \otimes B^{-1})(A \otimes B) = I$, thus $A \otimes B$ is invertible and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Conversely, suppose $A \otimes B$ is invertible. Since

$$A \otimes B = (A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I) ,$$

it follows that $A \otimes I$ and $I \otimes B$ are also invertible, so it will suffice to show that the invertibility of $A \otimes I$ implies that of $A$ (the proof for $B$ is similar).

We know that $A \otimes I$ and $(A \otimes I)^* = A^* \otimes I^* = A^* \otimes I$ are bounded below, and it will suffice to show that $A$ and $A^*$ are bounded below (*Introduction to Hilbert space*, p. 156, part vi of Exer. 11). Thus we are reduced to showing that the boundedness below of $A \otimes I$ implies that of $A$.

By supposition, there exists an $\varepsilon > 0$ such that $\|(A \otimes I)u\| \geqslant \varepsilon \, \|u\|$ for all $u \in \mathscr{H} \otimes \mathscr{K}$. Then, $\|(A \otimes I)x \otimes y\| \geqslant \varepsilon \, \|x\| \, \|y\|$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$, that is,

$$\varepsilon \, \|x\| \, \|y\| \leqslant \|Ax \otimes y\| = \|Ax\| \, \|y\| ,$$

whence $\|Ax\| \geqslant \varepsilon \, \|x\|$ (choose any nonzero $y$, then cancel).

(7) Proved in a paper by Arlen Brown and Carl Pearcy, *Spectra of tensor products of operators* [Proc. Amer. Math. Soc. **17** (1966), 162–166; *erratum*, *ibid.* **18** (1967), 1142; MR **32**#6218]. ◊

Corollary 1. — *If $U \in \mathscr{L}(\mathscr{H})$ and $V \in \mathscr{L}(\mathscr{K})$ are unitary then so is $U \otimes V$.*

*Proof.* $(U\otimes V)^*(U\otimes V) = (U^*\otimes V^*)(U\otimes V) = U^*U\otimes V^*V = I\otimes I = I$, and similarly $(U \otimes V)(U \otimes V)^* = I$. ◊

Corollary 2. — *If $s \mapsto U_s \in \mathscr{L}(\mathscr{H})$ and $t \mapsto V_t \in \mathscr{L}(\mathscr{K})$ are unitary representations of groups $G$ and $H$, respectively, then the mapping $(s,t) \mapsto U_s \otimes V_t$ is a unitary representation of the product group $G \times H$.*

*Proof.* The operators $U_s \otimes V_t$ are unitary (Cor. 1), and

$$U_{s_1 s_2} \otimes V_{t_1 t_2} = U_{s_1}U_{s_2} \otimes V_{t_1}V_{t_2} = (U_{s_1} \otimes V_{t_1})(U_{s_2} \otimes V_{t_2})\,;$$

since $(s_1,t_1)(s_2,t_2) = (s_1 s_2, t_1 t_2)$, this shows that $(s,t) \mapsto U_s \otimes V_t$ is a homomorphism. ◊

Corollary 2′. — *If $s \mapsto U_s \in \mathscr{L}(\mathscr{H})$ and $s \mapsto V_s \in \mathscr{L}(\mathscr{K})$ are unitary representations of the same group $\mathrm{G}$, then so is $s \mapsto U_s \otimes V_s$.*

*Proof.* Restrict the mapping of Cor. 2 to the "diagonal subgroup" (isomorphic to $\mathrm{G}$), consisting of the pairs $(s,s)$ of $\mathrm{G} \times \mathrm{G}$. ◊

When $G$ is a topological group, the unitary representations $s \mapsto U_s \in \mathscr{L}(\mathscr{H})$ of particular interest are those that are continuous when $\mathscr{L}(\mathscr{H})$ is equipped with the "strong operator topology", that is, if the mapping $s \mapsto U_s x$ is continuous for each $x \in \mathscr{H}$. (For a fuller description, see for example Def. 68.4 on p. 294–or Exer. 40.24 on p. 172– of *Lectures in functional analysis and operator theory* [Springer, 1974].) The following lemma illustrates the heart of the matter:

*Lemma. — If $x \to x_0$ in $\mathscr{H}$ and $y \to y_0$ in $\mathscr{K}$, then $x\otimes y \to x_0 \otimes y_0$ in $\mathscr{H} \times \mathscr{K}$.*

*Proof.* The idea is that if $x$ is made near $x_0$ and $y$ is made near $y_0$, then $x \otimes y$ will be near $x_0 \otimes y_0$. Given any $\varepsilon > 0$, suppose $\|x - x_0\| \leqslant \varepsilon$ and $\|y - y_0\| \leqslant \varepsilon$. From $x = (x - x_0) + x_0$ we see that $\|x\| \leqslant \varepsilon + \|x_0\|$. The identity

$$x \otimes y - x_0 \otimes y_0 = x \otimes (y - y_0) + (x - x_0) \otimes y_0\,,$$

yields the inequality

$$\begin{aligned}\|x \otimes y - x_0 \otimes y_0\| &\leqslant \|x \otimes (y - y_0)\| + \|(x - x_0) \otimes y_0\| \\ &= \|x\|\,\|y - y_0\| + \|x - x_0\|\,\|y_0\| \\ &\leqslant (\varepsilon + \|x_0\|)\varepsilon + \varepsilon\,\|y_0\|\,,\end{aligned}$$

which gives rigorous form to the idea. In particular, if $x_n \in \mathscr{H}$ and $y_n \in \mathscr{K}$ are sequences such that $\|x_n - x_0\| \to 0$ and $\|y_n - y_0\| \to 0$, then $\|x_n\| \leqslant \|x_n - x_0\| + \|x_0\|$ shows that the sequence $x_n$ is bounded, whence

$$\|x_n \otimes y_n - x_0 \otimes y_0\| \leqslant \|x_n\| \, \|y_n - y_0\| + \|x_n - x_0\| \, \|y_0\| \to 0 \,,$$

and so $x_n \otimes y_n \to x_0 \otimes y_0$ . ◊

Corollary 3. — *If* $s \mapsto U_s \in \mathscr{L}(\mathscr{H})$ *and* $t \mapsto V_t \in \mathscr{L}(\mathscr{K})$ *are strongly continuous unitary representations of topological groups* $G$ *and* $H$ *, respectively, then* $(s,t) \mapsto U_s \otimes V_t \in \mathscr{L}(\mathscr{H} \otimes \mathscr{K})$ *is a strongly continuous unitary representation of the product topological group* $G \times H$ *.*

*Proof.* If $(s,t) \to (s_0,t_0)$ (i.e., $s \to s_0$ and $t \to t_0$ ), then by the lemma

$$(U_s \otimes V_t) x \otimes y = U_s x \otimes V_t y \to U_{s_0} x \otimes V_{t_0} y = (U_{s_0} \otimes V_{t_0}) x \otimes y \,.$$

Thus $(s,t) \to U_s \otimes V_t$ is strongly continuous on $\mathscr{H} \odot \mathscr{K}$ . Our assertion will follow from uniform boundedness of the $U_s \otimes V_t$ (their norms are all equal to 1 ) and the density of $\mathscr{H} \odot \mathscr{K}$ in $\mathscr{H} \otimes \mathscr{K}$ ("Banach-Steinhaus principle"):

Let $(s,t) \to (s_0,t_0)$ . Given $w \in \mathscr{H} \otimes \mathscr{K}$ , we will show that $U_s \otimes V_t w \to U_{s_0} \otimes V_{t_0} w$ . Choose a sequence $w_n \in \mathscr{H} \odot \mathscr{K}$ with $\|w_n - w\| \to 0$. Then

$$\begin{aligned}
\|U_s \otimes V_t w - U_{s_0} \otimes V_{t_0} w\| &\leqslant \|U_s \otimes V_t w - U_s \otimes V_t w_n\| \\
&\quad + \|U_s \otimes V_t w_n - U_{s_0} \otimes V_{t_0} w_n\| \\
&\quad + \|U_{s_0} \otimes V_{t_0} w_n - U_{s_0} \otimes V_{t_0} w\| \\
&= \|w - w_n\| \\
&\quad + \|U_s \otimes V_t w_n - U_{s_0} \otimes V_{t_0} w_n\| \\
&\quad + \|w_n - w\| \,.
\end{aligned}$$

Given $\varepsilon > 0$ , choose an $n$ such that $\|w_n - w\| < \varepsilon/3$. For this $n$, letting $(s,t) \to (s_0,t_0)$ we have $\|U_s \otimes V_t w_n - U_{s_0} \otimes V_{t_0} w_n\| \to 0$ (by the first paragraph of the proof); hence this term can be made $< \varepsilon/3$ for $(s,t)$ sufficiently near $(s_0,t_0)$ , and then

$$\|U_s \otimes V_t w - U_{s_0} \otimes V_{t_0} w\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \,. \; ◊$$

Corollary 3′. — *If* $s \mapsto U_s \in \mathscr{L}(\mathscr{H})$ *and* $s \mapsto V_s \in \mathscr{L}(\mathscr{K})$ *are strongly continuous unitary representations of the same topological group* $G$ *, then so is* $s \mapsto U_s \otimes V_s$ *.*

*Remark.* — In Cor. 3′ (or Cor. 2′) suppose $T \in \mathscr{T} = \mathscr{H} \otimes \mathscr{K}$ (see Lemma 7) is a "fixed vector" for the representation $s \mapsto U_s \otimes V_s$ . Then $T = (U_s \otimes V_s)T = U_s T V_s{}^*$ for all $s \in G$, thus $U_s T = T V_s$ for all $s$ . Such operators $T$ are called "intertwining operators" for the two representations.

*Reflections*. The foregoing is drawn (and polished somewhat) from a set of lecture notes prepared for a 1958 Mathematics/Physics seminar on von Neumann algebras at the University of Iowa (Iowa City), organized by Professor of Physics Joseph Jauch and myself, not long before his return to Switzerland. It is the first of five chapters, the others being *The* $\mathscr{M} = \mathscr{M}''$ *theorem*, *Equivalence of projections*, *Finite projections*, and *Trace in a factor of Type* $\mathrm{II}_1$ .

I do not recall how I came to choose the definition of the tensor product of two Hilbert spaces (§1, Def. 1, §2, Def. 2) with its reliance on conjugate-linear mappings, assuring that $(x, y) \mapsto x \otimes y$ is linear in both $x$ and $y$ (§2, Lemma 9).

Another procedure is to commence with the algebraic tensor product of the two Hilbert spaces, equip it with an inner product that makes it a pre-Hilbert space, and pass to the completion (cf. Jacques Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien (Algèbres de von Neumann)*, 2nd. edn., Gauthiers-Villars, 1969, Ch. I, §2, No. 3, p.21 *ff*); but the algebraic tensor product of vector spaces is a demanding pre-requisite (cf. N. Bourbaki, *Algebra I*, Hermann/Addison-Wesley, 1974, Ch. III, §3, No. 1, Def. 1, p. 243).

The novelty of the present exposition is that it circumvents the algebraic tensor product by exploiting the (conjugate) self-duality of Hilbert space. A closely related approach is via the algebra of operators of Hilbert-Schmidt class, equipped with a Hilbert space structure (see Lemma 7 above and the paper of Brown and Pearcy referred to in the proof of Theorem 4). The Hilbert space $\mathscr{L}_{\mathrm{hs}}(\mathscr{H}, \mathscr{K})$ of Hilbert-Schmidt operators $T : \mathscr{H} \to \mathscr{K}$ is constructed in Exer. 11 on p. 136 of *Introduction to Hilbert space* (but no connection is made there with the concept of tensor product). For $T \in \mathscr{L}_c(\mathscr{K}, \mathscr{H})$, the definition of $T^\# \in \mathscr{L}_c(\mathscr{H}, \mathscr{K})$ and its association with a matrix is a variation on the adjoint $T^* \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ defined in the cited Exer. 11: for $x \in \mathscr{H}$ and $y \in \mathscr{K}$, their relation is expresed by

$$(T^\# x|y) = (Ty|x) = (y|T^* x) = \overline{(T^* x|y)}\,.$$

The relation shows explicitly that, whereas the mapping $T \mapsto T^*$ is conjugate-linear, the mapping $T \mapsto T^\#$ is linear. Thus, the mapping $T^*$ is linear, but the mapping $T \mapsto T^*$ is conjugate-linear, whereas the mapping $T^\#$ is conjugate-linear but the mapping $T \mapsto T^\#$ is linear.

The paper of Brown and Pearcy is concerned with the algebra of operators $\mathscr{L}(\mathscr{H})$, hence restricts attention to the Hilbert space $\mathscr{L}_{\mathrm{hs}}(\mathscr{H})$, which is an ideal of the algebra $\mathscr{L}(\mathscr{H})$, so that mappings of the form $T \mapsto ATB^*$ $\big(T \in \mathscr{L}_{\mathrm{hs}}(\mathscr{H}),\ A \in \mathscr{L}(\mathscr{H}),\ B \in \mathscr{L}(\mathscr{H})\big)$ may be viewed as operators on the Hilbert space $\mathscr{L}_{\mathrm{hs}}(\mathscr{H})$.

S. K. Berberian (10-20-2013)