The equation of motion for an incompressible fluid and its variants

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Abstract. In this paper, particular cases of an equation of motion in tension are being considered to which given conditions, characteristic of the Newtonian, nonviscous and ideal fluid, can be applied. As a result, existence of the differential equations not featured in the reference was revealed. The hierarchical schema of the differential equations was formed, new trajectories of a deduction of known special cases and their solutions were found. It is shown that the Navier-Stokes equation is a special case of other more general equation of motion of the Newtonian fluid.

Keywords: *the general equation, Navier-Stokes, Poiseuille, turbulence.*

1. Introduction

The mathematical description of fluid flow is based on the equation of motion in tension (Navier) which can be presented as [1, 2]:

$$X+\frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)=\frac{du_x}{dt}$$

$$Y+\frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)=\frac{du_y}{dt} \qquad (1.1)$$

$$Z+\frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)=\frac{du_z}{dt}\ ,$$

where p_{xx}, p_{yy}, p_{zz} - normal tensions, $\tau_{yx}, \tau_{zx}, \tau_{yz}$ - the tangential tensions, X, Y, Z - specific mass force, u_x, u_y, u_z - velocity projections, t - time.

The purpose of the present paper is to receive equations of motion by means of superimposition on (1.1) minimum numbers of certain conditions inherent for the Newtonian, nonviscous and ideal fluid.

2. Equations and short analysis

Let us converse system (1.1) having separated normal tensions from tangents. As $p_{xx}=-p_x$, $p_{yy}=-p_y$, $p_{zz}=-p_z$ (where $p_x, p_y, p_z -$ are pressure projections).

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\frac{1}{\rho}\left(\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)=\frac{du_x}{dt}$$

$$Y-\frac{1}{\rho}\frac{\partial p_y}{\partial y}+\frac{1}{\rho}\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{zy}}{\partial z}\right)=\frac{du_y}{dt} \tag{2.1}$$

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\frac{1}{\rho}\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}\right)=\frac{du_z}{dt}$$

We will converse the first line of the equation (2.1) having substituted expressions ($\tau=\mu\cdot grad\ u$) for the tangential tensions in the Newtonian fluid.

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\left[\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)\right]=\frac{du_x}{dt} \tag{2.2}$$

Let us add a zero to derivatives in brackets having presented it in the form of two identical summands with different signs.

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\left[\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}-\frac{\partial u_x}{\partial y}+\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}-\frac{\partial u_z}{\partial x}+\frac{\partial u_z}{\partial x}\right)\right]=\frac{du_x}{dt}\,.$$

As a result, we will gain:

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\left[\frac{\partial}{\partial y}\left(\left(curl\ u\right)_z+2\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\left(curl\ u\right)_y+2\frac{\partial u_z}{\partial x}\right)\right]=\frac{du_x}{dt}\,.$$

After similar transformations

$$Y-\frac{1}{\rho}\frac{\partial p_y}{\partial y}+\nu\left[\frac{\partial}{\partial x}\left(\left(curl\ u\right)_z+2\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\left(curl\ u\right)_x+2\frac{\partial u_y}{\partial z}\right)\right]=\frac{du_y}{dt} \tag{2.3}$$

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\nu\left[\frac{\partial}{\partial x}\left(\left(curl\ u\right)_y+2\frac{\partial u_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\left(curl\ u\right)_x+2\frac{\partial u_y}{\partial z}\right)\right]=\frac{du_z}{dt}\,,$$

or / $(curl\ u)_i=2\omega_i$, ω - angular velocity./

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+2\nu\left[\frac{\partial\omega_z}{\partial y}+\frac{\partial\omega_y}{\partial z}+\frac{\partial^2u_x}{\partial y^2}+\frac{\partial^2u_z}{\partial x\partial z}\right]=\frac{du_x}{dt}$$

$$Y-\frac{1}{\rho}\frac{\partial p_y}{\partial y}+2\nu\left[\frac{\partial\omega_x}{\partial z}+\frac{\partial\omega_z}{\partial x}+\frac{\partial^2 u_y}{\partial z^2}+\frac{\partial^2 u_x}{\partial x\partial y}\right]=\frac{du_y}{dt}$$

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+2\nu\left[\frac{\partial\omega_y}{\partial x}+\frac{\partial\omega_x}{\partial y}+\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_y}{\partial y\partial z}\right]=\frac{du_z}{dt},$$

where – the tangential tensions τ_{ij} , functions $(curl\ u)_i$

$$\tau_{xy}=\mu\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right),\qquad (curl\ u)_x=\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z},$$

$$\tau_{xz}=\mu\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right),\qquad (curl\ u)_y=\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x},$$

$$\tau_{yz}=\mu\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right),\qquad (curl\ u)_z=\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}.$$

Characteristic property of system (2.3) is the consideration of impact of the vortex /$(curl\ u)_i$/ and forward flow of fluid, and also direct dependence of pressure on fluid element.

From (2.3) by the data $(curl\ u)_i=0$ equations of motion for viscous irrotational flow follow.

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+2\nu\left(\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_z}{\partial x\partial z}\right)=\frac{du_x}{dt}$$

$$Y-\frac{1}{\rho}\frac{\partial p_y}{\partial y}+2\nu\left(\frac{\partial^2 u_y}{\partial z^2}+\frac{\partial^2 u_x}{\partial x\partial y}\right)=\frac{du_y}{dt}\tag{2.4}$$

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+2\nu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_y}{\partial y\partial z}\right)=\frac{du_z}{dt}.$$

By using cylindrical axis (*r*, *z*), it is possible to gain a special case of the equations (2.4) for a round pipe in a format like , $\frac{d^2u_z}{dr^2}=\frac{1}{2\mu}\cdot grad\ p$ from which Poiseuille equation follows [1, 2].

The system (2.3) has one more special case for a rotating vortex without translatory motion. By excepting the linear velocities in the left part, we will gain:

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\left[\frac{\partial}{\partial y}(curl\ u)_z+\frac{\partial}{\partial z}(curl\ u)_y\right]=\frac{du_x}{dt}$$

$$Y-\frac{1}{\rho}\frac{\partial p_y}{\partial y}+\nu\left[\frac{\partial}{\partial x}(curl\ u)_z+\frac{\partial}{\partial z}(curl\ u)_x\right]=\frac{du_y}{dt}\tag{2.5}$$

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\nu\left[\frac{\partial}{\partial x}(curl\ u)_y+\frac{\partial}{\partial y}(curl\ u)_x\right]=\frac{du_z}{dt}$$

The approach studied allows to gain the equation of motion within the limits of model of nonviscous medium. Assumed that $\nu = 0$, we will gain from (2.3) - (2.5).

$$X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} = \frac{du_x}{dt}$$

$$Y - \frac{1}{\rho}\frac{\partial p_y}{\partial y} = \frac{du_y}{dt} \qquad (2.6)$$

$$Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} = \frac{du_z}{dt}$$

From (2.6) it follows that pressure projections can differ in the absence of viscosity influence. This deduction contradicts the common stand that $p_x \neq p_y \neq p_z$ is possible under impact of viscosities only [2].

If we accept hydrostatic pressure distribution law ($p = p_x = p_y = p_z$) from (2.6), we will gain Euler's equation [2].

From (2.1) it is possible to gain the Nave-Stokes' equation, having used the several assumptions, one of which is possibility of the linear averaging of non-linear dependence $p = f(x, y, z, t)$ [1, 2].

Links between the equations studied can be presented in the form of the following plan (fig.1).

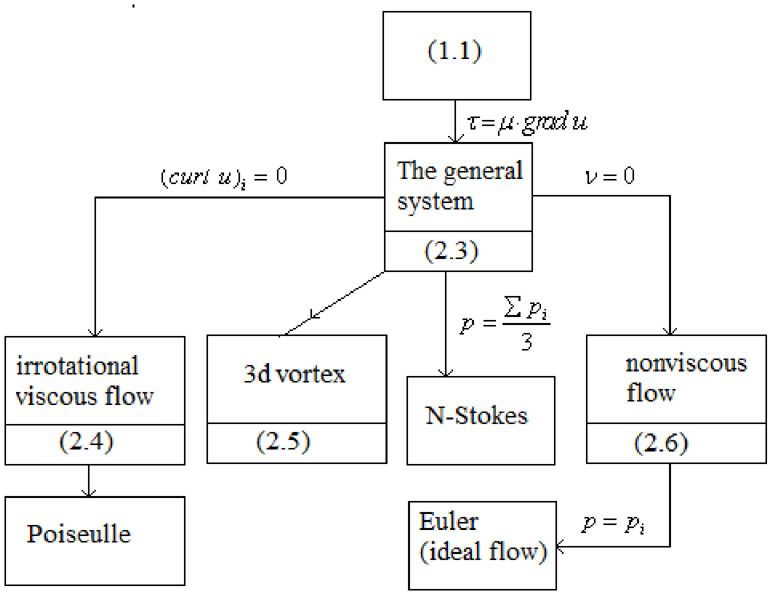

Fig. 1. Conditions, equations and dependence between them.

3. Discussion

3.1. In system (2.3), there are the summands characterising all kinds of flows of an incompressible fluid: translational and the vortex. It suggests that the given equations can be used for the turbulent flow description.

3.2. Use of definition for irrotational flow has allowed to find a special case (2.3) and the Poiseuille equation. It suggests that the system (2.4) might be used for laminar flow calculation.

3.3. The equation in the form of (2.1) is used at a deduction of the Nave-Stokes' equation at two basic assumptions:

3.3.1. Validly linear equation $p = \frac{1}{3}\left(p_x + p_y + p_z\right)$ to determine medial pressure of non-linear function $p = f(x, y, z, t)$ [1, 2]. This standard assumption is carried out at a small interval of averaging only and should be considered as approximate.0

3.3.2. Pressure in a point changes under the influence of viscosity only[1, 2]. This assumption will not be compatible with an equation of motion for a nonviscous fluid (2.5).

Taking into account both assumptions, the Nave-Strokes equation should be considered as approximate for a laminar flow.

4. References

[1].Genick Bar-Meir. Basic of Fluid Mechanics. V.0.3.4.0, 2013, 604 p.

[2].Loitsyansky L.G. Mechanics of Liguid and Gas. 5th edition. GRFML., Science. M, 1978.- 736 p.