ON THE SOLVABILITY OF SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH ANOMALOUS DIFFUSION

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Abstract: The article deals with the existence of solutions of a system of integro-differential equations in the case of anomalous diffusion with the Laplacian in a fractional power. The proof of existence of solutions is based on a fixed point technique. Solvability conditions for non Fredholm elliptic operators in unbounded domains are used.

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1. Introduction

The present work is devoted to the existence of stationary solutions of the following system of integro-differential equations

$$\frac{\partial u_m}{\partial t} = -D_m\left(-\frac{\partial^2}{\partial x^2}\right)^s u_m + \int_{-\infty}^{\infty} K_m(x-y) g_m(u(y,t))dy + f_m(x), \quad 1 \le m \le N \tag{1.1}$$

appearing in cell population dynamics. The space variable x here corresponds to the cell genotype, functions $u_m(x,t)$ describe the cell density distributions for various groups of cells as functions of their genotype and time,

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The right side of this system of equations describes the evolution of cell densities due to cell proliferation, mutations and cell influx or efflux. The anomalous diffusion terms with positive coefficients D_m correspond to the change of genotype due to small random mutations, and the nonlocal production terms describe large mutations. Functions $g_m(u)$ denote the rates of cell birth which depend on u (density dependent proliferation), and the kernels $K_m(x-y)$ express the proportions of newly born cells changing their genotype from y to x. We assume that they depend on the distance between the genotypes. The functions $f_m(x)$ describe the influx or efflux of cells for different genotypes.

The operator $\left(-\frac{\partial^2}{\partial x^2}\right)^s$ in system (1.1) describes a particular case of anomalous diffusion actively studied in the context of different applications in plasma physics and turbulence [11], [12], surface diffusion [13], [15], semiconductors [16] and so on. Anomalous diffusion can be understood as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for superdiffusion. Asymptotic behavior at infinity of the probability density function determines the value s of the power of the Laplacian [14]. The operator $\left(-\frac{\partial^2}{\partial x^2}\right)^s$ is defined by means of the spectral calculus. In the present work we will consider the case of $0 < s < 1/4$. A similar problem in the case of the standard Laplace operator in the diffusion term was studied recently in [26]. Note that the restriction on the power s here comes from the solvability conditions of our problem.

Let us set all $D_m = 1$ and establish the existence of solutions of the system of equations

$$-\left(-\frac{d^2}{dx^2}\right)^s u_m + \int_{-\infty}^{\infty} K_m(x-y)g_m(u(y))dy + f_m(x) = 0, \quad 0 < s < \frac{1}{4}, \tag{1.2}$$

with $1 \leq m \leq N$. Let us consider the case where the linear part of this operator fails to satisfy the Fredholm property. As a consequence, conventional methods of nonlinear analysis may not be applicable. We use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.3) contains the origin. As a consequence, such operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work deals with the studies of some properties of the operators of this kind. Note that elliptic problems with non Fredholm operators were treated actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The non Fredholm Schrödinger type operators were studied with the methods of the spectral and the scattering theory in [17], [19], [21]. The Laplace operator with drift from the point of view of non Fredholm operators was treated in [20] and linearized Cahn-Hilliard problems in [22] and [24]. Nonlinear non Fredholm elliptic problems were studied in [23] and [25]. Important applications to the theory of reaction-diffusion equations were developed in [8], [9]. Non Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is significantly different and the approach developed in these articles cannot be used. Front propagation equations with anomalous diffusion were studied largely in recent years (see e.g. [28], [29]).

We set $K_m(x) = \varepsilon_m \mathcal{K}_m(x)$ with $\varepsilon_m \geq 0$, such that

$$\varepsilon := max_{1 \leq m \leq N} \varepsilon_m$$

and suppose that the following assumption is satisfied.

Assumption 1. *Let* $1 \leq m \leq N$ *and consider* $0 < s < \frac{1}{4}$. *Let* $f_m(x) : \mathbb{R} \to \mathbb{R}$ *be nontrivial for some* m. *Let* $f_m(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and*

$$\left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}-s} f_m(x) \in L^2(\mathbb{R}).$$

Assume also that $\mathcal{K}_m(x) : \mathbb{R} \to \mathbb{R}$, *such that* $\mathcal{K}_m(x) \in L^1(\mathbb{R})$ *and*

$$\left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}-s} \mathcal{K}_m(x) \in L^2(\mathbb{R}).$$

Moreover,

$$\mathcal{K}^2 := \sum_{m=1}^{N} \|\mathcal{K}_m(x)\|^2_{L^1(\mathbb{R})} > 0$$

and

$$Q^2 := \sum_{m=1}^{N} \left\| \left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}-s} \mathcal{K}_m(x) \right\|^2_{L^2(\mathbb{R})} > 0.$$

Let us choose the space dimension $d = 1$, which is related to the solvability conditions for the linear Poisson type problem (4.1) stated in Lemma 6 below. We use the Sobolev spaces for $0 < s \le 1$, namely

$$H^{2s}(\mathbb{R}) := \left\{ \phi(x) : \mathbb{R} \to \mathbb{R} \mid \phi(x) \in L^2(\mathbb{R}), \ \left(-\frac{d^2}{dx^2} \right)^s \phi \in L^2(\mathbb{R}) \right\}$$

equipped with the norm

$$\|\phi\|^2_{H^{2s}(\mathbb{R})} := \|\phi\|^2_{L^2(\mathbb{R})} + \left\| \left(-\frac{d^2}{dx^2} \right)^s \phi \right\|^2_{L^2(\mathbb{R})}. \tag{1.4}$$

For a vector vector function

$$u(x) = (u_1(x), u_2(x), ..., u_N(x))^T$$

we will use the norm

$$\|u\|^2_{H^1(\mathbb{R},\mathbb{R}^N)} := \|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} + \sum_{m=1}^{N} \left\| \frac{du_m}{dx} \right\|^2_{L^2(\mathbb{R})}, \tag{1.5}$$

where

$$\|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} := \sum_{m=1}^{N} \|u_m\|^2_{L^2(\mathbb{R})}.$$

By means of the standard Sobolev inequality in one dimension (see e.g. Section 8.5 of [10]) we have

$$\|\phi\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}} \|\phi\|_{H^1(\mathbb{R})}. \tag{1.6}$$

When all the nonnegative parameters ε_m vanish, we obtain the linear Poisson type equations

$$\left(-\frac{d^2}{dx^2} \right)^s u_m = f_m(x), \quad 1 \le m \le N. \tag{1.7}$$

By virtue of Lemma 6 below along with Assumption 1 each equation (1.7) has a unique solution

$$u_{0,m}(x) \in H^{2s}(\mathbb{R}), \quad 0 < s < \frac{1}{4},$$

such that no orthogonality conditions are required. By means of Lemma 6, when $\frac{1}{4} \le s < 1$, certain orthogonality relations (4.3) and (4.4) are necessary to be able to solve problem (1.7) in $H^{2s}(\mathbb{R})$. By means of Assumption 1, since

$$\left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}} u_{0,m}(x) = \left(-\frac{d^2}{dx^2} \right)^{\frac{1}{2}-s} f_m(x) \in L^2(\mathbb{R}),$$

we get for the unique solution of linear problem (1.7) that $u_{0,m}(x) \in H^1(\mathbb{R})$, such that

$$u_0(x) := (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T \in H^1(\mathbb{R}, \mathbb{R}^N).$$

We seek the resulting solution of nonlinear system of equations (1.2) as

$$u(x) = u_0(x) + u_p(x), \tag{1.8}$$

where

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T.$$

Clearly, we arrive at the perturbative system of equations

$$\left(-\frac{d^2}{dx^2}\right)^s u_{p,m} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(u_0(y) + u_p(y)) dy, \quad 0 < s < \frac{1}{4}, \tag{1.9}$$

where $1 \leq m \leq N$. Let us introduce a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^1(\mathbb{R}, \mathbb{R}^N) \mid \|u\|_{H^1(\mathbb{R}, \mathbb{R}^N)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.10}$$

We seek the solution of problem (1.9) as the fixed point of the auxiliary nonlinear system of equations

$$\left(-\frac{d^2}{dx^2}\right)^s u_m = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(u_0(y) + v(y)) dy, \quad 0 < s < \frac{1}{4}, \tag{1.11}$$

with $1 \leq m \leq N$ in ball (1.10). For a given vector function $v(y)$ this is a system of equations with respect to $u(x)$. The left side of (1.11) involves the non Fredholm operator

$$\left(-\frac{d^2}{dx^2}\right)^s : H^{2s}(\mathbb{R}) \to L^2(\mathbb{R}).$$

Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, such operator has no bounded inverse. The similar situation appeared in articles [23] and [25] but as distinct from the present situation, the equations studied there required orthogonality conditions. The fixed point technique was used in [18] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear equation there had the Fredholm property (see Assumption 1 of [18], also [7]). We define the closed ball in the space of N dimensions as

$$I := \left\{z \in \mathbb{R}^N \mid |z| \leq \frac{1}{\sqrt{2}}(\|u_0\|_{H^1(\mathbb{R}, \mathbb{R}^N)} + 1)\right\} \tag{1.12}$$

along with the closed ball in the space of $C^2(I, \mathbb{R}^N)$ functions, namely

$$D_M := \{g(z) := (g_1(z), g_2(z), ..., g_N(z)) \in C^2(I, \mathbb{R}^N) \mid \|g\|_{C^2(I,\mathbb{R}^N)} \leq M\}, \tag{1.13}$$

where $M > 0$. Here the norms

$$\|g\|_{C^2(I,\mathbb{R}^N)} := \sum_{m=1}^{N} \|g_m\|_{C^2(I)}, \tag{1.14}$$

$$\|g_m\|_{C^2(I)} := \|g_m\|_{C(I)} + \sum_{n=1}^{N} \Big\| \frac{\partial g_m}{\partial z_n} \Big\|_{C(I)} + \sum_{n,l=1}^{N} \Big\| \frac{\partial^2 g_m}{\partial z_n \partial z_l} \Big\|_{C(I)}, \tag{1.15}$$

where $\|g_m\|_{C(I)} := max_{z \in I}|g_m(z)|$. Let us make the following assumption on the nonlinear part of system (1.2).

Assumption 2. *Let $1 \leq m \leq N$. Assume that $g_m(z) : \mathbb{R}^N \to \mathbb{R}$, such that $g_m(0) = 0$ and $\nabla g_m(0) = 0$. It is also assumed that $g(z) \in D_M$ and it does not vanish identically in the ball I.*

We introduce the operator T_g, such that $u = T_g v$, where u is a solution of system (1.11). Our first main proposition is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then for every $\rho \in (0, 1]$ there exists $\varepsilon^* > 0$, such that system (1.11) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon^*$. The unique fixed point $u_p(x)$ of this map T_g is the only solution of system (1.9) in B_ρ.*

Evidently, the resulting solution $u(x)$ of system (1.2) will be nontrivial because the source terms $f_m(x)$ are nontrivial for some $1 \leq m \leq N$ and all $g_m(0) = 0$ as assumed. We make use of the following trivial lemma.

Lemma 4. *For $R \in (0, +\infty)$ consider the function*

$$\varphi(R) := \alpha R^{1-4s} + \frac{\beta}{R^{4s}}, \quad 0 < s < \frac{1}{4}, \quad \alpha, \beta > 0.$$

It achieves the minimal value at $R^* := \dfrac{4\beta s}{\alpha(1-4s)}$, *which is given by*

$$\varphi(R^*) = \frac{(1-4s)^{4s-1}}{(4s)^{4s}} \alpha^{4s} \beta^{1-4s}.$$

Our second main result is about the continuity of the fixed point of the map T_g which existence was proved in Theorem 3 above with respect to the nonlinear vector function g.

Theorem 5. *Let $j = 1, 2$, the assumptions of Theorem 3 hold, such that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon_j^*$ and $\delta := min(\varepsilon_1^*, \varepsilon_2^*)$. Then for all $0 < \varepsilon < \delta$ the inequality*

$$\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le C\|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)} \tag{1.16}$$

holds, where $C > 0$ is a constant.

We proceed to the proof of our first main proposition.

2. The existence of the perturbed solution

Proof of Theorem 3. We choose arbitrarily $v(x) \in B_\rho$ and designate the term involved in the integral expression in the right side of system (1.11) as

$$G_m(x) := g_m(u_0(x) + v(x)), \quad 1 \le m \le N.$$

Let us use the standard Fourier transform

$$\widehat{\phi}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx. \tag{2.1}$$

Obviously, we have the inequality

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}} \|\phi(x)\|_{L^1(\mathbb{R})}. \tag{2.2}$$

Let us apply (2.1) to both sides of system (1.11) and obtain

$$\widehat{u}_m(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}}_m(p)\widehat{G}_m(p)}{|p|^{2s}}, \quad 1 \le m \le N.$$

Thus we express the norm as

$$\|u_m\|_{L^2(\mathbb{R})}^2 = 2\pi\varepsilon_m^2 \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s}} dp, \quad 1 \le m \le N. \tag{2.3}$$

As distinct from articles [23] and [25] involving the standard Laplace operator in the diffusion term, here we do not try to control the norm

$$\left\| \frac{\widehat{\mathcal{K}}_m(p)}{|p|^{2s}} \right\|_{L^\infty(\mathbb{R})}.$$

Instead, we estimate the right side of (2.3) using the analog of inequality (2.2) applied to functions $\mathcal{K}_m$ and G_m with $R > 0$ as

$$2\pi\varepsilon_m^2 \Big[\int_{|p| \le R} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s}} dp + \int_{|p| > R} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s}} dp \Big] \le$$

$$\leq \varepsilon_m^2 \|\mathcal{K}_m\|_{L^1(\mathbb{R})}^2 \left\{ \frac{1}{\pi} \|G_m(x)\|_{L^1(\mathbb{R})}^2 \frac{R^{1-4s}}{1-4s} + \frac{1}{R^{4s}} \|G_m(x)\|_{L^2(\mathbb{R})}^2 \right\}. \tag{2.4}$$

Due to the fact that $v(x) \in B_\rho$, we easily obtain

$$\|u_0 + v\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1.$$

Sobolev inequality (1.6) implies that

$$|u_0 + v| \leq \frac{1}{\sqrt{2}} (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1).$$

Let the dot denote the scalar product of two vectors in $\mathbb{R}^N$. Formula

$$G_m(x) = \int_0^1 \nabla g_m(t(u_0(x) + v(x))).(u_0(x) + v(x))dt, \quad 1 \leq m \leq N$$

with the ball I defined in (1.12) yields

$$|G_m(x)| \leq sup_{z\in I} |\nabla g_m(z)||u_0(x) + v(x)| \leq M|u_0(x) + v(x)|.$$

Thus

$$\|G_m(x)\|_{L^2(\mathbb{R})} \leq M\|u_0 + v\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq M(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1).$$

Apparently, for $t \in [0, 1]$ and $1 \leq m, j \leq N$, we have

$$\frac{\partial g_m}{\partial z_j}(t(u_0(x) + v(x))) = \int_0^t \nabla \frac{\partial g_m}{\partial z_j}(\tau(u_0(x) + v(x))).(u_0(x) + v(x))d\tau.$$

This implies

$$\left|\frac{\partial g_m}{\partial z_j}(t(u_0(x) + v(x)))\right| \leq sup_{z\in I} \left|\nabla \frac{\partial g_m}{\partial z_j}\right| |u_0(x) + v(x)| \leq$$

$$\leq \sum_{n=1}^N \left\| \frac{\partial^2 g_m}{\partial z_n \partial z_j} \right\|_{C(I)} |u_0(x) + v(x)|.$$

Therefore,

$$|G_m(x)| \leq |u_0(x)+v(x)| \sum_{n,j=1}^N \left\| \frac{\partial^2 g_m}{\partial z_n \partial z_j} \right\|_{C(I)} |u_{0,j}(x)+v_j(x)| \leq M|u_0(x)+v(x)|^2.$$

Hence,

$$\|G_m(x)\|_{L^1(\mathbb{R})} \leq M\|u_0 + v\|_{L^2(\mathbb{R},\mathbb{R}^N)}^2 \leq M(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2. \tag{2.5}$$

This enables us to obtain the upper bound for the right side of (2.4) as

$$\varepsilon_m^2 M^2 \|\mathcal{K}_m\|^2_{L^1(\mathbb{R})}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^2\left\{\frac{(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^2 R^{1-4s}}{\pi(1-4s)}+\frac{1}{R^{4s}}\right\},$$

with $R \in (0,+\infty)$. Lemma 4 gives us the minimal value of the expression above. Thus,

$$\|u_m\|^2_{L^2(\mathbb{R})} \le \varepsilon^2 \|\mathcal{K}_m\|^2_{L^1(\mathbb{R})}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^{2+8s}\frac{M^2}{(1-4s)(4\pi s)^{4s}},$$

such that

$$\|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \le \varepsilon^2 \mathcal{K}^2(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^{2+8s}\frac{M^2}{(1-4s)(4\pi s)^{4s}}. \tag{2.6}$$

Clearly, (1.11) yields

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_m(x) = \varepsilon_m \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s}\int_{-\infty}^{\infty}\mathcal{K}_m(x-y)G_m(y)dy, \quad 1 \le m \le N.$$

By means of the analog of inequality (2.2) applied to function G_m along with (2.5) we obtain

$$\left\|\frac{du_m}{dx}\right\|^2_{L^2(\mathbb{R})} \le \varepsilon_m^2 \|G_m\|^2_{L^1(\mathbb{R})}\left\|\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s}\mathcal{K}_m\right\|^2_{L^2(\mathbb{R})} \le$$

$$\le \varepsilon^2 M^2(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^4\left\|\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s}\mathcal{K}_m\right\|^2_{L^2(\mathbb{R})},$$

such that

$$\sum_{m=1}^{N}\left\|\frac{du_m}{dx}\right\|^2_{L^2(\mathbb{R})} \le \varepsilon^2 M^2(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^4 Q^2. \tag{2.7}$$

Therefore, by virtue of the definition of the norm (1.5) along with inequalities (2.6) and (2.7) we derive the estimate from above for $\|u\|_{H^1(\mathbb{R},\mathbb{R}^N)}$ as

$$\varepsilon M(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^2\left[\frac{\mathcal{K}^2(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^{8s-2}}{(1-4s)(4\pi s)^{4s}}+Q^2\right]^{\frac{1}{2}} \le \rho \tag{2.8}$$

for all $\varepsilon > 0$ sufficiently small. Hence, $u(x) \in B_\rho$ as well. If for a certain $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of system (1.11), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R},\mathbb{R}^N)$ solves

$$\left(-\frac{d^2}{dx^2}\right)^{s} w_m = 0, \quad 1 \le m \le N.$$

Because the operator $\left(-\frac{d^2}{dx^2}\right)^s$ considered on the whole real line does not possess nontrivial square integrable zero modes, $w(x)$ vanishes a.e. on $\mathbb{R}$. Thus, system (1.11) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ small enough.

Our goal is to establish that this map is a strict contraction. Let us choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above implies $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well. By means of (1.11) we have for $1 \le m \le N$

$$\left(-\frac{d^2}{dx^2}\right)^s u_{1,m} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(u_0(y) + v_1(y)) dy, \tag{2.9}$$

$$\left(-\frac{d^2}{dx^2}\right)^s u_{2,m} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(u_0(y) + v_2(y)) dy, \tag{2.10}$$

$0 < s < \frac{1}{4}$. We introduce

$$G_{1,m}(x) := g_m(u_0(x) + v_1(x)), \quad G_{2,m}(x) := g_m(u_0(x) + v_2(x)), \quad 1 \le m \le N$$

and apply the standard Fourier transform (2.1) to both sides of systems (2.9) and (2.10). This yields

$$\widehat{u}_{1,m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}}_m(p) \widehat{G}_{1,m}(p)}{|p|^{2s}}, \quad \widehat{u}_{2,m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}}_m(p) \widehat{G}_{2,m}(p)}{|p|^{2s}}.$$

Obviously,

$$\|u_{1,m} - u_{2,m}\|_{L^2(\mathbb{R})}^2 = \varepsilon_m^2 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_{1,m}(p) - \widehat{G}_{2,m}(p)|^2}{|p|^{4s}} dp.$$

Evidently, it can be estimated from above by virtue of inequality (2.2) by

$$\varepsilon^2 \|\mathcal{K}_m\|_{L^1(\mathbb{R})}^2 \left\{ \frac{1}{\pi} \|G_{1,m}(x) - G_{2,m}(x)\|_{L^1(\mathbb{R})}^2 \frac{R^{1-4s}}{1-4s} + \frac{\|G_{1,m}(x) - G_{2,m}(x)\|_{L^2(\mathbb{R})}^2}{R^{4s}} \right\},$$

with $R \in (0, +\infty)$. We will make use of the identity for $1 \le m \le N$

$$G_{1,m}(x) - G_{2,m}(x) = \int_0^1 \nabla g_m(u_0(x) + t v_1(x) + (1-t) v_2(x)).(v_1(x) - v_2(x)) dt.$$

Clearly, for $t \in [0,1]$

$$\|v_2(x) + t(v_1(x) - v_2(x))\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le t \|v_1(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} + (1-t) \|v_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le$$

$$\le \rho,$$

such that $v_2(x) + t(v_1(x) - v_2(x)) \in B_\rho$. Hence,

$$|G_{1,m}(x) - G_{2,m}(x)| \le sup_{z\in I}|\nabla g_m(z)||v_1(x) - v_2(x)| \le M|v_1(x) - v_2(x)|.$$

This yields

$$\|G_{1,m}(x) - G_{2,m}(x)\|_{L^2(\mathbb{R})} \le M\|v_1 - v_2\|_{L^2(\mathbb{R},\mathbb{R}^N)} \le M\|v_1 - v_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}.$$

Evidently, for $1 \le m, j \le N$, we can express $\dfrac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x))$ as

$$\int_0^1 \nabla \frac{\partial g_m}{\partial z_j}(\tau[u_0(x) + tv_1(x) + (1-t)v_2(x)]).[u_0(x) + tv_1(x) + (1-t)v_2(x)]d\tau,$$

such that for $t \in [0,1]$

$$\Big|\frac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x))\Big| \le$$

$$\le \sum_{n=1}^{N} \left\|\frac{\partial^2 g_m}{\partial z_n \partial z_j}\right\|_{C(I)} (|u_0(x)| + t|v_1(x)| + (1-t)|v_2(x)|).$$

We obtain the upper bound for $G_{1,m}(x) - G_{2,m}(x)$ in the absolute value as

$$M|v_1(x) - v_2(x)|\Big(|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)|\Big).$$

By means of the Schwarz inequality we arrive at the estimate from above for the norm $\|G_{1,m}(x) - G_{2,m}(x)\|_{L^1(\mathbb{R})}$ as

$$M\|v_1 - v_2\|_{L^2(\mathbb{R},\mathbb{R}^N)}\Big(\|u_0\|_{L^2(\mathbb{R},\mathbb{R}^N)} + \frac{1}{2}\|v_1\|_{L^2(\mathbb{R},\mathbb{R}^N)} + \frac{1}{2}\|v_2\|_{L^2(\mathbb{R},\mathbb{R}^N)}\Big) \le$$

$$\le M\|v_1 - v_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1). \tag{2.11}$$

Thus we arrive at the upper bound for the norm $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}$ given by

$$\varepsilon^2 \mathcal{K}^2 M^2 \|v_1 - v_2\|^2_{H^1(\mathbb{R},\mathbb{R}^N)}\Big\{\frac{1}{\pi}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 \frac{R^{1-4s}}{1-4s} + \frac{1}{R^{4s}}\Big\}.$$

By means of Lemma 4 we minimize the expression above over $R \in (0, +\infty)$ to obtain the estimate from above for $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}$ as

$$\varepsilon^2 \mathcal{K}^2 M^2 \|v_1 - v_2\|^2_{H^1(\mathbb{R},\mathbb{R}^N)} \frac{(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{8s}}{(1-4s)(4\pi s)^{4s}}. \tag{2.12}$$

By virtue of formulas (2.9) and (2.10), for $1 \leq m \leq N$ we have

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}}(u_{1,m}-u_{2,m}) = \varepsilon_m \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \int_{-\infty}^{\infty} \mathcal{K}_m(x-y)[G_{1,m}(y)-G_{2,m}(y)]dy.$$

Inequalities (2.2) and (2.11) yield

$$\left\|\frac{d}{dx}(u_{1,m}-u_{2,m})\right\|_{L^2(\mathbb{R})}^2 \leq \varepsilon^2 \|G_{1,m}-G_{2,m}\|_{L^1(\mathbb{R})}^2 \left\|\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s}\mathcal{K}_m\right\|_{L^2(\mathbb{R})}^2 \leq$$

$$\leq \varepsilon^2 M^2 \|v_1 - v_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^2 \left\|\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s}\mathcal{K}_m\right\|_{L^2(\mathbb{R})}^2,$$

such that

$$\sum_{m=1}^{N}\left\|\frac{d}{dx}(u_{1,m}-u_{2,m})\right\|_{L^2(\mathbb{R})}^2 \leq \varepsilon^2 M^2 \|v_1-v_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^2 Q^2. \tag{2.13}$$

By virtue of (2.12) and (2.13) the norm $\|u_1 - u_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}$ can be estimated from above by the expression

$$\varepsilon M(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)\left\{\frac{\mathcal{K}^2(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^{8s-2}}{(1-4s)(4\pi s)^{4s}} + Q^2\right\}^{\frac{1}{2}} \|v_1-v_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}. \tag{2.14}$$

This yields that the map $T_g : B_\rho \to B_\rho$ defined by system (1.11) is a strict contraction for all values of $\varepsilon > 0$ small enough. Its unique fixed point $u_p(x)$ is the only solution of system (1.9) in the ball B_ρ. The resulting $u(x) \in H^1(\mathbb{R},\mathbb{R}^N)$ given by (1.8) is a solution of system (1.2). Note that by means of (2.8) $u_p(x)$ tends to zero in the $H^1(\mathbb{R},\mathbb{R}^N)$ norm as $\varepsilon \to 0$. ■

Then we turn our attention to the proof of the second main statement of our article.

3. The continuity of the fixed point of the map T_g

Proof of Theorem 5. Obviously, for all $0 < \varepsilon < \delta$ we have

$$u_{p,1} = T_{g_1}u_{p,1}, \quad u_{p,2} = T_{g_2}u_{p,2}.$$

Hence

$$u_{p,1} - u_{p,2} = T_{g_1}u_{p,1} - T_{g_1}u_{p,2} + T_{g_1}u_{p,2} - T_{g_2}u_{p,2}.$$

Therefore,

$$\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le \|T_{g_1}u_{p,1}-T_{g_1}u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}+\|T_{g_1}u_{p,2}-T_{g_2}u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}.$$

Inequality (2.14) yields

$$\|T_{g_1}u_{p,1}-T_{g_1}u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le \varepsilon\sigma\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)},$$

with $\varepsilon\sigma < 1$ since the map $T_{g_1}: B_\rho \to B_\rho$ under our assumptions is a strict contraction. Here the positive constant

$$\sigma := M(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)\left\{\frac{\mathcal{K}^2(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^{8s-2}}{(1-4s)(4\pi s)^{4s}}+Q^2\right\}^{\frac{1}{2}}.$$

Hence, we obtain

$$(1-\varepsilon\sigma)\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le \|T_{g_1}u_{p,2}-T_{g_2}u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}. \tag{3.1}$$

Clearly, for our fixed point $T_{g_2}u_{p,2} = u_{p,2}$. Let us denote $\xi(x) := T_{g_1}u_{p,2}$. For $1 \le m \le N$, we arrive at

$$\left(-\frac{d^2}{dx^2}\right)^s \xi_m(x) = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y)g_{1,m}(u_0(y)+u_{p,2}(y))dy, \tag{3.2}$$

$$\left(-\frac{d^2}{dx^2}\right)^s u_{p,2,m}(x) = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y)g_{2,m}(u_0(y)+u_{p,2}(y))dy, \tag{3.3}$$

where $0 < s < \frac{1}{4}$. Let us designate here

$$G_{1,2,m}(x) := g_{1,m}(u_0(x)+u_{p,2}(x)), \quad G_{2,2,m}(x) := g_{2,m}(u_0(x)+u_{p,2}(x)).$$

We apply the standard Fourier transform (2.1) to both sides of (3.2) and (3.3). This yields

$$\widehat{\xi}_m(p) = \varepsilon_m\sqrt{2\pi}\frac{\widehat{\mathcal{K}}_m(p)\widehat{G}_{1,2,m}(p)}{|p|^{2s}}, \quad \widehat{u}_{p,2,m}(p) = \varepsilon_m\sqrt{2\pi}\frac{\widehat{\mathcal{K}}_m(p)\widehat{G}_{2,2,m}(p)}{|p|^{2s}}.$$

Evidently,

$$\|\xi_m(x)-u_{p,2,m}(x)\|^2_{L^2(\mathbb{R})} = \varepsilon_m^2 2\pi\int_{-\infty}^{\infty}\frac{|\widehat{\mathcal{K}}_m(p)|^2|\widehat{G}_{1,2,m}(p)-\widehat{G}_{2,2,m}(p)|^2}{|p|^{4s}}dp.$$

Apparently, it can be bounded from above by means of (2.2) by

$$\varepsilon^2\|\mathcal{K}_m\|^2_{L^1(\mathbb{R})}\left\{\frac{1}{\pi}\|G_{1,2,m}-G_{2,2,m}\|^2_{L^1(\mathbb{R})}\frac{R^{1-4s}}{1-4s}+\|G_{1,2,m}-G_{2,2,m}\|^2_{L^2(\mathbb{R})}\frac{1}{R^{4s}}\right\},$$

with $R \in (0, +\infty)$. We use the formula

$$G_{1,2,m}(x)-G_{2,2,m}(x) = \int_0^1 \nabla[g_{1,m}-g_{2,m}](t(u_0(x)+u_{p,2}(x))).(u_0(x)+u_{p,2}(x))dt,$$

such that

$$|G_{1,2,m}(x) - G_{2,2,m}(x)| \leq \|g_{1,m} - g_{2,m}\|_{C^2(I)}|u_0(x) + u_{p,2}(x)|.$$

Therefore,

$$\|G_{1,2,m} - G_{2,2,m}\|_{L^2(\mathbb{R})} \leq \|g_{1,m} - g_{2,m}\|_{C^2(I)}\|u_0 + u_{p,2}\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq$$
$$\leq \|g_{1,m} - g_{2,m}\|_{C^2(I)}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1).$$

Let us apply another useful representation formula with $1 \leq j \leq N$ and $t \in [0, 1]$, namely

$$\frac{\partial}{\partial z_j}(g_{1,m} - g_{2,m})(t(u_0(x) + u_{p,2}(x))) =$$
$$= \int_0^t \nabla\Big[\frac{\partial}{\partial z_j}(g_{1,m} - g_{2,m})\Big](\tau(u_0(x) + u_{p,2}(x))).(u_0(x) + u_{p,2}(x))d\tau.$$

Hence

$$\Big|\frac{\partial}{\partial z_j}(g_{1,m} - g_{2,m})(t(u_0(x) + u_{p,2}(x)))\Big| \leq$$
$$\leq \sum_{n=1}^{N}\left\|\frac{\partial^2(g_{1,m} - g_{2,m})}{\partial z_n \partial z_j}\right\|_{C(I)}|u_0(x) + u_{p,2}(x)|,$$

such that

$$|G_{1,2,m}(x) - G_{2,2,m}(x)| \leq \|g_{1,m} - g_{2,m}\|_{C^2(I)}|u_0(x) + u_{p,2}(x)|^2.$$

Thus,

$$\|G_{1,2,m} - G_{2,2,m}\|_{L^1(\mathbb{R})} \leq \|g_{1,m} - g_{2,m}\|_{C^2(I)}\|u_0 + u_{p,2}\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq$$
$$\leq \|g_{1,m} - g_{2,m}\|_{C^2(I)}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2. \tag{3.4}$$

This enables us to derive the upper bound for the norm $\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}$ as

$$\varepsilon^2\mathcal{K}^2(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2\|g_1 - g_2\|^2_{C^2(I,\mathbb{R}^N)}\Big[\frac{(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2R^{1-4s}}{\pi(1 - 4s)} + \frac{1}{R^{4s}}\Big].$$

This expression can be trivially minimized over $R \in (0, +\infty)$ by virtue of Lemma 4. We obtain the inequality

$$\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \varepsilon^2\mathcal{K}^2(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{2+8s}\frac{\|g_1 - g_2\|^2_{C^2(I,\mathbb{R}^N)}}{(1 - 4s)(4\pi s)^{4s}}.$$

Formulas (3.2) and (3.3) with $1 \leq m \leq N$ yield

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} \xi_m(x) = \varepsilon_m \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) G_{1,2,m}(y) dy,$$

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_{p,2,m}(x) = \varepsilon_m \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) G_{2,2,m}(y) dy,$$

such that by means of (2.2) and (3.4) the norm $\left\| \frac{d}{dx}\Big(\xi_m(x) - u_{p,2,m}(x)\Big) \right\|_{L^2(\mathbb{R})}^2$ can be estimated from above by

$$\varepsilon^2 \|G_{1,2,m} - G_{2,2,m}\|_{L^1(\mathbb{R})}^2 \left\| \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}_m \right\|_{L^2(\mathbb{R})}^2 \leq$$

$$\leq \varepsilon^2 \|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^4 \left\| \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}_m \right\|_{L^2(\mathbb{R})}^2 .$$

Then

$$\sum_{m=1}^{N} \left\| \frac{d}{dx}\Big(\xi_m(x) - u_{p,2,m}(x)\Big) \right\|_{L^2(\mathbb{R})}^2 \leq \varepsilon^2 \|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^4 Q^2.$$

Therefore, we arrive at $\|\xi(x) - u_{p,2}(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq$

$$\leq \varepsilon \|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)} (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 \left[\frac{\mathcal{K}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{8s-2}}{(1-4s)(4\pi s)^{4s}} + Q^2 \right]^{\frac{1}{2}}.$$

By virtue of inequality (3.1), the norm $\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}$ can be bounded from above by

$$\frac{\varepsilon}{1-\varepsilon\sigma} (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 \left[\frac{\mathcal{K}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{8s-2}}{(1-4s)(4\pi s)^{4s}} + Q^2 \right]^{\frac{1}{2}} \|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)},$$

which completes the proof of the theorem. ■

4. Auxiliary results

Below we state the solvability conditions proven easily in [27] by applying the standard Fourier transform (2.1) to the linear Poisson type equation with a square integrable right side

$$\left(-\frac{d^2}{dx^2}\right)^{s} \phi = f(x), \quad x \in \mathbb{R}, \quad 0 < s < 1. \tag{4.1}$$

We denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{4.2}$$

with a slight abuse of notations when the functions involved in (4.2) are not square integrable, like for instance the one involved in orthogonality condition (4.3) of Lemma 6 below. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x)$ is bounded, then the integral in the right side of (4.2) makes sense. The left side of relation (4.4) is well defined as well under the stated conditions. We have the following technical proposition.

Lemma 6. *Let* $f(x) : \mathbb{R} \to \mathbb{R}$ *and* $f(x) \in L^2(\mathbb{R})$.

1) When $0 < s < \frac{1}{4}$ *and in addition* $f(x) \in L^1(\mathbb{R})$, *equation (4.1) admits a unique solution* $\phi(x) \in H^{2s}(\mathbb{R})$.

2) When $\frac{1}{4} \le s < \frac{3}{4}$ *and additionally* $|x|f(x) \in L^1(\mathbb{R})$, *problem (4.1) possesses a unique solution* $\phi(x) \in H^{2s}(\mathbb{R})$ *if and only if the orthogonality relation*

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{4.3}$$

holds.

3) When $\frac{3}{4} \le s < 1$ *and in addition* $x^2 f(x) \in L^1(\mathbb{R})$, *equation (4.1) has a unique solution* $\phi(x) \in H^{2s}(\mathbb{R})$ *if and only if orthogonality conditions (4.3) and*

$$(f(x), x)_{L^2(\mathbb{R})} = 0 \tag{4.4}$$

hold.

Note that for the lower values of the power of the negative second derivative operator $0 < s < \dfrac{1}{4}$ under the conditions stated above no orthogonality relations are required to solve the linear Poisson type equation (4.1) in $H^{2s}(\mathbb{R})$.

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