On the smoothness of the conjugacy between circle maps with a break

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Abstract

For any $\alpha \in (0,1)$, $c \in \mathbb{R}_+ \backslash \{1\}$ and $\gamma > 0$, and Lebesgue almost all irrational $\rho \in (0,1)$, any two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break, with the same rotation number ρ and the same size of the breaks c, are conjugate to each other, via a C^1-smooth conjugacy whose derivative is uniformly continuous with modulus of continuity $\omega(x) = A|\log x|^{-\gamma}$, for some $A > 0$.

1 Introduction and statement of the result

The question of the smoothness of the conjugacy of circle diffeomorphisms to a rotation is a classic problem in dynamical systems. Arnol'd [1] proved that every analytic circle diffeomorphism with a Diophantine rotation number ρ, sufficiently close to the rigid rotation $R_\rho : x \mapsto x + \rho \mod 1$, is analytically conjugate to R_ρ. Arnol'd obtained this local rigidity result using the perturbative methods of KAM (Kolmogorov-Arnol'd-Moser) theory and conjectured that the result should also hold without the assumption of the closeness to the rotation. This global rigidity result was proven by Herman [3], almost two decades later, and is the subject of Herman's theory on the linearization of circle diffeomorphisms. Herman also proved that a C^∞-smooth circle diffeomorphism with a Diophantine rotation number ρ is C^∞-smoothly conjugate to R_ρ. The theory was further developed by Yoccoz [13] (see also [4]) who established, for C^r-smooth diffeomorphisms,

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the dependence of the smoothness of the conjugacy on the Diophantine properties of the rotation numbers. In a recent formulation [9], $C^{2+\alpha}$-smooth circle diffeomorphisms with a Diophantine rotation number ρ of class $D(\beta)$, with $0 \leq \beta < \alpha \leq 1$, are $C^{1+\alpha-\beta}$-smoothly conjugate to the rotation R_ρ. A number ρ is called Diophantine of class $D(\beta)$ if there exists $\mathtt{C} > 0$ and $\beta \geq 0$ such that $|\rho - p/q| \geq \mathtt{C}/q^{2+\beta}$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Since, for $\beta > 0$, the set of Diophantine numbers of class $D(\beta)$ is of full Lebesgue measure, $C^{1+\varepsilon}$-rigidity of circle diffeomorphisms, for some $\varepsilon > 0$, holds for almost all irrational rotation numbers. We use the term *rigidity* for the phenomenon that within a given class of maps that are topologically conjugate to each other, the conjugacy between any two maps is, in fact, smooth. Arnol'd proved that the latter rigidity result for circle diffeomorphisms cannot be extended to all irrational rotation numbers by constructing an example of an analytic circle diffeomorphism with an irrational rotation number for which the invariant measure is singular, which implies that the conjugacy to the corresponding rotation, with the same rotation number, is not absolutely continuous.

A natural question to ask is what are the rigidity properties of circle diffeomorphisms with a single singular point where the diffeomorphism condition is violated. Over the last 25 years, significant progress has been made in understanding the rigidity properties of circle diffeomorphisms with a singular point where the derivative vanishes (*critical circle maps*) or has a jump discontinuity (*circle diffeomorphisms or maps with a break*). Understanding the rigidity properties of circle maps with breaks is also of independent interest due to their relationship to generalized interval exchange transformations [2, 11]. These transformations were introduced recently by Marmi, Moussa and Yoccoz [11] and are obtained by replacing the linear branches with slope 1 of an interval exchange transformation, which cuts an interval in several pieces and permutes them, by smooth diffeomorphisms. A cyclic generalized interval exchange transformation of n intervals, i.e., a generalized interval exchange transformation with a cyclic permutation $\sigma : i \mapsto i + k \mod n$, of the indices $i = 1, \ldots, n$ of the intervals, is just a circle map with finitely many break points. Rigidity results for circle maps with breaks are, therefore, also relevant to generalized interval exchange transformations [8].

Recently, we proved a series of results that are at the core of rigidity theory for circle diffeomorphisms with a break [5–7, 10]. To state the results precisely, a C^r-smooth circle diffeomorphism with a break is a map $T : \mathbb{S}^1 \to \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, for which there exists $x_{\mathrm{br}} \in \mathbb{S}^1$ such that: (i) $T \in C^r(\mathbb{S}^1 \backslash \{x_{\mathrm{br}}\})$, (ii) T' is bounded away from zero and positive on $\mathbb{S}^1$, (iii) the one sided derivatives of T at x_{br} are not equal to each other, so that the size of the break

$$c := \sqrt{\frac{T'_-(x_{\mathrm{br}})}{T'_+(x_{\mathrm{br}})}} \neq 1. \tag{1.1}$$

The main technical tool to prove rigidity results for circle maps is *renormalization*. To define renormalizations of a circle map T, which is an orientation-preserving homeomorphisms of $\mathbb{S}^1$, we start with a marked point $x_0 \in \mathbb{S}^1$, and consider the marked orbit $x_i = T^i x_0$, with $i \in \mathbb{N}$. As shown by Poincaré, for every such map T, there is a unique rotation number $\rho \in [0,1)$, defined by $\rho := \lim_{n\to\infty} \mathcal{T}^n(x)/n \mod 1$, where $\mathcal{T}$ is any lift of T to $\mathbb{R}$. We consider orientation-preserving circle homeomorphisms with an irrational rotation number ρ, characterized by the continued fraction expansion

$$\rho = [k_1, k_2, k_3, \dots] := \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}} \in (0,1), \tag{1.2}$$

with an infinite sequence of *partial quotients* k_n, $n \in \mathbb{N}$, and an infinite sequence of *rational convergents* $p_n/q_n = [k_1, \dots, k_n]$. The denominators q_n can also be defined recursively via $q_n = k_n q_{n-1} + q_{n-2}$, with $q_0 = 1$ and $q_{-1} = 0$. The n-th renormalization of T, for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, is the map $f_n : [-1, 0] \to \mathbb{R}$, given by

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}, \tag{1.3}$$

where τ_n is the affine map from $[x_{q_{n-1}}, x_0]$ to $[-1, 0]$, for n even, and from $[x_0, x_{q_{n-1}}]$ to $[-1, 0]$, for n odd, that maps x_0 into 0 and $x_{q_{n-1}}$ into -1. Here, $x_{q_{-1}}$ is regarded as $x_0 - 1$.

Rigidity results for circle diffeomorphisms with a break rely on the following theorem that establishes the convergence of renormalizations for such maps.

Theorem 1 ([6]) *Let $\alpha \in (0,1)$ and let $c \in \mathbb{R}^+ \backslash \{1\}$. There exists $\lambda \in (0,1)$ such that, for every two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break T and $\widetilde{T}$, with the same irrational rotation number $\rho \in (0,1)$ and the same size of the break c, there exists $C > 0$ such that the renormalizations f_n and $\widetilde{f}_n$ of T and $\widetilde{T}$, respectively, satisfy $\|f_n - \widetilde{f}_n\|_{C^2} \leq C\lambda^n$, for all $n \in \mathbb{N}_0$.*

In [7], we have proven that this result implies C^1-rigidity of circle maps with breaks for almost all irrational rotation numbers. The main result of this paper is the following theorem that strengthens the main result of [6, 7].

Theorem 2 *Let $\alpha \in (0,1)$ and $c \in \mathbb{R}_+ \backslash \{1\}$ and $\gamma > 0$. There exist $A > 0$ and a set $S \subset (0,1)\backslash\mathbb{Q}$ of full Lebesgue measure such that the following holds. Any two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break T and $\widetilde{T}$, with the same size of the break c and the same rotation number $\rho \in S$, are C^1-smoothly conjugate to each other, i.e., there exists a C^1-smooth diffeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\varphi \circ T \circ \varphi^{-1} = \widetilde{T}$. The derivative φ' of the conjugacy φ is uniformly continuous with modulus of continuity $\omega(x) = A|\log x|^{-\gamma}$, i.e., it satisfies $|\varphi'(x) - \varphi'(y)| \leq \omega(|x - y|)$, for any $x, y \in \mathbb{S}^1$, such that $x \neq y$.*

Remark 1 The first part of the claim — the existence of the C^1-smooth conjugacy, for a set of full Lebesgue measure rotation numbers — has already been proven in [6, 7]. Here, we prove that, for any $\gamma > 0$ and for a subset S (depending on γ) of full Lebesgue measure,

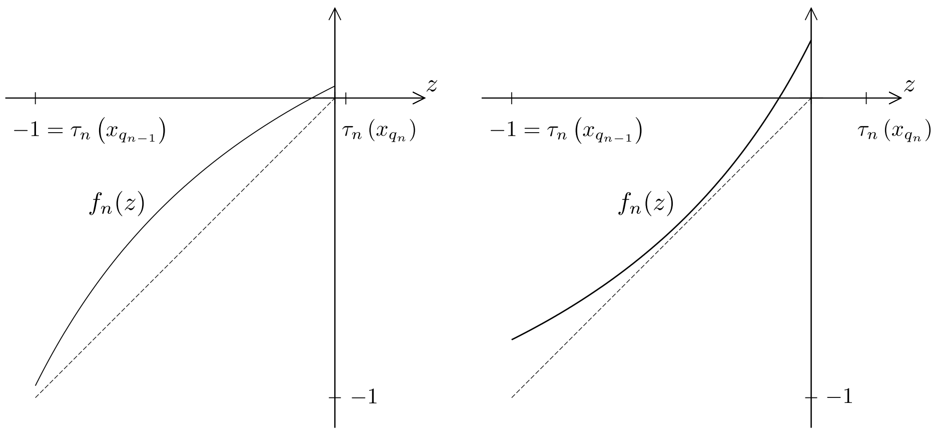

Figure 1: The graph of the n-th renormalization f_n of a circle map with a break: Case $0 < c < 1$ and n even, or $c > 1$ and n odd (left) and case $0 < c < 1$ and n odd, or $c > 1$ and n even (right).

the derivative of the conjugacy is uniformly continuous with modulus of continuity ω. We will show, in the proof of Theorem 2, that there exists $\lambda_1 \in (\lambda, 1)$ (depending on γ) such that the claim holds with S defined as follows. For $C_1 > 0$, let $S_1(C_1, \lambda_1)$ be the set of all irrational rotation numbers $\rho = [k_1, k_2, \dots] \in (0, 1)$ whose subsequence of partial quotients k_{n+1} for all n even satisfies the bound $k_{n+1} \leq C_1 \lambda_1^{-n}$, while the subsequence of partial quotients k_{n+1} for all n odd satisfies the bound $k_{n+1} \leq e^{C_1 \lambda_1^{-n}}$. Let $S_2(C_1, \lambda_1)$ be the set of all irrational rotation numbers $\rho = [k_1, k_2, \dots] \in (0, 1)$ whose subsequence of partial quotients k_{n+1} for all n odd satisfies the bound $k_{n+1} \leq C_1 \lambda_1^{-n}$, while the subsequence of partial quotients k_{n+1} for all n even satisfies the bound $k_{n+1} \leq e^{C_1 \lambda_1^{-n}}$. We define $S(C_1, \lambda_1) := S_1(C_1, \lambda_1)$, if $0 < c < 1$, and $S(C_1, \lambda_1) := S_2(C_1, \lambda_1)$, if $c > 1$. Finally, we define $S = S(\lambda_1) := \cup_{C_1 > 0} S(C_1, \lambda_1)$.

The distinction between the cases of even and odd n, in Remark 1, is related to the difference in the behavior of renormalizations. It was shown in [6], that, for sufficiently large n, renormalizations have a definite convexity: if $0 < c < 1$, f_n are concave if n is even and convex if n is odd; if $c > 1$, f_n are convex if n is even and concave if n is odd (Figure 1).

Remark 2 The claim of Theorem 2 cannot be extended to all irrational numbers [5]. In [5], we constructed examples of pairs of analytic circle diffeomorphisms with breaks, with the same irrational rotation number ρ and the same size of the break c, for which any conjugacy between them is not even Lipschitz continuous.

Remark 3 Theorem 2 cannot be strengthened by taking $\omega(x) = |x|^\varepsilon$, for any $\varepsilon > 0$ [10].

In fact, it was shown in [10] that there is a set of full Lebesgue measure irrational numbers S_{nr} such that, for any $r > 2$ and any $\rho \in S_{\mathrm{nr}}$, there is a pair of C^r-smooth circle diffeomorphisms with a break of size c, with the same rotation number ρ, which are not $C^{1+\varepsilon}$-smoothly conjugate to each other, for any $\varepsilon > 0$.

2 Proof of the result

A crucial role in the proof of this result is played by the geometry of dynamical partitions of the circle. These partitions are constructed from an orbit of a marked point $x_0 \in \mathbb{S}^1$. The n-th dynamical partition of a circle, associate to x_0, is given by

$$\mathcal{P}_n := \{T^i(\Delta_0^{(n-1)}) : 0 \leq i < q_n\} \cup \{T^i(\Delta_0^{(n)}) : 0 \leq i < q_{n-1}\}, \quad n \in \mathbb{N} \tag{2.1}$$

where $\Delta_0^{(n)} := [x_{q_n}, x_0]$, for n odd, and $\Delta_0^{(n)} := [x_0, x_{q_n}]$, for n even. We also define $\Delta_0^{(-1)} := [x_0 - 1, x_0]$ and $\mathcal{P}_0 := \{\Delta_0^{(-1)}\}$. These partitions are nested, i.e., intervals of partition $\mathcal{P}_{n+1}$ are subsets of the intervals of partition $\mathcal{P}_n$. The set of end points of the intervals of dynamical partition $\mathcal{P}_n$ will be denoted by Ξ_n. We denote by $\Delta_i^{(n)} := T^i(\Delta_0^{(n)})$.

Below, we consider circle maps with a break T and $\widetilde{T}$ which are $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, outside the break point $x_0 = x_{\mathrm{br}}$ and satisfy conditions (ii) and (iii). In particular, for these maps, there are constants $K_1, K_2 > 0$ such that $T'(x) \leq K_1, |T''(x)| \leq K_1$ and $T'(x) \geq K_2$, for all $x \in \mathbb{S}^1$ (at $x = x_0$, both left and right derivatives satisfy these inequalities). Since $V := \mathrm{Var}_{x\in\mathbb{S}^1} \ln T'(x) < \infty$, one can show that

(A) $|\ln(T^{q_n})'(x)| \leq V$,

(B) $|I_n| \leq K\bar{\lambda}^n$,

for any $I_n \in \mathcal{P}_n$, where $K > 0$ and $\bar{\lambda} = (1 + e^{-V})^{-1/2}$. In this paper, the absolute value sign $|\cdot|$ also denotes the length of an interval. Estimates (A) and (B) were proven in [12]. Estimate (A) is the Denjoy lemma. In particular, it implies property (B).

The following proposition contains some important estimates on the geometry of the iterates of the renormalizations of circle maps with a break that were proven in [6,7]. Let $c_n = c^{(-1)^n}$.

Proposition 3 ([6, 7]) *Let T be a $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphism with a break of size $c \in \mathbb{R}^+\backslash\{1\}$, with rotation number $\rho \in (0,1)\backslash\mathbb{Q}$. There exist $\epsilon > 0$ and $\varkappa \in (0,1)$ such that, for all $n \in \mathbb{N}_0$ and for $0 \leq j < k_{n+1}$,*

$$\begin{aligned} &|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \geq \epsilon\varkappa^{k_{n+1}}, \quad |\tau_n(\Delta_0^{(n+1)})| \geq \epsilon\varkappa^{k_{n+1}} \qquad (c_n < 1), \\ &|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \geq \epsilon k_{n+1}^{-2}, \quad |\tau_n(\Delta_0^{(n+1)})| \geq \epsilon\varkappa^{k_{n+2}} \qquad (c_n > 1). \end{aligned} \tag{2.2}$$

Notice that $|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| = f_n^{j+1}(-1) - f_n^j(-1)$. The difference between the cases of $c_n < 1$ and $c_n > 1$ corresponds to the difference in the behavior of renormalizations, explained in the previous section. For sufficiently large n, if $c_n < 1$, the renormalizations are concave and the iterates $f_n^j(-1)$ approach geometrically fast the end points of the interval $[-1, 0]$. If $c_n > 1$, the renormalizations are convex and the iterates $f_n^j(-1)$ approach at a slower rate the point of a parabolic almost tangency (where the derivative of f_n equals 1), somewhere in the interior of the interval $[-1, 0]$ (Figure 1).

The next proposition provides the central estimates on the geometry of dynamical partitions that will be used in the proof of our main result. These estimates are obtained by spreading the estimates that follow from Proposition 3 to the whole circle.

Proposition 4 *Let T be a $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphism with a break of size $c \in \mathbb{R}^+\backslash\{1\}$ and rotation number $\rho \in (0,1)\backslash\mathbb{Q}$. There exist $\epsilon_1 > 0$ and $\varkappa \in (0,1)$ such that, for all $n \in \mathbb{N}_0$, for every element I_{n+1} of partition $\mathcal{P}_{n+1}$ and the corresponding element $I_n \in \mathcal{P}_n$ satisfying $I_{n+1} \subset I_n$, we have*

$$
\begin{aligned}
&|I_{n+1}| \geq \epsilon_1 \varkappa^{k_{n+1}} |I_n| && (c_n < 1),\\
&|I_{n+1}| \geq \epsilon_1 k_{n+1}^{-2} |I_n| && (I_{n+1} \notin \mathcal{P}_{n+2}\ , c_n > 1),\\
&|I_{n+1}| \geq \epsilon_1 \varkappa^{k_{n+2}} |I_n| && (I_{n+1} = I_{n+2} \in \mathcal{P}_{n+2}\ , c_n > 1).
\end{aligned}
\tag{2.3}
$$

Proof. It suffices to prove the claim for sufficiently large n. Every interval $I_{n+1} \in \mathcal{P}_{n+1}$ is either $\Delta_i^{(n)}$, for some $0 \le i < q_{n+1}$, or it is $\Delta_l^{(n+1)}$, for some $0 \le l < q_n$. If $I_{n+1} = \Delta_i^{(n)}$, for some $0 \le i < q_{n-1}$, then $I_{n+1} \in \mathcal{P}_n$ and, thus, $I_n = I_{n+1}$. If $I_{n+1} = \Delta_i^{(n)}$ and $q_{n-1} \le i < q_{n+1}$, then $i = q_{n-1} + jq_n + l$, where $0 \le j < k_{n+1}$ and $0 \le l < q_n$. Thus, in this case, $I_{n+1} = T^l(\Delta^{(n)}_{q_{n-1}+jq_n}) \subset \Delta_l^{(n-1)} \in \mathcal{P}_n$. Similarly, if $I_{n+1} = \Delta_l^{(n+1)}$, for $0 \le l < q_n$, then $I_{n+1} = T^l(\Delta_0^{(n+1)}) \subset \Delta_l^{(n-1)} \in \mathcal{P}_n$. Therefore, in either case, there is an interval $\bar{I}_{n+1} \subset \Delta_0^{(n-1)}$, such that $I_{n+1} = T^l(\bar{I}_{n+1})$.

Since T is C^2-smooth outside the break point, we have

$$
\begin{aligned}
\frac{|I_{n+1}|}{|I_n|} &= \frac{|T^l(\bar{I}_{n+1})|}{|T^l(\Delta_0^{(n-1)})|} = \frac{|\bar{I}_{n+1}|}{|\Delta_0^{(n-1)}|} \prod_{i=0}^{l-1} \frac{T'(\zeta_i)}{T'(\xi_i)}\\
&\geq \frac{|\bar{I}_{n+1}|}{|\Delta_0^{(n-1)}|} \prod_{i=0}^{l-1} \left(1 - \frac{\max_{x\in\mathbb{S}^1} |T''(x)|}{\min_{x\in\mathbb{S}^1} |T'(x)|} |\Delta_i^{(n-1)}|\right)\\
&\geq \frac{|\bar{I}_{n+1}|}{|\Delta_0^{(n-1)}|} \exp\left(-2K_1K_2^{-1} \sum_{i=0}^{l-1} |\Delta_i^{(n-1)}|\right).
\end{aligned}
\tag{2.4}
$$

Here, $\zeta_i \in T^i(\bar{I}_{n+1})$ and $\xi_i \in T^i(\Delta_0^{(n-1)})$. We have used that $1 - x \ge e^{-2x}$, for $0 \le x < \frac{\ln 2}{2}$ and that $K_1K_2^{-1}|\Delta_i^{(n-1)}|$ can be made arbitrarily small, if n is sufficiently large, since the length of the longest element of partition $\mathcal{P}_n$ decreases exponentially with n (see property (B)). The claim now follows from Proposition 3, taking into account that $|\tau_n(I)| = |I|/|\Delta_0^{(n-1)}|$ and the fact that $\Delta_0^{(n+1)}$ is the only interval of partition $\mathcal{P}_{n+1}$ inside $\Delta_0^{(n-1)} \in \mathcal{P}_n$, which also belongs to $\mathcal{P}_{n+2}$. **QED**

The length of the shortest element of dynamical partition $\mathcal{P}_n$ can decrease arbitrarily fast with n. The next proposition shows that, if the rotation number $\rho \in S$, this speed is at most super-exponential.

Proposition 5 *Let T be a $C^{2+\alpha}$-smooth circle map with a break of size $c \in \mathbb{R}^+\backslash\{1\}$ and rotation number $\rho \in S$. There exists $C_2 > 0$ such that, for all $I_n \in \mathcal{P}_n$, we have $|\ln |I_n|| \leq C_2\lambda_1^{-n}$, for all $n \in \mathbb{N}_0$.*

Proof. It suffices to prove the claim for sufficiently large n. Since, for $\rho \in S$, there exists $C_1 > 0$ such that, $k_{m+1} \leq C_1\lambda_1^{-m}$, if $c_m < 1$ and $k_{m+1} \leq e^{C_1\lambda_1^{-m}}$, if $c_m > 1$, it follows from Proposition 4 that, in this case,

$$|I_n| \geq \prod_{\substack{m=0 \\ c_m<1}}^{n} \left(\epsilon_1 \varkappa^{k_{m+1}}\right) \prod_{\substack{m=0 \\ c_m>1}}^{n} \left(\frac{\epsilon_1}{k_{m+1}^2}\right) \geq \epsilon_1^{n+1} \prod_{m=0}^{n} \varkappa^{C_3\lambda_1^{-m}}, \tag{2.5}$$

where $C_3 = C_1 \max\left\{1, \frac{2}{\ln \varkappa^{-1}}\right\}$. Therefore, we obtain

$$\begin{aligned} |\ln |I_n|| &\leq (n+1)|\ln \epsilon_1| + |\ln \varkappa| \sum_{m=0}^{n} C_3\lambda_1^{-m} \\ &\leq (n+1)|\ln \epsilon_1| + |\ln \varkappa| \frac{C_3\lambda_1^{-n}}{1-\lambda_1}. \end{aligned} \tag{2.6}$$

The claim follows with $C_2 \geq 2C_3|\ln \varkappa|/(1-\lambda_1)$, if n is large enough. **QED**

Let T and $\widetilde{T}$ be two $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphisms with breaks x_0 and $\widetilde{x}_0$, respectively, with the same irrational rotation number $\rho \in S$ and the same size of the break $c \in \mathbb{R}_+\backslash\{1\}$, and let φ be the conjugacy between them satisfying $\varphi \circ T \circ \varphi^{-1} = \widetilde{T}$ and $\varphi(x_0) = \widetilde{x}_0$. For any interval $I \subset \mathbb{S}^1$, let $\widetilde{I} = \varphi(I)$ and let

$$\sigma(I) := \frac{|\varphi(I)|}{|I|}. \tag{2.7}$$

The following estimate on the ratios of lengths of the corresponding intervals of dynamical partitions $\mathcal{P}_n$ and $\widetilde{\mathcal{P}}_n$ plays a crucial role in our analysis.

Proposition 6 *There exist constants $\mathcal{C} > 0$ and $\mu \in (0,1)$ such that, for any two intervals $I, I' \in \mathcal{P}_n$, which are either adjacent to each other or $I, I' \subset J$ for some $J \in \mathcal{P}_{n-1}$, we have*

$$|\sigma(I) - \sigma(I')| \leq \mathcal{C}\mu^n, \tag{2.8}$$

for all $n \in \mathbb{N}$.

Proof. Using the main result of [6], in [7], we proved that, for $\rho \in S$, there exist constants $\mathcal{C}_0 > 0$ and $\mu \in (0,1)$ such that, for any two intervals $I, I' \in \mathcal{P}_n$, as in the statement of this proposition, we have $|\ln \sigma(I) - \ln \sigma(I')| \leq \mathcal{C}_0 \mu^n$, for all $n \in \mathbb{N}$. Since, for $\rho \in S$, the conjugacy φ is C^1-smooth (see Proposition 3.1 — the Criterion of smoothness — in [7]), estimate (2.8) follows immediately. **QED**

As a corollary, we obtain the following.

Proposition 7 *There exist constants $\mathcal{C} > 0$ and $\mu \in (0,1)$ such that the following holds. Let $\{I_n\}_{n\in\mathbb{N}}$ be any sequence of nested intervals $I_n \in \mathcal{P}_n$ of dynamical partitions $\mathcal{P}_n$ of T, satisfying $I_n \subset I_{n-1}$. Then,*

$$|\sigma(I_n) - \sigma(I_{n-1})| \leq \mathcal{C}\mu^n. \tag{2.9}$$

Proof. Each interval I_{n-1} of partition $\mathcal{P}_{n-1}$ is a union $I_{n-1} = \cup_{I'_n \subset I_{n-1}} I'_n$ of non-overlapping, except at the end points, intervals I'_n of partition $\mathcal{P}_n$ and, thus, we have $|I_{n-1}| = \sum_{I'_n \subset I_{n-1}} |I'_n|$. Denote $\widetilde{I}_n = \varphi(I_n)$. It follows directly from (2.8) that

$$(\sigma(I_n) - \mathcal{C}\mu^n)\,|I'_n| \leq |\widetilde{I}'_n| \leq (\sigma(I_n) + \mathcal{C}\mu^n)\,|I'_n|. \tag{2.10}$$

By summing up the inequalities (2.10) over all $I'_n \subset I_{n-1}$, the claim follows. **QED**

Proposition 8 *For any two different points $x, y \in \mathbb{S}^1$, there exist $m \in \mathbb{N}_0$, intervals $I_m, I'_m \in \mathcal{P}_m$ and an interval $I_{m+1} \in \mathcal{P}_{m+1}$, such that the following holds*

(i) $x \in I_m$, $y \in I'_m$ *and* $I_m \cap I'_m \neq \emptyset$,

(ii) $I_{m+1} \subset [x, y]$.

Here, $[x, y]$ denotes the shorter interval between two points on the circle.

Remark 4 Notice that the condition (i) implies that x and y either belong to the same element $I_m = I'_m$ or to the neighboring elements I_m and I'_m, respectively, of partition $\mathcal{P}_m$.

Proof. For sufficiently large n, there is an interval I_n of partition $\mathcal{P}_n$, such that $I_n \subset [x, y]$. This follows from the fact that the length of the maximal interval of dynamical partition $\mathcal{P}_n$ approaches zero, as $n \to \infty$ (see property (B)). This implies that the end points of

I_n, denoted by η_n^- and η_n^+, belong to $[x,y]$ as well. Hence, for sufficiently large n, at least two points in Ξ_n belong to $[x,y]$. Let m be the smallest number $n \in \mathbb{N}_0$ such that $[x,y]$ contains at most one point in Ξ_n. Such a number $m \in \mathbb{N}_0$ always exists since $\Xi_0 = \{x_0\}$. If $[x,y]$ contains exactly one point ξ_m of Ξ_m in its interior, then there are two elements of partition $\mathcal{P}_m$, I_m and I'_m, such that $[x,y] \subset I_m \cup I'_m$ and $\xi_m \in I_m \cap I'_m$. If $[x,y]$ contains no points of Ξ_m in its interior, then there is an interval I_m of $\mathcal{P}_m$ such that $[x,y] \subset I_m$. Therefore, condition (i) holds. By definition, $[x,y]$ contains at least two points in Ξ_{m+1}. Hence, there is an interval I_{m+1} of partition $\mathcal{P}_{m+1}$, such that $I_{m+1} \subset [x,y]$. Thus, condition (ii) holds as well. **QED**

Proof of Theorem 2. The first part of the claim, i.e., that, for $\rho \in S$, φ is C^1-smooth was proven in [6, 7]. We will now establish the regularity of the derivative φ'.

For any two different points $x, y \in \mathbb{S}^1$, by Proposition 8, there exist $m \in \mathbb{N}_0$, and intervals $I_m, I'_m \in \mathcal{P}_m$ and $J_{m+1} \in \mathcal{P}_{m+1}$ such that x and y either belong to the same element $I_m = I'_m$ or to the neighboring elements I_m and I'_m, respectively, of partition $\mathcal{P}_m$, and $J_{m+1} \subset [x,y]$. Furthermore, there are nested sequences $\{I_n\}_{n\in\mathbb{N}}, \{I'_n\}_{n\in\mathbb{N}}$ of elements I_n, I'_n of partitions $\mathcal{P}_n$, satisfying $I_n \subset I_{n-1}$ and $I'_n \subset I'_{n-1}$, such that $x = \cap_{n\in\mathbb{N}} I_n$ and $y = \cap_{n\in\mathbb{N}} I'_n$. Since $\rho \in S$ and φ is C^1-smooth, using Proposition 6 and Proposition 7, we obtain

$$\begin{aligned}
|\varphi'(x) - \varphi'(y)| &= \left|\lim_{n\to\infty} \sigma(I_n) - \lim_{n\to\infty} \sigma(I'_n)\right| = \lim_{n\to\infty} |\sigma(I_n) - \sigma(I'_n)| \\
&\le \lim_{n\to\infty} \Big(|\sigma(I_m) - \sigma(I'_m)| + \sum_{i=m}^{n-1} |\sigma(I_{i+1}) - \sigma(I_i)| \\
&\quad + \sum_{i=m}^{n-1} |\sigma(I'_{i+1}) - \sigma(I'_i)| \Big) \\
&\le \mathcal{C}\mu^m + 2\sum_{i=m}^{\infty} \mathcal{C}\mu^{i+1} = \frac{1+\mu}{1-\mu}\mathcal{C}\mu^m
\end{aligned} \tag{2.11}$$

It follows from Proposition 5 that

$$|\ln|x-y|| \le |\ln|J_{m+1}|| \le C_2 \lambda_1^{-(m+1)}. \tag{2.12}$$

For any $\gamma > 0$, if $\lambda_1 \in (\lambda, 1)$ is chosen such that $\lambda_1 > \mu^{\frac{1}{\gamma}}$, then, from (2.11) and (2.12), we obtain

$$|\varphi'(x) - \varphi'(y)| \le C_4 \mu^m \le A C_2^{-\gamma} \lambda_1^{\gamma(m+1)} \le A|\ln|x-y||^{-\gamma}, \tag{2.13}$$

where $C_4 = \frac{1+\mu}{1-\mu}\mathcal{C}$, provided that $A > 0$ is sufficiently large. The claim follows. **QED**

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