Inverse problems for some systems of parabolic equations

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Abstract: We study the system $\vec{u}_t - A\vec{u}_{xx} = \vec{h}(t)$, where $0 \le x \le \pi,\ t \ge 0$, assuming that $\vec{u}(0,t) = \vec{v}(t),\ \vec{u}(\pi,t) = \vec{0}$, and $\vec{u}(x,0) = \vec{g}(x)$. The coupling matrix A is a real, diagonalizable matrix for which all of the eigenvalues are positive reals. The question is: *What extra data determine the three unknown vector functions* $\{\vec{h}, \vec{v}, \vec{g}\}$ *uniquely?* This problem is solved and an analytical method for the recovery of the above three vector functions is presented.

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1. Introduction

Consider the system

$$\begin{aligned} &\vec{u}_t - A\vec{u}_{xx} = \vec{h}(t) \quad \text{for } (x,t) \in [0,\pi] \times [0,\infty), \\ \text{with} \quad &\vec{u}(0,t) = \vec{v}(t), \qquad \vec{u}(\pi,t) = \vec{0}, \qquad \text{and} \qquad \vec{u}(x,0) = \vec{g}(x), \end{aligned} \tag{1.1}$$

where $\vec{h}, \vec{v} : \mathbb{R}_{\ge 0} \to \mathbb{R}^N$ and $\vec{g} : [0,\pi] \to \mathbb{R}^N$ for some $N \ge 2$ are unknown, with

$$\vec{g}(x) = (g_1(x), g_2(x), ..., g_N(x))^T.$$

The solution of system (1.1) is a real vector function given by

$$\vec{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The regularity of $\vec{u}$ is related to the smoothness of $\{\vec{h}, \vec{v}, \vec{g}\}$. Similar to the work in [4], which was devoted to the studies of the single parabolic equation of this kind, in the present work we are not focused on the well-posedness of (1.1). We are interested in the following inverse problem:

What information about the solution $\vec{u}$ is sufficient
to uniquely determine the vector functions $\{\vec{h}, \vec{v}, \vec{g}\}$?

Inverse problems for the scalar heat equation have been studied extensively (see [1], [2], [4] and the references therein). An inverse source problem for the multidimensional heat equation in which the source was assumed to be a finite sum of point sources was considered in [3]. The inverse problem there was to find the location and the intensity of these point sources from the experimental data. The existence of stationary solutions of certain systems of parabolic equations was studied actively in recent years, see for instance [5] and [6] and the references therein.

We will use $\langle\ ,\ \rangle$ to denote the standard inner product on $L^2[0,\pi]$. That is,

$$\langle G, F\rangle = \int_0^\pi G(x)F(x)dx. \tag{1.2}$$

Clearly, (1.2) induces the following norm on $L^2[0,\pi]$:

$$\|F\| = \sqrt{\int_0^\pi F^2(x)dx}.$$

We extend the inner product notation to the situation where the first argument is a vector function, for which each component is an element of $L^2[0,\pi]$. In this case the result is obtained by computing the inner product of each component with the second argument. For example,

$$\begin{aligned}\langle \vec{g}, F\rangle &= \Big(\langle g_1, F\rangle, \dots, \langle g_N, F\rangle\Big)^T \\ &= \left(\int_0^\pi g_1(x)F(x)dx, \dots, \int_0^\pi g_N(x)F(x)dx\right)^T \\ &= \int_0^\pi \vec{g}(x)F(x)dx.\end{aligned} \tag{1.3}$$

Similarly,

$$\begin{aligned}\langle \vec{u}(\cdot,t), F\rangle &= \int_0^\pi \vec{u}(x,t)F(x)dx \\ &= \left(\int_0^\pi u_1(x,t)F(x)dx, \dots, \int_0^\pi u_N(x,t)F(x)dx\right)^T,\end{aligned}$$

giving a vector valued function of t.

Let $f_m(x) = \sqrt{\dfrac{2}{\pi}}\sin(mx)$ for $m \in \mathbb{N} = \{1, 2, \dots\}$. Then

$$f_m(0) = f_m(\pi) = 0, \quad \|f_m\| = 1 \quad \text{and} \quad -\frac{d^2 f_m}{dx^2}(x) = m^2 f_m(x) \ \text{ for } \ 0 \le x \le \pi,$$

so that $\{f_m(x)\}_{m=1}^{\infty}$ is the orthonormal set of the eigenfunctions of the one dimensional negative Dirichlet Laplacian on the interval $[0, \pi]$.

Let $y \in (0, \pi)$ such that $\dfrac{y}{\pi}$ is irrational. (This happens, for example, if y is rational.) Then it can be shown that

$$f_m(y) \neq 0 \tag{1.4}$$

for all $m \in \mathbb{N}$.

Let

$$\vec{u}_m(t) = \langle \vec{u}(\cdot, t), f_m \rangle$$

for $m \in \mathbb{N}$. Our main statement is as follows.

Theorem 1. *Suppose $N \ge 2$ and A is a constant real $N \times N$ diagonalizable matrix for which all of the eigenvalues are positive reals. Then knowing the functions*

$$\{\vec{u}_1(t), \vec{u}_3(t), \vec{u}(y, t)\}, \tag{1.5}$$

for all $t \ge 0$, is sufficient to uniquely determine the triple $\left\{\vec{h}, \vec{v}, \vec{g}\right\}$.

This theorem is a generalization of Theorem 1 of [4], which establishes the corresponding result for a single heat equation (i.e. for $N = 1$). Let us proceed to the proof of our main result.

2. Proof.

Proof of Theorem 1. From our assumptions, it follows that there exists an invertible real matrix P such that

$$PAP^{-1} = D = \text{diag}\,(d_1, \dots, d_N),$$

where $d_1, d_2, \dots, d_N > 0$ are the eigenvalues of A and, hence,

$$PA = DP. \tag{2.1}$$

By means of (2.1), multiplying the partial differential equation in (1.1) on the left by P gives

$$P\vec{u}_t - DP\vec{u}_{xx} = P\vec{h}(t). \tag{2.2}$$

Let us introduce new vector functions:

$$\tilde{u}(x,t) := P\vec{u}(x,t) \quad \text{and} \quad \tilde{h}(t) := P\vec{h}(t).$$

This allows us to write (2.2) (which is simply the PDE portion of the main system (1.1)) in terms of $\tilde{u}$ and $\tilde{h}$. Before doing so, we define

$$\tilde{v}(t) := P\vec{v}(t) \quad \text{and} \quad \tilde{g}(x) := P\vec{g}(x).$$

Now we can write the system in terms of the new variables:

$$\begin{aligned}&\frac{\partial \tilde{u}}{\partial t} - D\frac{\partial^2 \tilde{u}}{\partial x^2} = \tilde{h}(t),\\ \text{with} \quad &\tilde{u}(0,t) = \tilde{v}(t), \qquad \tilde{u}(\pi,t) = \vec{0} \quad \text{and} \quad \tilde{u}(x,0) = \tilde{g}(x).\end{aligned} \tag{2.3}$$

The reason that we have done this is that (2.3) consists of N fully decoupled scalar equations, allowing for solutions to be more easily obtained.

For $m \in \mathbb{N} = \{1, 2, \dots\}$ let

$$\tilde{u}_m(t) = \langle \tilde{u}(\cdot,t), f_m\rangle \quad \text{and} \quad \tilde{g}_m = \langle \tilde{g}, f_m\rangle \in \mathbb{R}^N,$$

where the inner product is defined in (1.3). It follows that

$$\tilde{g}(x) = \sum_{m=1}^{\infty} \tilde{g}_m f_m(x). \tag{2.4}$$

We look for the solution to (2.3) in the form

$$\tilde{u}(x,t) = \sum_{m=1}^{\infty} \tilde{u}_m(t) f_m(x) = \sum_{m=1}^{\infty} \Big\langle \tilde{u}(\cdot,t), f_m \Big\rangle f_m(x). \tag{2.5}$$

It is a standard result that such a solution exists. Taking the inner product of f_m with each side of the system of partial differential equations in (2.3) yields

$$\left\langle \frac{\partial \tilde{u}}{\partial t} - D\frac{\partial^2 \tilde{u}}{\partial x^2}, f_m \right\rangle = \tilde{h}(t)\langle \mathbf{1}, f_m\rangle, \tag{2.6}$$

where $\mathbf{1}(x) = 1$ for all $x \in [0,\pi]$. Letting

$$c_m = \langle \mathbf{1}, f_m\rangle = \int_0^{\pi} f_m(x)dx = \begin{cases} \sqrt{\frac{2}{\pi}}\frac{2}{m} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even,}\end{cases} \tag{2.7}$$

we rewrite (2.6) as

$$\int_0^{\pi} \frac{\partial \tilde{u}}{\partial t}(x,t) f_m(x)dx - \int_0^{\pi} D\frac{\partial^2 \tilde{u}}{\partial x^2}(x,t) f_m(x)dx = \tilde{h}(t)c_m.$$

Assuming the sufficient regularity of $\tilde{u}$, the first integral gives $\dfrac{d\tilde{u}_m}{dt}$. Using integration by parts twice on the second integral, we arrive at

$$\frac{d\tilde{u}_m(t)}{dt} + Dm^2\tilde{u}_m(t) = Df_m'(0)\tilde{v}(t) + c_m\tilde{h}(t), \tag{2.8}$$

for $m \in \mathbb{N}$. Equation (2.8) decouples into N scalar linear equations of the form $y' + Ky = a(t)$, which can be easily solved. The initial condition for (2.8) is

$$\tilde{u}_m(0) = \langle \tilde{u}(\cdot, 0), f_m\rangle = \langle \tilde{g}, f_m\rangle = \tilde{g}_m. \tag{2.9}$$

Recall that for a diagonal matrix, such as $D = \text{diag}\,(d_1, \ldots, d_N)$, and a scalar m^2t, exponentiation is termwise, so that

$$\mathrm{e}^{-Dm^2t} = \text{diag}\,\left(\mathrm{e}^{-d_1m^2t}, \ldots, \mathrm{e}^{-d_Nm^2t}\right).$$

From (2.8) we calculate that

$$\tilde{u}_m(t) = \mathrm{e}^{-Dm^2t}\tilde{g}_m + \int_0^t \mathrm{e}^{-Dm^2(t-s)}\left[Df_m'(0)\tilde{v}(s) + c_m\tilde{h}(s)\right] ds. \tag{2.10}$$

We now suppose that

$$\{\vec{u}_1(t), \vec{u}_3(t), \vec{u}(y,t)\}$$

(referred to as the data) are known, and we set about constructing the unknowns $\left\{\vec{h}, \vec{v}, \vec{g}\right\}$. Let

$$\tilde{F}_1(t) := \tilde{u}_1(t) - \mathrm{e}^{-Dt}\tilde{g}_1 \quad \text{and} \quad \tilde{F}_3(t) := \tilde{u}_3(t) - \mathrm{e}^{-9Dt}\tilde{g}_3. \tag{2.11}$$

Then Equation (2.10), for $m = 1$ and $m = 3$, gives

$$\begin{aligned} \tilde{F}_1(t) &= \int_0^t \mathrm{e}^{-D(t-s)}\left[Df_1'(0)\tilde{v}(s) + c_1\tilde{h}(s)\right] ds \\ \text{and} \qquad \tilde{F}_3(t) &= \int_0^t \mathrm{e}^{-9D(t-s)}\left[Df_3'(0)\tilde{v}(s) + c_3\tilde{h}(s)\right] ds \end{aligned} \tag{2.12}$$

respectively. By differentiating the formulas in (2.12) and rearranging, we obtain

$$\begin{aligned} Df_1'(0)\tilde{v}(t) + c_1\tilde{h}(t) &= \mathrm{e}^{-Dt}\frac{d}{dt}\left[\mathrm{e}^{Dt}\tilde{F}_1(t)\right], \\ \text{and} \qquad Df_3'(0)\tilde{v}(t) + c_3\tilde{h}(t) &= \mathrm{e}^{-9Dt}\frac{d}{dt}\left[\mathrm{e}^{9Dt}\tilde{F}_3(t)\right]. \end{aligned} \tag{2.13}$$

We treat (2.13) as a $2N$-dimensional linear system with unknowns $\tilde{v}(t)$ and $\tilde{h}(t)$. Its $2N \times 2N$ coefficient matrix M has the block form

$$M = \begin{pmatrix} Df_1'(0) & c_1I \\ Df_3'(0) & c_3I \end{pmatrix},$$

where I is the $N \times N$ identity matrix and $f'_m(0) = m\sqrt{\frac{2}{\pi}}$. Since each of the four blocks in this representation of M are diagonal, an easy computation gives $\det M = \left(\frac{32}{3\pi}\right)^N \det D \neq 0$. It now follows that system (2.13) admits a unique solution, meaning that $\tilde{v}$ and $\tilde{h}$ are uniquely determined by the right-hand sides of (2.13).

Note that for $m = 1, 3$, we have

$$\begin{aligned}\tilde{u}_m(t) = \langle \tilde{u}(\cdot, t), f_m \rangle = \langle P\vec{u}(\cdot, t), f_m \rangle &= \int_0^\pi P\vec{u}(x,t) f_m(x) dx \\ &= P\int_0^\pi \vec{u}(x,t) f_m(x) dx = P\vec{u}_m(t).\end{aligned}$$

Also, from the definition of $\tilde{g}_m$ in (2.9), we have

$$\tilde{g}_m = \tilde{u}_m(0) = P\vec{u}_m(0).$$

This means that $\tilde{F}_1(t)$ and $\tilde{F}_3(t)$, as defined in (2.11), can be calculated from the data in (1.5). This, in turn, means that $\tilde{v}(t)$ and $\tilde{h}(t)$ can be computed from the data. We then calculate

$$\vec{v}(t) = P^{-1}\tilde{v}(t) \quad \text{and} \quad \vec{h}(t) = P^{-1}\tilde{h}(t).$$

Now we work towards determining $\vec{g}$ in terms of the data. By combining (2.5) and (2.10) at $x = y$, we have

$$\tilde{u}(y,t) = \sum_{m=1}^{\infty} \mathrm{e}^{-Dm^2 t} \tilde{g}_m f_m(y) + \tilde{w}(y,t), \tag{2.14}$$

where

$$\tilde{w}(y,t) = \sum_{m=1}^{\infty} f_m(y) \int_0^t \mathrm{e}^{-Dm^2(t-s)}[Df'_m(0)\tilde{v}(s) + c_m \tilde{h}(s)]ds$$

an expression written in terms of known functions (since $\tilde{v}$ and $\tilde{h}$ have already been calculated from the data). Also, noting that $\tilde{u}(y,t) = P\vec{u}(y,t)$, it is clear that the left side of (2.14) is determined by the data (1.5). Thus, the only unknowns in (2.14) are $\tilde{g}_m$ for $m \in \mathbb{N}$.

Let $\tilde{q}(y,t) = \tilde{u}(y,t) - \tilde{w}(y,t)$. Then $q(y,t)$ is known as well and, from (2.14), satisfies

$$\begin{aligned}\tilde{q}(y,t) &= \sum_{m=1}^{\infty} \mathrm{e}^{-Dm^2 t} \tilde{g}_m f_m(y) \\ &= \mathrm{e}^{-Dt}\tilde{g}_1 f_1(y) + \mathrm{e}^{-4Dt}\tilde{g}_2 f_2(y) + \mathrm{e}^{-9Dt}\tilde{g}_3 f_3(y) + \dots\end{aligned} \tag{2.15}$$

We now perform a sequence of limits. The first limit is simply $\mathrm{e}^{Dt}\tilde{q}(y,t)$ as t approaches ∞, which equals $\tilde{g}_1 f_1(y)$. That is,

$$\tilde{g}_1 f_1(y) = \lim_{t\to\infty} \left[\mathrm{e}^{Dt}\tilde{q}(y,t)\right].$$

For the second limit, we subtract the first term of the series in (2.15) to the other side and multiply by e^{4Dt}, so that the limit gives $\tilde{g}_2 f_2(y)$. That is,

$$\tilde{g}_2 f_2(y) = \lim_{t\to\infty} \mathrm{e}^{4Dt}\left[q(y,t) - \mathrm{e}^{-Dt}\tilde{g}_1 f_1(y)\right].$$

Continuing in this fashion, we are able to calculate $\tilde{g}_m f_m(y)$ for each $m \in \mathbb{N}$. By Equation (1.4), each $f_m(y)$ is non-zero and so $\tilde{g}_m$ has now been determined for each $m \in \mathbb{N}$. Then, by (2.4) the vector function $\tilde{g}$ is determined. Finally, $\vec{g}(x) = P^{-1}\tilde{g}(x)$.

Thus, the triple $\left\{\vec{h}(t), \vec{v}(t), \vec{g}(x)\right\}$ is uniquely determined and can in fact be calculated from the given data $\{\vec{u}_1(t), \vec{u}_3(t), \vec{u}(y,t)\}$, $t \geq 0$. ■

Remark: We note that the initial data could be $\{\vec{u}_i(t), \vec{u}_j(t), \vec{u}(y,t)\}$, as long as the resulting matrix M is non-singular, which is the case as long as the 2×2 matrix

$$M^* = \begin{pmatrix} f_i'(0) & c_i \\ f_j'(0) & c_j \end{pmatrix}$$

is non-singular. Noting that $f_m'(0) = m\sqrt{\frac{2}{\pi}}$ and that c_m is given in (2.7), it follows that M^* is non-singular as long as $i \neq j$ and at least one of i and j is odd. In such a case, it would still be possible to calculate $\left\{\vec{h}(t), \vec{v}(t), \vec{g}(x)\right\}$ from the data.

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