SOLVABILITY OF SOME INTEGRO-DIFFERENTIAL EQUATIONS WITH ANOMALOUS DIFFUSION IN HIGHER DIMENSIONS

Vitali Vougalter

Department of Mathematics, University of Toronto
Toronto, Ontario, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

Abstract: The article deals with the studies of the existence of solutions of an integro-differential equation in the case of the anomalous diffusion with the negative Laplace operator in a fractional power in $\mathbb{R}^d, \ d=4,5$. The proof of the existence of solutions relies on a fixed point technique. Solvability conditions for non Fredholm elliptic operators in unbounded domains are used.

AMS Subject Classification: 35R11, 35A01, 35P30, 47F05
Key words: integro-differential equations, non Fredholm operators, Sobolev spaces

1. Introduction

In the present article we study the the existence of stationary solutions of the following nonlocal integro-differential equation

$$\frac{\partial u}{\partial t} = -D(-\Delta)^s u + \int_{\mathbb{R}^d} K(x-y)g(u(y,t))dy + f(x), \quad d=4,5, \tag{1.1}$$

with $\dfrac{3}{2}-\dfrac{d}{4}<s<1$, which is relevant to the cell population dynamics. The space variable x here corresponds to the cell genotype, $u(x,t)$ stands for the cell density as a function of their genotype and time. The right side of this problem describes the evolution of cell density by means of the cell proliferation, mutations and cell influx. The anomalous diffusion term in this context is corresponding to the change of genotype via small random mutations, and the integral term describes large mutations. Function $g(u)$ designated the rate of cell birth depending on u (density dependent proliferation), and the kernel $K(x-y)$ denotes the proportion of newly born cells changing their genotype from y to x. We assume here that it depends on the distance between the genotypes. Finally, the last term in the right side of (1.1) stands for the influx or efflux of cells for different genotypes.

The operator $(-\Delta)^s$ in problem (1.1) describes a particular case of the anomalous diffusion actively treated in the context of different applications in plasma physics and turbulence [7], [20], surface diffusion [14], [18], semiconductors [19] and so on. Anomalous diffusion can be described as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for the anomalous diffusion. Asymptotic behavior at infinity of the probability density function determines the value s of the power of our negative Laplace operator (see [17]). The operator $(-\Delta)^s$ is defined by viertue of the spectral calculus. In the present article we will consider the case of $\frac{3}{2} - \frac{d}{4} < s < 1$. A similar equation in the case of the standard Laplace operator in the diffusion term was studied recently in [32].

We set $D = 1$ and prove the existence of solutions of the problem

$$-(-\Delta)^s u + \int_{\mathbb{R}^d} K(x-y)g(u(y))dy + f(x) = 0, \quad \frac{3}{2} - \frac{d}{4} < s < 1, \tag{1.2}$$

where $d = 4, 5$. Let us consider the case when the linear part of this operator does not satisfy the Fredholm property. Consequently, the conventional methods of nonlinear analysis may not be applicable. We use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.3) contains the origin. Consequently, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article deals with the studies of certain properties of the operators of this kind. Note that elliptic equations with non Fredholm operators were studied actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The non Fredholm Schrödinger type operators were treated with the methods of the spectral and the scattering theory in [21], [27], [26]. The Laplacian with drift from the point of view of non Fredholm operators was considered in [29] and linearized Cahn-Hilliard problems in [24] and [30]. Nonlinear non Fredholm elliptic equations were treated in [28] and [31]. Important applications to the theory of reaction-diffusion problems were developed in [9], [10]. Operators without Fredholm property arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]).

However, the case of $a \neq 0$ is significantly different and the method developed in these works cannot be applied. Front propagation problems with anomalous diffusion were considered actively in recent years (see e.g. [22], [23]). The form boundedness criterion for the relativistic Schrödinger operator was proved in [16]. In work [15] the authors establish the imbedding theorems and study the spectrum of certain pseudodifferential operators.

Let us set $K(x) = \varepsilon \mathcal{K}(x)$, where $\varepsilon \geq 0$ and suppose that the assumption below holds.

Assumption 1. *Consider* $\dfrac{3}{2} - \dfrac{d}{4} < s < 1$,*where* $d = 4, 5$. *Let* $f(x) : \mathbb{R}^d \to \mathbb{R}$ *be nontrivial, such that* $f(x) \in L^1(\mathbb{R}^d)$ *and* $(-\Delta)^{\frac{3}{2}-s} f(x) \in L^2(\mathbb{R}^d)$. *Assume also that* $\mathcal{K}(x) : \mathbb{R}^d \to \mathbb{R}$ *and* $\mathcal{K}(x) \in L^1(\mathbb{R}^d)$. *In addition,* $(-\Delta)^{\frac{3}{2}-s}\mathcal{K}(x) \in L^2(\mathbb{R}^d)$, *such that*

$$Q := \left\| (-\Delta)^{\frac{3}{2}-s} \mathcal{K}(x) \right\|_{L^2(\mathbb{R}^d)} > 0.$$

We choose the space dimensions $d = 4, 5$, which is relevant to the solvability conditions for the linear Poisson type problem (4.34) formulated in Lemma 6 below. From the point of view of applications, the space dimensions are not limited to $d = 4, 5$ because the space variable is correspondent to the cell genotype but not to the usual physical space. Let us use the Sobolev inequality for the fractional Laplacian (see Lemma 2.2 of [12], also [13])

$$\|f(x)\|_{L^{\frac{2d}{d-6+4s}}(\mathbb{R}^d)} \leq c_s \|(-\Delta)^{\frac{3}{2}-s} f(x)\|_{L^2(\mathbb{R}^d)}, \quad \frac{3}{2} - \frac{d}{4} < s < 1, \quad d = 4, 5 \tag{1.4}$$

along with Assumption 1 above and the standard interpolation argument, which gives us

$$f(x) \in L^2(\mathbb{R}^d) \tag{1.5}$$

as well. On the real line our equation was studied in [34] only for $0 < s < \dfrac{1}{4}$ based on the solvability conditions for the analog of (4.34) when $d = 1$. In two dimensions the similar results were obtained in [35] with $0 < s < \dfrac{1}{2}$. In $\mathbb{R}^3$ our problem was treated in [33] for $\dfrac{1}{4} < s < \dfrac{3}{4}$. As distinct from the situations in lower dimensions $d = 1, 2$ and similarly to the present case of $d = 4, 5$, in three dimensions we were able to use the Sobolev inequality for the fractional Laplacian. For the technical purposes, we use the Sobolev spaces

$$H^{2s}(\mathbb{R}^d) := \left\{ u(x) : \mathbb{R}^d \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}^d),\ (-\Delta)^s u \in L^2(\mathbb{R}^d) \right\}, \quad 0 < s \leq 1,$$

where $d = 4, 5$, equipped with the norm

$$\|u\|^2_{H^{2s}(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \left\| (-\Delta)^s u \right\|^2_{L^2(\mathbb{R}^d)}. \tag{1.6}$$

By virtue of the standard Sobolev embedding in dimensions $d = 4, 5$, we have

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq c_e \|u\|_{H^3(\mathbb{R}^d)}, \tag{1.7}$$

where $c_e > 0$ is the constant of the embedding. Here

$$\|u\|^2_{H^3(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \left\|(-\Delta)^{\frac{3}{2}} u\right\|^2_{L^2(\mathbb{R}^d)}. \tag{1.8}$$

When the nonnegative parameter ε vanishes, we arrive at the linear Poisson type problem (4.34). By means of Lemma 6 below along with Assumption 1, equation (4.34) admits a unique solution

$$u_0(x) \in H^{2s}(\mathbb{R}^d), \quad \frac{3}{2} - \frac{d}{4} < s < 1,$$

such that no orthogonality relations are required. By virtue of Assumption 1, since

$$(-\Delta)^{\frac{3}{2}} u_0(x) = (-\Delta)^{\frac{3}{2}-s} f(x) \in L^2(\mathbb{R}^d),$$

we obtain for the unique solution of linear problem (4.34) that $u_0(x) \in H^3(\mathbb{R}^d)$. Let us seek the resulting solution of nonlinear equation (1.2) as

$$u(x) = u_0(x) + u_p(x). \tag{1.9}$$

Evidently, we derive the perturbative equation

$$(-\Delta)^s u_p(x) = \varepsilon \int_{\mathbb{R}^d} \mathcal{K}(x-y) g(u_0(y) + u_p(y)) dy, \tag{1.10}$$

where $\frac{3}{2} - \frac{d}{4} < s < 1, \ d = 4, 5.$ Let us denote a closed ball in the Sobolev space as

$$B_\rho := \{u(x) \in H^3(\mathbb{R}^d) \mid \|u\|_{H^3(\mathbb{R}^d)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.11}$$

We look for the solution of problem (1.10) as the fixed point of the auxiliary nonlinear equation

$$(-\Delta)^s u(x) = \varepsilon \int_{\mathbb{R}^d} \mathcal{K}(x-y) g(u_0(y) + v(y)) dy, \quad d = 4, 5, \tag{1.12}$$

with $\frac{3}{2} - \frac{d}{4} < s < 1$ in ball (1.11). For a given function $v(y)$ this is an equation with respect to $u(x)$. The left side of (1.12) involves the non Fredholm operator

$$(-\Delta)^s : H^{2s}(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Hence, this operator has no bounded inverse. The similar situation appeared in works [28] and [31] but as distinct from the present case, the equations studied there required orthogonality conditions. The fixed point technique was used in [25] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem there had the Fredholm property (see Assumption 1 of [25], also [8]). The existence of pulses for local and nonlocal reaction- diffusion equations was established via the Leray-Schauder method in [11] using the operators which possessed the Fredholm property as well. Let define the interval on the real line

$$I := \big[-c_e\|u_0\|_{H^3(\mathbb{R}^d)} - c_e,\ c_e\|u_0\|_{H^3(\mathbb{R}^d)} + c_e\big] \tag{1.13}$$

along with the closed ball in the space of $C_2(I)$ functions, namely

$$D_M := \{g(z) \in C_2(I) \mid \|g\|_{C_2(I)} \le M\}, \quad M > 0. \tag{1.14}$$

Here the norm

$$\|g\|_{C_2(I)} := \|g\|_{C(I)} + \|g'\|_{C(I)} + \|g''\|_{C(I)}, \tag{1.15}$$

with $\|g\|_{C(I)} := max_{z\in I}|g(z)|$. We make the following technical assumption on the nonlinear part of equation (1.2).

Assumption 2. *Let $g(z) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$ and $g'(0) = 0$. We also assume that $g(z) \in D_M$ and it is not equal to zero identically on the interval I.*

Let us explain why we impose condition $g'(0) = 0$. Assume here that the Fourier image of the kernel $\mathcal{K}(x)$ is positive in the whole $\mathbb{R}^2$, which is common in many biological applications. If $g'(0) < 0$, then the essential spectrum of the corresponding operator is in the left-half plane. This operator is Fredholm, and conventional methods of nonlinear analysis are applicable here. If $g'(0) \ge 0$, then the operator does not satisfy the Fredholm property, and the goal of this work is to establish the existence of solutions in such case where usual methods are not applicable. The method developed in the present article can be used for $g'(0) = 0$ but not for $g'(0) > 0$. This is the reason we impose such condition on the nonlinearity.

We introduce the operator T_g, such that $u = T_g v$, where u is a solution of equation (1.12). Our first main result is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then equation (1.12) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon*$ for some $\varepsilon* > 0$. The unique fixed point $u_p(x)$ of this map T_g is the only solution of problem (1.10) in B_ρ.*

Apparently, the resulting solution of equation (1.2) given by (1.9) will be nontrivial becuase the source term $f(x)$ is nontrivial and $g(0) = 0$ due to our assumptions. We make use of the following trivial statement.

Lemma 4. *For $R \in (0, +\infty)$ and $d = 4, 5$ consider the function*

$$\varphi(R) := \alpha R^{d-4s} + \frac{1}{R^{4s}}, \quad \frac{3}{2} - \frac{d}{4} < s < 1, \quad \alpha > 0.$$

It attains the minimal value at $R^* := \left(\frac{4s}{\alpha(d-4s)}\right)^{\frac{1}{d}}$*, which is given by*

$$\varphi(R^*) = \left(\frac{\alpha}{4s}\right)^{\frac{4s}{d}} \frac{d}{(d-4s)^{\frac{d-4s}{d}}}.$$

Our second main statement deals with the continuity of the fixed point of the map T_g which existence was established in Theorem 3 above with respect to the nonlinear function g.

Theorem 5. *Let $j = 1, 2$, the assumptions of Theorem 3 hold, such that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon_j^*$ and $\delta := min(\varepsilon_1^*, \varepsilon_2^*)$. Then for all $0 < \varepsilon < \delta$ the bound*

$$\|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^d)} \le C\|g_1 - g_2\|_{C_2(I)}, \quad d = 4, 5 \tag{1.16}$$

holds, where $C > 0$ is a constant.

Let us proceed to the proof of our first main proposition.

2. The existence of the perturbed solution

Proof of Theorem 3. Let us choose arbitrarily $v(x) \in B_\rho$ and denote the term involved in the integral expression in the right side of problem (1.12) as

$$G(x) := g(u_0(x) + v(x)).$$

We use the standard Fourier transform

$$\widehat{\phi}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \phi(x) e^{-ipx} dx, \quad d = 4, 5. \tag{2.17}$$

Evidently, we have the bound

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{\frac{d}{2}}} \|\phi(x)\|_{L^1(\mathbb{R}^d)}. \tag{2.18}$$

We apply (2.17) to both sides of problem (1.12). This gives us

$$\widehat{u}(p) = \varepsilon(2\pi)^{\frac{d}{2}} \frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{|p|^{2s}}.$$

Hence, for the norm we obtain

$$\|u\|^2_{L^2(\mathbb{R}^d)} = (2\pi)^d \varepsilon^2 \int_{\mathbb{R}^d} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s}} dp. \tag{2.19}$$

As distinct from works [28] and [31] with the standard Laplacian in the diffusion term, here we do not try to control the norm

$$\left\| \frac{\widehat{\mathcal{K}}(p)}{|p|^{2s}} \right\|_{L^\infty(\mathbb{R}^d)}.$$

Instead, let us estimate the right side of (2.19) by means of the analog of inequality (2.18) applied to functions $\mathcal{K}$ and G with $R > 0$ as

$$(2\pi)^d \varepsilon^2 \int_{|p|\le R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s}} dp + (2\pi)^d \varepsilon^2 \int_{|p|>R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s}} dp \le$$

$$\le \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)} \left\{ \frac{1}{(2\pi)^d} \|G(x)\|^2_{L^1(\mathbb{R}^d)} |S^d| \frac{R^{d-4s}}{d-4s} + \frac{1}{R^{4s}} \|G(x)\|^2_{L^2(\mathbb{R}^d)} \right\}. \tag{2.20}$$

Here and further down S^d stands for the unit sphere centered at the origin and $|S^d|$ for its Lebesgues measure. Due to the fact that $v(x) \in B_\rho$, we derive

$$\|u_0 + v\|_{L^2(\mathbb{R}^d)} \le \|u_0\|_{H^3(\mathbb{R}^d)} + 1.$$

Sobolev embedding (1.7) yields

$$|u_0 + v| \le c_e(\|u_0\|_{H^3(\mathbb{R}^d)} + 1).$$

Equality $G(x) = \displaystyle\int_0^{u_0+v} g'(z)dz$ with the interval I defined in (1.13) gives us

$$|G(x)| \le sup_{z\in I}|g'(z)||u_0 + v| \le M|u_0 + v|.$$

Hence,

$$\|G(x)\|_{L^2(\mathbb{R}^d)} \le M\|u_0 + v\|_{L^2(\mathbb{R}^d)} \le M(\|u_0\|_{H^3(\mathbb{R}^d)} + 1).$$

Clearly, $G(x) = \displaystyle\int_0^{u_0+v} dy\Big[\int_0^y g''(z)dz\Big]$. This yields

$$|G(x)| \le \frac{1}{2} sup_{z\in I}|g''(z)||u_0 + v|^2 \le \frac{M}{2}|u_0 + v|^2,$$

such that

$$\|G(x)\|_{L^1(\mathbb{R}^d)} \le \frac{M}{2}\|u_0 + v\|^2_{L^2(\mathbb{R}^d)} \le \frac{M}{2}(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2. \tag{2.21}$$

Hence, we obtain the estimate from above for the right side of (2.20) as

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)} M^2 (\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2 \left\{ \frac{(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2 |S^d| R^{d-4s}}{4(2\pi)^d (d-4s)} + \frac{1}{R^{4s}} \right\},$$

with $R \in (0, +\infty)$. Lemma 4 gives us the minimal value of the expression above. Thus, $\|u\|^2_{L^2(\mathbb{R}^d)} \leq$

$$\leq \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)} M^2 (\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^{2+\frac{8s}{d}} \left(\frac{|S^d|}{16s} \right)^{\frac{4s}{d}} \frac{d}{(2\pi)^{4s}(d-4s)}. \tag{2.22}$$

Obviosly, by means of (1.12) we have

$$(-\Delta)^{\frac{3}{2}} u(x) = \varepsilon (-\Delta)^{\frac{3}{2}-s} \int_{\mathbb{R}^d} \mathcal{K}(x-y) G(y) dy.$$

By virtue of the analog of estimate (2.18) applied to function G along with (2.21) we arrive at

$$\|(-\Delta)^{\frac{3}{2}} u\|^2_{L^2(\mathbb{R}^d)} \leq \varepsilon^2 \|G\|^2_{L^1(\mathbb{R}^d)} Q^2 \leq \varepsilon^2 \frac{M^2}{4} (\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^4 Q^2. \tag{2.23}$$

Thus, by means of the definition of the norm (1.8) along with inequalities (2.22) and (2.23) we obtain the upper bound for $\|u\|_{H^3(\mathbb{R}^d)}$ given by

$$\varepsilon (\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2 M \left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)} (\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^{\frac{8s}{d}-2} d}{(2\pi)^{4s}(d-4s)} \left(\frac{|S^d|}{16s} \right)^{\frac{4s}{d}} + \frac{Q^2}{4} \right]^{\frac{1}{2}} \leq \rho$$

for all $\varepsilon > 0$ small enough. This means that $u(x) \in B_\rho$ as well. If for a certain $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of problem (1.12), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^d)$ satisfies

$$(-\Delta)^s w = 0.$$

Since the operator $(-\Delta)^s$, $\frac{3}{2} - \frac{d}{4} < s < 1$ considered on the whole $\mathbb{R}^d$ does not possess nontrivial square integrable zero modes, $w(x)$ vanishes in $\mathbb{R}^d$. Hence, problem (1.12) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ sufficiently small.

Our goal is to prove that this map is a strict contraction. We choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above yields $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well. By virtue of equation (1.12) we obtain

$$(-\Delta)^s u_1(x) = \varepsilon \int_{\mathbb{R}^d} \mathcal{K}(x-y) g(u_0(y) + v_1(y)) dy, \tag{2.24}$$

$$(-\Delta)^s u_2(x) = \varepsilon \int_{\mathbb{R}^d} \mathcal{K}(x-y) g(u_0(y) + v_2(y)) dy, \tag{2.25}$$

where $\frac{3}{2} - \frac{d}{4} < s < 1, \ d = 4, 5$. Let us define

$$G_1(x) := g(u_0(x) + v_1(x)), \quad G_2(x) := g(u_0(x) + v_2(x))$$

and apply our standard Fourier transform (2.17) to both sides of problems (2.24) and (2.25). This gives us

$$\widehat{u_1}(p) = \varepsilon (2\pi)^{\frac{d}{2}} \frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{|p|^{2s}}, \quad \widehat{u_2}(p) = \varepsilon (2\pi)^{\frac{d}{2}} \frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{|p|^{2s}}.$$

Evidently,

$$\|u_1 - u_2\|^2_{L^2(\mathbb{R}^d)} = \varepsilon^2 (2\pi)^d \int_{\mathbb{R}^d} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^{4s}} dp.$$

Apparently, it can be bounded from above by means of inequality (2.18), since

$$\varepsilon^2 (2\pi)^d \int_{|p| \le R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^{4s}} dp +$$

$$+ \varepsilon^2 (2\pi)^d \int_{|p| > R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^{4s}} dp \le$$

$$\le \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)} \left\{ \frac{|S^d|}{(2\pi)^d} \|G_1(x) - G_2(x)\|^2_{L^1(\mathbb{R}^d)} \frac{R^{d-4s}}{d-4s} + \frac{\|G_1(x) - G_2(x)\|^2_{L^2(\mathbb{R}^d)}}{R^{4s}} \right\},$$

with $R \in (0, +\infty)$. Let us use the equality

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} g'(z) dz.$$

Thus,

$$|G_1(x) - G_2(x)| \le sup_{z \in I} |g'(z)| |v_1(x) - v_2(x)| \le M |v_1(x) - v_2(x)|.$$

Therefore,

$$\|G_1(x) - G_2(x)\|_{L^2(\mathbb{R}^d)} \le M \|v_1 - v_2\|_{L^2(\mathbb{R}^d)} \le M \|v_1 - v_2\|_{H^3(\mathbb{R}^d)}.$$

Clearly,

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} dy \Big[\int_0^y g''(z) dz \Big].$$

We derive the estimate from above for $G_1(x)-G_2(x)$ in the absolute value as

$$\frac{1}{2}sup_{z\in I}|g''(z)||(v_1-v_2)(2u_0+v_1+v_2)|\le\frac{M}{2}|(v_1-v_2)(2u_0+v_1+v_2)|.$$

By means of the Schwarz inequality we arrive at the upper bound for the norm $\|G_1(x)-G_2(x)\|_{L^1(\mathbb{R}^d)}$ given by

$$\frac{M}{2}\|v_1-v_2\|_{L^2(\mathbb{R}^d)}\|2u_0+v_1+v_2\|_{L^2(\mathbb{R}^d)}\le$$

$$\le M\|v_1-v_2\|_{H^3(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1). \tag{2.26}$$

Hence, we obtain the estimate from above for the norm $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R}^d)}$ as

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}M^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^d)}\left\{\frac{|S^d|}{(2\pi)^d}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2\frac{R^{d-4s}}{d-4s}+\frac{1}{R^{4s}}\right\}.$$

Lemma 4 enables us to minimize the expression above over $R\in(0,+\infty)$ to derive the upper bound for $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R}^d)}$ given by

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}M^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^d)}\frac{|S^d|^{\frac{4s}{d}}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s}{d}}}{(2\pi)^{4s}(4s)^{\frac{4s}{d}}}\frac{d}{d-4s}. \tag{2.27}$$

By means of formulas (2.24) and (2.25) we have

$$(-\Delta)^{\frac{3}{2}}(u_1-u_2)(x)=\varepsilon(-\Delta)^{\frac{3}{2}-s}\int_{\mathbb{R}^d}\mathcal{K}(x-y)[G_1(y)-G_2(y)]dy.$$

Using inequalities (2.18) and (2.26) we derive

$$\|(-\Delta)^{\frac{3}{2}}(u_1-u_2)\|^2_{L^2(\mathbb{R}^d)}\le\varepsilon^2Q^2\|G_1-G_2\|^2_{L^1(\mathbb{R}^d)}\le$$

$$\le\varepsilon^2Q^2M^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2. \tag{2.28}$$

Due to (2.27) and (2.28) the norm $\|u_1-u_2\|_{H^3(\mathbb{R}^d)}$ can be bounded from above by the expression $\varepsilon M(\|u_0\|_{H^3(\mathbb{R}^d)}+1)\times$

$$\times\left\{\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}|S^d|^{\frac{4s}{d}}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s}{d}-2}}{(2\pi)^{4s}(4s)^{\frac{4s}{d}}}\frac{d}{d-4s}+Q^2\right\}^{\frac{1}{2}}\|v_1-v_2\|_{H^3(\mathbb{R}^d)}. \tag{2.29}$$

Thus, the map $T_g: B_\rho\to B_\rho$ defined by problem (1.12) is a strict contraction for all values of $\varepsilon>0$ sufficiently small. Its unique fixed point $u_p(x)$ is the only solution of equation (1.10) in the ball B_ρ. The resulting $u(x)\in H^3(\mathbb{R}^d)$ given by (1.9) is a solution of problem (1.2). ∎

Let us proceed to proving the second main statement of our work.

3. The continuity of the fixed point of the map T_g

Proof of Theorem 5. Clearly, for all $0<\varepsilon<\delta$ we have

$$u_{p,1}=T_{g_1}u_{p,1}, \quad u_{p,2}=T_{g_2}u_{p,2}.$$

Thus

$$u_{p,1}-u_{p,2}=T_{g_1}u_{p,1}-T_{g_1}u_{p,2}+T_{g_1}u_{p,2}-T_{g_2}u_{p,2}.$$

Hence,

$$\|u_{p,1}-u_{p,2}\|_{H^3(\mathbb{R}^d)}\le \|T_{g_1}u_{p,1}-T_{g_1}u_{p,2}\|_{H^3(\mathbb{R}^d)}+\|T_{g_1}u_{p,2}-T_{g_2}u_{p,2}\|_{H^3(\mathbb{R}^d)}.$$

By virtue of estimate (2.29), we derive

$$\|T_{g_1}u_{p,1}-T_{g_1}u_{p,2}\|_{H^3(\mathbb{R}^d)}\le \varepsilon\sigma\|u_{p,1}-u_{p,2}\|_{H^3(\mathbb{R}^d)},$$

where $\varepsilon\sigma<1$ since the map $T_{g_1}:B_\rho\to B_\rho$ under the given assumptions is a strict contraction and the positive constant

$$\sigma:=M(\|u_0\|_{H^3(\mathbb{R}^d)}+1)\left\{\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}|S^d|^{\frac{4s}{d}}(\|u_0\|_{H^3(\mathbb{R})^d}+1)^{\frac{8s}{d}-2}}{(2\pi)^{4s}(4s)^{\frac{4s}{d}}}\frac{d}{d-4s}+Q^2\right\}^{\frac{1}{2}}.$$

Thus, we arrive at

$$(1-\varepsilon\sigma)\|u_{p,1}-u_{p,2}\|_{H^3(\mathbb{R}^d)}\le \|T_{g_1}u_{p,2}-T_{g_2}u_{p,2}\|_{H^3(\mathbb{R}^d)}. \tag{3.30}$$

Note that for our fixed point $T_{g_2}u_{p,2}=u_{p,2}$ and denote $\xi(x):=T_{g_1}u_{p,2}$. Apparently

$$(-\Delta)^s\xi(x)=\varepsilon\int_{\mathbb{R}^d}\mathcal{K}(x-y)g_1(u_0(y)+u_{p,2}(y))dy, \tag{3.31}$$

$$(-\Delta)^s u_{p,2}(x)=\varepsilon\int_{\mathbb{R}^d}\mathcal{K}(x-y)g_2(u_0(y)+u_{p,2}(y))dy, \tag{3.32}$$

where $\frac{3}{2}-\frac{d}{4}<s<1$. Denote $G_{1,2}(x):=g_1(u_0(x)+u_{p,2}(x))$ and $G_{2,2}(x):=g_2(u_0(x)+u_{p,2}(x))$. Let us apply the standard Fourier transform (2.17) to both sides of problems (3.31) and (3.32). This gives us

$$\widehat{\xi}(p)=\varepsilon(2\pi)^{\frac{d}{2}}\frac{\widehat{\mathcal{K}}(p)\widehat{G}_{1,2}(p)}{|p|^{2s}}, \quad \widehat{u}_{p,2}(p)=\varepsilon(2\pi)^{\frac{d}{2}}\frac{\widehat{\mathcal{K}}(p)\widehat{G}_{2,2}(p)}{|p|^{2s}}.$$

Clearly,

$$\|\xi(x)-u_{p,2}(x)\|^2_{L^2(\mathbb{R}^d)}=\varepsilon^2(2\pi)^d\int_{\mathbb{R}^d}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}_{1,2}(p)-\widehat{G}_{2,2}(p)|^2}{|p|^{4s}}dp.$$

Evidently, it can be estimated from above using (2.18), such that

$$\varepsilon^2(2\pi)^d\int_{|p|\le R}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}_{1,2}(p)-\widehat{G}_{2,2}(p)|^2}{|p|^{4s}}dp+$$

$$+\varepsilon^2(2\pi)^d\int_{|p|>R}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}_{1,2}(p)-\widehat{G}_{2,2}(p)|^2}{|p|^{4s}}dp\le$$

$$\le\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}\left\{\frac{|S^d|}{(2\pi)^d}\|G_{1,2}-G_{2,2}\|^2_{L^1(\mathbb{R}^d)}\frac{R^{d-4s}}{d-4s}+\frac{\|G_{1,2}-G_{2,2}\|^2_{L^2(\mathbb{R}^d)}}{R^{4s}}\right\},$$

where $R\in(0,+\infty)$. Let us use the equality

$$G_{1,2}(x)-G_{2,2}(x)=\int_0^{u_0(x)+u_{p,2}(x)}[g_1'(z)-g_2'(z)]dz.$$

Hence

$$|G_{1,2}(x)-G_{2,2}(x)|\le sup_{z\in I}|g_1'(z)-g_2'(z)||u_0(x)+u_{p,2}(x)|\le$$

$$\le\|g_1-g_2\|_{C_2(I)}|u_0(x)+u_{p,2}(x)|.$$

Thus

$$\|G_{1,2}-G_{2,2}\|_{L^2(\mathbb{R}^d)}\le\|g_1-g_2\|_{C_2(I)}\|u_0+u_{p,2}\|_{L^2(\mathbb{R}^d)}\le$$

$$\le\|g_1-g_2\|_{C_2(I)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1).$$

Another useful identity would be

$$G_{1,2}(x)-G_{2,2}(x)=\int_0^{u_0(x)+u_{p,2}(x)}dy\Big[\int_0^y(g_1''(z)-g_2''(z))dz\Big].$$

Clearly,

$$|G_{1,2}(x)-G_{2,2}(x)|\le\frac{1}{2}sup_{z\in I}|g_1''(z)-g_2''(z)||u_0(x)+u_{p,2}(x)|^2\le$$

$$\le\frac{1}{2}\|g_1-g_2\|_{C_2(I)}|u_0(x)+u_{p,2}(x)|^2.$$

Therefore,

$$\|G_{1,2}-G_{2,2}\|_{L^1(\mathbb{R}^d)}\le\frac{1}{2}\|g_1-g_2\|_{C_2(I)}\|u_0+u_{p,2}\|^2_{L^2(\mathbb{R}^d)}\le$$

$$\le\frac{1}{2}\|g_1-g_2\|_{C_2(I)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2. \tag{3.33}$$

This yields the estimate from above for the norm $\|\xi(x)-u_{p,2}(x)\|^2_{L^2(\mathbb{R}^d)}$ as

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2\|g_1-g_2\|^2_{C_2(I)}\Big[\frac{|S^d|(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2}{4(2\pi)^d}\frac{R^{d-4s}}{d-4s}+\frac{1}{R^{4s}}\Big].$$

Such expression can be easily minimized over $R\in(0,+\infty)$ by virtue of Lemma 4. We derive the estimate $\|\xi(x)-u_{p,2}(x)\|^2_{L^2(\mathbb{R}^d)}\le$

$$\le\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{2+\frac{8s}{d}}\|g_1-g_2\|^2_{C_2(I)}\frac{|S^d|^{\frac{4s}{d}}}{(16s)^{\frac{4s}{d}}(2\pi)^{4s}}\frac{d}{d-4s}.$$

Formulas (3.31) and (3.32) give us

$$(-\Delta)^{\frac{3}{2}}\xi(x)=\varepsilon(-\Delta)^{\frac{3}{2}-s}\int_{\mathbb{R}^d}\mathcal{K}(x-y)G_{1,2}(y)dy,$$

$$(-\Delta)^{\frac{3}{2}}u_{p,2}(x)=\varepsilon(-\Delta)^{\frac{3}{2}-s}\int_{\mathbb{R}^d}\mathcal{K}(x-y)G_{2,2}(y)dy.$$

Using (2.18) and (3.33), the norm $\|(-\Delta)^{\frac{3}{2}}[\xi(x)-u_{p,2}(x)]\|^2_{L^2(\mathbb{R}^d)}$ can be bounded from above by

$$\varepsilon^2\|G_{1,2}-G_{2,2}\|^2_{L^1(\mathbb{R}^d)}Q^2\le\frac{\varepsilon^2Q^2}{4}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^4\|g_1-g_2\|^2_{C_2(I)}.$$

Therefore, $\|\xi(x)-u_{p,2}(x)\|_{H^3(\mathbb{R}^d)}\le\varepsilon\|g_1-g_2\|_{C_2(I)}\times$

$$\times(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2\left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s}{d}-2}|S^d|^{\frac{4s}{d}}}{(16s)^{\frac{4s}{d}}(2\pi)^{4s}}\frac{d}{d-4s}+\frac{Q^2}{4}\right]^{\frac{1}{2}}.$$

By means of inequality (3.30), the norm $\|u_{p,1}-u_{p,2}\|_{H^3(\mathbb{R}^d)}$ can be estimated from above by

$$\frac{\varepsilon}{1-\varepsilon\sigma}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2\times$$

$$\times\left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^d)}(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^{\frac{8s}{d}-2}|S^d|^{\frac{4s}{d}}}{(16s)^{\frac{4s}{d}}(2\pi)^{4s}}\frac{d}{d-4s}+\frac{Q^2}{4}\right]^{\frac{1}{2}}\|g_1-g_2\|_{C_2(I)},$$

which completes the proof of the theorem. ■

4. Auxiliary results

Below we state and prove the solvability conditions for the linear Poisson type equation with a square integrable right side

$$(-\Delta)^s u=f(x),\quad x\in\mathbb{R}^d,\quad d=4,5,\quad 0<s<1. \tag{4.34}$$

This statement was established in the one of our previous works but we provide the argument here for the convenience of the readers. We designate the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \quad d = 4, 5, \tag{4.35}$$

with a slight abuse of notations when the functions involved in (4.35) are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x) \in L^\infty(\mathbb{R}^d)$, then the integral in the right side of (4.35) is well defined. The technical statement below is easily proved by applying the standard Fourier transform (2.17) to both sides of equation (4.34).

Lemma 6. *Let* $0 < s < 1,\ f(x) : \mathbb{R}^d \to \mathbb{R},\ d = 4, 5$ *and* $f(x) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. *Then equation (4.34) admits a unique solution* $u(x) \in H^{2s}(\mathbb{R}^d)$.

Proof. First of all, let us note that under the given conditions any square integrable solution of equation (4.34) will belong to $H^{2s}(\mathbb{R}^d)$ as well. Indeed, if $u(x) \in L^2(\mathbb{R}^d)$ satisfies (4.34) with a square integrable right side, we have $(-\Delta)^s u \in L^2(\mathbb{R}^d)$, such that by means of the definition of the norm (1.6), we obtain $u(x) \in H^{2s}(\mathbb{R}^d)$.

To establish the uniqueness of solutions for our problem, we suppose that equation (4.34) possesses two solutions $u_{1,2}(x) \in H^{2s}(\mathbb{R}^d)$. Then their difference $w(x) = u_1(x) - u_2(x) \in H^{2s}(\mathbb{R}^d)$ satisfies the homogeneous equation

$$(-\Delta)^s w = 0.$$

Since the operator $(-\Delta)^s : H^{2s}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ does not have any nontrivial zero modes, $w(x)$ will vanish in $\mathbb{R}^d$.

Let us apply Fourier transform (2.17) to both sides of problem (4.34). This yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s}} = \frac{\widehat{f}(p)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}^d \ | \ |p|\le 1\}} + \frac{\widehat{f}(p)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}^d \ | \ |p|>1\}}, \tag{4.36}$$

where χ_A denotes the characteristic function of a set $A \subseteq \mathbb{R}^d$. Clearly, the second term in the right side of (4.36) can be estimated in the absolute value from above by $|\widehat{f}(p)| \in L^2(\mathbb{R}^d)$ due to the one of our assumptions. By means of inequality (2.18), we estimate the norm

$$\left\| \frac{\widehat{f}(p)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}^d \ | \ |p|\le 1\}} \right\|^2_{L^2(\mathbb{R}^d)} \le \frac{\|f\|^2_{L^1(\mathbb{R}^d)}}{(2\pi)^d} \int_0^1 |S^d||p|^{d-1-4s}d|p| =$$

$$= \frac{\|f\|^2_{L^1(\mathbb{R}^d)}}{(2\pi)^d}\frac{|S^d|}{d-4s} < \infty$$

by virtue of our assumptions. ∎

Note that by proving the lemma above we establish the solvability of equation (4.34) in $H^{2s}(\mathbb{R}^d), \ d = 4, 5$ for all values of the power of the negative Laplace operator $0 < s < 1$, such that no orthogonality conditions are required for the right side $f(x)$.

References

[1] G.L. Alfimov, E.V. Medvedeva, D.E. Pelinovsky, *Wave systems with an infinite number of localized traveling waves*, Phys. Rev. Lett., **112** (2014), 054103, 5pp.

[2] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces*, J. Math. Pures Appl., (9) **76** (1997), No. 1, 55–81.

[3] C. Amrouche, F. Bonzom, Mixed exterior Laplace's problem, J. Math. Anal. Appl., **338** (2008), No. 1, 124–140.

[4] P. Bolley, T.L. Pham, *Propriétés d'indice en théorie hölderienne pour des opérateurs différentiels elliptiques dans R^n*, J. Math. Pures Appl., (9) **72** (1993), No. 1, 105–119.

[5] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Hölderienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), No. 1-2, 315–334.

[6] N. Benkirane, *Propriétés d'indice en théorie hölderienne pour des opérateurs elliptiques dans R^n*, CRAS, **307**, Série I (1988), No. 11, 577–580.

[7] B. Carreras, V. Lynch, G. Zaslavsky. *Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence model*, Phys. Plasmas, **8** (2001), 5096–5103.

[8] S. Cuccagna, D. Pelinovsky, V. Vougalter, *Spectra of positive and negative energies in the linearized NLS problem*, Comm. Pure Appl. Math., **58** (2005), No. 1, 1–29.

[9] A. Ducrot, M. Marion, V. Volpert, *Systemes de réaction-diffusion sans propriété de Fredholm*, CRAS, **340** (2005), No. 9, 659–664.

[10] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non-Fredholm operators*, Adv. Differential Equations, **13** (2008), No. 11-12, 1151–1192.

[11] N. Eymard, V. Volpert, V. Vougalter, *Existence of pulses for local and nonlocal reaction-diffusion equations*, J. Dynam. Differential Equations, **29** (2017), No. 3, 1145–1158.

[12] H. Hajaiej, X. Yu, Z. Zhai, *Fractional Gagliardo-Nirenberg and Hardy inequalities under Lorentz norms*, J. Math. Anal. Appl., **396** (2012), No. 2, 569–577.

[13] E.H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math., (2) **118** (1983), No. 2, 349–374.

[14] P. Manandhar, J. Jang, G.C. Schatz, M.A. Ratner, S. Hong. *Anomalous surface diffusion in nanoscale direct deposition processes*, Phys. Rev. Lett., **90** (2003), 4043–4052.

[15] V.G. Maz'ja, M. Otelbaev. *Imbedding theorems and the spectrum of a a certain pseudodifferential operator*, (Russian) Sibirsk. Mat. Z., **18** (1977), No. 5, 1073–1087, 1206.

[16] V.G. Maz'ya, I.E. Verbitsky. *The form boundedness criterion for the relativistic Schrödinger operator*, Ann. Inst. Fourier (Grenoble), **54** (2004), No. 2, 317–339.

[17] R. Metzler, J. Klafter. *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep., **339** (2000), 1–77.

[18] J. Sancho, A. Lacasta, K. Lindenberg, I. Sokolov, A. Romero. *Diffusion on a solid surface: anomalous is normal*, Phys. Rev. Lett., **92** (2004), 250601.

[19] H. Scher, E. Montroll. *Anomalous transit-time dispersion in amorphous solids*, Phys. Rev. B, **12** (1975), 2455–2477.

[20] T. Solomon, E. Weeks, H. Swinney. *Observation of anomalous diffusion and Lévy flights in a two-dimensional rotating flow*, Phys. Rev. Lett., **71** (1993), 3975–3978.

[21] V. Volpert. Elliptic partial differential equations. Volume 1: Fredholm theory of elliptic problems in unbounded domains, Monographs in Mathematics, **101**. Birkhäuser/Springer, (2011), 639 pp.

[22] V.A . Volpert, Y. Nec, A.A. Nepomnyashchy. *Exact solutions in front propagation problems with superdiffusion*, Phys. D, **239** (2010), No. 3–4, 134–144.

[23] V.A. Volpert, Y. Nec, A.A. Nepomnyashchy. *Fronts in anomalous diffusion-reaction systems*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **371** (2013), No. 1982, 20120179, 18 pp.

[24] V. Volpert, V. Vougalter. *On the solvability conditions for a linearized Cahn-Hilliard equation*, Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9.

[25] V. Vougalter, *On threshold eigenvalues and resonances for the linearized NLS equation*, Math. Model. Nat. Phenom., **5** (2010), No. 4, 448–469.

[26] V. Vougalter, V. Volpert. *Solvability relations for some non Fredholm operators*, Int. Electron. J. Pure Appl. Math., **2** (2010), No. 1, 75–83.

[27] V. Vougalter, V. Volpert, *Solvability conditions for some non-Fredholm operators*, Proc. Edinb. Math. Soc., (2) **54** (2011), No. 1, 249–271.

[28] V. Vougalter, V. Volpert. *On the existence of stationary solutions for some non-Fredholm integro-differential equations*, Doc. Math., **16** (2011), 561–580.

[29] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms*, Commun. Pure Appl. Anal., **11** (2012), No. 1, 365–373.

[30] V. Vougalter, V. Volpert. *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order*, Math. Model. Nat. Phenom., **7** (2012), No. 2, 146–154.

[31] V. Vougalter, V. Volpert. *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems*, Anal. Math. Phys., **2** (2012), No. 4, 473–496.

[32] V. Vougalter, V. Volpert. *Existence of stationary solutions for some nonlocal reaction-diffusion equations*, Dyn. Partial Differ. Equ., **12** (2015), No. 1, 43–51.

[33] V. Vougalter, V. Volpert. *Existence of stationary solutions for some integro-differential equations with anomalous diffusion*, J. Pseudo-Differ. Oper. Appl., **6** (2015), No. 4, 487–501.

[34] V. Vougalter, V. Volpert. *Solvability of some integro-differential equations with anomalous diffusion. Regularity and stochasticity of nonlinear dynamical systems*, Nonlinear Syst. Complex., **21**, *Springer, Cham*, (2018), 1–17.

[35] V. Vougalter, V. Volpert. *Solvability of some integro- differential equations with anomalous diffusion in two dimensions*, J. Math. Sci. (N. Y.), **235** (2018), no. 3, Problems in mathematical analysis, No. 95 (Russian), 243–255.