Spectral theory of Schrödinger operators over circle diffeomorphisms

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Abstract

We initiate the study of Schrödinger operators with ergodic potentials defined over circle map dynamics, in particular over circle diffeomorphisms. For analytic circle diffeomorphisms and a set of rotation numbers satisfying Yoccoz's $\mathcal{H}$ arithmetic condition, we discuss an extension of Avila's global theory. We also prove a sharp Gordon-type theorem which implies that for every C^{1+BV} circle diffeomorphism, with a Liouville rotation number and an invariant measure μ, for μ-almost all $x \in \mathbb{T}^1$, the corresponding Schrödinger operator has purely continuous spectrum for every Hölder continuous potential V.

1 Introduction and the statements of the results

Spectral theory of discrete ergodic 1D Schrödinger operators has seen a considerable development in the last several decades. The general setup involves Schrödinger operators H on the space of square-summable sequences $\ell^2(\mathbb{Z})$, defined by

$$(H_x u)_n := u_{n-1} + u_{n+1} + V(T^n x)u_n, \qquad u \in \ell^2(\mathbb{Z}), \tag{1.1}$$

where T is an ergodic automorphism of a phase space (M, μ), $x \in M$ and $V : M \to \mathbb{R}$. Aside from the beautiful general results that hold for all ergodic operators, or under minimal general assumptions[1], most attention has been devoted to two families: random

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[1]some also for multi-dimensional analogues and further generalizations

potentials (T being a shift operator on a product measure space) and almost periodic, particularly, quasiperiodic potentials (T being an irrational rotation of the torus). Both of these families have strong origins in physics and both have led to deep mathematics. While it would be very interesting to understand the features of potentials over other base dynamics, it has proved surprisingly difficult, and there are few results for other underlying dynamical systems. We refer the reader to the reviews [9, 18] for further history and discussion of these results. In this paper, we initiate the study of Schrödinger operators with potentials over circle maps, i.e., orientation-preserving homeomorphisms of a circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.

As an irrational rotation is a basic example of a circle map, it is natural to view corresponding potentials as generalizations of one-frequency quasiperiodic potentials. The theory of the latter has seen dramatic advances in the last twenty years (see e.g. [6, 22, 23] and references therein) and continues to develop rapidly. It remains the only ergodic family with established transitions between the spectral types with changes of parameters, that can often be proved and analyzed in a sharp arithmetic way or through analytic behavior of certain dynamical quantities.

Poincaré established that, for every orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, there is a unique rotation number $\rho \in (0, 1)$, given by the (x-independent) limit

$$\rho := \lim_{n\to\infty} \frac{\mathcal{T}^n(x) - x}{n} \mod 1, \tag{1.2}$$

where $\mathcal{T}$ is any lift of T to $\mathbb{R}$, and $x \in \mathbb{R}$. Poincaré also proved that if the rotation number ρ of an orientation-preserving circle homeomorphism T is irrational, T is topologically semi-conjugate to the rotation $R_\rho : x \mapsto x + \rho \mod 1$, i.e. there is a continuous map $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$ such that

$$T \circ \varphi = \varphi \circ R_\rho. \tag{1.3}$$

It follows that $T^n \circ \varphi = \varphi \circ R_\rho^n$, for every $n \in \mathbb{Z}$, i.e. the orbit $x_n = T^n x$ of x can be viewed as the image of the orbit $\theta_n = R_\rho^n \theta$ of a preimage θ of $x = \varphi(\theta)$ under φ. This establishes a relation between Schrödinger operators over circle maps and one-frequency quasiperiodic Schrödinger operators.

It is a fundamental problem to understand the rigidity of Schrödinger operators over circle maps. Rigidity is a phenomenon that systems that are a priori equivalent in a weak sense are actually equivalent in a much stronger sense. An important problem is to determine the classes of these operators with, in a sense, equivalent spectral properties.

In general, the semi-conjugacy φ may not even be invertible. However, as shown by Denjoy [11], if T is a C^{1+BV} circle diffeomorphism, i.e. a C^1-smooth circle diffeomorphism with a derivative of bounded variation, φ is a homeomorphism, termed the topological conjugacy. Herman's theory [16] — further developed by Yoccoz [31] — establishes certain level of smoothness of the conjugacy for diffeomorphisms of higher smoothness. In particular, as proved by Herman [16], analytic circle diffeomorphisms with Diophantine rotation numbers are analytically conjugate to a rotation. These results are at the core of rigidity theory of circle diffeomorphisms. An optimal condition for analytic linearization has been obtained by Yoccoz [31]. He established that for a set $\mathcal{H}$ of irrational numbers satisfying an arithmetic condition — known as Yoccoz's $\mathcal{H}$ arithmetic condition — every analytic circle diffeomorphism, with a rotation number in this set, is analytically conjugate to the rotation.

Some of the most interesting recent advances in the theory of one-frequency quasiperiodic operators have been developed for *analytic* potentials [1, 6, 21, 25]. An analytic conjugacy maps potentials $V_n = f(T^n x)$ with analytic f into potentials of the form $W_n = g(R_\rho^n x)$ with analytic g, allowing for some results concerning the spectrum of the Schrödinger operators over (analytic) circle diffeomophisms to be obtained directly from the corresponding results for one-frequency quasiperiodic operators. However, when the conjugacy is not analytic the resulting g will not be analytic either, potentially leading to counter-intuitive properties (e.g.[30]). An important aspect of the study of the spectrum of Schrödinger operators over circle diffeomorphisms is to understand what properties hold in the absence of an analytic conjugacy. In this paper, we address one of the most basic questions of this nature: absence of the point spectrum for Liouville rotation numbers, already leading to non-trivial analysis.

Finally, the study of the smoothness of the conjugacy between circle diffeomorphisms with irrational rotation numbers and the corresponding rotations has been one of the prime examples of small denominator analysis that has eventually led to the first sharp arithmetic transition results [31]. The study of spectral properties of quasiperiodic operators has also recently led to sharp arithmetic transitions [4, 19, 20, 21, 25]. The study of ergodic Schrödinger operators with circle map dynamics is expected to lead to further interplay between those small denominator problems.

In this paper, we consider a class of Schrödinger operators $H = H(x) = H(T, V, x)$ of the form (1.1) where $T : \mathbb{T}^1 \to \mathbb{T}^1$ is an orientation-preserving homeomorphism of the circle, $V : \mathbb{T}^1 \to \mathbb{R}$, and $x \in \mathbb{T}^1$. Since T, V will usually be fixed we will often suppress them from the notation.

Ergodic Schrödinger operators are intimately related with a family of cocycles — dynamical systems associated with each eigen-equation $Hu = Eu$. In the case of Schrödinger operators over circle maps with an irrational rotation number, the cocycle is given by

$$(T, A) : (x, y) \mapsto (Tx, A(x, E)y), \tag{1.4}$$

where $A \in SL(2, \mathbb{R})$, $x \in \mathbb{T}^1$, $y \in \mathbb{R}^2$. If $u = (u_n)$ is a sequence satisfying $Hu = Eu$, then

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_n(x, E) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \text{ where } A_n(x, E) := \begin{pmatrix} E - V(T^n x) & -1 \\ 1 & 0 \end{pmatrix} \tag{1.5}$$

is the transfer matrix. Thus,

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = P_n(x, E) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}, \tag{1.6}$$

where $P_n(x, E) := \prod_{i=n-1}^{0} A_i(x, E) = A_{n-1}(x, E) \dots A_0(x, E)$. Thus, $P_n(x, E)$ is the product of the values of a matrix valued function $A(\cdot, E) : \mathbb{T}^1 \to \mathrm{SL}(2, \mathbb{R})$ along the orbit $x_i = T^i x$ of x, under the action of T. We also define $P_{-n}(x, E) = P_n(T^{-n}x, E)^{-1}$ and $P_0 = I$.

One way to divide the spectrum of one-frequency quasiperiodic Schrödinger operators into different regimes is through the Lyapunov exponent [1]. One regime corresponds to positive Lyapunov exponent; another corresponds to zero Lyapunov exponent stable under complexification; and the third one (critical) corresponds to zero Lyapunov exponent unstable under complexification. An operator is called acritical if there are no critical energies E in the spectrum.

In the case of more general Schrödinger operators over circle maps with an irrational rotation number, we can still define the Lyapunov exponent. If the rotation number ρ of T is irrational, T is uniquely ergodic [13]. We will denote by μ the unique invariant probability measure of T.

We define the Lyapunov exponent

$$L(E) := \lim_{n \to \infty} L_n(E), \tag{1.7}$$

where

$$L_n(E) := \int L_n(x, E)\, d\mu, \qquad L_n(x, E) := \frac{1}{n} \ln \|P_n(x, E)\|. \tag{1.8}$$

Due to submultiplicativity of $P_n(x, E)$, $L(E)$ exists. Since T is ergodic, by Kingman's ergodic theorem, for almost every x,

$$L(E) = \lim_{n \to \infty} \frac{1}{n} \ln \|P_n(x, E)\|. \tag{1.9}$$

This paper can be divided into two parts. In the first part, we discuss the spectral properties of Schrödinger operators with large, small, as well as typical potentials, over analytic circle diffeomorphisms, with badly approximable rotation numbers.

Theorem 1.1 *Let T be an analytic circle diffeomorphism with rotation number ρ satisfying Yoccoz's $\mathcal{H}$ arithmetic condition and $V : \mathbb{T}^1 \to \mathbb{R}$ be analytic. Then*

(i) There exists $\lambda_0(T, V) > 0$ such that for $\lambda < \lambda_0(T, V)$ operator $H(T, \lambda V)$ has purely absolutely continuous spectrum.

(ii) *There exist* $\lambda_1(T,V) < \infty$ *such that for* $\lambda > \lambda_1(T,V)$ *the Lyapunov exponent* $L(E) > 0$ *for all* E.

(iii) *For a (measure-theoretically) typical analytic* $V : \mathbb{T}^1 \to \mathbb{R}$, *the operator* $H(T,V)$ *is acritical.*

In the second part, we prove our main result and exclude localization for well-approximable rotation numbers.

For $\rho \in \mathbb{R}$ let $\|\rho\| := \text{dist}(\rho, \mathbb{Z})$ be the distance to the nearest integer. Let

$$\beta = \beta(\rho) := \limsup_{n\to\infty} -\frac{\ln \|\rho n\|}{n}. \tag{1.10}$$

Definition 1.2 A number $\rho \in \mathbb{R}\backslash\mathbb{Q}$ is called *Liouville* if $\beta(\rho) = \infty$.

Definition 1.3 The class of C^{1+BV} diffeomorphisms consists of C^1-smooth diffeomorphisms T with T' of bounded variation.

In particular, a C^2-smooth diffeomorphism of a circle is also of class C^{1+BV}. As a corollary of our main result, we have the following claim.

Theorem 1.4 *For every Liouville number* $\rho \in (0,1)$, *and every* C^{1+BV} *circle diffeomorphism* T, *with rotation number* ρ *and the invariant measure* μ, *for* μ*-almost all* $x \in \mathbb{T}^1$, *the corresponding Schrödinger operator* $H(T,V,x)$ *has purely continuous spectrum for every Hölder-continuous potential* $V : \mathbb{T}^1 \to \mathbb{R}$.

Remark 1 As shown in a parallel work [28], an analogous claim holds for sufficiently smooth circle diffeomorphisms with a single singular point where the derivative vanishes (critical circle maps) or has a jump discontinuity (circle maps with a break). The rigidity theory of these maps has been an important topic in circle dynamics — in the context of an extension of Herman's theory — which experienced a considerable development in recent years [15, 26, 27].

Remark 2 It is an interesting question whether, unlike the case of Schrödinger operators with Hölder continuous potential over rotations, for some circle diffeomorphisms T with Liouville rotation numbers, there are phases $x \in \mathbb{T}^1$ such that $H(T,V,x)$ has eigenvalues. Clearly, if the conjugacy to the corresponding rotation is sufficiently regular (e.g. Hölder continuous), then there could be no such phases, for any Hölder continuous potential. The existence of such phases for Schrödinger operators over rotations is known only for unbounded (and therefore discontinuous) potentials (e.g. [19]).

Remark 3 For C^{1+BV} circle diffeomorphisms T, ϕ is a topological conjugacy, so $H(T,V,x)$ is unitarily equivalent to $H(R_\rho, V_1, y)$ with a continuous $V_1 : \mathbb{T}^1 \to \mathbb{R}$. Even though absence of point spectrum of $H(R_\rho, V_1, y)$ holds for all y, for topologically generic ρ (depending on V_1), this is insufficient to conclude such an absence for $H(T,V,x)$ because V_1 depends on T.

Theorem 1.4 is a corollary of the following sharp result. Different components of the spectrum $\Sigma_{T,V}$ of an operator $H(T,V,x)$ are denoted by Σ_{ac} (absolutely continuous), Σ_{sc} (singular continuous) and Σ_{pp} (pure point). By ergodicity, all those sets are μ-a.e. x-independent, however Σ_{pp} and Σ_{sc} may depend on x (e.g. [21]). We also denote the collection of eigenvalues of $H(T,V,x)$ by $S_{pp}(x)$. Finally, we set $\mathcal{H} = \ell^2(Z)$, $\mathcal{H}_{sc}(x)$ the corresponding singular continuous subspace, and $P_A(x)$ the operator of spectral projection on a Borel set A, corresponding to $H(T,V,x)$.

Theorem 1.5 *Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be any C^{1+BV} circle diffeomorphism with an irrational rotation number $\rho \in (0,1)$ and an invariant measure μ. For μ-almost all $x \in \mathbb{T}^1$, and any α-Hölder continuous real-valued function $V : \mathbb{T}^1 \to \mathbb{R}$ on the circle, with $\alpha \in (0,1]$,*

(i) $S_{pp}(x) \cap \{E : 0 \le L(E) < \alpha\beta(\rho)\} = \emptyset$,

(ii) $P_{\{E:0<L<\alpha\beta(\rho)\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Remark 4 This theorem is optimal in the sense that there exist $H(T,V,x)$ with $\alpha = 1$ and eigenvalues at E with $L(E) \ge \beta(\rho)$ [3, 20].

While Theorem 1.1 is a direct corollary of the main results of [1, 2, 29], the proof of Theorem 1.4 and Theorem 1.5 require new techniques from one-dimensional dynamics, previously not used in the spectral theory of Schrödinger operators. In the next section, we discuss global theory of typical Schrödinger operators over analytic circle diffeomorphisms. In the third section, we prove a sharp Gordon theorem (that could be of independent use) and give a proof of Theorem 1.5.

2 Typical operators over analytic circle diffeomorphisms

The Yoccoz set $\mathcal{H}$ of rotation numbers can be defined as follows.

For $\alpha \in (0,1)$, and $x \in \mathbb{R}$, we define

$$r_\alpha(x) := \begin{cases} \alpha^{-1}(x - \ln\alpha^{-1} + 1), & \text{if} \quad x \ge \ln\alpha^{-1}, \\ e^x, & \text{if} \quad x \le \ln\alpha^{-1}. \end{cases} \tag{2.1}$$

For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $k > 0$, we set

$$\Upsilon_k(\alpha) := r_{\alpha_{k-1}} \circ \cdots \circ r_{\alpha_0}(0), \tag{2.2}$$

where $\alpha_0 = \alpha - [\alpha]$, $\alpha_n = G^n(\alpha_0)$, and G is the Gauss map $G : x \mapsto \left\{\frac{1}{x}\right\}$. Here, $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional part of a number, respectively. We also define

$$\mathcal{H}_{k,n} := \{\alpha \in \mathcal{B}, B(\alpha_n) \leq \Upsilon_k(\alpha_{n-k})\}, \tag{2.3}$$

where $B(\alpha) = \sum_{n\geq 0} \beta_{n-1} \ln \alpha_n^{-1}$ is the Brjuno function, $\beta_n = \prod_{j=0}^{n} \alpha_j$, and $\mathcal{B}$ is the set of Brjuno numbers α for which $B(\alpha) < \infty$. We define

$$\mathcal{H} := \cap_{m\geq 0}(\cup_{k\geq 0}\mathcal{H}_{k,k+m}). \tag{2.4}$$

Clearly, $\mathcal{H} \subset \mathcal{B}$.

In this section, we restrict our considerations to $\rho \in \mathcal{H}$. The study of the spectrum of Schrödinger operators with analytic potentials over analytic circle diffeomorphisms with rotation numbers that do not satisfy the Yoccoz's $\mathcal{H}$ arithmetic condition corresponds to the study of one-frequency quasiperiodic Schrödinger operators with not-necessarily analytic potentials, and involves difficult problems. In the case when T is an analytic circle diffeomorphism with a rotation number $\rho \in \mathcal{H}$ satisfying this condition, however, there is an analytic conjugacy φ to the rotation R_ρ, and many results follow directly from the corresponding results for the one-frequency quasiperiodic Schrödinger operators with analytic potentials. Indeed, we have the following correspondence

$$H(T, V, x) = H(R_\rho, V \circ \varphi, \varphi^{-1}x), \tag{2.5}$$

In the case when $T = R_\rho$, to simplify the notation, we denote the spectrum $\Sigma_{R_\rho,V}$ of $H(R_\rho, V, x)$ by $\Sigma_{\rho,V}$.[2]

As in [1], we can classify energies in the spectrum $\Sigma_{T,V}$ of $H(T, V, x)$ in the following way. An energy E in the spectrum $\Sigma_{T,V}$ of $H(T, V, x)$ is said to be

(i) supercritical if $L(E) > 0$, so $\sup_{x\in\mathbb{T}^1} \|P_n(x, E)\|$ grows exponentially;

(ii) subcritical if there is a uniform subexponential bound on the growth of $\|P_n(z, E)\|$ through some band $|\mathrm{Im}\ z| < \varepsilon$;

(iii) critical otherwise.

Clearly, the notions of supercritical, subcritical and acritical are also independent of x.

It is well-known that, contrary to the case of the almost Mathieu operator, in the case of general quasiperiodic potentials, and $T = R_\rho$, the coexistence of these regimes is possible. Hence, the supercritical and subcritical are properties of the individual energies, not of the whole operators. Although a given potential may display both subcritical

[2] Since orientation-preserving circle homeomorphisms with an irrational rotation number are minimal, we have $\Sigma_{T,V,x} = \Sigma_{T,V}$.

and supercritical energies, in order to go from one regime to the other, it may not be necessary to go through the critical regime. The reason is that the spectrum may be a Cantor set and the transition may happen through a gap. Avila showed that this is a prevalent behavior, when T is a circle rotation [1]. The same holds when T is analytic circle diffeomorphism with an irrational rotation number satisfying Yoccoz's $\mathcal{H}$ arithmetic condition.

The operator $H(T, V, x)$ is said to be acritical, if no energy E in the spectrum $\Sigma_{T,V}$ of $H(T, V, x)$ is critical. Since the spectrum $\Sigma_{T,V}$ and the notion of acritical do not depend on x, we will simply say that the operator $H(T, V)$ is acritical if no energy E in the spectrum $\Sigma_{T,V}$ of $H(T, V)$ is critical.

Proof of Theorem 1.1. When $T = R_\rho$, it follows from [29] that large potentials fall into the supercritical regime. It further follows from [2] and [7] that small potentials fall into the subcritical regime. When T is an analytic circle diffeomorphism with rotation number satisfying Yoccoz's $\mathcal{H}$ arithmetic condition, we similarly obtain that large potentials fall into the supercritical regime and that small potentials fall into the subcritical regime. This follows from the above correspondence (2.5), the fact that in this case the conjugacy φ is analytic (so $V \circ \varphi$ is analytic, whenever V is), and the fact that the supremum norms of V and $V \circ \varphi$ are the same. This implies parts (i) and (ii) of the claim. Since the composition operator is an isometry, part (iii) of the claim follows from the main theorem of [1]. QED

3 Absence of eigenvalues for well-approximable rotation numbers

3.1 A sharp Gordon theorem

Gordon's trick [14] has been fruitfully used to prove absence of point spectra of 1D operators since [5] (see e.g. [10]). A sharp version was used in [4] to treat the singular continuous part of arithmetic spectral transition for the almost Mathieu operator. Here, we give an abstract formulation, for any bounded (not necessarily ergodic) potential.[3]

Consider Schrödinger operator on $\ell^2(\mathbb{Z})$ given by

$$(Hu)_n = u_{n+1} + u_{n-1} + V(n)u_n. \tag{3.1}$$

For $\beta > 0$, we say that a real sequence $\{V(n)\}_{n\in Z}$ has β-repetitions if there is a sequence of positive integers $q_n \to \infty$ such that

$$\max_{0\le j<q_n} |V(j) - V(j \pm q_n)| \le e^{-\beta q_n}. \tag{3.2}$$

[3] In fact, only the boundedness of the Cesaro average of $\ln V(n)$ is required.

As in (1.5), we can define the transfer matrix $A_n(E)$ and, as in (1.6), the n-step transfer-matrix $P_n(E)$. Let

$$\Lambda(E) := \limsup_{|n|\to\infty} \frac{\ln \|P_n(E)\|}{n}. \tag{3.3}$$

Clearly, for bounded V, $\Lambda(E) < \infty$, for every E.

Theorem 3.1 *Suppose that V has β-repetitions with $\beta > \Lambda(E)$. Then E is not an eigenvalue of operator* (3.1).

Remark 5 This theorem is sharp in the sense that there are operators (3.1) (found e.g. within the almost Mathieu family) with $\beta \le \Lambda(E)$ and eigenvalues [3, 4, 20].

Remark 6 As usual with the Gordon-type arguments, we actually prove more: absence of decaying solutions to $H\Psi = E\Psi$, in fact that $\liminf_{n\to\infty} |\Psi_n| \ge 1/2$ if $\Psi(0) = 1$.

Remark 7 The small but crucial difference with the usual Gordon-type proof is to study the characteristic polynomial not of the periodic approximation but of the q-step transfer-matrix itself.

Proof. Since E is fixed, we will suppress it from the notations. Let $q = q_n$. By a standard telescoping argument[4], for any $\epsilon > 0$ and sufficiently large n, we have

$$\|P_{-q} - P_q^{-1}\| < e^{(\Lambda-\beta+\epsilon)q}, \tag{3.4}$$

$$\|P_{2q}v - P_q^2 v\| < e^{(\Lambda-\beta+\epsilon)q}\|P_q v\|. \tag{3.5}$$

Assume there is a decaying u such that $Hu = Eu$. Let $v = (u_0, u_{-1})^T$ and assume $\|v\| = 1$. Then, for sufficiently large n we have $\max(\|P_q v\|, \|P_{-q}v\|, \|P_{2q}v\|) < 1/2$. Since, by the characteristic equation, $P_q - TrP_q I + P_q^{-1} = 0$, using (3.4) (assuming $\epsilon < \beta - \Lambda$) and applying the characteristic equation to v, we obtain $|TrP_q| < 1$, for n large enough. Then, applying another form of the characteristic equation, $P_q^2 - TrP_q P_q + I = 0$, again to v and using (3.5), we obtain, for large enough n, $\|P_{2q}v\| > 1/2$, which is a contradiction. **QED**

Consider the Schrödinger operator (3.1) with $V_n = V(T^n x)$ where $V : \mathbb{T}^1 \to \mathbb{R}$ is a bounded real-valued function on the circle and T is an orientation-preserving homeomorphism of a circle with an irrational rotation number ρ. Let the Lyapunov exponent $L(E)$ be defined as in (1.7). We then have

[4]The core of the argument is the identity $P_q - \widetilde{P}_q = \sum_{i=0}^{q-1} A_{q-1}\dots A_{i+1}(A_i - \widetilde{A}_i)\widetilde{A}_{i-1}\dots\widetilde{A}_0$.

Theorem 3.2 *Assume that for some $x \in \mathbb{T}^1$, $C > 0$ and $\bar{\beta} > 0$, there is a sequence of positive integers $q_n \to \infty$ such that*

$$\sup_{0 \le i < q_n} |V_{i \pm q_n}(x) - V_i(x)|\} < Ce^{-\bar{\beta} q_n}. \tag{3.6}$$

If $L(E) < \bar{\beta}$, then E is not an eigenvalue of the Schrödinger operator $H(T, V, x)$.

Proof. In order to apply Theorem 3.1, it suffices to prove $\limsup_{|n| \to \infty} \frac{\ln \|P_n(E)\|}{n} \le L(E)$. This is a result of Furman [12], and also a well-known corollary of subadditivity, compactness, and unique ergodicity (see e.g. [24] for a short proof). **QED**

For a sequence $q_n \to \infty$, let

$$\hat{\beta} := \limsup_{n \to \infty} \frac{\ln(\sup_{0 \le i < q_n} |x_i - x_{i \pm q_n}|)^{-1}}{q_n}, \tag{3.7}$$

where $x_i = T^i x$.

Let $S_{pp}, P_A, \mathcal{H}, \mathcal{H}_{sc}$ be as in Theorem 1.5.

Theorem 3.3 *Let $V : \mathbb{T}^1 \to \mathbb{R}$ be a α-Hölder continuous real-valued function on the circle, with $\alpha \in (0, 1]$. Then, we have*

(i) $S_{pp}(x) \cap \{E : 0 \le L(E) < \alpha\hat{\beta}\} = \emptyset$,

(ii) $P_{\{E: 0 < L < \alpha\hat{\beta}\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Proof. It suffices to prove the part (i) of the claim, i.e. to exclude the point spectrum. Part (ii) of the claim then follows from Kotani's theory [8] since, the set $\{E : L(E) > 0\}$, does not support any absolutely continuous spectrum.

If $L < \alpha\hat{\beta}$, then $v_i = V(T^i x)$ satisfy the assumption (3.6) of Theorem 3.2 for any $\bar{\beta}$ satisfying $L < \bar{\beta} < \alpha\hat{\beta}$. The claim follows. **QED**

In order to establish Theorem 1.5, all we need is an appropriate bound on $\hat{\beta}(x)$. For $T = R_\rho$ we have $\hat{\beta} = \beta(\rho)$ for all x. This is no longer true in general. However, our goal is to show that for C^{1+BV} diffeomorphisms $\hat{\beta} \ge \beta(\rho)$ for μ-a.e. x, which is sufficient.

3.2 Dynamical partitions of a circle and renormalization

The construction of a set of full measure for which Theorem 1.5 holds is based on the dynamical partitions of a circle. These partitions are obtained by using the continued fraction expansion of the rotation number $\rho \in (0,1)$ of the map T. Every irrational $\rho \in (0,1)$ can be written uniquely as

$$\rho = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}} =: [k_1, k_2, k_3, \dots], \tag{3.8}$$

with an infinite sequence of *partial quotients* $k_n \in \mathbb{N}$. Conversely, every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, \dots, k_n]$, obtained by the finite truncations of the continued fraction expansion (3.8). It is well-known that p_n/q_n form a sequence of best rational approximations of an irrational ρ, i.e., there are no rational numbers, with denominators smaller or equal to q_n, that are closer to ρ than p_n/q_n. The rational convergents can also be defined recursively by $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

To define the dynamical partitions of an orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, with an irrational rotation number ρ, we start with an arbitrary point $x_0 \in \mathbb{T}^1$, and consider the orbit $x_i = T^i x_0$, with $i \in \mathbb{N}$. The subsequence x_{q_n}, $n \in \mathbb{N}$, indexed by the denominators q_n of the sequence of rational convergents of the rotation number ρ, are called the sequence of *dynamical convergents*. It follows from the simple arithmetic properties of the rational convergents that the sequence of dynamical convergents x_{q_n}, $n \in \mathbb{N}$, for the rigid rotation R_ρ has the property that its subsequence with n odd approaches x_0 from the left and the subsequence with n even approaches x_0 from the right. Since all circle homeomorphisms with the same irrational rotation number are combinatorially equivalent, the order of the dynamical convergents of T is the same.

The intervals $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$. We also define $\Delta_i^{(n)} = T^i(\Delta_0^{(n)})$. Certain number of images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, under the iterates of a map T, cover the whole circle without overlapping beyond the end points and form the n-th *dynamical partition* of the circle

$$\mathcal{P}_n := \{T^i(\Delta_0^{(n-1)}) : 0 \le i < q_n\} \cup \{T^i(\Delta_0^{(n)}) : 0 \le i < q_{n-1}\}. \tag{3.9}$$

The intervals $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ will be called the *fundamental intervals* of $\mathcal{P}_n$. These partitions are nested, in the sense that intervals of partition $\mathcal{P}_{n+1}$ are obtained by dividing intervals of partition $\mathcal{P}_n$ into finitely many intervals.

The n-th *renormalization* of an orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, with rotation number ρ, with respect to a point $x_0 \in \mathbb{T}^1$, is a function $f_n : [-1, 0] \to \mathbb{R}$, obtained from the restriction of T^{q_n} to $\Delta_0^{(n-1)}$, by rescaling the coordinates. More precisely, if τ_n is the affine change of coordinates that maps $x_{q_{n-1}}$ to -1 and x_0 to 0, then

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}\,. \tag{3.10}$$

If we identify x_0 with zero, then τ_n is just the multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here, and in what follows, $|I|$ denotes the length of an interval I on $\mathbb{T}^1$.

3.3 A set of full invariant measure

In this section, we construct a set of full invariant measure, for which we have an appropriate control on the distances between an orbit of a point under the map and an orbit of the $\pm q_n$-th iterate of the point, i.e, the control of the quantity $\hat{\beta}$ in (3.7). Let σ_n, $n \in \mathbb{N}$, be any increasing subsequence of $\mathbb{N}$ such that the corresponding sequence k_{σ_n+1} of partial quotients diverges to infinity. We will assume that such a subsequence exists since if the sequence of partial quotients is bounded, then $\beta = 0$. Let η_n be any sequence converging to zero such that $\eta_n k_{\sigma_n+1}$ diverges to infinity as well, as $n \to \infty$.

For each $n \in \mathbb{N}$, let

$$E_{n,0} := \left\{ I \in \mathcal{P}_{\sigma_n+1} | I \subset \Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)},\ |\tau_{\sigma_n}(I)| \leq \frac{1}{\eta_n k_{\sigma_n+1}} \right\}. \qquad (3.11)$$

and let

$$E_{n,i} := T^i(E_{n,0}), \quad \text{for} \quad i = 1, \ldots, q_{\sigma_n} - 1. \qquad (3.12)$$

We define

$$E_n := \bigcup_{i=0}^{q_{\sigma_n}-1} E_{n,i}, \qquad (3.13)$$

and

$$E := \limsup_{n\to\infty} E_n = \bigcap_{n\geq 1} \bigcup_{j\geq n} E_j. \qquad (3.14)$$

Proposition 3.4 $\mu(E) = 1$.

Proof. The number of the elements I of partition $\mathcal{P}_{\sigma_n+1}$ inside of $\Delta_0^{(\sigma_n-1)}$, that do not belong to $E_{n,0}$ is bounded from above by $\eta_n k_{\sigma_n+1}$, since the length of each of the corresponding rescaled intervals $\tau_{\sigma_n}(I) \subset \tau_{\sigma_n}(\Delta_0^{(\sigma_n-1)})$ is larger than $(\eta_n k_{\sigma_n+1})^{-1}$ and the length of their union is less than or equal to 1. Since the invariant measure of the intervals $\tau_{\sigma_n}^{-1}([f_{\sigma_n}^{i-1}(-1), f_{\sigma_n}^{i}(-1)])$ is independent of i and equal to $\mu(\Delta_0^{(\sigma_n)})$, for $i = 1, \ldots, k_{\sigma_n+1}$, and $\Delta_0^{(\sigma_n+1)} \subset \tau_{\sigma_n}^{-1}([f_{\sigma_n}^{i-1}(-1), f_{\sigma_n}^{i}(-1)])$, for $i = k_{\sigma_n+1} + 1$, we have

$$\mu(E_{n,0})/\mu(\tau_{\sigma_n}^{-1}([-1,0])) \geq 1 - \frac{\eta_n k_{\sigma_n+1} \mu(\Delta_0^{(\sigma_n)})}{k_{\sigma_n+1}\mu(\Delta_0^{(\sigma_n)}) + \mu(\Delta_0^{(\sigma_n+1)})} \geq 1 - \eta_n. \qquad (3.15)$$

By the invariance of the measure μ, $\mu(E_{n,i})/\mu(\Delta_i^{(\sigma_n-1)}) \geq 1 - \eta_n$. Since

$$\sum_{i=0}^{q_{\sigma_n}-1} \mu(\Delta_i^{(\sigma_n-1)}) + \sum_{i=0}^{q_{\sigma_n-1}-1} \mu(\Delta_i^{(\sigma_n)}) = q_{\sigma_n}\mu(\Delta_0^{(\sigma_n-1)}) + q_{\sigma_n-1}\mu(\Delta_0^{(\sigma_n)}) = 1, \qquad (3.16)$$

$q_{\sigma_n-1} \le q_{\sigma_n}$ and $\mu(\Delta_0^{(\sigma_n)}) = \mu(\tau_{\sigma_n}^{-1}([-1, f_{\sigma_n}(-1)]))$, we have

$$\mu(E_n) \ge (1-\eta_n)\frac{k_{\sigma_n+1}}{k_{\sigma_n+1}+1}. \tag{3.17}$$

Since $\mu(\cup_{j\ge n} E_j) \ge \mu(E_i)$, for any $i \ge n$, and $\mu(E_i) \to 1$ as $i \to \infty$, it follows that $\mu(\cup_{j\ge n} E_j) = 1$, for any $n \in \mathbb{N}$. The claim follows. **QED**

3.4 Distance of dynamical convergents

In this section, we consider circle maps (orientation-preserving homeomorphisms of a circle) T with an irrational rotation number and bounded variation $V = \mathrm{Var}_{\mathbb{T}^1} \ln T' < \infty$. Consider dynamical partitions of a circle defined by an arbitrary point $\chi_0 \in \mathbb{T}^1$. The following proposition holds for all intervals $I_0 \subset \Delta_0^{(n-1)}$ such that $I_0 \in \mathcal{P}_{n+1}$, and the corresponding intervals $I_i = T^i(I_0)$, $i \in \mathbb{Z}$.

Proposition 3.5 *If T is $C^{1+BV}(\mathbb{T}^1)$ orientation-preserving circle diffeomorphism with an irrational rotation number, there exists $C_1 > 0$ such that $|I_i| \le C_1 |\Delta_i^{(n-1)}| \frac{|I_0|}{|\Delta_0^{(n-1)}|}$, for all $i = 0, \dots, q_n - 1$, and all $n \in \mathbb{N}$.*

Proof. For $i = 0, \dots, q_n - 1$, there exist $\zeta_{i-1} \in I_{i-1} \subset \Delta_{i-1}^{(n-1)}$ and $\xi_{i-1} \in \Delta_{i-1}^{(n-1)}$ such that

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} = \frac{|T(I_{i-1})|}{|T(\Delta_{i-1}^{(n-1)})|} = \frac{T'(\zeta_{i-1})}{T'(\xi_{i-1})}\frac{|I_{i-1}|}{|\Delta_{i-1}^{(n-1)}|}. \tag{3.18}$$

This implies the estimate

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} \le \left(1 + \frac{|T'(\zeta_{i-1}) - T'(\xi_{i-1})|}{T'(\xi_{i-1})}\right)\frac{|I_{i-1}|}{|\Delta_{i-1}^{(n-1)}|}. \tag{3.19}$$

By iterating this inequality, we obtain that, for some $\zeta_j, \xi_j \in \Delta_j^{(n-1)}$,

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} \le \prod_{j=0}^{i-1}\left(1 + \frac{|T'(\zeta_j) - T'(\xi_j)|}{\min_{\xi\in\mathbb{T}^1} T'(\xi)}\right)\frac{|I_0|}{|\Delta_0^{(n-1)}|}. \tag{3.20}$$

Using the obvious inequality $1 + x \le e^x$, we obtain

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} \le \exp\left(\sum_{j=0}^{i-1} \frac{|T'(\zeta_j) - T'(\xi_j)|}{\min_{\xi\in\mathbb{T}^1} T'(\xi)}\right)\frac{|I_0|}{|\Delta_{i-1}^{(n-1)}|}. \tag{3.21}$$

Since, for $i = 0, \ldots, q_n - 1$, the intervals $\Delta_i^{(n-1)}$ do not overlap except possibly at the end points, we have

$$\sum_{j=0}^{q_n-1} |T'(\zeta_j) - T'(\xi_j)| \leq \max_{\chi \in \mathbb{T}^1} T'(\chi) \sum_{j=0}^{q_n-1} |\ln T'(\zeta_j) - \ln T'(\xi_j)| \leq V \max_{\chi \in \mathbb{T}^1} T'(\chi), \qquad (3.22)$$

where $V = \mathrm{Var}_{\mathbb{T}^1} \ln T'$. Since T' is bounded both from below and from above by positive constants, the claim follows. **QED**

Let $l_n = \max_{\xi \in \mathbb{T}^1} |T^{q_n}\xi - \xi|$. If T is $C^{1+BV}(\mathbb{T}^1)$ orientation-preserving circle diffeomorphism, the Denjoy theory implies that, for some $C > 0$,

(A) $\ln(T^{q_n})'(\xi) \leq V$, for any $\xi \in \mathbb{T}^1$,

(B) $l_n \leq C\lambda^n$, where $\lambda = \frac{1}{1+e^{-2V}}$.

Proposition 3.6 *If T is $C^{1+BV}(\mathbb{T}^1)$ orientation-preserving circle diffeomorphism then, there exists $C_2 > 0$ such that, for all $x \in E$, there are infinitely many $n \in \mathbb{N}$ such that*

$$|T^{q_{\sigma_n}}x - x| \leq \frac{C_2 |\Delta_i^{(\sigma_n - 1)}|}{\eta_n k_{\sigma_n+1}}, \qquad (3.23)$$

where $\Delta_i^{(\sigma_n-1)}$ is an element of partition $\mathcal{P}_{\sigma_n}$ containing x.

Proof. For every $x \in E$, there are infinitely many n, such that $x \in E_n$. Furthermore, there exists an element I_i of partition $\mathcal{P}_{\sigma_n+1}$ inside $E_{n,i} \subset \Delta_i^{(\sigma_n-1)}$, for some $i = 0, \ldots, q_{\sigma_n} - 1$, such that $x \in I_i$. It follows from the definition of $E_{n,0}$ and Proposition 3.5 that there exists $\chi \in E_{n,i}$, such that $I_i = [\chi, T^{q_{\sigma_n}}\chi]$ and $|I_i| \leq C_1 |\Delta_i^{(\sigma_n-1)}|/(\eta_n k_{\sigma_n+1})$. Then, $|x - \chi| \leq |T^{q_{\sigma_n}}\chi - \chi| \leq C_1 |\Delta_i^{(\sigma_n-1)}|/(\eta_n k_{\sigma_n+1})$.

Since there exists $\zeta \in I_i$ such that

$$T^{q_{\sigma_n}}x = T^{q_{\sigma_n}}\chi + (T^{q_{\sigma_n}})'(\zeta)(x - \chi), \qquad (3.24)$$

we obtain the following estimate

$$|T^{q_{\sigma_n}}x - x| \leq |T^{q_{\sigma_n}}\chi - \chi| + |\chi - x| + (T^{q_{\sigma_n}})'(\zeta)|x - \chi|. \qquad (3.25)$$

If T is $C^{1+BV}(\mathbb{T}^1)$ orientation-preserving circle diffeomorphism, by the Denjoy estimate (A), we have $\ln(T^{q_n})'(\xi) \leq V$, for all $\xi \in \mathbb{T}^1$. The claim now follows. **QED**

Let $x_i = T^i_{\Omega_\rho} x$ and let $I_i := [x_{i-q_n}, x_i]$, if n is even, or $I_i := [x_i, x_{i-q_n}]$, if n is odd. Let $\chi_0 \in \mathbb{T}^1$, $\chi_j = T^j\chi_0$, and let $\Delta_j^{(n-1)}(\chi_0) := [T^{q_{n-1}}\chi_j, \chi_j]$, if n is even, or $\Delta_j^{(n-1)}(\chi_0) := [\chi_j, T^{q_{n-1}}\chi_j]$, if n is odd.

Proposition 3.7 *Let* $x \in \Delta_j^{(n-1)}(\chi_0)$. *There exists* $C_3 \geq 1$ *such that*

$$|I_i| \leq C_3 |\Delta_i^{(n-1)}(\chi_{j-q_n})| \frac{|I_{q_n}|}{|\Delta_j^{(n-1)}(\chi_0)|}, \tag{3.26}$$

for all $i = 0, \ldots, q_n - 1$.

Proof. To be specific, let us assume that n is even, i.e., $\rho > p_n/q_n$; the proof in the other case is similar.

It follows from the mean value theorem that, for $i = 0, \ldots, q_n - 1$, there exist $\xi_i \in \Delta_i^{(n-1)}(\chi_{j-q_n}) \cup \Delta_i^{(n)}(\chi_{j-q_n})$ and $\zeta_i \in \Delta_i^{(n-1)}(\chi_{j-q_n})$, such that

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} = \frac{|T^{-1}(I_{i+1})|}{|T^{-1}(\Delta_{i+1}^{(n-1)}(\chi_{j-q_n}))|} = \frac{|I_{i+1}|}{|\Delta_{i+1}^{(n-1)}(\chi_{j-q_n})|} \frac{T'(\zeta_i)}{T'(\xi_i)}. \tag{3.27}$$

This implies the estimate

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} \leq \frac{|I_{i+1}|}{|\Delta_{i+1}^{(n-1)}(\chi_{j-q_n})|} \left(1 + \frac{|T'(\zeta_i) - T'(\xi_i)|}{|T'(\xi_i)|}\right). \tag{3.28}$$

By iterating the latter inequality, we obtain

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} \leq \frac{|I_{q_n}|}{|\Delta_j^{(n-1)}(\chi_0)|} \exp\left(\sum_{k=i}^{q_n-1} \frac{|T'(\zeta_k) - T'(\xi_k)|}{\min_{\xi\in\mathbb{T}^1} |T'(\xi)|}\right). \tag{3.29}$$

Since the intervals $\Delta_i^{(n-1)}(\chi_{j-q_n})$, for $i = 0, \ldots, q_n - 1$, belong to the same partition of a circle, taking into the account the order of points ζ_k and ξ_k, for $k = i, \ldots, q_n - 1$, we obtain

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} \leq \frac{|I_{q_n}|}{|\Delta_j^{(n-1)}(\chi_0)|} \exp\left(\frac{\max_{\xi\in\mathbb{T}^1} |T'(\xi)|}{\min_{\xi\in\mathbb{T}^1} |T'(\xi)|} 2V\right). \tag{3.30}$$

The claim follows. **QED**

Proposition 3.6, Proposition 3.7 and Denjoy estimate (A) imply the following lemma.

Lemma 3.8 *If* T *is* $C^{1+BV}(\mathbb{T}^1)$ *orientation-preserving circle diffeomorphism with an irrational rotation number* ρ *then, there exists* $C_4 > 0$ *such that, for all* $x \in E$, *there are infinitely many* $n \in \mathbb{N}$ *such that, for all* $i = 0, \ldots, 2q_{\sigma_n} - 1$,

$$|x_i - x_{i-q_{\sigma_n}}| \leq \frac{C_4 l_{\sigma_n - 1}}{\eta_n k_{\sigma_n + 1}}. \tag{3.31}$$

Proof. For $i = q_{\sigma_n}$, the claim holds directly from Proposition 3.6, with $C_4 \geq C_2$. Proposition 3.6 and Proposition 3.7 together imply (3.31) for $i = 0, \ldots, q_{\sigma_n} - 1$, with $C_4 \geq C_2 C_3$. Using the Denjoy estimate (A), the bound (3.31) can be extended to $i = q_{\sigma_n} + 1, \ldots, 2q_{\sigma_n} - 1$, with $C_4 \geq C_2 C_3 e^V$, since $|x_{i+q_{\sigma_n}} - x_i| \leq e^V |x_i - x_{i-q_{\sigma_n}}|$, for $i = 1, \ldots, q_{\sigma_n} - 1$. **QED**

3.5 Proof of the main result

Proof of Theorem 1.5. Let p_n/q_n be the sequence of rational convergents of ρ, given by the truncations of the continued fraction algorithm of ρ. If ρ satisfies $L < \alpha\beta$, then $\beta > 0$ and there is an increasing sequence σ_n, such that k_{σ_n+1} diverges to infinity. Let η_n be any sequence converging to zero such that $\eta_n k_{\sigma_n+1}$ diverges to infinity as well. We use these sequences to construct the set E, as in section 3.3. For every $x \in E$, by Lemma 3.8, there are infinitely many n, such that estimate (3.31) holds. This implies $\hat{\beta} \geq \beta$. The claim now follows from Theorem 3.3. **QED**

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