SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON FREDHOLM OPERATORS WITH THE BI-LAPLACIAN

Vitali Vougalter[1], **Vitaly Volpert**[2,3]

[1] Department of Mathematics, University of Toronto
Toronto, Ontario, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

[2] Institute Camille Jordan, UMR 5208 CNRS, University Lyon 1
Villeurbanne, 69622, France
[3] RUDN University, ul. Miklukho-Maklaya 6, Moscow, 117198, Russia
e-mail: volpert@math.univ-lyon1.fr

Abstract: We study solvability of some linear nonhomogeneous elliptic problems and prove that under reasonable technical conditions the convergence in $L^2(\mathbb{R}^d)$ of their right sides implies the existence and the convergence in $H^4(\mathbb{R}^d)$ of the solutions. The equations involve the fourth order non Fredholm differential operators and we use the methods of spectral and scattering theory for Schrödinger type operators similarly to our preceding work [26].

AMS Subject Classification: 35J30, 35P30, 47F05
Key words: solvability conditions, non Fredholm operators, Sobolev spaces

1. Introduction

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and $V(x)$ is a function tending to 0 at infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$, which corresponds to the left side of equation (1.1) contains the origin. As a consequence, this operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. In the present work we will study some properties of the operators of this kind. We recall that elliptic problems with non-Fredholm operators were treated extensively in recent years (see [16], [17], [18], [19], [20], [21],

[23], [24], also [5]) along with their potential applications to the theory of reaction-diffusion equations (see [7], [8]). In the particular case when $a = 0$ the operator A satisfies the Fredholm property in some properly chosen weighted spaces [1], [2], [3], [4], [5]. However, the case when $a \neq 0$ is considerably different and the method developed in these works is not applicable.

One of the important questions concerning equations with non-Fredholm operators is their solvability. We address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Since the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. We call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of equation $Au = f$ (see [15]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. Solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of the non Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In this case, solution in the sense of sequences may not imply the existence of the usual solution. In the present work we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent. Solvability in the sense of sequences for the sums of non Fredholm Schrödinger type operators was studied in [25].

In the present article we would like to exploit these ideas for the fourth order differential operators without the Fredholm property. In the first part of the work we study the equation

$$\Delta^2 u - a^2 u = f(x), \ x \in \mathbb{R}^d, \ d \in \mathbb{N}, \tag{1.2}$$

where $a > 0$ is a constant and the right side is square integrable. The problem analogous to (1.2) but with the standard Laplace operator in the context of the solvability in the sense of sequences was considered in [26]. Note that for the operator $\Delta^2 - a^2 : H^4(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ the essential spectrum fills the semi-axis $[-a^2, \ \infty)$ such that its inverse from $L^2(\mathbb{R}^d)$ to $H^4(\mathbb{R}^d)$ is not bounded.

Let us write down the corresponding sequence of equations with $n \in \mathbb{N}$ as

$$\Delta^2 u_n - a^2 u_n = f_n(x), \ x \in \mathbb{R}^d, \ d \in \mathbb{N}, \tag{1.3}$$

where the right sides converge to the right side of (1.2) in $L^2(\mathbb{R}^d)$ as $n \to \infty$. The inner product of two functions

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \tag{1.4}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x)$ is bounded, then clearly the integral in the right side of (1.4) makes sense, like for instance in the case of functions involved in the orthogonality relations of Theorems 1 and 2 below. Let use the spaces $H^2(\mathbb{R}^d)$ and $H^4(\mathbb{R}^d)$ equipped with the norms

$$\|u\|^2_{H^2(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u\|^2_{L^2(\mathbb{R}^d)} \tag{1.5}$$

and

$$\|u\|^2_{H^4(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta^2 u\|^2_{L^2(\mathbb{R}^d)} \tag{1.6}$$

respectively. Throughout the article, the sphere of radius $r > 0$ in $\mathbb{R}^d$ centered at the origin will be denoted by S^d_r. First of all, we formulate the solvability relations for problem (1.2).

Theorem 1. *Let $a > 0,\ f(x) \in L^2(\mathbb{R}^d),\ d \in \mathbb{N}$ and $xf(x) \in L^1(\mathbb{R}^d)$.*

a) When $d = 1$, problem (1.2) admits a unique solution $u(x) \in H^4(\mathbb{R})$ if and only if

$$\left(f(x), \frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0 \tag{1.7}$$

holds.

b) When $d \geq 2$, equation (1.2) possesses a unique solution $u(x) \in H^4(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e. \tag{1.8}$$

holds.

Then we turn our attention to the issue of the solvability in the sense of sequences for our problem.

Theorem 2. *Let $a > 0,\ n \in \mathbb{N}$ and $f_n(x) \in L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Let in addition $xf_n(x) \in L^1(\mathbb{R}^d),\ n \in \mathbb{N}$, such that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$.*

a) When $d = 1$, let the orthogonality relations

$$\left(f_n(x), \frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0 \tag{1.9}$$

hold for all $n \in \mathbb{N}$. Then problems (1.2) and (1.3) admit unique solutions $u(x) \in H^4(\mathbb{R})$ and $u_n(x) \in H^4(\mathbb{R})$ respectively, such that $u_n(x) \to u(x)$ in $H^4(\mathbb{R})$ as $n \to \infty$.

b) When $d \geq 2$, let the orthogonality relations

$$\left(f_n(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e. \tag{1.10}$$

hold for all $n \in \mathbb{N}$. Then problems (1.2) and (1.3) have unique solutions $u(x) \in H^4(\mathbb{R}^d)$ and $u_n(x) \in H^4(\mathbb{R}^d)$ respectively, such that $u_n(x) \to u(x)$ in $H^4(\mathbb{R}^d)$ as $n \to \infty$.

We use the hat symbol to denote the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{-ipx} dx, \ p \in \mathbb{R}^d, \ d \in \mathbb{N}. \tag{1.11}$$

(1.11) will be needed to establish the statements of our Theorems 1 and 2.

In the second part of the work we study the equation

$$(-\Delta + V(x))^2 u - a^2 u = f(x), \ x \in \mathbb{R}^3, \ a > 0, \tag{1.12}$$

with the square integrable right side. The corresponding sequence of approximate equations for $n \in \mathbb{N}$ is given by

$$(-\Delta + V(x))^2 u_n - a^2 u_n = f_n(x), \ x \in \mathbb{R}^3, \ a > 0, \tag{1.13}$$

with the right sides converging to the right side of (1.12) in $L^2(\mathbb{R}^3)$ as $n \to \infty$. Let us make the following technical assumptions on the scalar potential involved in problem above. Note that the conditions on $V(x)$, which is shallow and short-range will be analogous to those formulated in Assumption 1.1 of [17] (see also [18], [19]). However, for the technical purposes we will add a few extra regularity assumptions. The essential spectrum of our Schrödinger operator fills the nonnegative semi-axis (see e.g. [10]).

Assumption 3. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate*

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\delta}}$$

with some $\delta > 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}} \frac{9}{8} (4\pi)^{-\frac{2}{3}} \|V\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)} \|V\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} < 1 \quad and \quad \sqrt{c_{HLS}} \|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi. \tag{1.14}$$

Moreover, $|\nabla V(x)|, \ \Delta V(x) \in L^\infty(\mathbb{R}^3)$.

Here and further down C will stand for a finite positive constant and c_{HLS} given on p.98 of [12] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x) f_1(y)}{|x-y|^2} dx dy \Big| \leq c_{HLS} \|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

By virtue of Lemma 2.3 of [17], under Assumption 3 above on the potential function, the operator $-\Delta + V(x)$ on $L^2(\mathbb{R}^3)$ is self-adjoint and unitarily equivalent to $-\Delta$ via the wave operators (see [11], [14])

$$\Omega^{\pm} := s - \lim_{t\to\mp\infty} e^{it(-\Delta+V)} e^{it\Delta},$$

where the limit is understood in the strong L^2 sense (see e.g. [13] p.34, [6] p.90). Hence $(-\Delta + V(x))^2 : H^4(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ has only the essential spectrum

$$\sigma_{ess}((-\Delta + V(x))^2 - a^2) = [-a^2, \infty)$$

and no nontrivial $L^2(\mathbb{R}^3)$ eigenfunctions. Its functions of the continuous spectrum satisfy

$$(-\Delta + V(x))^2 \varphi_k(x) = |k|^4 \varphi_k(x), \quad k \in \mathbb{R}^3, \tag{1.15}$$

in the integral formulation the Lippmann-Schwinger equation for the perturbed plane waves (see e.g. [13] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} (V\varphi_k)(y) dy \tag{1.16}$$

and the orthogonality conditions

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3. \tag{1.17}$$

In particular, when the vector $k = 0$, we have $\varphi_0(x)$. Let us denote the generalized Fourier transform with respect to these functions using the tilde symbol as

$$\tilde{f}(k) := (f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \ k \in \mathbb{R}^3. \tag{1.18}$$

(1.18) is a unitary transform on $L^2(\mathbb{R}^3)$. The integral operator involved in (1.16) is being denoted as

$$(Q\varphi)(x) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} (V\varphi)(y) dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

Let us consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. Under Assumption 3, by virtue of Lemma 2.1 of [17] the operator norm $\|Q\|_\infty$ is estimated from above by the quantity $I(V)$, which is the left side of the first inequality in (1.14), such that $I(V) < 1$. We have the following proposition dealing with the solvability of equation (1.12).

Theorem 4. *Let the constant $a > 0$, Assumption 3 holds, $f(x) \in L^2(\mathbb{R}^3)$ and in addition $xf(x) \in L^1(\mathbb{R}^3)$. Then problem (1.12) admits a unique solution $u(x) \in H^4(\mathbb{R}^3)$ if and only if*

$$(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \quad k \in S^3_{\sqrt{a}} \quad a.e. \tag{1.19}$$

holds.

Our final main statement is devoted to the solvability in the sense of sequences of problem (1.12).

Theorem 5. *Let the constant $a > 0$, Assumption 3 holds, $n \in \mathbb{N}$ and $f_n(x) \in L^2(\mathbb{R}^3)$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$. Let in addition $xf_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, such that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$ and the orthogonality relations*

$$(f_n(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0,\ k \in S^3_{\sqrt{a}} \quad a.e. \tag{1.20}$$

hold for all $n \in \mathbb{N}$. Then equations (1.12) and (1.13) have unique solutions $u(x) \in H^4(\mathbb{R}^3)$ and $u_n(x) \in H^4(\mathbb{R}^3)$ respectively, such that $u_n(x) \to u(x)$ in $H^4(\mathbb{R}^3)$ as $n \to \infty$.

Note that (1.19) and (1.20) are the orthogonality conditions to the functions of the continuous spectrum of our Schrödinger operator, as distinct from the Limiting Absorption Principle in which one needs to orthogonalize to the standard Fourier harmonics (see e.g. Lemma 2.3 and Proposition 2.4 of [9]).

2. Solvability in the sense of sequences in the no potential case

Proof of Theorem 1. Obviously, if $u(x) \in L^2(\mathbb{R}^d)$ is a solution of (1.2) with a square integrable right side, it belongs to $H^4(\mathbb{R}^d)$ as well. Indeed, directly from (1.2) we have $\Delta^2 u(x) \in L^2(\mathbb{R}^d)$. This implies that $u(x) \in H^4(\mathbb{R}^d)$ via the norm definition (1.6).

To prove the uniqueness of solutions for our equation, let us suppose that (1.2) has two square integrable solutions $u_1(x)$ and $u_2(x)$. Then their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^d)$ as well. Evidently, it solves the equation

$$\Delta^2 w = a^2 w.$$

Since the operator Δ^2 has no nontrivial square integrable eigenfunctions in the whole space, we have that $w(x) = 0$ vanishes in $\mathbb{R}^d$.

We apply the standard Fourier transform (1.11) to both sides of problem (1.2) and arrive at

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^4 - a^2}. \tag{2.21}$$

Clearly, the right side of (2.21) can be easily written as

$$\frac{\widehat{f}(p)}{2a(|p|^2 - a)} - \frac{\widehat{f}(p)}{2a(|p|^2 + a)}. \tag{2.22}$$

Obviously, the second term in (2.22) can be bounded from above in the absolute value by

$$\frac{|\widehat{f}(p)|}{2a^2} \in L^2(\mathbb{R}^d)$$

due to the one of our assumptions. Let us recall the proof of the part a) of Lemma 5 of [24] and the argument to establish the result of the part a) of Lemma 6 of [24]. Hence, the first term in (2.22) is square integrable if and only if (1.7) holds in one dimension and (1.8) for $d \geq 2$. ■

Let us turn our attention to establishing the solvability in the sense of sequences for our equation in the no potential case.

Proof of Theorem 2. Let us suppose $u(x)$ and $u_n(x),\ n \in \mathbb{N}$ are the unique solutions of problems (1.2) and (1.3) in $H^4(\mathbb{R}^d),\ d \in \mathbb{N}$ respectively and it is known that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Then it can be easily verified that $u_n(x) \to u(x)$ in $H^4(\mathbb{R}^d)$ as $n \to \infty$ as well. Indeed, from equations (1.2) and (1.3) we easily obtain that

$$\Delta^2(u_n(x) - u(x)) = a^2(u_n(x) - u(x)) + (f_n(x) - f(x)).$$

Clearly, this gives us

$$\|\Delta^2(u_n(x) - u(x))\|_{L^2(\mathbb{R}^d)} \leq a^2\|u_n(x) - u(x)\|_{L^2(\mathbb{R}^d)} + \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)} \to 0$$

as $n \to \infty$ due to our assumptions. By virtue of the result of the parts a) and b) of Theorem 1, problem (1.3) has a unique solution $u_n(x) \in H^4(\mathbb{R}^d),\ n \in \mathbb{N}$. Let us recall the statement of the part a) of Lemma 3.3 of [26]. Hence, under the given conditions we arrive at the limiting orthogonality relations

$$\left(f(x), \frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0$$

in one dimension and

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R})} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

for $d \geq 2$. Therefore, by virtue of the results of the parts a) and b) of Theorem 1 above, problem (1.2) possesses a unique solution $u(x) \in H^4(\mathbb{R}^d)$. Let us apply the standard Fourier transform (1.11) to both sides of (1.2) and (1.3). This yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^4 - a^2}, \quad \widehat{u}_n(p) = \frac{\widehat{f}_n(p)}{|p|^4 - a^2}, \quad n \in \mathbb{N}. \tag{2.23}$$

This allows us to write $\widehat{u}_n(p) - \widehat{u}(p)$ as

$$\frac{\widehat{f}_n(p) - \widehat{f}(p)}{2a(|p|^2 - a)} - \frac{\widehat{f}_n(p) - \widehat{f}(p)}{2a(|p|^2 + a)}, \quad n \in \mathbb{N}. \tag{2.24}$$

Obviously, the second term in (2.24) can be estimated from above in the absolute value by $\frac{|\widehat{f}_n(p) - \widehat{f}(p)|}{2a^2}$. Thus

$$\left\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{2a(|p|^2 + a)} \right\|_{L^2(\mathbb{R}^d)} \leq \frac{\|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)}}{2a^2} \to 0, \quad n \to \infty$$

via the one of our assumptions. Let us recall the proof of the part a) of Theorem 1.1 of [26] for $d = 1$ and the argument to establish the statement of the part a) of Theorem 1.2 of [26] for $d \geq 2$. Therefore, the first term in (2.24) tends to zero in the $L^2(\mathbb{R}^d),\ d \in \mathbb{N}$ norm as $n \to \infty$ as well. This gives us that

$$\|u_n(x) - u(x)\|_{H^4(\mathbb{R}^d)} \to 0, \quad n \to \infty,$$

which completes the proof of our theorem. ■

3. Solvability in the sense of sequences with a scalar potential

Proof of Theorem 4. First of all we observe that it is sufficient to solve problem (1.12) in $H^2(\mathbb{R}^3)$, since such solution will belong to $H^4(\mathbb{R}^3)$ as well. Indeed, it can be trivially shown that

$$(-\Delta + V(x))^2 u = \Delta^2 u + V^2(x) u - 2V(x)\Delta u - u\Delta V(x) - 2\nabla V(x).\nabla u, \tag{3.25}$$

where $u(x)$ is a solutions of (1.12) belonging to $H^2(\mathbb{R}^3)$. The dot symbol in the last term in the right side of (3.25) stands for the standard scalar product of two vectors in $\mathbb{R}^3$. We observe that the left side of identity (3.25) is square integrable, which easily follows from (1.12) with $f(x) \in L^2(\mathbb{R}^3)$ as assumed. The second term in the right side of (3.25) belongs to $L^2(\mathbb{R}^3)$ since the scalar potential $V(x)$ is bounded via Assumption 3. The third term in the right side of (3.25) is square integrable since $V(x) \in L^\infty(\mathbb{R}^3)$ and $\Delta u \in L^2(\mathbb{R}^3)$ as assumed. The last two terms in the right side of (3.25) belong to $L^2(\mathbb{R}^3)$ because $\Delta V(x)$ and $|\nabla V(x)|$ are bounded due to Assumption 3. Hence, by virtue of equality (3.25) we have $\Delta^2 u(x) \in L^2(\mathbb{R}^3)$, which implies that $u(x) \in H^4(\mathbb{R}^3)$.

To establish the uniqueness of solutions for our equation, we suppose that there exist $u_1(x),\ u_2(x) \in H^4(\mathbb{R}^3)$ satisfying (1.12). Then their difference $w(x) := u_1(x) - u_2(x) \in H^4(\mathbb{R}^3)$ solves the equation

$$(-\Delta + V(x))^2 w = a^2 w.$$

But the operator $(-\Delta + V(x))^2 : H^4(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ does not have any nontrivial eigenfunctions as discussed above. Therefore, $w(x) = 0$ a.e. in $\mathbb{R}^3$.

Let us apply the generalized Fourier transform (1.18) with respect to the functions of the continuous spectrum of our Schrödinger operator to both sides of problem. (1.12). This yields

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|^4 - a^2}. \tag{3.26}$$

The right side of formula (3.26) can be easily written as

$$\tilde{g}_1(k) + \tilde{g}_2(k) = \frac{\tilde{f}(k)}{2a(|k|^2 - a)} - \frac{\tilde{f}(k)}{2a(|k|^2 + a)}. \tag{3.27}$$

Obviously, the second term in the right side of (3.27) can be easily bounded from above in the absolute value by

$$\frac{|\tilde{f}(k)|}{2a^2} \in L^2(\mathbb{R}^3),$$

because $f(x) \in L^2(\mathbb{R}^3)$ due to the one of our assumptions. Moreover, the image of $\tilde{g}_2(k)$ under the inverse of the transform (1.18) will satisfy the equation

$$-\Delta g_2(x) + V(x)g_2(x) + ag_2(x) = -\frac{f(x)}{2a}. \tag{3.28}$$

Since $V(x) \in L^\infty(\mathbb{R}^3)$ and $f(x) \in L^2(\mathbb{R}^3)$ as assumed and $g_2(x) \in L^2(\mathbb{R}^3)$, we deduce from equation (3.28) that $\Delta g_2(x) \in L^2(\mathbb{R}^3)$, such that $g_2(x) \in H^2(\mathbb{R}^3)$. Evidently, the function $g_1(x)$ solves the equation

$$-\Delta g_1(x) + V(x)g_1(x) - ag_1(x) = \frac{f(x)}{2a} \tag{3.29}$$

and its image under transform (1.18) is the first term in the right side of equality (3.27). Let us recall the result stated in the part a) of Theorem 1.2 of [17]. Hence $g_1(x) \in L^2(\mathbb{R}^3)$ (and equivalently $g_1(x) \in H^2(\mathbb{R}^3)$, since the right side of equation (3.29) is square integrable and the scalar potential function $V(x)$ is bounded as assumed) if and only if orthogonality condition (1.19) holds. Since $u(x) \in H^2(\mathbb{R}^3)$, it will belong to $H^4(\mathbb{R}^3)$ as well as discussed above. ∎

Let us proceed to the proof of our last main statement concerning the solvability in the sense of sequences.

Proof of Theorem 5. First of all, we establish that if $u(x)$ and $u_n(x),\ n \in \mathbb{N}$ are the unique $H^4(\mathbb{R}^3)$ solutions of (1.12) and (1.13) respectively and $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^3)$ as $n \to \infty$, then we have $u_n(x) \to u(x)$ in $H^4(\mathbb{R}^3)$ as $n \to \infty$ as well.

Indeed, from (1.12) and (1.13) we easily obtain that

$$(-\Delta + V(x))^2(u_n(x) - u(x)) = a^2(u_n(x) - u(x)) + (f_n(x) - f(x)), \quad n \in \mathbb{N}.$$

Thus

$$\|(-\Delta + V(x))^2(u_n(x) - u(x))\|_{L^2(\mathbb{R}^3)} \le a^2\|u_n(x) - u(x)\|_{H^2(\mathbb{R}^3)} +$$

$$+\|f_n(x) - f(x)\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

as assumed. Evidently, we have the identity

$$(-\Delta + V(x))^2(u_n - u) = \Delta^2(u_n - u) + V^2(x)(u_n - u) - 2V(x)\Delta(u_n - u) -$$

$$-(u_n - u)\Delta V(x) - 2\nabla V(x).\nabla(u_n - u), \tag{3.30}$$

where $u(x)$ and $u_n(x)$ are the solutions of equations (1.12) and (1.13) respectively belonging to $H^4(\mathbb{R}^3)$. Since $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^3)$ as $n \to \infty$, as assumed, we have here

$$u_n(x) \to u(x), \quad \nabla u_n(x) \to \nabla u(x), \quad \Delta u_n(x) \to \Delta u(x)$$

in $L^2(\mathbb{R}^3)$ as $n \to \infty$ and $V(x), |\nabla V(x)|, \Delta V(x)$ are bounded functions due to our Assumption 3 above. Therefore, the second, the third and the last two terms in the right side of identity (3.30) tend to zero in $L^2(\mathbb{R}^3)$ as $n \to \infty$, which implies that $\Delta^2 u_n \to \Delta^2 u$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$ as well. By means of norm definition (1.6) we obtain that $u_n(x) \to u(x)$ in $H^4(\mathbb{R}^3)$ as $n \to \infty$.

By virtue of Theorem 4 above, equation (1.13) admits a unique solution $u_n(x) \in H^4(\mathbb{R}^3),\ n \in \mathbb{N}$. We recall the statement of the part a) of Lemma 3.3 of ([26]). Hence, under our assumptions we obtain the limiting orthogonality relation

$$(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \quad k \in S^3_{\sqrt{a}} \quad a.e.$$

Note that the functions of the continuous spectrum of our Schrödinger operator are bounded under the given conditions due to Corollary 2.2 of [17]. Then by means of Theorem 4 above equation (1.12) possesses a unique solution $u(x) \in H^4(\mathbb{R}^3)$. Let us apply the generalized Fourier transform (1.18) to both sides of equations (1.12) and (1.13). We arrive at

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|^4 - a^2}, \quad \tilde{u}_n(k) = \frac{\tilde{f}_n(k)}{|k|^4 - a^2}, \quad n \in \mathbb{N}. \tag{3.31}$$

This allows us to express $\tilde{u}_n(k) - \tilde{u}(k)$ as

$$\tilde{g}_{1,n}(k) - \tilde{g}_1(k) + \tilde{g}_{2,n}(k) - \tilde{g}_2(k) = \frac{\tilde{f}_n(k) - \tilde{f}(k)}{2a(|k|^2 - a)} - \frac{\tilde{f}_n(k) - \tilde{f}(k)}{2a(|k|^2 + a)}, \quad n \in \mathbb{N}, \tag{3.32}$$

where

$$\tilde{g}_{1,n}(k)=\frac{\tilde{f}_n(k)}{2a(|k|^2-a)}, \quad \tilde{g}_{2,n}(k)=-\frac{\tilde{f}_n(k)}{2a(|k|^2+a)},$$

$\tilde{g}_1(k)$ and $\tilde{g}_2(k)$ are given in formula (3.27). Clearly, $g_{2,n}(x)$ is a solution of the equation

$$-\Delta g_{2,n}(x)+V(x)g_{2,n}(x)+ag_{2,n}(x)=-\frac{f_n(x)}{2a} \tag{3.33}$$

and $g_2(x)$ solves (3.28). Evidently, $\tilde{g}_{2,n}(k)$ can be easily estimated from above in the absolute value by

$$\frac{|\tilde{f}_n(k)|}{2a^2}\in L^2(\mathbb{R}^3), \tag{3.34}$$

since $f_n(x)$ is square integrable as assumed. From (3.32) we easily deduce that

$$|\tilde{g}_{2,n}(k)-\tilde{g}_2(k)|\leq\frac{|\tilde{f}_n(k)-\tilde{f}(k)|}{2a^2},$$

such that

$$\|g_{2,n}(x)-g_2(x)\|_{L^2(\mathbb{R}^3)}\leq\frac{\|f_n(x)-f(x)\|_{L^2(\mathbb{R}^3)}}{2a^2}\to 0$$

as $n\to\infty$ as assumed. By means of (3.33) along with (3.28) we arrive at

$$\|\Delta(g_{2,n}(x)-g_2(x))\|_{L^2(\mathbb{R}^3)}\leq(\|V(x)\|_{L^\infty(\mathbb{R})}+a)\|g_{2,n}(x)-g_2(x)\|_{L^2(\mathbb{R}^3)}+$$

$$+\frac{1}{2a}\|f_n(x)-f(x)\|_{L^2(\mathbb{R}^3)}\to 0$$

as $n\to\infty$ according to our assumptions. Therefore, by means of the norm definition (1.5) we have that $g_{2,n}(x)\to g_2(x)$ in $H^2(\mathbb{R}^3)$ as $n\to\infty$.

Obviously, $g_{1,n}(x)$ solves the equation

$$-\Delta g_{1,n}(x)+V(x)g_{1,n}(x)-ag_{1,n}(x)=\frac{f_n(x)}{2a} \tag{3.35}$$

and $g_1(x)$ is a solution to (3.29). By virtue of the result of the part a) of Theorem 1.4 of [26] under the given conditions, we have $g_{1,n}(x)\in H^2(\mathbb{R}^3)$, such that $g_{1,n}(x)\to g_1(x)$ in $H^2(\mathbb{R}^3)$ as $n\to\infty$. Therefore, $u_n(x)\to u(x)$ in $H^2(\mathbb{R}^3)$ as $n\to\infty$. This implies that $u_n(x)\to u(x)$ in $H^4(\mathbb{R}^3)$ as $n\to\infty$ as discussed above. ■

References

[1] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator. An approach in weighted Sobolev spaces*, J. Math, Pures Appl., **76** (1997), 55–81.

[2] C. Amrouche, F. Bonzom, Mixed exterior Laplace's problem, J. Math. Anal. Appl., **338** (2008), 124–140.

[3] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Holderienne pour des opérateurs différentiels elliptiques dans* R^n, J. Math. Pures Appl., **72** (1993), 105–119.

[4] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Hölderienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), No. 1-2, 315-334.

[5] N. Benkirane, *Propriété d'indice en théorie Holderienne pour des opérateurs elliptiques dans* R^n, CRAS, **307**, Série I (1988), 577–580.

[6] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag, Berlin (1987).

[7] A. Ducrot, M. Marion, V. Volpert, *Systemes de réaction-diffusion sans propriété de Fredholm*, CRAS, **340** (2005), 659–664.

[8] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non Fredholm operators*, Advances Diff. Equations, **13** (2008), No. 11-12, 1151–1192.

[9] M. Goldberg, W. Schlag, *A limiting absorption principle for the three-dimensional Schrödinger equation with* L^p *potentials*, Int. Math. Res. Not., (2004), No. 75, 4049–4071.

[10] B.L.G. Jonsson, M. Merkli, I.M. Sigal, F. Ting, *Applied Analysis*, In preparation.

[11] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann., **162** (1965/1966), 258–279.

[12] E. Lieb, M. Loss, *Analysis. Graduate Studies in Mathematics*, **14**, American Mathematical Society, Providence (1997).

[13] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, III: Scattering Theory*, Academic Press (1979).

[14] I. Rodnianski, W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math., **155** (2004), No. 3, 451–513.

[15] V. Volpert. Elliptic partial differential equations. Volume 1. Fredholm theory of elliptic problems in unbounded domains. Birkhauser, 2011.

[16] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski, *Solvability conditions for elliptic problems with non-Fredholm operators*, Appl.Math., **29** (2002), No. 2, 219–238.

[17] V. Vougalter, V. Volpert, *Solvability conditions for some non-Fredholm operators*, Proc. Edinb. Math. Soc. (2), **54** (2011), No.1, 249–271

[18] V. Vougalter, V. Volpert. *On the solvability conditions for some non Fredholm operators,* Int. J. Pure Appl. Math., **60** (2010), No. 2, 169–191.

[19] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11** (2012), No. 1, 365–373.

[20] V. Vougalter, V. Volpert. *Solvability relations for some non Fredholm operators,* Int. Electron. J. Pure Appl. Math., **2** (2010), No. 1, 75–83.

[21] V. Volpert, V. Vougalter. *On the solvability conditions for a linearized Cahn-Hilliard equation,* Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9.

[22] V. Vougalter. *Sharp semiclassical bounds for the moments of eigenvalues for some Schrödinger type operators with unbounded potentials,* Math. Model. Nat. Phenom., **8** (2013), No. 1, 237–245.

[23] V. Vougalter, V. Volpert. *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order,* Math. Model. Nat. Phenom., **7** (2012), No. 2, 146–154.

[24] V. Vougalter, V. Volpert. *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), No.4, 473–496.

[25] V. Vougalter, V. Volpert. *On the solvability in the sense of sequences for some non-Fredholm operators*, Dyn. Partial Differ. Equ., **11** (2014), No.2, 109–124.

[26] V. Volpert, V. Vougalter. *Solvability in the sense of sequences to some non-Fredholm operators*, Electron. J. Differential Equations, **2013**, No. 160, 16pp.

[27] V. Vougalter, V. Volpert. *Existence of stationary solutions for some integro-differential equations with superdiffusion,* Rend. Semin. Mat. Univ. Padova, **137** (2017), 185–201.