Investigating $f(\mathcal{R})$ gravity and cosmologies

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Abstract

The $f(\mathcal{R})$ theory of gravity is an extended theory of gravity that is based on general relativity in the simplest case of $f(\mathcal{R}) = \mathcal{R}$. This theory extends such a function of the Ricci scalar into arbitrary functions that are not necessarily linear, i.e. could be of the form $f(\mathcal{R}) = \alpha\mathcal{R}^2$. The action for such a theory would be $S_{EH} = \frac{1}{2k}\int f(\mathcal{R}) + \mathcal{L}^m \; d^4x\sqrt{-g}$, where S_{EH} is the Einstein-Hilbert action for our theory, g is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{L}^m$ is the Lagrangian density for matter. In this paper, we will look at some of the physical implications of such a theory, and the importance of such a theory in cosmology and in understanding the geometric nature of such $f(\mathcal{R})$ theories of gravity.

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1 Introduction to formalisms

$f(\mathcal{R})$ theories of gravity are extended theories of gravity which simplify to general relativity in the most elementary case of the function $f(\mathcal{R})$. The idea of understanding this theory stemmed in two ways – one by considering a variation of the Einstein-Hilbert action with the metric (the metric formalism), and the other by varying the metric and an independent connection (called the Palatini formalism)[1]. Each of these methods describe a form of an extended theory of gravity by considering some function of the Ricci scalar – the field equations in each case (i.e. the field equations for an $f(\mathcal{R})$ theory of gravity for some form of the function) can be derived by either of these two formalisms. We can recover the Brans-Dicke theory in these formalisms by setting the Brans-Dicke parameter suitably [2]. The nature of the actions of $f(\mathcal{R})$ theories of gravity was first studied extensively in [3].

We will first look at the variation of the Einstein-Hilbert action in the usual way. We will start by noting that general relativity starts by describing an action of the form

$$S_{EH} = \frac{1}{2k} \int \mathcal{R} + \mathcal{L}^m \; d^4x\sqrt{-g}$$

Where $k = 8\pi G$. In the $f(\mathcal{R})$ theories of gravity, the Ricci scalar $\mathcal{R}$ is replaced by a function of the Ricci scalar $f(\mathcal{R})$. In order to find the field equations for this theory, we need the individual variations of the metric and the connection coefficients, which would be as follows:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}$$

$$\delta\Gamma^{\gamma}_{\mu\nu} = \frac{1}{2}g^{\gamma\lambda}(\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\lambda\mu} - \nabla_\lambda g_{\mu\nu})$$

The variation of the Ricci scalar can also be computed by these variations. This would be

$$\delta\mathcal{R} = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\nabla_\mu\nabla_\nu\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}$$

The variation condition is set by demanding that the variation of the Einstein-Hilbert action is zero – therefore, we have

$$\delta S_{EH} = \delta \int f(\mathcal{R})\; d^4x\sqrt{-g} = 0$$

From this, the required field equations would be

$$f'(\mathcal{R})R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(\mathcal{R}) - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]f'(\mathcal{R}) = kT_{\mu\nu} \tag{1}$$

The nature of the field equations in $f(\mathcal{R})$ gravity is not the same as in GR. The field equations in GR are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu}$$

[1] Interestingly, the formalism in this method was introduced by Einstein, and not Palatini.

By taking the trace of the above equation, we see that we get an alternative form of the field equations in GR as

$$R_{\mu\nu} = k(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{2}$$

Where T is the trace of the energy-momentum tensor. We see that (2) is such that a non-zero value of $T_{\mu\nu}$[2] would *always* correspond to a non-zero value of the Ricci tensor. This is to say that if $T = 0$, then necessarily $\mathcal{R} = 0$, which is seen in the algebraic relation

$$\mathcal{R} = kT$$

However, this is not so in the case of $f(\mathcal{R})$ gravity when $f(\mathcal{R}) \neq \mathcal{R}$. This can be seen by looking at the trace of (1), which is of the form

$$f'(\mathcal{R})\mathcal{R} + 3\Box f' - 2f(\mathcal{R}) = kT \tag{3}$$

It is therefore quite straightforward to see that a case where $T = 0$ does not necessarily imply $R = 0$. This is a primary difference between GR and $f(\mathcal{R})$ gravity. This nature of the $f(\mathcal{R})$ theories of gravity implies that there exist many different solutions than does GR. It is going to be convenient to look at an elementary case of $f(\mathcal{R})$ theory of gravity where we have a non-zero trace of the energy-momentum tensor and a fixed value of R – this would give us an algebraic equation of the Ricci scalar. This would admit two different types of solutions – one when $\mathcal{R} = 0$, and the other when $\mathcal{R} = k$, where k is a constant. This is an indication that there are many possibilities of solutions to a single case of $f(\mathcal{R})$ gravity. This is, in general a reason why showing a direct way of testing an $f(\mathcal{R})$ theory of gravity is difficult – there exist many solutions to each of these theories.

Another way of describing an $f(\mathcal{R})$ theory of gravity is by considering a variation independent of the metric and a connection, i.e. by introducing a new connection [4]. In this way, the action in the Palatini formalism can be described as

$$S_{Palatini} = \int f(\mathcal{R}')\ d^4x\sqrt{-g} + S_m(g_{\mu\nu}, \Psi) \tag{4}$$

In this action, $\mathcal{R}'$ denotes the Ricci scalar defined by the new connection. We will denote the covariant derivative by the new connection as $\bar{\nabla}$. In this new formalism, the field equations are of the form

$$f'(\mathcal{R}')\mathcal{R}'_{\mu\nu} - \frac{1}{2}f(\mathcal{R}')g_{\mu\nu} = kT_{\mu\nu} \tag{5}$$

$$\bar{\nabla}_\alpha(\sqrt{-g}f'(\mathcal{R}')g^{\mu\nu}) = 0 \tag{6}$$

Look at the simplest case of $f(R') = R'$ – then, from (5), we would see that we would get the usual field equations in GR, and (6) would yield the standard Levi-Civita connection. From (6), we can introduce a metric defined by $\tilde{g}_{\mu\nu}$ such that it is related by a transformation $\tilde{g}_{\mu\nu} = f'(\mathcal{R}')g_{\mu\nu}$.

$$f'(\mathcal{R}')\mathcal{R}' - 2f(\mathcal{R}) = kg^{\mu\nu}T_{\mu\nu} \tag{7}$$

[2]Since the trace of the energy-momentum tensor is $T = T^\alpha_\alpha$, it is quite easy to see that for non-zero values of $T_{\mu\nu}$ the relation (2) would mean that $R_{\mu\nu}$ would also be non-zero.

Which is quite simple to get. By some manipulations on a conformal[3] metric to $g_{\mu\nu}$ (which would involve understanding the Christoffel symbols and the Ricci scalars in the initial and primed connection systems), the final equation (5) would take a form independent of the new connection term we used in (5) and (6). In the case of a vacuum solution to (7), we would have $T = 0$, for which the case of quadratic solutions would contain conformal invariance, as seen in [6].

2 An example: the case of quadratic $f(\mathcal{R})$

One of the most elementary examples of an $f(\mathcal{R})$ theory of gravity is the case when

$$f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2 \tag{8}$$

In this case, the first thing to do would be to define the field equations for this theory. The action for this theory can be written as the form

$$\delta \int \mathcal{R} + \alpha\mathcal{R}^2 \; d^4x\sqrt{-g} \tag{9}$$

By varying this action, we get an expansion of the terms $\mathcal{R}$ and $\alpha\mathcal{R}^2$ as the following, where the variation of the Ricci scalar would reduce into the usual variation that includes the Einstein tensor $G_{\mu\nu}$:

$$\int \delta g^{\mu\nu} G_{\mu\nu} \; d^4x\sqrt{-g} \; + \; \delta \int \alpha\mathcal{R}^2 \; d^4x\sqrt{-g} = 0 \tag{10}$$

Here, we can define the term

$$\alpha \, \delta \int \mathcal{R}^2 \; d^4x\sqrt{-g}$$

as

$$2\alpha \left(-\frac{1}{4} \int g_{\mu\nu}\delta g^{\mu\nu}\mathcal{R}^2 \; d^4x\sqrt{-g} \; + \; \int \{\delta g^{\mu\nu}\mathcal{R}_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\}\mathcal{R} \; d^4x\sqrt{-g} \right)$$

Here, the terms in the paranthesis are the results of varying the quadratic term of the Ricci scalar $\mathcal{R}^2$. We can define a new metric and its trace as the variation $\gamma^{\mu\nu} = -\delta g^{\mu\nu}$ and $\gamma = g_{\mu\nu}h^{\mu\nu}$, and transform the integrals into

$$\delta \int \mathcal{R}^2 \; d^4x\sqrt{-g} \; = \; \int \mathcal{R}\{2R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}\}\delta g^{\mu\nu} \; d^4x\sqrt{-g} + 2 \int \mathcal{R}\{g_{\mu\nu}\Box - \nabla_\mu\nabla_\mu\}\delta g^{\mu\nu} \; d^4x\sqrt{-g} \tag{11}$$

We can use (11) in equation (10) to derive the required field equations in our $f(\mathcal{R})$ theory as

$$G_{\mu\nu} + 2\alpha\mathcal{R}\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R} \right) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)2\alpha\mathcal{R} = T_{\mu\nu} \tag{12}$$

[3]A conformal metric $\gamma_{\mu\nu}$ would be a metric that is based on the metric as $\gamma_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where Ω is the conformal factor relating γ and g.

This is the required set of field equations for our theory of gravity where $f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2$. Further, the trace of (12) is of the form

$$\Box\mathcal{R} - \chi\mathcal{R} = \chi T$$

Where χ is a constant equal to $\frac{1}{6\alpha}$.
$f(\mathcal{R})$ theories of gravity such as the one we saw above have many implications in cosmology. Such theories have been investigated for a long time, and there has been a considerable advance in understanding cosmological implications of $f(\mathcal{R})$ theories of gravity in works such as [8, 15, 16]. We will now discuss cosmology in terms of such theories of gravity.

3 Cosmology and $f(\mathcal{R})$ gravity

3.1 Dynamics of cosmology in $f(\mathcal{R})$ gravity

Measurements in observational cosmology show that the rate of expansion (defined by the Hubble factor $H = \frac{\dot{a}(t)}{a(t)}$, where $a(t)$ is the acceleration of the universe) is positive, meaning that there is a speed-up of acceleration. Supernovae measurements [14] show that the universe has a positive acceleration, in contrast to previous predictions that the universe must slow down over time. Such measurements using Type Ia supernovae and stellar distance indicators show that the acceleration of the Universe is caused by an additional term in the physical content of the Universe in the form of a field adding up to the matter content and as an additional source of gravity. In $f(\mathcal{R})$ gravity, we can define the nature of the model in terms of f' to understand the degrees of freedom. We can do this by first looking at the trace of the field equations in $f(\mathcal{R})$ gravity (3). By identifying the derivative f' as some scalar ϕ and a potential $V(\phi)$ in terms of the derivative f'. Using this, we can define the trace of the field equations in $f(\mathcal{R})$ gravity as

$$\Box\phi = \frac{3V' + 8\pi G}{3}T \tag{13}$$

The Friedmann equations in this theory would be of the form

$$H(H + \frac{\dot{\phi}}{\phi}) - \frac{f(\mathcal{R}) - \phi\mathcal{R}}{6\phi} = \frac{8\pi G}{3}(\rho_{\text{total}}) \tag{14}$$

Where ρ_{total} is the total density of matter content and is composed of the matter and radiation densities, or $\rho_{\text{total}} = \rho_{\text{matter}} + \rho_{\text{radiation}}$. The dynamics of such $f(\mathcal{R})$ cosmologies would therefore be composed of a collection $\{\phi,\ \dot{\phi},\ H,\ a\}$. There are several papers that discuss the idea of explaining cosmic speed-up in terms of a description of the "effective" dark matter terms that can be formulated into our theory. Studying the resulting equations of state would show that such a term would have to be less than $-\frac{1}{3}$ in order to incorporate such speed-up.
$f(\mathcal{R})$ theories of gravity have been studied for quite some time, and they have many implications particularly in the aspect of terms in the form of $1/\mathcal{R}$. We will study a particular case of

$$f(\mathcal{R}) = \mathcal{R} - \frac{\alpha^2}{n\mathcal{R}} \tag{15}$$

3.2 The case of Palatini $f(\mathcal{R}) = \mathcal{R} - \alpha^2/n\mathcal{R}$ cosmologies

In this section, we will investigate in particular the case of Palatini $n = 3$ (15), which would give an $f(\mathcal{R})$ theory of gravity of the form [7]

$$f(\mathcal{R}) = \mathcal{R} - \frac{\alpha^2}{3\mathcal{R}} \tag{16}$$

In this case, our theory would have the field equations (1) as

$$\left(1 + \frac{\alpha^2}{3\mathcal{R}^2}\right) R_{\mu\nu} - \frac{1}{2} g_{mu\nu} \left(\mathcal{R} - \frac{\alpha^2}{3\mathcal{R}}\right) = -kT_{\mu\nu} \tag{17}$$

By suitably contracting this equation, (17) takes a quadratic form in $\mathcal{R}$ as

$$\mathcal{R}(\mathcal{R} - kT) = \alpha^2 \tag{18}$$

The solutions of (18) under suitable conditions[4] can be seen to reduce into the form of

$$\mathcal{R} = \frac{kT - \sqrt{k^2T^2 + 4\alpha^2}}{2} \tag{19}$$

The vacuum solution of (19) would give a de Sitter model defining our universe in the case of an $f(\mathcal{R})$ theory taking the form (16). We can further study the nature of the universe in terms of this theory by considering the FLRW metric, which is of the form

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$$

In terms of the parameter α, we can identify that the acceleration factor would be identical to the form $a(t) = e^{Ht} + b$, where $b \equiv b(t)$ is an additional term that we wish to look at in the later phase of the evolution of the near-de Sitter universe, and H would be defined in terms of α by some factor. We can then show that b would take the form

$$b = -\psi e^{-Ht} \tag{20}$$

Where ψ is a coefficient that can be found to be of the following form, where we define α in later times as $3H^2$:

$$\psi = \frac{8\pi G\, \rho}{3\alpha} e^{-Ht}$$

We have done this manipulation to capture that we are describing the evolution of a universe into de Sitter space. (20) shows that the evolution of the universe is exponential into de Sitter space. At early times the universe would be described by the usual field equations in GR – however, at later times this universe would evolve into a de Sitter universe, and eventually (20) shows this evolution to be exponential.

There have also been several papers[5] that considered a form of modified theory of gravity to replace the idea of dark energy with a function of the Ricci scalar so that such cosmic speed-up is explained without dark energy. We will discuss this aspect of a modified form of theory in the following section.

[4] When the dominant energy condition holds true for our theory.

[5] most notably Carroll *et al*'s paper in Physical Review D on a model of modified gravity that considered a function of the Ricci scalar such that it replaces the idea of unaccounted matter contribution to the expansion of the universe (dark energy) with only gravitational effects involving a modified term in the action for our theory.

3.3 Cosmic speed-up with $f(\mathcal{R})$ gravity

Carroll *et al* in [9] introduced the idea of a function $f(\mathcal{R})$ replacing the terms for dark energy to explain cosmic speed-up. It was shown that for a function

$$f(\mathcal{R}) = \mathcal{R} - \frac{\mu^4}{\mathcal{R}}, \tag{21}$$

we can define a theory of gravity such that the need for dark energy was eliminated, and at the same time formulate a model of a non-static universe. The action for our theory when $f(\mathcal{R})$ takes the form of (21) would be

$$S = \frac{1}{2k} \int \left(\mathcal{R} - \frac{\mu^4}{\mathcal{R}} \right) d^4x\sqrt{-g} \; + \; S_m(g_{\mu\nu}, \; \Psi) \tag{22}$$

We can use the standard set of field equations (1) to derive the field equations for the case when $f(\mathcal{R})$ is as (21). This would be of the form

$$\left(1 + \frac{\mu^4}{\mathcal{R}^2}\right) R_{\mu\nu} - \frac{\mathcal{R}}{2}\left(1 - \frac{\mu^4}{\mathcal{R}^2}\right) g_{\mu\nu} + \frac{\mu^4}{\mathcal{R}^2}[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu] \tag{23}$$

The Friedmann equations in GR are extended into non-zero values of μ in our theory of gravity – namely, the Friedmann equations in our $f(\mathcal{R})$ theory go back to the original Friedmann equations in GR when $\mu = 0$, as can be seen from the Friedmann equations in (21) theory[6] as:

$$3 \; H^2 = \tau\mu^4 = \frac{\rho_\mathrm{m}}{k} \tag{24}$$

Where τ is a coefficient comprising of the Hubble factor. When $\mu = 0$, the usual Friedmann equations can be seen to arise from pure GR. The coefficient τ can be seen to be of the form

$$\tau = \frac{2H\ddot{H} + 15H^2\dot{H} + 2\dot{H}^2 + 6H^4}{12(\dot{H} + 2H^2)^3}$$

(24) is the 00 component of the field equations. We can switch the terms of our degree of freedom into the Einstein frame, where a scalar ϕ is introduced. We will define a new conformal metric $h_{\mu\nu}$ in terms of g as [12, 13]:

$$h_{\mu\nu} = p(\phi)g_{\mu\nu} \tag{25}$$

Through this, we can analyse the physical nature of the universe in this frame. In the original matter frame, the energy-momentum tensor for a perfect fluid would be

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}$$

Where we define the pressure as $P = \omega\rho$, where ω is a coefficient that determines the form of matter, i.e. determines if the density is added by radiation or matter. From (25), we define the factor p as $p(\phi) = \exp(\gamma)$ in terms of the usual degree of freedom:

$$\gamma = \sqrt{\frac{2}{3}}\frac{\phi}{M_{pl}}$$

[6] In this theory, we consider the energy-momentum tensor to be that of a perfect fluid.

Where M_{pl} is the Planck mass, $\sqrt{8\pi G}$. We introduce a potential $V(\phi)$ as $\mu^2 M_{pl}^2\left(\frac{p-1}{p^2}\right)$. By analyzing the nature of this term in the Einstein frame in our cosmological setting, we can deduce that the final nature of $f(\mathcal{R})$ theories of the generalised form of (21)

$$f(\mathcal{R}) = \mathcal{R} - \frac{\mu^{2(n+1)}}{\mathcal{R}^n} \tag{26}$$

would be such that there is an intrinsic form of speed-up of cosmologies described by such theories, eliminating the need of an additional contribution by an unaccounted form of matter. Such an effective form of matter would be of the form

$$-1 < \omega_{\text{effective}} < -2/3$$

In [9], it was shown that such a theory of the form (26) with suitable settings on μ and n could yield an explanation for the accelerated expansion of the universe, eliminating the need for a matter term that accounts for the perturbed value of the acceleration of the universe. As seen in the previous case in section 3.2, the description of the universe may vary at different points of time. In a similar way, GR describes, in our present case, some parts of the history of the universe, with the inflatory aspects being explained by our theory using purely gravitational effects.

4 Conclusion

Modified theories of gravitation have many implications, from both a purely geometric background and their implications in cosmology. In this review, we have considered a look at $f(\mathcal{R})$ theories of gravity, their construction and some of their cosmological effects in the field of explaining the nature of the universe and their physical dynamics, in particular the acceleration of the universe. We will conclude this paper by noting that there are many numerical implications of modified theories of gravity, and as we have seen, those models that are quadratic have many roots in explaining the physical nature of cosmologies.

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