Decribing cosmological entropy using the Newman-Penrose formalism

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Abstract. The gravitational entropy proposal describes the evolution of a cosmology in terms of the Weyl curvature, where the beginning of the universe implies a low entropy state due to the homogeneous and isotropic nature of the universe, implying a perfect Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. The Weyl curvature of the FLRW solution is zero due to conformal flatness, and as time progresses, gravitational effects play a significant role in determining structure formations, which account for anisotropies. These anisotropies in turn imply a departure from the FLRW metric, and therefore the Weyl curvature increases. By using the Weyl curvature as a quantification of entropy in some sense, Penrose introduced the Weyl curvature hypothesis [6], which considered the Weyl curvature as the description of a gravitational analog of the second law of thermodynamics. In this paper, we will look at gravitational entropy proposals and their ranges, focusing on the Weyl invariant based proposal [1], the Newmann-Penrose formalism [18] approach towards cosmologies as introduced by Clifton, Ellis and Tavakol [11] and the Weyl scalar Ψ_2 based proposal by Gregoris and Ong [7] (since we will only consider Petrov type D spacetimes in this paper).

INTRODUCTION

The universe in the present time has higher anisotropies than in the early times, when the universe was far more homogeneous and isotropic than now. This can be attributed to the second law of thermodynamics, which refers to the entropy of a system always increasing with time, and therefore the initial state of the universe had a much lower entropy configuration than in later states. Penrose's Weyl curvature hypothesis [6] aims at describing the universe using gravitational entropy, which is governed by a set of conditions that allow a consistent gravitational entropy. The hypothesis is based on the observation that at the early stages of the universe, since the anisotropies formed by gravitational "clumping" of matter would be lesser than present, the Weyl curvature decreases as you move closer towards times very close to the initial singularity, where the Weyl curvature disappears due to the FLRW model being conformally flat. This observation led Penrose to conjecture that a Weyl curvature based approach towards defining a sort of gravitational form of entropy would explain the evolution of the universe. This has been mathematically formulated in [1], where the equivalence between the holographic entropy of black holes (the Hawking-Bekenstein entropy relation) and the gravitational entropy for black holes was used as the basis for describing gravitational entropy for several models. Recall that the Hawking-Bekenstein entropy is given by

$$S_{HB} = \frac{Ac^3}{4G\hbar} \tag{1}$$

Where A is the area of the event horizon. Since the gravitational entropy must be equal to the Hawking-Bekenstein entropy in the case of black holes, we can define a surface interal using a scalar containing the Weyl curvature invariant terms to define the gravitational entropy. This allows us to find the coefficient in the integral, which can be done by considering the Schwarzschild solution (as we will see in section 2, the simplest case of the scalar field being the square root of the Weyl invariant (constructed by forming the "square" of the Weyl tensor $W = C_{abcd}C^{abcd}$) cannot be considered due to blow-up of Penrose's description near isotropic singularities under such a scalar Wainwright and Anderson (1984). We therefore apply a factor containing the Kretschmann invariant $K = R_{abcd}R^{abcd}$, and using this form of the scalar, the gravitational entropy is maximal). There are two reasons why black holes are of particular interest while discussing gravitational entropy. Firstly, black holes contain the highest entropy for their physical configuration, relating the entropy to the area of the horizon by only a constant factor. Secondly, since black holes contain vacuum solutions, the entropy contribution can be considered of a gravitational origin, which fuels to the interest in describing black holes via gravitational entropy.

However, their proposal fails for the de Sitter spacetime, where we have a cosmological constant Λ with a cosmological horizon, in which case the entropy would resemble the Hawking-Bekenstein relation but with a cosmological horizon radius rather than an event horizon. In the Weyl invariant based formulation, such solutions are not described by the gravitational entropy proposal, since we would get zero entropy (due to a vanishing Weyl tensor) rather than the Gibbons-Hawking entropy as we would expect.

A 2013 proposal by Clifton, Ellis and Tavakol [Clifton, Ellis and Tavakol (2013)] describes gravitational entropy using the Newmann-Penrose formalism, where instead of basing gravitational entropy on the black hole entropy equivalence, the basis is by considering spacetimes with a non-vanishing Weyl curvature via a tetrad. In this case, while the Hawking-Bekenstein entropy can be reproduced in the cases of gravitational entropy for black holes, the approach also applies for several cosmological models, allowing a description of gravitational entropy in such cases.

In order for gravitational entropy to be consistent with our formulation of Weyl curvature, any proposal describing gravitational entropy must satisfy the following conditions:

1. The gravitational entropy S_g vanishes only in cases when the Weyl tensor is zero $S_g = 0 \leftrightarrow C_{abcd} = 0$ – i.e. gravitational entropy vanishes only for conformally flat spacetimes,
2. S_g must also account for anisotropies,
3. S_g must increase monotonically with time, and
4. S_g must be the Hawking-Bekenstein entropy in the case of black hole solutions

This paper is made of three sections – first, an introduction to the Weyl invariant based proposal. Second, we will introduce the CET formulation of gravitational entropy, where we use the tetrad formalism to find different forms of the Weyl scalar. The third section will focus on a gravitational entropy proposal that uses the Newman-Penrose formalism to describe gravitational entropy, and we will consider an example of a spacetime to illustrate the nature of this spacetime.

WEYL INVARIANT BASED PROPOSAL

The thermodynamic entropy of black holes was first described in terms of the area of the event horizon by a thought experiment by Bekenstein, which considered a box of gas being lowered into a black hole. Bekenstein realised that the entropy of the black hole must be related to the variation in the area of the event horizon, and this was forwarded by Hawking in their work on the entropy of black holes. Since the entropy of a particle falling into a black hole seems to "disappear" from the view point of an observer outside the black hole, the total entropy of the black hole can be described in terms of the variation using the Hawking-Bekenstein entropy [**?**]:

$$S_{BH} = S_t + S_{HB}$$

This was the "generalised version of the second law of thermodynamics", and was in agreement with the entropy variation constraint from thermodynamics

$$\frac{\Delta S}{\Delta t} \geq 0$$

Rudjord and Gron [1] use the equivalence condition between the gravitational entropy and the Hawking-Bekenstein entropy in the case of black holes as a starting point, extending the analogy to other spacetimes, such as the Schwarzschild de Sitter spacetime. In this approach, we start by noting that

$$S_g \equiv S_{HB} = \frac{Ac^3}{4G\hbar}$$

Then, we define a surface integral over the horizon:

$$S_g = k_s \int_\sigma \Psi \mathbf{e}_r \cdot d\sigma \tag{2}$$

Where $\Psi\mathbf{e}$ is a vector field in the radial direction and k_s is a constant, which we will determine soon. The scalar Ψ is composed of the Weyl invariant in the most elementary case, but due to isotropic singularities [9], it must contain a factor involving either the Ricci invariant $R = R_{ab}R^{ab}$ or the Kretschmann invariant $K = R_{abcd}R^{abcd}$, of which we will choose the latter – the scalar Ψ would now be of the form

$$\Psi = \sqrt{\frac{C_{abcd}C^{abcd}}{R_{abcd}R^{abcd}}}$$

Suppose we wanted to compute the integral (2) for the Schwarzschild spacetime, described by the metric

$$ds^2_{sch} = -dt^2 e^{2\alpha(r)} + dr^2 e^{-2\alpha(r)} + r^2 d\Omega^2 \tag{3}$$

In this case, the Weyl invariant and the Kretschmann invariant are equal, reducing Ψ to 1, implying that the case of the Schwarzschild solution is that of maximal gravitational entropy.

In order to compute the integral, we must introduce a spatial metric involving the spatial components of the metric, which would be in a diagonalised form due to the nature of the metric ($g_{ij} = 0$ for $i \neq j$). Using this, we can find the element $d\sigma$ to find the integral. But, also note that the singularity at $r = 0$ in the solution cannot be included in the integral, otherwise the integral would diverge. Therefore, we consider a small sphere of radius ε around the singularity and let it tend to zero while computing the integral to find the gravitational entropy. Using the divergence theorem, we can transform the surface integral into a volume integral:

$$S_g = k_s \int_V |\nabla \cdot \Psi| dV \tag{4}$$

Using this, we can find out the value of the entropy density variation, which would be of the form $s_d = k_s|\nabla \cdot \Psi|$. Here, the absolute brackets are necessary to prevent a negative value of the entropy density. But first, we have to solve the integral (2). Since the scalar Ψ is equal to 1, the components $\Psi_{R_{sch}}$ and Ψ_{R_ε} would be simply $R^2_{sch} - R^2_\varepsilon$. This would reduce to the familiar form

$$S_g = 4\pi k_s R^2_{sch} \tag{5}$$

Since $S_g = S_{HB}$, the constant k_s can be equated to the factors of A in S_{HB}, giving us

$$k_s = \frac{c^3}{4G\hbar} \tag{6}$$

This method can be applied to a range of astrophysical models, including black holes and wormholes and works quite consistently. However, the description of entropy attributed to cosmological horizons is not described by this proposal. A step further was taking using the description of gravitational entropy using the Newman-Penrose formalism. In this approach, we write the law of thermodynamics in terms of gravitational analogs:

$$T_g dS_g = dE_g + p_g dV \tag{7}$$

Where T_g, E_g and p_g are the gravitational analogs of temperature, energy and pressure respectively. Note that these are not contributions from matter fields on (M, g) and therefore do not contribute to the energy-momentum tensor in any way. We use the terms ρ_g and T_g in terms of the Weyl scalars in order to define gravitational entropy via a volume integral, and the thermodynamic variables are based on the nature of the spacetime – under different conditions, these variables take up differnet forms based on the Weyl scalar in consideration.

CET PROPOSAL

The CET proposal is based on the algebraic square root of the Bel-Robinson tensor, which is defined in terms of the Weyl tensor and its dual component. While this proposal works only for spacetimes that are based on general relativity and are Petrov type D or N (which we have discussed below), i.e. either Coulumb-like or wave-like fields. Also, on applying the CET proposal to cosmologies, there are cases that violate the Weyl curvature hypothesis due to a non-ncreasing gravitational entropy.

In order to find the directional derivatives and the Weyl scalars, the first step is to construct a tetrad, built of three unique null vectors and one null vector being the conjugate of the complex null vector in the tetrad. With this, we have the following tetrad, where m^a is the complex null vector and $\bar{m}^a$ is the conjugate null vector of m^a forming the tetrad:

$$\begin{aligned} l^a &= \tfrac{1}{\sqrt{2}}(u^a - z^a) \\ n^a &= \tfrac{1}{\sqrt{2}}(u^a + z^a) \\ m^a &= \tfrac{1}{\sqrt{2}}(x^a - iy^a) \\ \bar{m}^a &= \tfrac{1}{\sqrt{2}}(x^a + iy^a) \end{aligned}$$

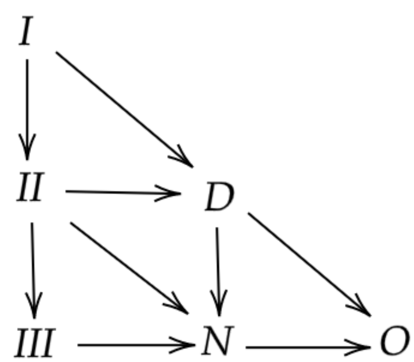
Where x^a, y^a and z^a are spacelike unit vectors, and u^a is a timelike unit vector. The tetrad here $\{l^a, n^a, m^a, \bar{m}^a\}$ follows

$$l_a l^a = n_a n^a = m_a m^a = \bar{m}_a \bar{m}^a = 0$$

We see that the use of the Weyl scalars using the above tetrad requires us to be familiar with the nature of the Weyl scalar in each spacetime. Some spacetimes have certain components non-zero, and in the case of the CET proposal the Ψ_2 and Ψ_4 components are non-zero. Following the requirement for describing the Petrov classification, we can define the Petrov types of spacetimes as:

- $\Psi_0 = 0$ – Petrov type I
- $\Psi_0 = \Psi_1 = 0$ – Petrov type II
- $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ – Petrov type D
- $\Psi_0 = \Psi_1 = \Psi_2 = 0$ – Petrov type III
- $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ – Petrov type N
- $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$ – Petrov type O

The Petrov classifications can be further shown as:



The Petrov classification – the arrows represent classes. Note how both Petrov D and N type spacetimes are parts of Petrov O type spacetimes.

CET uses the nature of non-vanishing Ψ_2 and Ψ_4 to describe the gravitational entropy of different cosmologies. Consider the case of the FLRW solution, described by the metric

$$ds^2_{FLRW} = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{8}$$

Where $a(t)$ is the scale factor and k is the sectional curvature term. In this spacetime, the metric is conformally flat and the Weyl tensor vanishes, and therefore all the Weyl scalars are zero. Due to this, the FLRW solution is considered to be of Petrov type O. Following CET, we define the gravitational entropy to be

$$S_g = \int \frac{\rho_g}{T_g} dV \tag{9}$$

The terms ρ_g is based on the Weyl scalar Ψ_2, and therefore vanishes. The temperature T_g is non-zero in this case, taking up a form based on the scale factor, therefore setting $S_g = 0$ in the case of FLRW solution. While this may not seem like a surprise, this calculation showed us the motivation to the general class of Petrov type D spacetimes, of which there are many cosmological examples [2, 4, 9, 15].

We can consider the case of Bianchi type I spacetime with the condition of local rotational symmetry (referred to as LRS Bianchi type I spacetimes) to illustrate the above example in non-vanishing cases of Ψ_2, i.e. Petrov type D spacetimes. In the case of non-LRS Bianchi type I spacetimes, the metric in terms of the tuples $x^\mu = (t,x,y,z)$ is:

$$ds^2_{\text{Bianchi I}} = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2 \tag{10}$$

The condition of being LRS implies that $b^2(t) = c^2(t)$, reducing the metric to

$$ds^2_{\text{LRS Bianchi I}} = -dt^2 + a^2(t)dx^2 + b^2(t)(dy^2 + dz^2) \tag{11}$$

This is a Petrov type D spacetime, since Ψ_2 is non-zero, and therefore the gravitational entropy cannot be zero in this case.

We will now look at the case of using the Newman-Penrose formalism via [7], where we consider the Weyl scalars and DW, where $D \equiv n^\mu \nabla_\mu$ and W is the Weyl tensor. This formulation again considers the tetrad approach, and runs on a track parallel to the CET proposal, considering these parameters instead of the gravitational analogs which are equivalent in formulation.

A PETROV TYPE D SPACETIME PROPOSAL

The proposal by Gregoris and On [7] can be broken down into three steps – first, we consider a tetrad for our spacetime. Next, we find the Weyl scalar Ψ_2 and DW using the tetrad. Finally, we find the gravitational entropy via an integral over the surface of the horizon with appropriate limits. In order to illustrate this, consider the case of a Schwarzschild black hole described by the metric (3). Choosing a tetrad [7]

$$\begin{aligned} l_a &= \tfrac{1}{\sqrt{2}}\left(\sqrt{e^{2\alpha(r)}}dt - e^{-2\alpha(r)}dr\right) \\ n_a &= \tfrac{1}{\sqrt{2}}\left(\sqrt{e^{2\alpha(r)}}dt + e^{-2\alpha(r)}dr\right) \\ m_a &= \tfrac{1}{\sqrt{2}}\left(rd\theta + ir\sin\theta d\phi\right) \\ \bar{m}^a &= \tfrac{1}{\sqrt{2}}\left(rd\theta - ir\sin\theta d\phi\right) \end{aligned}$$

Using this tetrad, we can construct the Weyl scalar Ψ_2 and DW (for simplicity, we will consider the function $e^{2\alpha(r)} = f(r)$):

$$\begin{aligned} \Psi_2 &= \tfrac{L(r)}{12r^2} \\ DW &= \tfrac{L(r)\sqrt{2f(r)}}{8r^3} \end{aligned}$$

The gravitational entropy then is given by the integration of the term $F = \frac{DW}{\Psi_2}$ with absolute brackets to prevent a negative value. The gravitational entropy then can be found out using F. However, before calculating the integral, it is important to note that DW vanishes on the horizon, and therefore we must consider a term to account for this, which can be done by introducing a term $1/\sqrt{f(r)}$. With this in mind, we construct the integral under appropriate limits

$$S_g = \delta \int \left|\frac{DW}{\Psi_2}\right| \frac{dV}{\sqrt{f(r)}} = \frac{A}{4G\hbar} \tag{12}$$

This satisfies our previously imposed condition that the gravitational entropy must reduce to the Hawking-Bekenstein entropy. The constant δ is found out to be $\frac{1}{3\sqrt{2}}$, which can be found by investigating the relation between coefficients in DW and Ψ_2.

Note that this only applies for the cases of spacetimes where the spin coefficients μ and ρ reduce to this form. In the other cases, we must consider the indiviudal forms of μ and ρ – therefore, the aspect of considering only Petrov type D spacetimes is in the case when the form DW/Ψ_2 is important. The proposal is actually based on μ and ρ, and this becomes clear in cases such as FLRW spacetime when we are not considering static and spherically symmetric black holes where the coefficients are equal. An interesting feature of this proposal is that the upper limit over the integral is always a horizon. That is, the upper limit on the integral must specify the event horizon or the cosmological horizon when applicable in order for the integral to result in the Hawking-Bekenstein entropy. In the case of the de

Sitter spacetime, we have a cosmological horizon at $R = \sqrt{3/\Lambda}$, and therefore this proposal incorporates the de Sitter entropy in the form of Gibbons-Hawking entropy [3]. Further, this proposal applies for spacetimes where Ψ_2 is non-vanishing (in our consideration – otherwise we have to consider the spin coefficients individually as stated before), since that is the base of the proposal. Comparing with the CET proposal, the factors DW and Ψ are similar to the factors ρ_g and T_g respectively – however, this proposal is much more wider for non-vanishing Ψ_2 spacetimes, since the only requirement is that of a horizon as the upper limit on the integral.

CONCLUSION

The gravitational entropy landscape has several aspects of developments, particularly those involving the computation of gravitational entropy in spacetimes with a cosmological horizon. The Weyl invariant based proposal does not provide the source of origin of the Gibbons-Hawking entropy for de Sitter spacetime in the form of a gravitational contribution, which poses a problem with regard to the aspect of the Weyl curvature hypothesis. Further, the removal of the singularity via the subtraction of a sphere around the singularity is an *ad hoc* condition. The CET proposal provides a consistent description in many cosmological models, but only does so for Petrov type D and N spacetimes, and has been shown to have a negative entropy variation [17]. The Cartan invariant based proposal provides a good description of gravitational entropy in the case of Petrov type D spacetimes, and also fulfills our requirement for the understanding of cosmological horizons, and the only conditions on this proposal are that the Weyl scalar Ψ_2 is non-vanishing (in the case of $\mu = \rho \propto DW/\Psi_2$ – the case $\mu \neq \rho$ has also been described in [7]) and the requirement of a horizon for the upper limit of the integral. Further idea of how the cosmological implications of this proposal develop will be understood in the future, and extensions of this proposal into higher order theories of gravity such as Gauss-Bonnet and Lovelock gravity will be provided soon.

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