Splitting and coexistence of 2-D strange attractors in a general family of Expanding Baker Maps

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2023

Abstract. We consider a two-parameter family $\Gamma_{a,\theta}$ of Expanding Baker Maps on the plane, being $a > 1$ and $0 < \theta < \pi$ an expansion rate and a rotation angle, respectively. We prove that $\Gamma_{a,\theta}$ exhibits strange attractors for every a sufficiently close to 1. We also study how such attractors may split into other ones of a larger number of connected pieces as a decreases to 1 and θ/π is a rational number. The study of the family $\Gamma_{a,\theta}$ is strongly motivated by the rich dynamics observed for the quadratic family $T_{a,b}(x, y) = (a + y^2, x + by)$.

1. Introduction

Chaos, as the behaviour of the evolution of a certain process, can be greatly explained when the modelling family of dynamical systems exhibits, in a observable way, strange attractors.

Definition 1.1. *An* ***attractor*** *for a transformation f defined on a manifold $\mathcal{M}$ is a transitive f-invariant compact set $\mathcal{Z}$ whose stable set*

$$W^s(\mathcal{Z}) = \{P \in \mathcal{M} : d(f^n(P), \mathcal{Z}) \to 0 \text{ as } n \to \infty\}$$

has nonempty interior. An attractor is said to be ***strange*** *if it contains a dense orbit $\{f^n(P_0) : n \geq 0\}$ with some positive Lyapunov exponent, i.e. there exist a unit vector v and a constant $c > 0$ such that, for every $n \geq 0$,*

$$\|Df^n(P_0)(v)\| \geq e^{cn}.$$

The supremum λ of such c is called a Lyapunov exponent*. When there exist k positive Lyapunov exponents λ_k (or, at least, such that $\lambda_1 + \cdots + \lambda_k > 0$), the strange attractor is said to be a k-**dimensional strange attractor**.*

The observability of strange attractors is expressed in terms of their persistence with respect to small changes in the parameters of the family. The abundance of chaotic dynamics in nature suggests that this persistence shows at least in a probabilistic sense.

Definition 1.2. *Let $f_\mu\colon \mathcal{M} \to \mathcal{M}$ be a family of maps such that f_{μ_0} has a strange attractor. This attractor is said to be* ***persistent*** *if, for every $\delta > 0$, strange attractors still exist for values of the parameter μ belonging to a positive Lebesgue measure set $E \subset B(\mu_0, \delta)$. If $E = B(\mu_0, \delta)$ for some $\delta > 0$, then the strange attractor is said to be* ***fully persistent****.*

Numerous works in the literature numerically simulate chaotic behaviours that suggest the presence of strange attractors. However, analytical results proving the existence of such attractors and their persistence in generic families of dynamical systems are lacking and usually laborious. A first proof of the persistence of strange attractors for a family of diffeomorphisms was given in [1] for the Hénon family

$$H_{a,b}(x, y) = (1 - ax^2 + y, bx). \tag{1}$$

The starting point of such intricate proof is to note that for $b = 0$ the dynamics of family (1) reduces to that of the quadratic family

$$f_a(x) = 1 - ax^2, \tag{2}$$

which is said to be a *limit family* of family (1). In [2] the authors had previously proved that this quadratic family has persistent strange attractors for $a \in (2 - \varepsilon, 2]$. After a lot of hard work, it was proved in [1] that these strange attractors persist on a branch of the unstable manifold of the saddle point of family (1) when $0 < b < \varepsilon \ll 1$.

The first proof of the persistence of strange attractors in a wider scenario was given in [3], where it is proved the persistence of strange attractors in a generic family of diffeomorphisms $f_\mu\colon \mathcal{M} \to \mathcal{M}$ on a surface $\mathcal{M}$, that unfold a homoclinic tangency. This proof is strongly based on the existence of families of return maps associated to the unfolding of homoclinic tangencies. See also [4], [5], [6], and [7]. Under an appropriate change of coordinates, these return maps are defined in a neighbourhood of the homoclinic point, and are very similar to the Hénon family (1). Hence their attractors are called *Hénon-like attractors*. A proof on the existence of strange attractors for families of vector fields on $\mathbb{R}^3$, which are not Hénon-like, can be seen in [8].

The proof of the persistence of strange attractors given in [2] for family (2) is complicated because the zero derivative at the critical point $x = 0$ hinders the expansivity of the orbits close to it, and sinks can thus appear. This difficulty does not arise when considering the one-parameter family of one-dimensional piecewise linear maps (*tent maps*) $\lambda_\mu\colon [0, 1] \to [0, 1]$, with $1 < \mu \le 2$, given by

$$\lambda_\mu(x) = \begin{cases} \mu x & \text{if } x \le 1/2, \\ \mu(1 - x) & \text{if } x \ge 1/2. \end{cases} \tag{3}$$

In this case, the interval $[(2 - \mu)\mu/2, \mu/2]$ is an invariant set for λ_μ, when $\mu \in (1, 2]$. This invariant set is a strange attractor for every $\mu \in (\sqrt{2}, 2]$. Strange attractors with several pieces may also be obtained for values of the parameter $\mu \in (1, \sqrt{2}]$ by using renormalization techniques (see for instance [9], [10] or [11] for related details).

In many cases, the dynamics of the tent maps given in (3) is similar (i.e. conjugate) to the dynamics of family (2). It is well known that this is the case for $\mu = a = 2$. This fact is the first step in [2] to prove that family (2) displays persistent strange attractors for values of the parameter sufficiently close to $a = 2$. In summary, the study of the dynamics of piecewise linear families is a first step towards understanding the more complicated dynamics of quadratic families.

All the above-mentioned attractors are one-dimensional. In order to get abundance of strange attractors with two positive Lyapunov exponents, we can consider a generic two-parameter family $f_{a,b}\colon \mathcal{M} \to \mathcal{M}$ of three-dimensional diffeomorphisms unfolding a generalised homoclinic tangency as it was originally defined in [12].

For the case in which the unstable manifold involved in the homoclinic tangency has dimension one, the limit family corresponds to Hénon family. But, and this fact explains why we are not working with generalizations of the Hénon family, if the unstable manifold has dimension two then the limit family is conjugate to the family of two-dimensional endomorphisms defined in $\mathbb{R}^2$ by

$$T_{a,b}(x, y) = (a + y^2, x + by). \tag{4}$$

Therefore, if one tries to show the persistence of two-dimensional strange attractors when such a homoclinic tangency is unfolded, the first step should be to prove, as was done in the one-dimensional setting, the persistence of strange attractors for the limit family (4). Only after this does it make sense to lift the dynamics to the closure of the unstable manifold, which is the candidate to contain the two-dimensional strange attractor arising in the unfolding of the tangency.

Different types of strange attractors for family (4) were numerically detected in [13]. In particular, the different regions of the parameter space according to the number of positive Lyapunov exponents are plotted (see Figure 1 in [13]). In [14], a curve of parameters

$$G = \left\{ (a(s), b(s)) = (-\frac{1}{4}s^3(s^3 - 2s^2 + 2s - 2), -s^2 + s) : s \in \mathbb{R} \right\}$$

was constructed in such a way that $T_{a,b}$ has an invariant region in $\mathbb{R}^2$ homemorphic to a triangle for all $(a, b) \in G$. This curve contains the point $(-4, -2)$ (by taking $s = 2$), and the map $T_{-4,-2}$ is conjugate to the non-invertible piecewise affine map

$$\Lambda(x, y) = \begin{cases} (x + y, x - y) & \text{if } (x, y) \in \mathcal{T}_0, \\ (2 - x + y, 2 - x - y) & \text{if } (x, y) \in \mathcal{T}_1 \end{cases}$$

defined on the triangle $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$, where

$$\mathcal{T}_0 = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq x\},$$
$$\mathcal{T}_1 = \{(x, y) : 1 \leq x \leq 2,\ 0 \leq y \leq 2 - x\}.$$

As was pointed in [14], this map Λ enjoys the same nice properties as tent map given in (3). In particular, the consecutive pre-images $\{\Lambda^{-n}(\mathcal{C})\}_{n\in\mathbb{N}}$ of the critical line $\mathcal{C} = \{(x, y) \in \mathcal{T} : x = 1\}$ define a sequence of partitions (whose diameter tends to zero as n goes to infinity) of $\mathcal{T}$ leading the authors to conjugate Λ (and therefore $T_{-4,-2}$) to a one-sided shift on two symbols. Furthermore, for every initial point $(x_0, y_0) \in \mathcal{T}$ whose orbit never visits the critical line, the Lyapunov exponent of Λ along the orbit of (x_0, y_0) is positive (in fact, it is equal to $\frac{1}{2}\log 2$) in all non-zero direction, and the same holds for the limit return map T_{-4-2}. Finally, an absolutely continuous ergodic invariant measure for Λ can be constructed, and therefore the same holds for T_{-4-2}. These basically were the main reasons why the authors in [14] called Λ the 2-D tent map.

As a first approach to the study of the dynamics of $T_{a(s),b(s)}$ (with $s \neq 2$) the family $\{\Lambda_t\}_{t\in[0,1]}$ of piecewise linear maps of $\mathcal{T}$ given by

$$\Lambda_t(x,y) = t\cdot\Lambda(x,y) = \begin{cases} (t(x+y), t(x-y)) & \text{if } (x,y)\in\mathcal{T}_0, \\ (t(2-x+y), t(2-x-y)) & \text{if } (x,y)\in\mathcal{T}_1, \end{cases} \tag{5}$$

was introduced in [15]. This family can be seen as the composition of linear maps defined by the matrices

$$A_t = \begin{pmatrix} t & t \\ t & -t \end{pmatrix}$$

with the fold of the whole plane along the line $\mathcal{C} = \{(x,y)\in\mathbb{R}^2 : x = 1\}$. This fold can be defined by

$$\mathcal{F}_{\mathcal{C}}(x,y) = \begin{cases} (x,y) & \text{if } x < 1, \\ (2-x, y) & \text{if } x \geq 1. \end{cases} \tag{6}$$

Note that $\mathcal{O} = (0,0)$ is a fixed point for every Λ_t in (5). In addition, for $t > 1/\sqrt{2}$, each Λ_t has another fixed point $P_t \in \mathcal{T}_1$ given by

$$P_t = (x_t, y_t) = \left(\frac{2t(2t+1)}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1}\right).$$

Clearly, the affine change of coordinates

$$X = \frac{x - x_t}{1 - x_t}, \quad Y = -\frac{y - y_t}{1 - x_t}$$

moves $\mathcal{O}$ onto the point P_t and keeps the critical line $X = 1$. These coordinates transform each Λ_t in (5) into the composition of $\mathcal{F}_{\mathcal{C}}$ with the linear application defined by the matrix

$$\widetilde{A}_t = \begin{pmatrix} -t & -t \\ t & -t \end{pmatrix}$$

whose eigenvalues are $\sqrt{2}te^{\pm i\theta}$ with $\theta = 3\pi/4$. That is, P_t is an unstable focus and then, in the complex plane, the family Λ_t in (5) can be expressed as

$$\Gamma_{a,\theta}(x+iy) = \begin{cases} ae^{i\theta}(x+iy) & \text{if } x < 1, \\ ae^{i\theta}(2-x+iy) & \text{if } x \geq 1 \end{cases} \tag{7}$$

with $a = \sqrt{2}t$ and $\theta = 3\pi/4$.

The study of the dynamics exhibited by family (5) is mainly justified when one compares its attractors (numerically obtained in [16]) with the attractors (numerically obtained in [13]) for the family (4) with $(a,b)\in G$. For different values of the parameters, both families of maps display convex strange attractors, connected (but not simply connected) strange attractors, and disconnected strange attractors (formed by numerous connected pieces).

A first analytical proof of the existence of a convex strange attractor for Λ_t was given in [17] for all $t \in (t_0, 1]$, where $t_0 = 2^{-1/2}(1+\sqrt{2})^{1/4}$ and, as it was seen in [14] for $t = 1$, it was also proved that the attractor supports a unique ergodic invariant probability measure for all $t \in (t_0, 1]$. The existence of persistent strange attractors with several pieces for $2^{-1/2} < t < 2^{-2/5}$ is proved from [18], [19] and [20]. The proof is a consequence of a renormalization procedure that allows us to understand how connected invariant compact sets (formed by a unique piece) may split giving rise to others formed by an increasing number n of pieces. Then, from Theorem 1.2 in [20], it follows that these new disconnected invariant compact sets contain strange attractors formed by n pieces. Moreover, it is proved in [19] the coexistence of any number of strange attractors.

The proof of the previous results is strongly based on the existence of compact sets $\mathcal{K}$ that are strictly invariant, that is, $\Lambda_t(\mathcal{K}) = \mathcal{K}$. Every attractor $\mathcal{Z}$ for a transformation f must be strictly invariant compact minimal set. Once proved the existence of this attractor $\mathcal{Z}$, the expansitivity of Λ_t allows to conclude that $\mathcal{Z}$ is a strange attractor. In the case for family (4), or even for $T_{a(s),b(s)}$ with $(a(s), b(s)) \in G$, this is not that simple because, like in the one-dimensional case, the maps $T_{a,b}$ are not expansive near their critical line. Nevertheless, the numerical simulations carried out in [13], which show the possible existence of two-dimensional strange attractors and similar to those found for family (5), strengthen the idea that a certain possible (though surely laborious) exclusion of parameters (like in [2] for the one-dimensional case) could conclude the persistence of strange attractors.

Although the process of exclusion of parameters seems to be the essential (and most complicated) step in the proof of the persistence of strange attractors for family (4), other peculiarities should be considered when studying the dynamics of this family. Contrary to what happens for family (5), at the fixed points of $T_{a,b}$ with eigenvalues $e^{\pm\theta \mathrm{i}}$, a Hopf bifurcation might lead to the presence of an attracting closed curve. Then, the angle θ conditions the dynamics on this curve. This is one of the reasons why in this paper we will study the general two-parameter family

$$\{\Gamma_{a,\theta} : (a, \theta) \in (1, \infty) \times (0, \pi)\} \tag{8}$$

defined by (7) for $a > 1$ and $0 < \theta < \pi$. For $\theta = 0$ or $\theta = \pi$, the dynamics looks like one-dimensional. For θ and $-\theta$ the respective dynamics are conjugate.

The strictly invariant compact sets of family (8) will report the strictly invariant compact sets of family (4). As seen in references [13]-[20] for the particular case $\theta = 3\pi/4$, both sets will have the same type of connection: simply connected, connected and successive ruptures in pieces. Having thus understood the topology and arrangement of these sets for (4), it remains to study the effect of the quadratic nonlinearity on the inner dynamics: existence of a dense and expansive orbit in the corresponding strictly invariant compact sets.

Note that $\Gamma_{a,\theta}$ is the composition of the fold

$$\mathcal{F}_{\mathcal{C}}(x+\mathrm{i}y) = 1 - |1-x| + \mathrm{i}y \tag{9}$$

and the expanding linear map $A_{a,\theta}(z) = ae^{\mathrm{i}\theta}z$ being $z = x + \mathrm{i}y$. For this reason, we say that each $\Gamma_{a,\theta}$ is an *Expanding Baker Map* (EBM for short) and write

$$\Gamma_{a,\theta} = EBM(\mathcal{C}, \mathcal{O}, A_{a,\theta})$$

where $\mathcal{C}$ denotes the line $x = 1$ along which the plane is folded towards the half-plane containing $\mathcal{O}$. The successive images $\mathcal{L}_j = A^j_{a,\theta}(\mathcal{C})$ of $\mathcal{C}$ are given implicitly by the equations

$$x \cos j\theta + y \sin j\theta = a^j. \tag{10}$$

Later on it will be necessary to refer to EBMs of at least two folds, that we will denote by $EBM(\mathcal{C}, \mathcal{L}_{j_1}, \ldots, \mathcal{L}_{j_k}, A_{a,\theta})$ where $\mathcal{C}, \mathcal{L}_{j_1}, \ldots, \mathcal{L}_{j_k}$ define the lines to fold written from left to right in the order that these folds act. See Section 2.

Some initial results for the family $\Gamma_{a,\theta}$ were given in [21] (see Theorems A and B). There it was proved the existence of $a_\theta > 1$ such that $\Gamma_{a,\theta}$ has a closed polygon $\mathcal{K}_{a,\theta}$ which is strictly $\Gamma_{a,\theta}$-invariant for every $a \in (1, a_\theta]$. When $\mathcal{K}_{a,\theta}$ is minimal (it does not contain any other set with the same mentioned properties), then it is actually a two-dimensional strange attractor. The existence of these attractors follows next from the existence of invariant compact sets with nonempty interior.

Theorem A. *Let $0 < \theta < \pi$. For every $a \in (1, a_\theta]$ there exists a finite family $\mathbf{Z}_{a,\theta}$ of 2-D strange attractors for $\Gamma_{a,\theta}$ with the following properties:*

(i) *If $\mathcal{Z}$ is an attractor for $\Gamma_{a,\theta}$, then $\mathcal{Z} \in \mathbf{Z}_{a,\theta}$.*

(ii) *For every $\mathcal{Z} \in \mathbf{Z}_{a,\theta}$ there exists an ergodic absolutely continuous invariant measure μ for $\Gamma_{a,\theta}$ supported on $\mathcal{Z}$.*

(iii) *For every $\mathcal{Z} \in \mathbf{Z}_{a,\theta}$ there exists a natural number p and a decomposition*

$$\mathcal{Z} = X_0 \cup X_1 \cup \cdots \cup X_{p-1}$$

of $\mathcal{Z}$ in such a way that $\Gamma_{a,\theta}(X_i) = X_{i+1 \bmod p}$ for all $i = 0, \ldots, p-1$. The measure μ supported on $\mathcal{Z}$ is mixing (up to the eventual period p) from which $\Gamma^p_{a,\theta}$ is topologically mixing on every X_i.

(iv) *If $\mathcal{Z} \in \mathbf{Z}_{a,\theta}$, then $\mathcal{Z}$ traps almost every point in $W^s(\mathcal{Z})$, i.e. for almost every point $P \in W^s(\mathcal{Z})$, there exists $j \in \mathbb{N}$ with $\Gamma^j_{a,\theta}(P) \in \mathcal{Z}$. Moreover, the set $\bigcup_{\mathcal{Z} \in \mathbf{Z}_{a,b}} W^s(\mathcal{Z})$ covers a full Lebesgue measure set of $\mathcal{K}_{a,\theta}$.*

(v) *If U is a compact $\Gamma_{a,\theta}$-invariant set with nonempty interior, then there exists $\mathcal{Z} \in \mathbf{Z}_{a,\theta}$ such that $\mathcal{Z} \subset U$. Moreover, if U_1 and U_2 are compact $\Gamma_{a,\theta}$-invariant sets with disjoint nonempty interiors, then there exist two different 2-D strange attractors $\mathcal{Z}_i \in \mathbf{Z}_{a,\theta}$ with $\mathcal{Z}_i \subset U_i$ for $i = 1, 2$.*

Theorem A will be proved in the same way as Theorem 1.2 in [20]. In fact, once again, the proof of Theorem A is strongly based on the results given in [22], [23] and [24]. Nevertheless, we include here a sketch of the proof of Theorem A because certain crucial considerations on the so called weighted multiplicity must be checked for all $\theta \in (0,\pi)$. Finally, we point out that the proof of Theorem A mainly relies on the existence of an invariant set for $\Gamma_{a,\theta}$, namely $\mathcal{K}_{a,\theta}$ (which is in fact strictly invariant), hence the restriction $1 < a \leq a_\theta$ on the space of parameters.

From now on, we will consider $\theta = 2\pi k/n \in (0,\pi)$ with $k,n \in \mathbb{N}$ and $\gcd(k,n) = 1$. The following results show how for a sufficiently close to 1 the invariant set $\mathcal{K}_{a,\theta}$ (and, consequently, the attractor it contains) may split into another set formed by pieces permuted by $\Gamma_{a,\theta}$. Thus, a connected strange attractor bifurcates for a first value $a_1(\theta)$ into a disconnected strange attractor formed by at least n pieces. When n is odd, this new attractor splits into other strange attractors of n^2 pieces for some $a_2 < a_1$. When $n = 2\nu$ with ν odd, likewise a second splitting occurs for some $a_2 < a_1$, giving rise in this case to another strange attractors of $n^2/2$ pieces. However, when ν is even, the original attractor of at least n pieces splits into two different strange attractors, each of which having $n^2/2$ pieces. We will call this phenomenon *doubling of attractors*. This process goes on when $\nu/2$ is even. Since for $\theta = \pi/2$ it was proved in [19] the existence of a decreasing sequence $\{a_i\}$ of values of splitting of strange attractors, we obtain an infinite cascade of doubling of attractors for any $n = 2^s$ with $s \geq 2$.

The next result establishes the existence of a value $a_1(\theta)$ for which the first splitting takes place:

Theorem B. *Let $\theta = 2\pi k/n \in (0,\pi)$ with $k,n \in \mathbb{N}$ and* $\gcd(k,n) = 1$.

(a) There exists $a_1 = a_1(\theta) \in (1,\infty)$ and $\mathcal{D} \subset \mathbb{R}^2$ such that, for every $a \in (1,a_1)$ it holds that

  (i) $\Gamma^j_{a,\theta}(\mathcal{D}) \cap \mathcal{D} = \varnothing$ for every $j = 1,\ldots,n-1$.

  (ii) $\Gamma^n_{a,\theta}(\mathcal{D}) \subset \mathcal{D}$.

(b) For $n \geq 4$, the restriction of $\Gamma^n_{a,\theta}$ to $\mathcal{D}$ is conjugate by means of an affine change in coordinates to an 2-fold EBM.

A set verifying statements (i) and (ii) is said to be a *restrictive domain* of $\Gamma^n_{a,\theta}$. See Definition 2.4. According to Theorem A, the map $\Gamma_{a,\theta}$ displays a strange attractor with at least n pieces, each of which contained in a different $\Gamma^j_{a,\theta}(\mathcal{D})$ with $j = 0,1,\ldots,n-1$. Note that from statement (a) it follows that for every pair $i,j = 1,\ldots,n$ it holds that $\Gamma^i_{a,\theta}(\mathcal{D}) \cap \Gamma^j_{a,\theta}(\mathcal{D}) = \varnothing$ whenever $i \neq j$.

As a consequence of the proof of Theorem B we obtain the following result:

Corollary 1.3. *Under the hypothesis of Theorem B, the set $\mathcal{D}$ can be constructed in such a way that every attractor for $\Gamma_{a,\theta}$ is contained in the forward orbit of $\mathcal{D}$.*

In order to illustrate this corollary and statement a) of Theorem B, see Figures 12a, 13a, and 14a at the end of this paper.

For values of $a < a_1(\theta)$ sufficiently close to 1, the restriction of $\Gamma^n_{a,\theta}$ to $\mathcal{D}$ has an unstable focus P with eigenvalues $a^n e^{2\theta_1 \mathrm{i}}$ with $\theta_1 = 2\pi/n$. The translation of the origin of coordinates to P and a suitable change in coordinates allows to express $\Gamma^n_{a,\theta}$ as the map $\Psi_{a,\theta_1} = EBM(\mathcal{C}, \mathcal{L}_{a,\theta_1}, \mathcal{O}, A')$, for which:

(i) $\mathcal{C}$ is the critical line $x = 1$.

(ii) $\mathcal{L}_{a,\theta_1}$ is a line that crosses $\mathcal{C}$ at an angle θ_1 and its distance $r(a)$ to the origin satisfies $\lim_{a\to 1^+} r(a) = 1$.

(iii) $A'(z) = a^n e^{2\theta_1 \mathrm{i}} z$.

Let $\mathbb{F}^k_{\sigma,\varphi}$ be the set of maps $\Psi_{\sigma,\varphi} = EBM(\mathcal{C}, \mathcal{L}_{\sigma,\varphi}, A')$ satisfying (i), (ii) y (iii) for σ, φ, and k instead of a, θ_1, and n. Thus, $\Psi_{a,\theta_1} \in \mathbb{F}^n_{a,\theta_1}$. We will consider $EBM(\mathcal{C}, A')$ to be included in $EBM(\mathcal{C}, \mathcal{L}_{\sigma,\varphi}, A')$, since this trivially holds when $\mathcal{L}_{\sigma,\varphi} = \mathcal{C}$ or $\mathcal{L}_{\sigma,\varphi}$ does not intersect a certain invariant set on which the dynamics is studied.

Let us state a first result on renormalization:

Theorem C. *Let $\theta = 2\pi k/n \in (0, \pi)$ with $k, n \in \mathbb{N}$ and $n \geq 3$ and $\gcd(k, n) = 1$. The following statements hold:*

(a) If n is odd, then there exists $a_2 = a_2(\theta) < a_1(\theta)$ such that for all $a \in (1, a_2)$ there exists a restrictive domain $\mathcal{D}_1 \subsetneq \mathcal{D}$ for Ψ^n_{a,θ_1}.

(b) If $n = 2\nu$ is even, then one of the following statements hold:

(i) If ν is odd, there exists $a_2 = a_2(\theta) < a_1(\theta)$ such that for all $a \in (1, a_2)$ the map Ψ^ν_{a,θ_1} has a restrictive domain $\mathcal{D}_1 \subsetneq \mathcal{D}$.

(ii) If ν is even, there exists $a_2 = a_2(\theta) < a_1(\theta)$ such that for all $a \in (1, a_2)$ the map Ψ^ν_{a,θ_1} has two disjoint restrictive domains $\mathcal{D}^\pm_1 \subsetneq \mathcal{D}$. Moreover, the restriction of Ψ^ν_{a,θ_1} to each one of these domains belongs to $\mathbb{F}^\nu_{a^n,\theta_1}$.

Figures 12, 13, and 14 shed some light on statements a), b-i) and b-ii) of Theorem C, respectively.

Statement (ii) of (b) still holds for $\Psi^{\nu/2}_{a,\theta_1}$ provided that $\nu/2$ is even, and successively for any power $n = 2^s$ with $s \geq 2$. Therefore, as we have stated above, as a consequence of the previous Theorems and Theorem B in [19] (See Lemma 2.7 in Section 2) we obtain the following result:

Corollary 1.4. *Let $\theta = 2\pi k/n \in (0, \pi)$ with $k, n \in \mathbb{N}$ and $\gcd(k, n) = 1$. If $n = 2^s$ for some $s \geq 2$, then there exists a decreasing sequence $\{a_j(\theta)\}_{j\geq 1}$ such that $\Gamma_{a,\theta}$ exhibits 2^{j-1} strange attractors simultaneously for all $a \in [a_{j+1}, a_j)$ and for every $j \geq 1$.*

Thus, in this paper we extend to $\theta = 2\pi k/n \in (0, \pi)$ with $k, n \in \mathbb{N}$, $\gcd(k, n) = 1$ and $n \geq 4$ all the results proved in [18], [19], and [20] for the case $\theta = 3\pi/4$. In particular, the existence of a cascade of doubling of strange attractors for a dense set of values $\theta = \pi k/2^j$ with k an odd natural number and $j \geq 1$. Some questions still remain open for the rest of values of θ. For instance, we may ask if the renormalization from statements a) and b-i) of Theorem C can be indefinitely iterated like in statement b-ii) thereof. In particular, we may ask what happens with the dynamics in the special case $n = 3$. Finally, one should think about the dynamics when θ/π is not rational.

This article is organized in the following manner. In Section 2 are gathered some definitions and basic results on EBMs. Section 3 is devoted to the proof of Theorem A. In Section 4 we prove Theorem B considering three different cases independently: $n \geq 5$, $n = 4$, and $n = 3$. In Section 5 Theorem C is proved. In Section 6 some open questions are posed.

2. Expanding Baker Maps

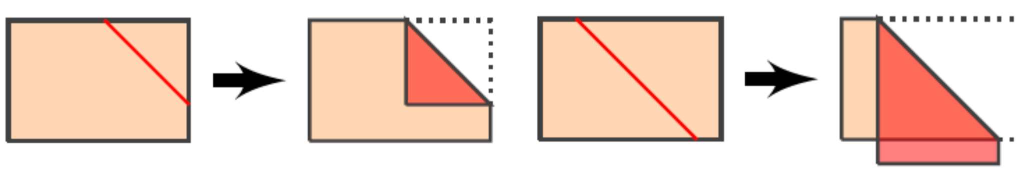
In this section we will introduce the concept of EBM and renormalizable EBM. These maps are characterized for a very particular dynamics: They fold some domain and, after that, they expand the folded region. For our purposes it is enough to define EBMs on compact and convex domains of $\mathbb{R}^2$.

To begin with, let us state the definition of *fold* and *good fold.*

Definition 2.1. *Let $\mathcal{K} \subseteq \mathbb{R}^2$ be a set with nonempty interior and let $\mathcal{L}$ be a line in $\mathbb{R}^2$ intersecting the interior of $\mathcal{K}$. The line $\mathcal{L}$ splits $\mathcal{K}$ into two sets $\mathcal{K}_0$ and $\mathcal{K}_1$, i.e. $\mathcal{K}_0 \cup \mathcal{K}_1 = \mathcal{K}$ and $\mathcal{K}_0 \cap \mathcal{K}_1 = \mathcal{L} \cap \mathcal{K}$. Let $P \in \mathcal{K}_0$. The **fold** with respect to $\mathcal{L}$ onto P for $\mathcal{K}$ is the map $\mathcal{F}_{\mathcal{L},P}\colon \mathcal{K} \to \mathbb{R}^2$ given by*

$$\mathcal{F}_{\mathcal{L},P}(Q) = \begin{cases} Q & \text{if } Q \in \mathcal{K}_0, \\ \widetilde{Q} & \text{if } Q \in \mathcal{K}_1, \end{cases}$$

*where $\widetilde{Q}$ denotes the symmetric point of Q with respect to $\mathcal{L}$. If $\mathcal{F}_{\mathcal{L},P}(\mathcal{K}) = \mathcal{K}_0$, we say that $\mathcal{F}_{\mathcal{L},P}$ is a **good fold** for $\mathcal{K}$.*



(a) A good fold (b) A not good fold

Figure 1: Examples of folds

Now, let us write $\mathcal{L} = \mathcal{L}^1$ and let $\mathcal{L}^2$ be a line with $\mathcal{L}^2 \cap int(\mathcal{K}_0) \neq \varnothing$ and $P \notin \mathcal{L}^2$. Then, $\mathcal{L}^2$ divides $\mathcal{K}_0$ into two subsets $\mathcal{K}_{00}$ and $\mathcal{K}_{01}$ ($\mathcal{K}_{00}$ denotes the one containing P). Let us assume that $\mathcal{F}_{\mathcal{L}^2,P}(\mathcal{K}_0) = \mathcal{K}_{00}$ (i.e, $\mathcal{F}_{\mathcal{L}^2,P}$ is a good fold). Repeating these arguments, we may successively define a sequence of good folds $\mathcal{F}_{\mathcal{L}^1,P}, \mathcal{F}_{\mathcal{L}^2,P}, \ldots, \mathcal{F}_{\mathcal{L}^n,P}$, where

$$\mathcal{F}_{\mathcal{L}^1,P} : \mathcal{K} \to \mathcal{K}_0$$
$$\mathcal{F}_{\mathcal{L}^j,P} : \mathcal{K}_{0\overset{j-1}{\ldots}0} \to \mathcal{K}_{0\overset{j}{\ldots}0}$$

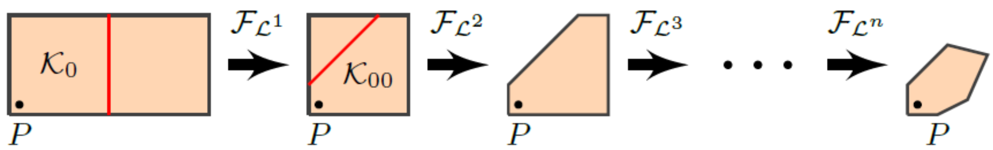


Figure 2: Sequence of goods folds

with $\mathcal{K}_{0\overset{j-1}{\cdots}0} \subset \mathcal{K}_{0\overset{j}{\cdots}0}$ and $P \in \mathcal{K}_{0\overset{j}{\cdots}0}$ for every $j = 1, 2, \ldots, n$.

We now may introduce the concepts of EBM and renormalizable EBM.

Definition 2.2. *Let $\mathcal{K} \subset \mathbb{R}^2$ be a set with nonempty interior. Let P be a point in $\mathcal{K}$ and $\{\mathcal{F}_{\mathcal{L}^1,P}, \ldots, \mathcal{F}_{\mathcal{L}^n,P}\}$ a sequence of good folds of $\mathcal{K}$ with $P \in \mathcal{K}_{0\overset{j}{\cdots}0}$ for every $j = 1, \ldots, n$. Let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an expanding linear map, i.e. $|\det A| > 1$. Let us consider*

$$\widetilde{A} : Q \in \mathbb{R}^2 \to \widetilde{A}(Q) = P + A(Q - P)$$

*and assume that $\widetilde{A}(\mathcal{K}_{0\overset{j}{\cdots}0}) \subset \mathcal{K}$. We define the **Expanding Baker Map** associated to P, A, $\mathcal{L}^1$,...,$\mathcal{L}^n$ as the map $\Gamma : \mathcal{K} \to \mathcal{K}$ given by*

$$\Gamma = \widetilde{A} \circ \mathcal{F}_{\mathcal{L}^n,P} \circ \ldots \circ \mathcal{F}_{\mathcal{L}^1,P}$$

For short, we will denote

$$\Gamma = EBM(\mathcal{L}^1, \ldots, \mathcal{L}^n, P, A)$$

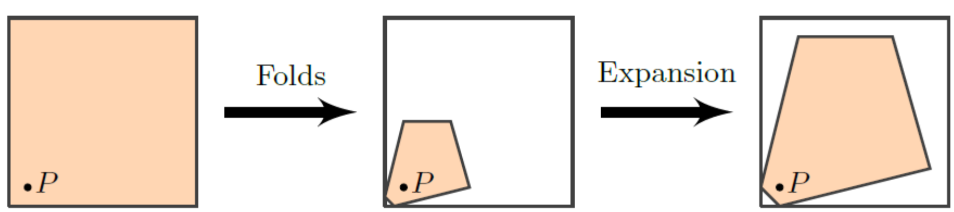


Figure 3: An example of EBM

We will usually take P as the origin of coordinates $\mathcal{O}$ and omit it in the notation of a fold ($\mathcal{F}_{\mathcal{L}}$ instead of $\mathcal{F}_{\mathcal{L},\mathcal{O}}$). The linear map A will be of the type

$$A_{a,\theta} = \begin{pmatrix} a\cos\theta & -a\sin\theta \\ a\sin\theta & a\cos\theta \end{pmatrix}, \qquad (11)$$

(or $A_{a,\theta}(z) = ae^{i\theta}z$ in the complex variable) with $a > 1$.

The following lemma will be useful in Section 4:

Lemma 2.3. *Let $\mathcal{L}$ be a line in $\mathbb{R}^2$ such that $\mathcal{O} \notin \mathcal{L}$. Let $\mathcal{K}_0$ and $\mathcal{K}_1$ be the two half-planes limited by $\mathcal{L}$, with $\mathcal{O} \in int(\mathcal{K}_0)$. Let $\widetilde{\mathcal{L}} = A_{a,\theta}^{-1}(\mathcal{L})$. Then,*

$$\mathcal{F}_{\mathcal{L}} \circ A_{a,\theta}(Q) = A_{a,\theta} \circ \mathcal{F}_{\widetilde{\mathcal{L}}}(Q),$$

for all $Q \in \mathcal{K}_0$ such that $\mathcal{F}_{\widetilde{\mathcal{L}}}(Q) \in \mathcal{K}_0$.

Proof. Let $\mathcal{K}_{00}$ and $\mathcal{K}_{01}$ be the two subsets into which $\mathcal{K}_0$ is divided by $\widetilde{\mathcal{L}} \cap \mathcal{K}_0$, with $\mathcal{O} \in int(\mathcal{K}_{00})$. Then, $A_{a,\theta}(\mathcal{K}_{0i}) \subset \mathcal{K}_i$ for $i = 0, 1$.

For $Q \in \mathcal{K}_{00}$, it holds that $\mathcal{F}_{\widetilde{\mathcal{L}}}(Q) = Q$ and $A_{a,\theta}(Q) \in \mathcal{K}_0$. Thus,

$$\mathcal{F}_{\mathcal{L}} \circ A_{a,\theta}(Q) = A_{a,\theta}(Q) = A_{a,\theta} \circ \mathcal{F}_{\widetilde{\mathcal{L}}}(Q).$$

Now, let $Q \in \mathcal{K}_{01}$ and assume that $\mathcal{F}_{\widetilde{\mathcal{L}}}(Q) \in \mathcal{K}_0$. By definition of $\mathcal{F}_{\widetilde{\mathcal{L}}}$, the points Q and $\mathcal{F}_{\widetilde{\mathcal{L}}}(Q)$ are symmetric with respect to $\widetilde{\mathcal{L}}$, and so are $A_{a,\theta}(Q)$ and $A_{a,\theta} \circ \mathcal{F}_{\widetilde{\mathcal{L}}}(Q)$ with respect to $A_{a,\theta}(\widetilde{\mathcal{L}}) = \mathcal{L}$ because of the orthogonality of the rotation matrix $a^{-1}A_{a,\theta}$. Since $A_{a,\theta}(Q) \in \mathcal{K}_1$, then

$$\mathcal{F}_{\mathcal{L}} \circ A_{a,\theta}(Q) = A_{a,\theta} \circ \mathcal{F}_{\widetilde{\mathcal{L}}}(Q)$$

and the lemma is proved. ■

Definition 2.4. *Let Γ be a map defined in a certain domain $\mathcal{K}$. We say that $\mathcal{D} \subset \mathcal{K}$ is a* ***restrictive domain*** *if $\mathcal{D} \neq \mathcal{K}$ and there exists $n = n(\mathcal{D}) \in \mathbb{N}$ such that*

(i) $\Gamma^j(\mathcal{D}) \cap \mathcal{D} = \varnothing$ *for every $j = 1, \ldots, n-1$, and*

(ii) $\Gamma^n(\mathcal{D}) \subset \mathcal{D}$.

Apart from several extensions to higher dimension (see [25], [26], and [27] among others), the notion of renormalizable maps comes from the one-dimensional framework. A one-dimensional map f belonging to some family $\mathbb{F}$ (for instance, the family of unimodal maps defined on an interval) is said to be renormalizable if there exists a restrictive domain $\mathcal{D}$ such that f^n restricted to $\mathcal{D}$ is, up to an afine change in coordinates, a member of $\mathbb{F}$. In this sense, the concept of renormalizable EBM arises.

Definition 2.5. *An EBM Γ defined on certain domain $\mathcal{K}$ is said to be* ***renormalizable*** *if there exists a restrictive domain $\mathcal{D}$ (with an associated natural number $n = n(\mathcal{D})$) such that the restriction $\Gamma^n_{|\mathcal{D}}$ of Γ^n to $\mathcal{D}$ is, up to an affine change of coordinates, an EBM defined in $\mathcal{K}$.*

Definition 2.6. *Let Γ be a renormalizable EBM with restrictive domain $\mathcal{D}$ (with an associated natural number $n = n(\mathcal{D})$). If $\Gamma^n_{|\mathcal{D}}$ is a renormalizable EBM, we say that Γ is* ***twice renormalizable****. Similarly, we can speak of* ***k-times renormalizable EBMs*** *for any $k \geq 3$ and* ***infinitely many times renormalizable EBMs****.*

The following result for the map $\Gamma_{a,\theta}$ with $\theta = \pi/2$ was proved in [19] (see Lemma 2.2) and will be useful in Sections 4 and 5.

Lemma 2.7. *For $\theta = \pi/2$, the following statements hold:*

(a) For every $a \in [2^{\frac{1}{4}}, 2^{\frac{1}{2}}]$, the map $\Gamma_{a,\theta}$ has a strongly topologically mixing strange attractor with two positive Lyapunov exponents.

(b) For every $n \geq 2$ and every $a \in [2^{2^{-(n+1)}}, 2^{2^{-n}}]$, the map $\Gamma_{a,\theta}$ is n-times renormalizable and displays 2^n strange attractors with two Lyapunov exponents.

3. Existence of 2-D Strange Attractors: Proof of Theorem A

The proof of Theorem A is similar to the proof of Theorem 1.2 in [20]. In fact, the proof would be the same if we restrict ourselves to the case in which θ/π is a rational number. This is because the formula given in (18) is easily obtained in the rational case without using the *Geometric Estimate* given in [22] (see page 700). See also Remark 3.3. Let us point out here that formula (18) is, as we will see soon, the key to applying the results in [22]. Since the Geometric Estimate was not used in the proof of Theorem 1.2 in [20] we think we must give a sketch of the proof of Theorem A paying special attention to these new details.

Fix $0 < \theta < \pi$ and then fix $a \in (1, a_\theta]$. In Theorem B of [21] we show the existence of a (strictly) invariant set $\mathcal{K}_{a,\theta}$. We consider the sets

$$\mathcal{K}_0 = \{(x, y) \in \mathcal{K}_{a,\theta} : x \leq 1\}, \quad \mathcal{K}_1 = \{(x, y) \in \mathcal{K}_{a,\theta} : x \geq 1\}. \tag{12}$$

Let $\mathcal{P} = \{int(\mathcal{K}_0), int(\mathcal{K}_1)\}$ and $Y = int(\mathcal{K}_0) \cup int(\mathcal{K}_1)$. Then, $\Gamma_{a,\theta}\colon Y \to \overline{Y} = \mathcal{K}_{a,\theta}$ is an expanding piecewise analytic map of the plane according to the corresponding definition given in [22].

Therefore, according to the main result in [22] we have the following result.

Proposition 3.1. *Let $0 < \theta < \pi$. For every $a \in (1, a_\theta]$ there exist absolutely continuous invariant measures for $\Gamma_{a,\theta}$. Moreover:*

(i) Each one of these ACIMs is a convex combination of a fixed, finite collection of ergodic ones.

(ii) For every ergodic measure μ of $\Gamma_{a,\theta}$, there exist a constant $\kappa < 1$, a natural number p, and a decomposition

$$supp(\mu) = X = X_0 \cup X_1 \cup \ldots \cup X_{p-1}$$

of the support of μ in such a way that $\Gamma_{a,\theta}(X_i) = X_{i+1 \bmod p}$ for all $i = 0, \ldots, p-1$. Moreover, the measure μ is mixing (up to the eventual period p) and then the map $\Gamma^p_{a,\theta}$ is topologically mixing on any X_i.

Now, and here is where the differences between the proof of Theorem A and Theorem 1.2 in [20] arise, the crucial argument is to control the weighted multiplicity of a piecewise analytic mapping of the plane. This weighted multiplicity is defined in Section 1 of [22]. In our context (piecewise linear maps), we may introduce this notion as follows. Let us begin with the definition of *circular sectors*.

Definition 3.2. *Given any $P \in \mathcal{K}_{a,\theta}$ and any $r > 0$ a* ***circular sector*** *S_P at P is the interior of any bounded set whose boundary is formed by two different straight segments l_1 and l_2 starting at P and the circle $C(P, r)$ centered at P with radius r. For any circular sector S_P we define $\overrightarrow{S}_P$ as the set of unit vectors which are in the cone generated by the straight segments l_1 and l_2 at P.*

The partition $\overline{\mathcal{P}} = \{\mathcal{K}_0, \mathcal{K}_1\}$, see (12), defines at each point $P \in \mathcal{K}_{a,\theta}$ a collection $\mathcal{S}_P\overline{\mathcal{P}}$ of at most two circular sectors S_P given by $int(\mathcal{K}_0) \cap B(P, r)$ and $int(\mathcal{K}_1) \cap B(P, r)$

for $r > 0$ small enough. Here we also consider as a possible circular sector the whole ball $B(P, r)$. The *local weighted multiplicity* of $\Gamma_{a,\theta}$ in a point $P \in \mathcal{K}_{a,\theta}$ can be now defined by

$$\texttt{mult}(\overline{\mathcal{P}}, \Gamma_{a,\theta}, P) = \sum_{S_P \in \mathcal{S}_P \overline{\mathcal{P}}} \frac{\Lambda_+(\Gamma_{a,\theta} \mid \overrightarrow{S}_P)}{\texttt{jac}(\Gamma_{a,\theta})_{\mathcal{A}}(P)} \tag{13}$$

where $\mathcal{A}$ is the element of $\overline{\mathcal{P}}$ containing S_P and

$$\Lambda_+(\Gamma_{a,\theta} \mid \overrightarrow{S}_P) := \sup_{v \in \overrightarrow{S}_P} \|(\Gamma_{a,\theta})'_{\mathcal{A}}(P)(v)\| \ .$$

Now, let us define

$$\texttt{mult}(\overline{\mathcal{P}}^n, \Gamma^n_{a,\theta}) := \sup_{P \in \mathcal{K}_{a,\theta}} \texttt{mult}(\overline{\mathcal{P}}^n, \Gamma^n_{a,\theta}, P) = \sup_{P \in \mathcal{K}_{a,\theta}} \sum_{S_P \in \mathcal{S}_P \overline{\mathcal{P}}^n} \frac{\Lambda_+(\Gamma^{\mathcal{A}}_{a,\theta} \mid \overrightarrow{S}_P)}{\texttt{jac}(\Gamma^{\mathcal{A}}_{a,\theta})(P)} \tag{14}$$

where $\overline{\mathcal{P}}^n$ is made of sets $\mathcal{A}$ which are given by sequence of nonempty intersections

$$\mathcal{A}_0 \cap \Gamma^{-1}_{a,\theta}(\mathcal{A}_1) \cap \ldots \cap \Gamma^{-n+1}_{a,\theta}(\mathcal{A}_{n-1}), \qquad \mathcal{A}_0, \ldots, \mathcal{A}_{n-1} \in \overline{\mathcal{P}}$$

and we also take the notation

$$\Gamma^{\mathcal{A}}_{a,\theta} = (\Gamma_{a,\theta})_{\mathcal{A}_{n-1}} \circ \ldots \circ (\Gamma_{a,\theta})_{\mathcal{A}_0}$$

for any of such sets $\mathcal{A} \in \overline{\mathcal{P}}^n$. Observe that in the formula given at (14) and for every $P \in \mathcal{K}_{a,\theta}$, $\mathcal{S}_P\overline{\mathcal{P}}^n$ is the collection of circular sectors given (for $r > 0$ small enough) by the nonempty intersections $int(\mathcal{A}) \cap B(P, r)$, whenever $\mathcal{A} \in \overline{\mathcal{P}}^n$.

As we said before, related to formula (13), we have $\Lambda_+(\Gamma_{a,\theta} \mid \overrightarrow{S}_P) = a$ and $\texttt{jac}(\Gamma_{a,\theta})_{\mathcal{A}}(P) = a^2$. Therefore, taking $n = 1$ in (14), it is easy to see that

$$\texttt{mult}(\overline{\mathcal{P}}, \Gamma_{a,\theta}) = \frac{2}{a} \tag{15}$$

because the supremum taking part of the definition of $\texttt{mult}(\overline{\mathcal{P}}, \Gamma_{a,\theta})$ is achieved at points x in the critical set.

With respect to formula (14) it follows that $\Lambda_+(\Gamma^{\mathcal{A}}_{a,\theta} \mid \overrightarrow{S}_P) = a^n$ and $\texttt{jac}(\Gamma^{\mathcal{A}}_{a,\theta})(P) = a^{2n}$, for every $\theta \in (0, \pi), P \in \mathcal{K}_{a,\theta}, n \in \mathbb{N}$ and $\mathcal{A} \in \overline{\mathcal{P}}^n$. Therefore, for every $n \in \mathbb{N}$, we obtain that

$$\texttt{mult}(\overline{\mathcal{P}}^n, \Gamma^n_{a,\theta}) = \frac{R_{n,a,\theta}}{a^n} \tag{16}$$

where we have introduced the sequence

$$R_{n,a,\theta} = \max_{P \in \mathcal{K}_{a,\theta}} \text{card}\{\mathcal{A} \in \overline{\mathcal{P}}^n : \ P \in \mathcal{A}\} \tag{17}$$

Observe that $R_{1,a,\theta} = 2$ and therefore equation (15) follows from equation (16).

Following the *Geometric Estimate* given in [22], we may conclude that

$$\limsup_{n \to \infty} \frac{1}{n} \log \texttt{mult}(\overline{\mathcal{P}}^n, \Gamma^n_{a,\theta}) < 0$$

This crucial bound gives, in our case,

$$\limsup_{n \to \infty} \frac{1}{n} \log R_{n,a,\theta} < \log a. \tag{18}$$

Remark 3.3. *If θ/π is rational then $R_{n,a,\theta}$ remains bounded as n goes to infinity. This is because the slopes of the straight lines forming part of the boundary of any set $\mathcal{A} \in \overline{\mathcal{P}}^n$ belong to a finite set of real numbers. Therefore, equation (18) easily follows without using the Geometric Estimate.*

Let us now consider from [23] the definition of the dilatation coefficient of $\Gamma_{a,\theta}$ given by

$$\delta(\Gamma_{a,\theta}) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{P\in\Gamma^n_{a,\theta}(Y)} \|D\Gamma^{-n}_{a,\theta}(P)\|,$$

where $Y = U_1 \cap U_2 = int(\mathcal{K}_0) \cap int(\mathcal{K}_1)$ and the norm of the derivative is taken along each smooth branch of $\Gamma^{-n}_{a,\theta}$. In our case, it follows that $\delta(\Gamma_{a,\theta}) = -\log a$. Therefore, from (18) we have

$$\lim_{n\to\infty} \frac{1}{n} \log R_{n,a,\theta} + \delta(\Gamma_{a,\theta}) < 0$$

This inequality allows us to apply Lemma 2.2 in [23] to assert that some iterate of the map $\Gamma_{a,\theta}$ satisfies conditions $(PE1) - (PE5)$ in [23] (see page 226). Therefore, from Proposition 3.4, Theorem 5.1 (ii) and Proposition 5.1 in [23] we have the following result:

Proposition 3.4. *Let $0 < \theta < \pi$. For every $a \in (1, a_\theta]$ and for every ACIM μ of $\Gamma_{a,\theta}$ the interior of the support of μ has full μ-measure. Moreover, each ACIM μ of $\Gamma_{a,\theta}$ is finite.*

Finally, see Theorem 3 in [24], we also have the following result:

Proposition 3.5. *Let $0 < \theta < \pi$. For every $a \in (1, a_\theta]$ there exist finitely many absolutely continuous ergodic probability measures $\mu_1, \dots \mu_l$, for $\Gamma_{a,\theta}$. Moreover, the basin of each measure μ_i is an open set modulo sets with null Lebesgue measure, and the union $\bigcup_{i=1}^{l} \mathtt{Basin}(\mu_i)$ has full Lebesgue measure in $\mathcal{K}_{a,\theta}$. Moreover, $\bigcup_{i=1}^{l} int(\mathtt{Basin}(\mu_i))$ has full Lebesgue measure in $\mathcal{K}_{a,\theta}$.*

The rest of the proof of Theorem A follows exactly in the same way as the proof of Theorem 1.2 in [20]. In fact, it is enough to consider the measures $\{\mu_1, \dots, \mu_l\}$ given by Proposition 3.5, define $\mathcal{Z}_i$ as the support of μ_i for $i = 1, \dots, l$ and take

$$\mathbf{Z}_{a,\theta} = \{\mathcal{Z}_i : i = 1, \dots, l\}.$$

In the same way as Lemma 5.1 in [20] was proven, we have here the following result:

Proposition 3.6. *For every $i = 1, \dots, l$ the following statements hold:*

(i) The interior of $\mathcal{Z}_i$ traps every point in $\mathtt{Basin}(\mu_i)$, i.e. for every $P \in \mathtt{Basin}(\mu_i)$, there exists $j \in \mathbb{N}$ such that $\Gamma^j_{a,\theta}(P) \in int(\mathcal{Z}_i)$.

(ii) The set $\mathcal{Z}_i$ is a 2-D strange attractor for $\Gamma_{a,\theta}$.

This last result is the key to completing the proof of Theorem A in the same way as the proof of Theorem 1.2 was completed using Lemma 6.1 in [20].

4. Splitting of Attractors: Proof of Theorem B

Let us consider the two-parameter family $\Gamma_{a,\theta}$ given in (8) and assume that $\theta = 2\pi k/n \in (0,\pi)$ with $k,n \in \mathbb{N}$ and $\gcd(k,n) = 1$. Recall that $\Gamma_{a,\theta} = EBM(\mathcal{C}, A_{a,\theta})$ is the composition of the map $\mathcal{F}_{\mathcal{C}}$ which folds the plane along the line $x = 1$ (see (9) and (6)) and the linear map defined by the matrix $A_{a,\theta}$ given in (11).

For each $j = 0,\ldots,n-1$ we will denote by $\mathcal{S}^j$ the ray that starts from the origin and extends indefinitely in the direction of the vector $(\cos 2\pi j/n, \sin 2\pi j/n)$. Thus, the plane is divided into n regions $\mathcal{R}^0,\ldots,\mathcal{R}^{n-1}$, where $\mathcal{R}^j$ is the region bounded by $\mathcal{S}^j$ and $\mathcal{S}^{j+1}$ for every $j = 0,\ldots,n-1$, setting $\mathcal{S}^n = \mathcal{S}^0$. See Figure 4. The dynamics of $\Gamma_{a,\theta}$ on these regions depends on k and the following inclusion holds for every $j = 0,\ldots,n-1$:

$$\Gamma_{a,\theta}(\mathcal{R}^j \cap \{x \le 1\}) \subset \mathcal{R}^{j+k \bmod n}.$$

As a consequence, given a point $Q \in \mathcal{R}^j \cap \{x \le 1\}$, if $\Gamma^i_{a,\theta}(Q) \in \{x \le 1\}$ for every $i = 1,\ldots,n-1$, then $\Gamma^n_{a,\theta}(Q) \in \mathcal{R}^j$. In order to define $\mathcal{D}$, we will first construct the set of points $Q \in \mathcal{R}^0 \cap \{x \le 1\}$ such that $\Gamma^i_{a,\theta}(Q) \in \{x \le 1\}$ for every $i = 1,\ldots,n-1$.

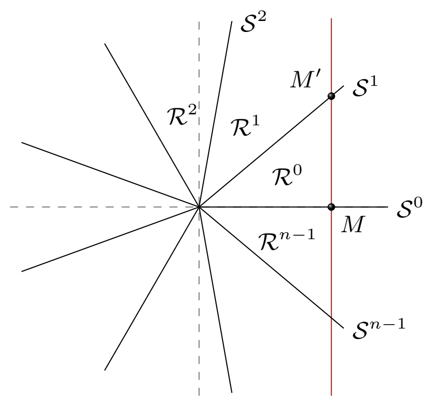


Figure 4: Regions for $x \le 1$

From now on, given a point B or a set $\mathcal{B}$, we denote $B_j = \Gamma^j_{a,\theta}(B)$ and $\mathcal{B}_j = \Gamma^j_{a,\theta}(\mathcal{B})$ for each $j \ge 0$. Furthermore, $\Omega(B^1,\ldots,B^n)$ denotes the polygon with vertices $B^1,\ldots,B^n$ and with sides $\overline{B^1B^2}$, $\overline{B^2B^3}$, $\ldots$, $\overline{B^nB^1}$, where $\overline{BC}$ is the segment joining B and C.

We will divide the proof of Theorem B into three cases. First we will prove the result for $n \ge 5$ in order the point M' (see Figure 4) is on the right half-plane. In this situation, the arguments of the proof hold regardless of the value of n. Then, we will consider the case $n = 4$, for which M' belongs to $x = 0$. Actually, this case was already studied previously and was collected separately in Lemma 2.7 because of its interest for the successive renormalizations given in Corollary 1.4. Finally, we will consider the special case $n = 3$ to which only statement (a) applies.

4.1. Case $n \geq 5$

Fix $n \geq 5$. In this case, the set $\mathcal{R}^0 \cap \{x \leq 1\}$ is the triangle $\Omega(\mathcal{O}, M, M')$ with $M = (1, 0)$ and $M' = (1, \tan 2\pi/n)$.

Lemma 4.1. *There exists $a_M \in (1, \infty)$ such that $M_j \in \{x \leq 1\}$ for every $j = 1, \ldots, n-1$ and every $a \in (1, a_M)$.*

Proof. Assuming that $M_i \in \{x \leq 1\}$ for every $i = 1, \ldots, j-1$, the point M_j is given by

$$M_j = (a^j \cos j\theta, a^j \sin j\theta).$$

Therefore, it is enough to find a value a_M such that $a^j \cos j\theta \leq 1$ for every $j = 1, \ldots, n-1$ and every $a \in (1, a_M)$. In fact, we can ignore the iterates j for which $M_j \in \{x \leq 0\}$ because, in that case, $\cos j\theta \leq 0$ and hence $a^j \cos j\theta \leq 1$. Let $J \subset \{1, \ldots, n-1\}$ be the set of indices j such that $M_j \in \{x > 0\}$. For every $j \in J$ it holds that $0 < \cos j\theta < 1$. We define

$$a_M = \min_{j \in J} \sqrt[j]{\sec j\theta}. \tag{19}$$

Then, $a_M > 1$ and therefore $a^j \cos j\theta < a_M^j \cos j\theta < 1$ for each $j \in J$ and for every $a \in (1, a_M)$. ■

Let us now consider $a \in (1, a_M)$ and denote by m the first natural number such that $M_m \in \mathcal{S}^1$. By definition,

$$\cos m\theta = \cos \frac{2\pi}{n}, \quad \sin m\theta = \sin \frac{2\pi}{n}\ . \tag{20}$$

On the other hand, the point M_m belongs to the line $\mathcal{L}_m$ given in (10). Thus, the line $\mathcal{L}_m$ is also given by

$$x \cos \frac{2\pi}{n} + y \sin \frac{2\pi}{n} = a^m. \tag{21}$$

See Figure 5a. We define the triangle

$$\Delta = \Omega(\mathcal{O}, M_m, K) \tag{22}$$

where K is the intersection point of $\mathcal{L}_m$ and $\mathcal{S}^0$, i.e.

$$K = (a^m \sec 2\pi/n, 0). \tag{23}$$

The first aim is to determine the values of a such that $\Delta_n \subset \Delta$. By definition, $\mathcal{F}_{\mathcal{C}}(\Delta) = \Omega(\mathcal{O}, M_m, H, M)$ where H is the intersection point of $\mathcal{C}$ and $\mathcal{L}_m$ and is given by

$$H = (1, a^m \csc 2\pi/n - \cot 2\pi/n). \tag{24}$$

Hence,

$$\Delta_1 = \Omega(\mathcal{O}, M_{m+1}, H_1, M_1).$$

According to Lemma 4.1, if $m+1 \leq n-1$, then the points M_{m+1} and M_1, and therefore H_1, belong to $\{x \leq 1\}$. In fact, this is also true for M_{m+j}, M_j, and H_j whenever $m+j \leq n-1$. Hence,

$$\Delta_j = \Omega(\mathcal{O}, M_{m+j}, H_j, M_j) \subset \{x \leq 1\}$$

for every $j = 1, \ldots, n-m+1$ and

$$\Delta_{n-m} = \Omega(\mathcal{O}, M_n, H_{n-m}, M_{n-m}).$$

However, $\Delta_{n-m} \not\subset \{x \leq 1\}$ because $M_n = (a^n, 0)$ with $a > 1$. See Figure 5b. Note that, according to (20),

$$\cos(n-m)\theta = \cos\frac{2\pi}{n}, \quad \sin(n-m)\theta = -\sin\frac{2\pi}{n}. \tag{25}$$

Since $H_j \in \{x \leq 1\}$ for each $j = 1, \ldots, n-m-1$, it follows

$$H_{n-m} = (a^n, a^n \cot\frac{2\pi}{n} - a^{n-m}\csc\frac{2\pi}{n}) \,. \tag{26}$$

Therefore, the point H_{n-m} also belongs to the line $x = a^n$, so that $H_{n-m} \in \{x > 1\}$. From (20) and from the definition of a_M in (19) it follows that

$$a^n \cot\frac{2\pi}{n} - a^{n-m}\csc\frac{2\pi}{n} < 0. \tag{27}$$

We denote $\widetilde{H}_{n-m}$ the point of intersection of line $\mathcal{C}$ with the segment $\overline{H_{n-m}M_{n-m}}$. If $\mathcal{F}_{\mathcal{C}}(H_{n-m}) \in \Delta_{n-m}$, then

$$\mathcal{F}_{\mathcal{C}}(\Delta_{n-m}) = \Omega(\mathcal{O}, M, \widetilde{H}_{n-m}, M_{n-m})$$

and

$$\Delta_{n-m+1} = \Omega(\mathcal{O}, M_1, \widetilde{H}_{n-m+1}, M_{n-m+1}).$$

Furthermore, Δ_{n-m+1} is contained in certain region $\mathcal{R}^j$.

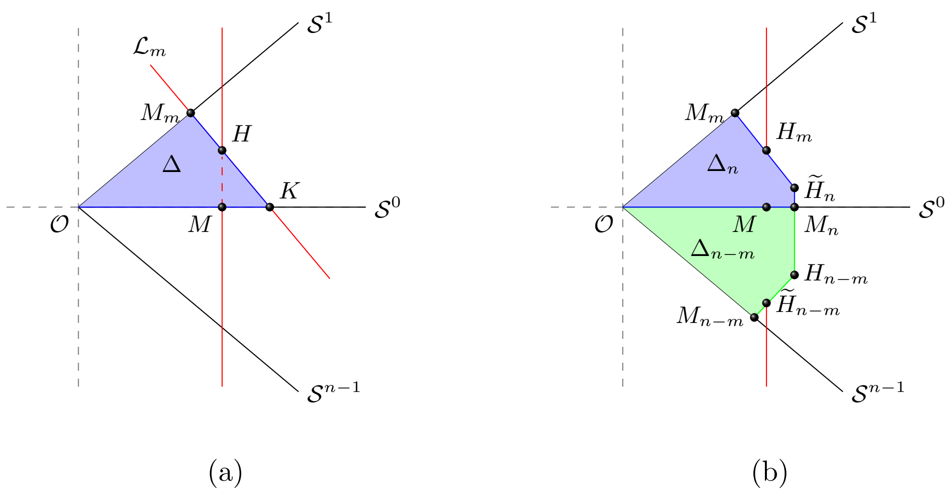


Figure 5

Lemma 4.2. *There exists $a_H \in (1, \infty)$ such that $\mathcal{F}_\mathcal{C}(H_{n-m}) \in \Delta_{n-m}$ for every $a \in (1, a_H)$.*

Proof. From (26) we deduce that

$$\mathcal{F}_\mathcal{C}(H_{n-m}) = (2 - a^n, a^n \cot \frac{2\pi}{n} - a^{n-m} \csc \frac{2\pi}{n}).$$

In order to prove that $\mathcal{F}_\mathcal{C}(H_{n-m}) \in \Delta_{n-m}$ it is sufficient to show that $\mathcal{F}_\mathcal{C}(H_{n-m})$ belongs to the region

$$\mathcal{R}^{n-1} \cap \{x \leq 1\} = \{(x, y) : 0 \leq x \leq 1, -x \tan \frac{2\pi}{n} \leq y \leq 0\}.$$

As we have seen in (27), $\mathcal{F}_\mathcal{C}(H_{n-m}) \in \{y \leq 0\}$. On the other hand, for sufficiently small $a \in (1, a_M)$ it holds that $a^n < 2$. Therefore, it only remains to prove that

$$-(2 - a^n) \tan \frac{2\pi}{n} \leq a^n \cot \frac{2\pi}{n} - a^{n-m} \csc \frac{2\pi}{n}$$

or, equivalently,

$$a^n \cos^2 \frac{2\pi}{n} - a^{n-m} \cos \frac{2\pi}{n} + (2 - a^n) \sin^2 \frac{2\pi}{n} \geq 0 \tag{28}$$

since $n \geq 5$ and both $\sin 2\pi/n$ and $\cos 2\pi/n$ are positive. For $a = 1$, inequality (28) is strictly satisfied:

$$\cos^2 \frac{2\pi}{n} - \cos \frac{2\pi}{n} + \sin^2 \frac{2\pi}{n} = 1 - \cos \frac{2\pi}{n} > 0.$$

By continuity, there exists $a_H > 1$ such that (28) is fulfilled and, consequently, $\mathcal{F}_\mathcal{C}(H_{n-m}) \in \Delta_{n-m}$ for every $a \in (1, a_H)$. ■

Notice that both a_M in Lemma 4.1 and a_H in Lemma 4.2 depend on θ. From now on we denote $a_\theta = \min\{a_H, a_M\}$.

Proposition 4.3. *For every $\theta = 2\pi k/n$ with $\gcd(k, n) = 1$ and $n \geq 5$ and for every $a \in (1, a_\theta)$, it holds that*

(i) $\Delta_n \subset \Delta$,

(ii) $\Delta_j \cap \{x > 1\} = \varnothing$ *for every* $j \in \{1, \ldots, n-1\} \setminus \{n - m\}$, *where m is the smallest natural number such that $M_m \in \mathcal{S}^1$.*

Proof. Recall that $\Delta = \Omega(\mathcal{O}, M_m, K)$. As we have seen below, $\Delta_j \cap \{x > 1\} = \varnothing$ for each $j \in \{1, \ldots, n-m-1\}$ and $\Delta_{n-m} \cap \{x > 1\} \neq \varnothing$. Furthermore, since $a < a_\theta \leq a_H$ it holds that

$$\Delta_{n-m+1} = \Omega(\mathcal{O}, M_1, \widetilde{H}_{n-m+1}, M_{n-m+1})$$

(see Lemma 4.2).

Now, since $a < a_\theta \leq a_M$, then the points M_1 and M_{n-m+1}, and therefore $\widetilde{H}_{n-m+1}$, belong to $\{x \leq 1\}$. This is also true for M_j, M_{n-m+j} and $\widetilde{H}_{n-m+j}$ whenever $n - m + j \leq n - 1$. Hence,

$$\Delta_j = \Omega(\mathcal{O}, M_{j-(n-m)}, \widetilde{H}_j, M_j) \subset \{x \leq 1\}$$

for every $j = n - m + 1, \ldots, n - 1$ and statement (ii) is proved. For $j = n$, it turns out that

$$\Delta_n = \Omega(\mathcal{O}, M_m, \widetilde{H}_n, M_n).$$

See Figure 5b. To prove that $\Delta_n \subset \Delta$ it is sufficient to note that $\widetilde{H}_n$ belongs to $\mathcal{L}_m$ and both $\widetilde{H}_n$ and M_n are on the vertical line $x = a^n$. Since a^n is less than the abscissa of K in (23), statement (i) is proved and the proof ends. ■

Note that

$$\mathcal{F}_{\mathcal{C}}(\Delta) = \mathcal{F}_{\mathcal{C}}(\Delta_n) = \Omega(\mathcal{O}, M_m, H, M)$$

and consequently, $\Delta_j = \Gamma^j_{a,\theta}(\Delta) = \Gamma^j_{a,\theta}(\Delta_n)$. Therefore, we get the following corollary.

Corollary 4.4. *For every $\theta = 2\pi k/n$ with $\gcd(k,n) = 1$ and $n \geq 5$ and for every $a \in (1, a_\theta)$, it holds that*

(i) $\Delta_{2n} = \Delta_n$,

(ii) $\Delta_{n+j} \cap \{x > 1\} = \varnothing$ *for each $j \in \{1, \ldots, n-1\} \setminus \{n - m\}$, where m is the smallest natural number such that $M_m \in \mathcal{S}^1$.*

Let us now study the dynamics of $\Gamma^n_{a,\theta}$ on Δ_n. To that end, we denote by $\mathcal{L}_{-(n-m)} = A^{-(n-m)}_{a,\theta}(\mathcal{C})$ the line given by

$$x\cos(n-m)\theta - y\sin(n-m)\theta = a^{-(n-m)}. \tag{29}$$

We will prove that the n-th iterate of $\Gamma_{a,\theta}$ on Δ_n is the two-fold EBM that results from composing the linear map $A^n_{a,\theta}$ with the folds $\mathcal{F}_{\mathcal{L}_{-(n-m)}}$ and $\mathcal{F}_{\mathcal{C}}$.

Proposition 4.5. *There exists $\overline{a}_\theta \in (1, a_\theta]$ such that for every $(a,\theta) \in (1, \overline{a}_\theta) \times (0, \pi)$ the map $\Gamma^n_{a,\theta}$ restricted to Δ_n is equal to $EBM(\mathcal{C}, \mathcal{L}_{-(n-m)}, \mathcal{O}, a^n I)$, where m is the smallest natural number such that $M_m \in \mathcal{S}^1$.*

Proof. From the proof of Proposition 4.3, it follows that

$$\Gamma^n_{a,\theta}(\Delta_n) = A^m_{a,\theta} \circ \mathcal{F}_{\mathcal{C}} \circ A^{n-m}_{a,\theta} \circ \mathcal{F}_{\mathcal{C}}(\Delta_n).$$

We will see that Lemma 2.3 can be used to get

$$\mathcal{F}_{\mathcal{C}} \circ A^{n-m}_{a,\theta} = A^{n-m}_{a,\theta} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)}}$$

Then

$$\Gamma^n_{a,\theta}(\Delta_n) = A^n_{a,\theta} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)}} \circ \mathcal{F}_{\mathcal{C}}(\Delta_n)$$

and, finally, since $n\theta = 2k\pi$,

$$\Gamma^n_{a,\theta}(\Delta_n) = a^n I \circ \mathcal{F}_{\mathcal{L}_{-(n-m)}} \circ \mathcal{F}_{\mathcal{C}}(\Delta_n). \tag{30}$$

That is, $\Gamma^n_{a,\theta} = EBM(\mathcal{C}, \mathcal{L}_{-(n-m)}, \mathcal{O}, a^n I)$.

In order to apply Lemma 2.3, we need to prove that $\mathcal{F}_{\mathcal{L}_{-(n-m)}}$ is a good fold defined on $\Delta_n \cap \{x \leq 1\}$. Indeed, the equation of $\mathcal{L}_{-(n-m)}$ in (29) can be expressed by

$$x\cos\frac{2\pi}{n} + y\sin\frac{2\pi}{n} = a^{-(n-m)}. \tag{31}$$

Thus, it is parallel to $\mathcal{L}_m$ given in (21) and its slope is $-\cot 2\pi/n < 0$. Both are perpendicular to $\mathcal{S}^1$ and, since $0 < a^{-(n-m)} < a^m$, the line $\mathcal{L}_{-(n-m)}$ cuts the set Δ_n and, in particular, to the critical line $\mathcal{C}$ at the point

$$V = (1, a^{m-n} \csc \frac{2\pi}{n} - \cot \frac{2\pi}{n}) \tag{32}$$

forming an angle $2\pi/n$. Then, the fold with respect to $\mathcal{L}_{-(n-m)}$ is

$$\mathcal{F}_{\mathcal{L}_{-(n-m)}} = a^{m-n} e^{-i2\pi/n} I \circ \mathcal{F}_{\mathcal{C}} \circ a^{n-m} e^{i2\pi/n} I. \tag{33}$$

As a consequence, it is directly verified that the line $\mathcal{F}_{\mathcal{L}_{-(n-m)}}(\mathcal{L}_m)$ is given by

$$x \cos \frac{2\pi}{n} + y \sin \frac{2\pi}{n} = a^m (2a^{-n} - 1).$$

This line intersects $\mathcal{C}$ at the point

$$H'_m = (1, a^{m-n}(2 - a^n) \csc \frac{2\pi}{n} - \cot \frac{2\pi}{n}),$$

so that $\mathcal{F}_{\mathcal{L}_{-(n-m)}}$ is a good fold defined on $\Delta_n \cap \{x \leq 1\}$, as long as

$$a^{m-n}(2 - a^n) - \cos \frac{2\pi}{n} \geq 0. \tag{34}$$

Since $a^{m-n}(2 - a^n) - \cos 2\pi/n > 0$ for $a = 1$, there exits $\overline{a}_\theta > 1$ such that (34) holds for $1 < a \leq \overline{a}_\theta$. The result is proved. ■

Finally, we can prove Theorem B. We take $\varepsilon > 0$ and choose $a_1(\theta) < \overline{a}_\theta$ such that

$$a^{m-n}(2 - a^n) - \cos \frac{2\pi}{n} \geq \varepsilon > 0 \tag{35}$$

for every $1 < a \leq a_1(\theta)$. Then

$$\Gamma^n_{a,\theta} = a^n I \circ \mathcal{F}_{\mathcal{L}_{-(n-m)}} \circ \mathcal{F}_{\mathcal{C}}$$

is well defined on Δ_n. In particular, it is well defined on

$$\Delta^\varepsilon_n = \Delta_n \cap \{y \geq \varepsilon\}.$$

Note that $\mathcal{F}_{\mathcal{L}_{-(n-m)}} \circ \mathcal{F}_{\mathcal{C}}(\Delta^\varepsilon_n) \subseteq \Delta^\varepsilon_n \cap \{x \leq 1\}$ and

$$\Gamma^n_{a,\theta}(\Delta^\varepsilon_n) \subseteq \Delta_n \cap \{y \geq a^n \varepsilon\} \subsetneq \Delta^\varepsilon_n.$$

Therefore, $\Delta^\varepsilon_n \subsetneq \Delta_n$ is a $\Gamma^n_{a,\theta}$-invariant set. Moreover, Δ^ε_n traps the orbit of every point in $\Delta_n \cap \{y > 0\}$. It is clear that Δ^ε_n does not intersect $\mathcal{S}^0$ so that Δ^ε_n and $\Gamma_{a,\theta}(\Delta^\varepsilon_n)$ are disjoint (see Figure 5b). Then $D = \Delta^\varepsilon_n$ verifies each statement of Theorem B. The proof is complete for $n \geq 5$. ■

From the fact that $\mathcal{D} = \Delta^\varepsilon_n$ traps the orbits of every point in $\Delta_n \cap \{y > 0\}$, we can deduce Corollary 1.3 straightforwardly.

4.2. Case $n = 4$

This case can be followed directly from the proof of Lemma 2.7; however, it is considered here for the sake of completeness. The proof is similar to that for $n \geq 5$.

Since $\theta = \pi k/2 \in (0, \pi)$, then $k = 1$ and $\theta = \pi/2$. Let us define

$$\Delta_n = [0, a^4] \times [-a^3, 0]$$

with $a^4 < 2$. Then, $\mathcal{F}_{\mathcal{C}}(\Delta_n) \subset \Delta_n \cap \{x \leq 1\}$, $\Gamma^j_{a,\theta}(\Delta_n) \subset \{x \leq 1\}$ for $j = 1, 2$ and $\Gamma^3_{a,\theta}(\Delta_n) = \Delta_n$.

A new fold is necessary to define $\Gamma^4_{a,\theta}$ restricted to Δ_n in such a way that, by applying Lemma 2.3,

$$\Gamma^4_{a,\theta} = EBM(\mathcal{C}, \mathcal{L}_{-3}, \mathcal{O}, a^4 I)$$

where $\mathcal{L}_{-3}$ is the line $y = a^{-3}$. Note that $\mathcal{F}_{\mathcal{L}_{-3}}$ is a good fold on $\Delta_n \cap \{x \leq 1\}$ whenever $2/a^3 - a > 0$ (or equivalently, $a^4 < 2$). Hence, for $1 < a < a_1 < 2^{1/4}$, the set $\mathcal{D} = \Delta_n$ verifies Theorem B.

4.3. Case $n = 3$

First of all, since $\theta = 2\pi k/3 \in (0, \pi)$, then $k = 1$ and $\theta = 2\pi/3$.

Let $M = (1, 0)$ and $A = (1, \frac{\sqrt{3}}{3}(2a + 1))$ be the intersection points of the critical line $\mathcal{C}$ and the lines $\mathcal{S}^0$ and $\mathcal{C}_1$ respectively. We define

$$\widetilde{\Delta} = \Omega(\mathcal{O}, M, A, M_1)\ .$$

See Figure 6. By construction, $\widetilde{\Delta} \subset \{x \leq 1\}$ and $\widetilde{\Delta}_1 = \Omega(\mathcal{O}, M_1, A_1, M_2)\ .$

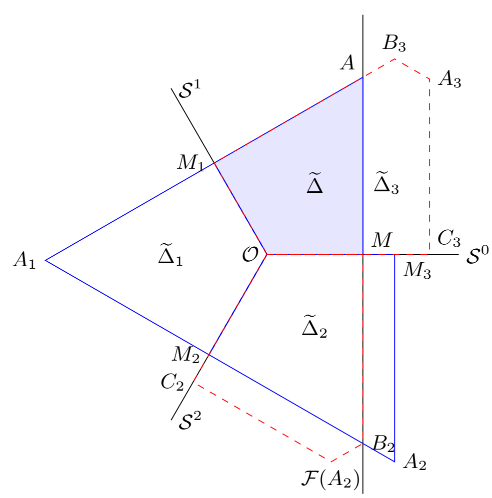


Figure 6: Orbit of $\widetilde{\Delta}$

It is clear that $\widetilde{\Delta}_1 \subset \{x \leq 1\}$. Then, $\widetilde{\Delta}_2 = \Omega(\mathcal{O}, M_2, A_2, M_3)$ being

$$A_2 = \left(a^3, -\frac{\sqrt{3}}{3}a^2(a+2)\right)$$

Since $a > 1$, $\widetilde{\Delta}_2$ intersects $\{x > 1\}$. Hence, to obtain $\widetilde{\Delta}_3$ we first need to define $\mathcal{F}_{\mathcal{C}}(\widetilde{\Delta}_2)$. Note that

$$\mathcal{F}_{\mathcal{C}}(\widetilde{\Delta}_2) \subset \Omega(\mathcal{O}, C_2, \mathcal{F}(A_2), B_2, M)$$

where B_2 is the intersection point between $\mathcal{C}$ and $\mathcal{C}_2$, given by

$$B_2 = (1, -\frac{\sqrt{3}}{3}(2a^2 + 1)) ,$$

and C_2 is the intersection point between S^2 and the straight line parallel to $\mathcal{C}_2$ through the point $\mathcal{F}_{\mathcal{C}}(A_2)$. Therefore, see Figure 6,

$$\widetilde{\Delta}_3 \subset \Omega(\mathcal{O}, C_3, A_3, B_3, M_1),$$

where

$$A_3 = (a^4 + a^3 - a, -\frac{\sqrt{3}}{3}a^4 + \frac{\sqrt{3}}{3}a^3 + \sqrt{3}a)$$

Let us define $\Delta = \Omega(\mathcal{O}, C_3, A_3, B_3, M_1)$. We will prove that Δ is $\Gamma^3_{a,\theta}$-invariant.

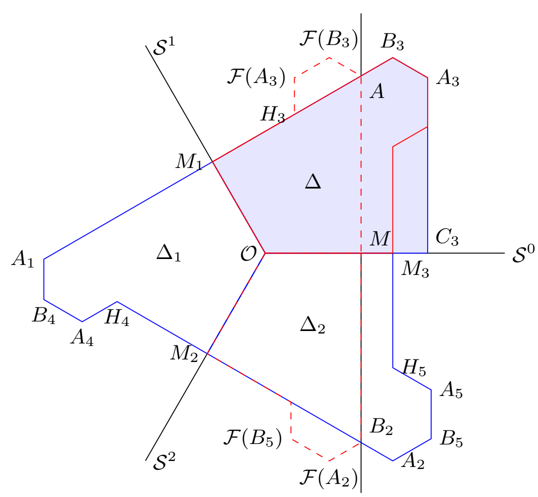


Figure 7: Orbit of Δ

Proposition 4.6. *There exists $a_\theta \in (1, \infty)$ such that, for every $a \in (1, a_\theta)$, it holds that*

(i) $\Delta_3 \subset \Delta$,

(ii) $\Delta_j \cap \{x > 1\} = \varnothing$ *for* $i = 1, 2$.

Proof. See Figure 6 and Figure 7. Note that

$$\mathcal{F}_{\mathcal{C}}(\Delta) = (\mathcal{O}, M, A, \mathcal{F}_{\mathcal{C}}(B_3), \mathcal{F}(A_3), H_3, M_1)$$

where H_3 is the intersection point between $\mathcal{C}_1$ and the vertical line through the point $\mathcal{F}_{\mathcal{C}}(A_3)$. Thus,

$$\Delta_1 = (\mathcal{O}, M_1, A_1, B_4, A_4, H_4, M_2)$$

and

$$\Delta_2 = (\mathcal{O}, M_2, A_2, B_5, A_5, H_5, M_3)$$

where

$$A_5 = (a^5 + a^3 - a^2, \frac{a^2\sqrt{3}}{3}(2a^4 + a^3 - 3a - 3))$$

We claim that

$$\mathcal{F}_{\mathcal{C}}(\Delta_2) \subseteq \Omega(\mathcal{O}, C_2, \mathcal{F}_{\mathcal{C}}(A_2), B_2, M)$$

and hence, $\Delta_3 = \Gamma^3_{a,\theta}(\Delta) \subseteq \Delta$. To prove the claim is enough to prove that $\mathcal{F}_{\mathcal{C}}(A_5) \in \Delta_2$. In fact, $\mathcal{F}_{\mathcal{C}}(A_5) \in \Delta_2$ if and only if $\mathcal{F}_{\mathcal{C}}(A_5) \in \{y - x\tan 2\theta \leq 0\}$ and this holds whenever

$$a^6 + 2a^5 - 3a^2 - 3 \leq 0,$$

Thus, $\Delta_3 = \Gamma^3_{a,\theta}(\Delta) \subseteq \Delta$ as long as $1 < a < 1,1762\ldots$ ■

As in the previous cases, the proof of the first statement of Theorem B for $n = 3$ concludes by taking $\mathcal{D} = \Delta \cap \{y \geq \varepsilon\}$ for $\varepsilon > 0$ arbitrarily small.

Remark 4.7. *The restriction of* $\Gamma^3_{a,\theta}$ *to* Δ *is the composition* $A_{a,\theta} \circ \mathcal{F}_{\mathcal{C}} \circ A^2_{a,\theta} \circ \mathcal{F}_{\mathcal{C}}$ *with* $\theta = 2\pi/3$. *In order to show that* $\Gamma^3_{a,\theta|_\Delta}$ *is the EBM given by* $EBM(\mathcal{C}, \mathcal{L}_{-2}, \mathcal{O}, a^3 I)$, *where* $\mathcal{L}_{-2} = A^{-2}_{a,\theta}(\mathcal{C})$ *is the line* $-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = \frac{1}{a^2}$ *we need that* $\mathcal{F}_{\mathcal{C}} \circ A^2_{a,\theta} = A^2_{a,\theta} \circ \mathcal{F}_{\mathcal{L}_{-2}}$. *But, unfortunately, Lemma 2.3 cannot be applied, because* $\mathcal{F}_{\mathcal{L}_{-2}}$ *is not a good fold on* $\mathcal{F}_{\mathcal{C}}(\Delta)$.

5. Renormalization and Coexistence of Attractors: Proof of Theorem C and Corollary 1.4.

Throughout this section we will suppose that $\Gamma_{a,\theta}$ satisfies the assumptions in Theorem B. In particular, we will assume that inequality (35) holds.

Since $\Gamma^2_{a,\pi/2}$ is the cartesian product of two one-dimensional tent maps, the renormalization for the case $n = 4$ follows from Lemma 2.2 of [19] and the results were already collected in Lemma 2.7. For $n \geq 5$ we cannot take advantage of this fact and we will see how the construction of the restrictive domains for the successive renormalizations get more complicated, depending on whether n is odd or even. In the first case we will obtain a single restrictive domain as stated in statement (a) of Theorem C. In the second case we have $n = 2\nu$. Then, when ν is odd, we also obtain a restrictive domain for a first step in the renormalization, statement (b-i). When ν is even, two disjoint restrictive domains are obtained and the renormalization can be continued, at least one more step, in each of these domains, statement (b-ii). It is clear then that when n is a power of 2 there is a sequence of renormalizations leading to a sequence of doubling attractors, as stated in Corollary 1.4. We emphasize that the results obtained in previous publications [18]-[20] correspond to the case $n = 2^3$. For the general case $n \neq 2^m$, obstructions to the renormalization arise from the difficulty in defining good folds over the corresponding restrictive domain.

We will assume that $n \geq 5$ and denote $\theta_1 = 2\pi/n$. As we have seen in Proposition 4.5, the restriction of $\Gamma^n_{a,\theta}$ to the region Δ_n can be shown as the EBM given by

$$G_{a,\theta} = a^n I \circ \mathcal{F}_{\mathcal{L}_{-(n-m)}} \circ \mathcal{F}_{\mathcal{C}},$$

which is defined as the composition of two folds: the first one along the critical line $\mathcal{C}$ and the second one along the line $\mathcal{L}_{-(n-m)}$ whose equation given in (31) is of the form $y = -\alpha x + \beta$ with $\alpha = \cot\theta_1 > 0$ and $\beta = a^{m-n}\csc\theta_1 > 0$. These critical lines cross each other at the angle θ_1, so that the respective folds are conjugated by $a^{n-m}e^{-i\theta_1}I$, that is,

$$\mathcal{F}_{\mathcal{L}_{-(n-m)}} = a^{m-n}e^{\mathrm{i}\theta_1}I \circ \mathcal{F}_{\mathcal{C}} \circ a^{n-m}e^{-\mathrm{i}\theta_1}I.$$

Therefore, setting $\lambda = a^{m-n}$, we have (see Figure 8a) that

$$G_{a,\theta} = \begin{cases} G^{0,0}_{a,\theta} = a^n I & \text{in } \mathcal{K}_{00} \\ G^{0,1}_{a,\theta} = a^n\lambda e^{i\theta_1}I \circ \mathcal{F}_{\mathcal{C}} \circ \lambda^{-1}e^{-i\theta_1}I & \text{in } \mathcal{K}_{01} \\ G^{1,0}_{a,\theta} = a^n I \circ \mathcal{F}_{\mathcal{C}} & \text{in } \mathcal{K}_{10} \\ G^{1,1}_{a,\theta} = a^n\lambda e^{i\theta_1}I \circ \mathcal{F}_{\mathcal{C}} \circ \lambda^{-1}e^{-i\theta_1}I \circ \mathcal{F}_{\mathcal{C}} & \text{in } \mathcal{K}_{11} \end{cases} \tag{36}$$

Noting that $\mathcal{F}_{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{C},P}$ are reversed folds (the same for $\mathcal{F}_{\mathcal{L}_{-(n-m)}}$ and $\mathcal{F}_{\mathcal{L}_{-(n-m)},P}$), the next lemma is directly obtained from definition of $G_{a,\theta}$.

Lemma 5.1. *The following statements hold:*

(i) In $\mathcal{K}^*_{01} = \{Q \in \mathcal{K}_{01} : \mathcal{F}_{\mathcal{C},P}(Q) \in \mathcal{K}_{11}\}$ *it holds that* $G_{a,\theta} = G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{C},P}$.

(ii) In $\mathcal{K}^*_{00} = \{Q \in \mathcal{K}_{00} : \mathcal{F}_{\mathcal{L}_{-(n-m)},P}(Q) \in \mathcal{K}^*_{01}\}$ *it holds that*

$$G_{a,\theta} = G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{C},P} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P}.$$

(iii) In $\mathcal{K}^*_{10} = \{Q \in \mathcal{K}_{10} : \mathcal{F}_{\mathcal{C}}(Q) \in \mathcal{K}^*_{00}\}$ *it holds that* $G_{a,\theta} = G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{C},P} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P} \circ \mathcal{F}_{\mathcal{C}}$.

Proof.

(i) If $Q \in \mathcal{K}^*_{01}$ then

$$\begin{aligned} G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{C},P}(Q) &= a^n\lambda e^{i\theta_1}I \circ \mathcal{F}_{\mathcal{C}} \circ \lambda^{-1}e^{-i\theta_1}I \circ \mathcal{F}_{\mathcal{C}} \circ \mathcal{F}_{\mathcal{C},P}(Q) \\ &= a^n\lambda e^{i\theta_1}I \circ \mathcal{F}_{\mathcal{C}} \circ \lambda^{-1}e^{-i\theta_1}I(Q) = G^{0,1}_{a,\theta}(Q) \end{aligned}$$

But $G^{0,1}_{a,\theta}(Q) = G_{a,\theta}(Q)$ because $Q \in \mathcal{K}_{01}$.

(ii) If $Q \in \mathcal{K}^*_{00}$ then $\mathcal{F}_{\mathcal{L}_{-(n-m)},P}(Q) \in \mathcal{K}^*_{01}$ and, according to (i),

$$G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{C},P} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P}(Q) = G_{a,\theta} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P}(Q).$$

Since $\mathcal{F}_{\mathcal{L}_{-(n-m)},P}(Q) \in \mathcal{K}_{01}$ then

$$\begin{aligned} G_{a,\theta} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P}(Q) &= G^{0,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P}(Q) \\ &= G^{0,0}_{a,\theta} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)}} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P}(Q) = G^{0,0}_{a,\theta}(Q), \end{aligned}$$

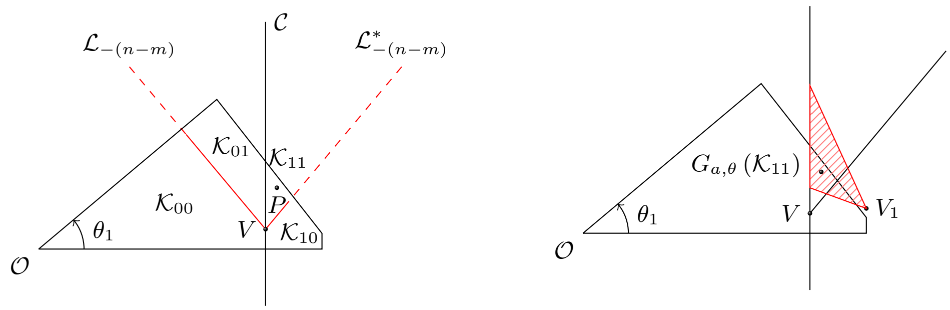
but $G^{0,0}_{a,\theta}(Q) = G_{a,\theta}(Q)$ because $Q \in \mathcal{K}_{00}$.

(iii) If $Q \in \mathcal{K}^*_{10}$ then $\mathcal{F}_{\mathcal{C}}(Q) \in \mathcal{K}^*_{00}$ and, according to (ii),

$$G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{C},P} \circ \mathcal{F}_{\mathcal{L}_{-(n-m),P}} \circ \mathcal{F}_{\mathcal{C}}(Q) = G^{0,0}_{a,\theta}(\mathcal{F}_{\mathcal{C}}(Q)) = G^{1,0}_{a,\theta}(Q),$$

with $G^{1,0}_{a,\theta}(Q) = G_{a,\theta}(Q)$ because $Q \in \mathcal{K}_{10}$.

■



(a) Domains of definition of $G_{a,\theta}$ (b) Intersection of $\mathcal{K}_{11}$ and $G_{a,\theta}(\mathcal{K}_{11})$

Figure 8

The differential map of $G_{a,\theta}$ is constant in the four regions $\mathcal{K}_{ij}$ with $i,j \in \{0,1\}$, and we will see next that it has complex eigenvalues only in $\mathcal{K}_{11}$. More concretely, in this case the differential map is given by the matrix $a^n e^{i2\theta_1} I$; see (8). In order to take advantage of this analogy, in $\mathcal{K}_{11}$ we will search for a fixed point P of $G_{a,\theta}$ and then perform a certain change of variables in which P is the origin of coordinates.

In $\mathcal{K}_{11}$, the map $G_{a,\theta}$ can be written in the complex variable as

$$G^{1,1}_{a,\theta}(z) = a^n(2\lambda e^{i\theta_1} - 2e^{i2\theta_1} + e^{i2\theta_1} z). \tag{37}$$

Then, the fixed point is

$$z_0 = x_0 + iy_0 = \frac{2e^{i2\theta_1} - 2\lambda e^{i\theta_1}}{e^{i2\theta_1} - a^{-n}} \tag{38}$$

where

$$x_0 = 2\,\frac{a^{2n} - a^m(a^n-1)\cos\theta_1 - a^n\cos 2\theta_1}{1 - 2a^n\cos 2\theta_1 + a^{2n}}, \tag{39}$$
$$y_0 = 2\,\frac{a^m(1+a^n)\sin\theta_1 - a^n\sin 2\theta_1}{1 - 2a^n\cos 2\theta_1 + a^{2n}}.$$

The set $\mathcal{K}_{11}$ is limited by $\mathcal{C}$ and the reflection $\mathcal{L}^*_{-(n-m)}$ of $\mathcal{L}_{-(n-m)}$ with respect to $\mathcal{C}$, whose equation is

$$(2-x)\cos\theta_1 + y\sin\theta_1 = a^{-(n-m)}. \tag{40}$$

Therefore, the point P is an interior point of $\mathcal{K}_{11}$ if and only if

$$r_1 = x_0 - 1 \tag{41}$$
$$r_2 = (2-x_0)\cos\theta_1 + y_0\sin\theta_1 - a^{m-n}$$

are positive. In this case, each r_j represents the distance of P to the respective lines limiting $\mathcal{K}_{11}$. Their values are obtained by a straightforward substitution of (39) in (41). That is,

$$r_1 = \frac{a^n - 2a^m \cos\theta_1 + 1}{a^{2n} - 2a^n \cos 2\theta_1 + 1}(a^n - 1), \qquad (42)$$
$$r_2 = \frac{a^m + a^{m-n} - 2\cos\theta_1}{a^{2n} - 2a^n \cos 2\theta_1 + 1}(a^n - 1).$$

Then, since $a > 1$, we have the following lemma:

Lemma 5.2. *The map $G_{a,\theta}$ has a fixed point P in $\mathcal{K}_{11}$ if and only if the following conditions hold:*

(i) $a^n - 2a^m \cos\theta_1 + 1 > 0$

(ii) $a^m + a^{m-n} - 2\cos\theta_1 > 0$

Remark 5.3. *For any θ_1 both conditions of Lemma 5.2 hold as $a \to 1$. In particular, for all $a \in (1, a_1)$, being a_1 given in Theorem B. Indeed, according to (34), it holds that* $\cos\theta_1 < a^{m-n}(2 - a^n)$. *Then, by Lemma 5.2, and since $m \leq n-1$, we obtain*

$$\begin{aligned} a^n - 2a^m \cos\theta_1 + 1 &> a^n + 2a^{2m}(1 - 2a^{-n}) + 1 \\ &> 1 + a^{n-2}(2a^n + a^2 - 4) \end{aligned}$$

where the latter term vanishes for $a = 1$ and increases as $a \to \infty$. On the other hand,

$$a^m + a^{m-n} - 2\cos\theta_1 > 3a^m(1 - a^{-n}) > 0.$$

From this remark we obtain the following result:

Proposition 5.4. *Let $\theta = 2\pi k/n \in (0, \pi)$ with $k, n \in \mathbb{N}$ and $\gcd(k, n) = 1$. For every $a \in (1, a_1)$, the map $G_{a,\theta}$ has a fixed point $P \in \mathcal{K}_{11}$ whose coordinates are given in (39). Moreover, the Jacobian of $G_{a,\theta}$ at P is the matrix $a^n e^{i2\theta_1} I$. In particular,*

$$G_{a,\theta}(Q) = P + a^n e^{i2\theta_1}(Q - P)$$

for all $Q \in \mathcal{K}_{11}$.

Proof. The existence of the fixed point $P \in \mathcal{K}_{11}$ is an immediate consequence of Lemma 5.2 and Remark 5.3.

To calculate the differential of $G_{a,\theta}$ at P, substitute the expression of $G^{1,1}_{a,\theta}$ given in (36) for its differential. Then, we obtain

$$a^n \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is equal to

$$a^n \begin{pmatrix} \cos 2\theta_1 & -\sin 2\theta_1 \\ \sin 2\theta_1 & \cos 2\theta_1 \end{pmatrix}.$$

Finally, since $G^{1,1}_{a,\theta}$ is a composition of affine maps and fixes P, then it can be written as in the statement of this Proposition. ■

We will now consider the folds $\mathcal{F}_{\mathcal{C},P}$ and $\mathcal{F}_{\mathcal{L}^*_{-(n-m)},P}$. Both folds coincide with the identity on $\mathcal{K}_{11}$. It also holds that

$$\mathcal{F}_{\mathcal{L}^*_{-(n-m)},P} = \mathcal{F}_{\mathcal{C},P} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P} \circ \mathcal{F}_{\mathcal{C}}. \tag{43}$$

Going back to Lemma 5.1 to define $\mathcal{K}^* = \mathcal{K}^*_{00} \cup \mathcal{K}^*_{01} \cup \mathcal{K}_{11} \cup \mathcal{K}^*_{10}$. Then, we obtain the following result:

Proposition 5.5. *In $\mathcal{K}^*$ it holds that*

$$G_{a,\theta} = G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{L}^*_{-(n-m)},P} \circ \mathcal{F}_{\mathcal{C},P}. \tag{44}$$

Proof. The first equality is trivially verified in $\mathcal{K}_{11}$ because both $\mathcal{F}_{\mathcal{L}^*_{-(n-m)},P}$ and $\mathcal{F}_{\mathcal{C},P}$ are the identity map. In $\mathcal{F}_{\mathcal{C},P}(\mathcal{K}^*_{01})$, the fold $\mathcal{F}_{\mathcal{L}^*_{-(n-m)},P}$ coincides with the identity map. Therefore, by statement i) of Lemma 5.1,

$$G^{0,1}_{a,\theta} = G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{C},P} = G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{L}^*_{-(n-m)},P} \circ \mathcal{F}_{\mathcal{C},P}.$$

In $\mathcal{K}^*_{00}$, by statement ii) of Lemma 5.1 we have that

$$G_{a,\theta} = G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{C},P} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P}.$$

Since from (43) it follows that

$$\mathcal{F}_{\mathcal{C},P} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P} = \mathcal{F}_{\mathcal{L}^*_{-(n-m)},P} \circ \mathcal{F}_{\mathcal{C},P},$$

we obtain (44). Finally, in $\mathcal{K}^*_{10}$, statement iii) of Lemma 5.1 and (43) lead to

$$G_{a,\theta} = G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{C},P} \circ \mathcal{F}_{\mathcal{L}_{-(n-m)},P} \circ \mathcal{F}_{\mathcal{C}} = G^{1,1}_{a,\theta} \circ \mathcal{F}_{\mathcal{L}^*_{-(n-m)},P}.$$

We obtain (44) by noting that $\mathcal{F}_{\mathcal{C},P}$ is the identity map in $\mathcal{K}^*_{10}$. ■

Remark 5.6. *It is clear that the line $y = 0$ is invariant for $G_{a,\theta}$ and that every line of the form $y = \varepsilon > 0$ is moved upwards by $G_{a,\theta}$. Also, every line forming an angle θ_1 with $y = 0$ located at the right of the origin is moved rightwards. Therefore, it easily follows that the possible attractors for $\Gamma^n_{a,\theta}$ on Δ_n must be in $\Delta_n \cap \mathcal{K}^*$.*

Since Δ_n is an invariant set for $G_{a,\theta}$, according to Remark 5.6, the study of the dynamics of $\Gamma^n_{a,\theta}$ on Δ_n reduces to that on $\mathcal{K}^*$. We will use the expression of $G_{a,\theta}$ given in (44). The composition of the folds $\mathcal{F}_{\mathcal{L}^*_{-(n-m)},P} \circ \mathcal{F}_{\mathcal{C},P}$ takes all $\mathcal{K}^*$ on $\mathcal{K}_{11}$, which is the region limited by the lines $\mathcal{C}$ and $\mathcal{L}^*_{-(n-m)}$. These lines intersect at a point V given in (32) and cross each other at an angle θ_1. The image V_1 of V is obtained after rotating around P an angle $2\theta_1$ and multiplying by a, or directly from $G_{a,\theta}(V) = G^{0,0}_{a,\theta}(V)$ to obtain

$$V_1 = (a^n, a^m \csc\theta_1 - a^n \cot\theta_1) \tag{45}$$

Then, the image $G_{a,\theta}(\mathcal{K}_{11})$ of $\mathcal{K}_{11}$ intersects $\mathcal{K}_{11}$ according to Figure 8b.

Let $\mathcal{A} = G_{a,\theta}(\mathcal{A}) \subset \mathcal{K}^*$ be a strictly $G_{a,\theta}$-invariant set. Then, $G_{a,\theta}(\mathcal{A}) \subset G_{a,\theta}(\mathcal{K}^*) = G_{a,\theta}(\mathcal{K}_{11})$ and therefore $\mathcal{A} \subset \mathcal{K}^* \cap G_{a,\theta}(\mathcal{K}_{1,1}) = \mathcal{R}_1$. Repeating this process iteratively we have that $\mathcal{A} \subset \mathcal{K}^* \cap G_{a,\theta}(\mathcal{R}_1) = \mathcal{R}_2$ and in general $\mathcal{A} \subset \mathcal{R}_n$ with $\mathcal{R}_n = \mathcal{K}^* \cap G_{a,\theta}(\mathcal{R}_{n-1})$. Thus, $A \subset \cap_{n=1}^{\infty} \mathcal{R}_n$. Following the arguments in [21] we can prove that there exists a natural number N such that $\mathcal{R}_n = \mathcal{R}_{n+1}$ for all $n \geq N$, so that $\mathcal{R} = \cap_{n=1}^{N} \mathcal{R}_n$ is an strictly invariant set. However, we are interested in studying next how this strictly invariant set can split into several pieces.

5.1. Renormalization

We will translate $\mathcal{O}$ on the fixed point $P = (x_P, y_P)$ of the map $G_{a,\theta}$ given in (39). Afterwards, the change of variables $(x, y) \to (\frac{x}{1-x_P}, \frac{y}{1-x_P})$ turns $\mathcal{C}$ into another critical line whose equation is $x = 1$ in the new coordinates and which will be still denoted by $\mathcal{C}$. The line $\mathcal{L}^*_{-(n-m)}$ is turned into another line $\mathcal{L}_{a,\theta}$ that crosses $\mathcal{C}$ at an angle θ_1 and whose distance to the origin is

$$r(a) = \frac{r_2(a)}{r_1(a)} = \frac{a^m(a^n - 2a^{n-m}\cos\theta_1 + 1)}{a^n(a^n - 2a^m\cos\theta_1 + 1)}. \tag{46}$$

As usual, we will denote by $\mathcal{K}_{00}$ the region limited by $\mathcal{C}$ and $\mathcal{L}_{a,\theta}$ that contains the new origin. Note that $\lim_{a\to 1} r(a) = 1$.

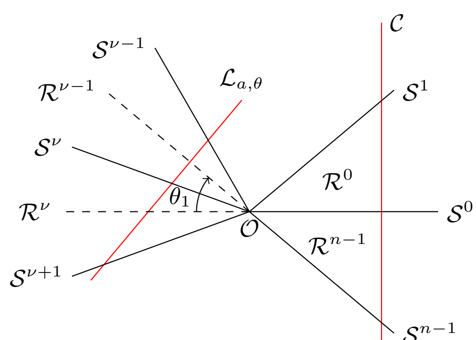
After performing these changes of variables, and according to Propositions 5.4 and 5.5, the dynamics of the restriction of $\Gamma^n_{a,\theta}$ to Δ_n for $a \in (1, a_1)$ is equivalent to that of the family of EBMs

$$\Psi_{a,\theta} = a^n e^{\mathrm{i}2\theta_1} I \circ \mathcal{F}_{\mathcal{L}_{a,\theta}} \circ \mathcal{F}_{\mathcal{C}}. \tag{47}$$

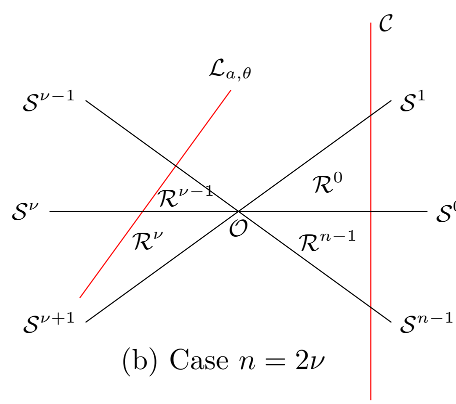
As at the beginning of section 4, for each $j = 0, \ldots, n-1$ let us again denote by $\mathcal{S}^j$ the ray that starts from the origin and has $(\cos j\theta_1, \sin j\theta_1)$ as its director vector. Then, the plane is divided into n regions $\mathcal{R}^0, \ldots, \mathcal{R}^{n-1}$, where $\mathcal{R}^j$ is the region bounded by $\mathcal{S}^j$ and $\mathcal{S}^{j+1}$ for each $j = 0, \ldots, n-1$, setting $\mathcal{S}^n = \mathcal{S}^0$. It is clear that

$$\Psi_{a,\theta}(\mathcal{R}^j \cap \mathcal{K}_{00}) \subset \mathcal{R}^{j+2 \bmod n}.$$

We will denote in this section $M_k = \Psi^k_{a,\theta}(M_0)$. Note that $M_k \in \mathcal{S}^{2k \bmod n}$. In Figure 9 we represent the position of the regions $\mathcal{R}^j$ with respect to the line $\mathcal{L}_{a,\theta}$ depending on whether n is odd (Figure 9a) or even (Figure 9b). We will study each case separately.



(a) Case $n = 2\nu + 1$



(b) Case $n = 2\nu$

Figure 9

5.1.1. Case n odd. Let $n = 2\nu + 1$. Then, $\theta = 2\theta_1 = k2\pi/n$ with $k = 2$ verifies the hypotheses of Theorem B. We will revisit its proof with $\delta = a^n$ instead of a and taking again $\theta_1 = 2\pi/n$. In this case, $m = \nu + 1$. Since the angle formed by $\mathcal{L}_{a,\theta}$ and $\mathcal{C}$ is θ_1, the line $\mathcal{L}_{a,\theta}$ is orthogonal to the bisector of the region $\mathcal{R}^{\nu-1}$, which is limited by $\mathcal{S}^{\nu-1}$ and $\mathcal{S}^{\nu}$. Let r be such that $\mathcal{R}^{\nu-1} = \Psi^r_{a,\theta}(\mathcal{R}^0)$. If ν is odd, then $r = (\nu-1)/2$, otherwise $r = 3\nu/2$. Note that in the latter case the iterates $\Psi^j_{a,\theta}(\mathcal{R}^0)$ with $j = 0, \ldots, r$ describe a full rotation.

Let us take on the rays $\mathcal{S}^{\nu-1}$ and $\mathcal{S}^{\nu}$ the points $M_r = \Psi^r_{a,\theta}(M_0)$ and $M_{r+m} = \Psi^r_{a,\theta}(M_m)$, respectively. Whether ν is even or odd, the distance of these points from the origin is less than δ^n. The orthogonal lines through M_r and M_{r+m} to the respective rays $\mathcal{S}^{\nu-1}$ and $\mathcal{S}^{\nu}$ intersect each other at a point H_r; see Figure 10.

Lemma 5.7. *There exists $\delta_M > 1$ such that for all $1 < \delta = a^n < \delta_M$ it holds that*

(i) The point $M = (\delta^s \cos j\theta_1, \delta^s \sin j\theta_1)$ for all $j = 0, 1, \ldots, n-1$ and $0 \le s < n$ belongs to the half-plane $\{x \le 1\}$.

(ii) The line $\mathcal{L}_{a,\theta}$ intersects the polygon $\Omega(\mathcal{O}, M_r, H_r, M_{r+m})$ at two points H_r^- and H_r^+.

Proof. Statement i) is proved proceeding as in Lemma 4.1. In fact, the result for $\{x < 1\}$ is obvious taking $\delta = 1$ and $j \neq 0$. By continuity it still holds as $\delta \to 1$. Since $r(a) \to 1$ as $a \to 1$ (see (46)), this argument also concludes statement ii). ■

Before proving statement a) of Theorem C, note that except for $\mathcal{F}_{\mathcal{L}_{a,\theta}}$, the map $\Psi_{a,\theta}$ belongs to family (8) with $a = \delta$, $\theta = 2\theta_1 = k2\pi/n$ and $k = 2$. Therefore, we will try to construct the set $\mathcal{D}_1$ from another set $\overline{\Delta} = \Omega(\mathcal{O}, M_m, K)$ as in (22), with $m = \nu + 1$. Then,

$$\mathcal{F}_{\mathcal{C}}(\overline{\Delta}) = \Omega(\mathcal{O}, M_m, H, M)$$

as in Figure 5a, and

$$\overline{\Delta}_1 = \Omega(\mathcal{O}, M_{m+1}, H_1, M_1).$$

In the case $r = (\nu-1)/2$ we obtain

$$\overline{\Delta}_{\nu-1} = \Omega(\mathcal{O}, M_{m+r}, H_r, M_r).$$

Now, in order to obtain $\overline{\Delta}_\nu$, it is necessary to consider the fold $\mathcal{F}_{\mathcal{L}_{a,\theta}}$ that leads to

$$\mathcal{F}_{\mathcal{L}_{a,\theta}}(\overline{\Delta}_{\nu-1}) = \Omega(\mathcal{O}, M_{m+r}, H_r^+, H_r^-, M_r)$$

where H_r^+ and H_r^- are the points given in Lemma 5.7. Then,

$$\overline{\Delta}_\nu = \Omega(\mathcal{O}, M_{m+r+1}, H_{r+1}^+, H_{r+1}^-, M_{r+1})$$

and

$$\overline{\Delta}_{n-m} = \Omega(\mathcal{O}, M_n, H_{n-m}^+, H_{n-m}^-, M_{n-m}).$$

See Figure 10 and compare with Figure 5b. Of course, the set $\overline{\Delta}_{n-m}$ is contained

$$\Delta_{n-m} = \Omega(\mathcal{O}, M_n, H_{n-m}, M_{n-m}).$$

Note that $\overline{\Delta}_{n-m}$ would coincide with Δ_{n-m} if the fold $\mathcal{F}_{\mathcal{L}_{a,\theta}}$ had not intervened. After applying to both Δ_{n-m} and $\overline{\Delta}_{n-m}$ the fold $\mathcal{F}_{\mathcal{C}}$ and the iteration $\Psi^m_{a,\theta}$, we obtain

$$\overline{\Delta}_n \subset \Delta_n \subset \Delta = \overline{\Delta}.$$

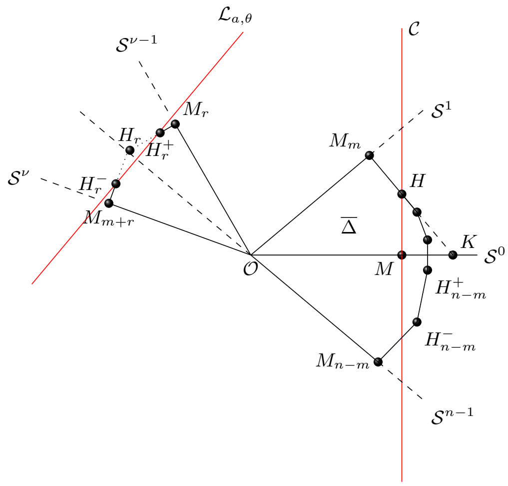


Figure 10: Case $n = 2\nu + 1$: The restriction of $\Psi_{a,\theta}^n$ to $\overline{\Delta}$ is a 3-fold EBM

In the case $r = 3\nu/2$, the fold $\mathcal{F}_{\mathcal{L}_{a,\theta}}$ would have not intervened before the $(n-m)$th iterate, so that $\overline{\Delta}_{n-m} = \Delta_{n-m}$. In fact, $\overline{\Delta}_{n-m+j} = \Delta_{n-m+j}$ for $j = 0, 1, \ldots, \nu/2$. Since $n - m + \nu/2 = r$, we have that $\mathcal{F}_{\mathcal{L}_{a,\theta}}(\overline{\Delta}_{n-m+\nu/2}) \subset \overline{\Delta}_{n-m+j}$, and therefore $\overline{\Delta}_{n-m+j} = \Delta_{n-m+j}$ for $j = 0, 1, \ldots, m$.

In conclusion, we obtain for the map $\Psi_{a,\theta}$ given in (47) the following result, which is similar to Proposition 4.3 and, consequently, to Corollary 4.4.

Proposition 5.8. *There exists $1 < \delta_1 < \delta_M$ such that as long as $1 < \delta = a^n < \delta_1$ it holds that*

(i) $\Psi_{a,\theta}^n(\overline{\Delta}_n) \subset \overline{\Delta}_n$

(ii) $\Psi_{a,\theta}^j(\overline{\Delta}_n) \cap \{x > 1\} = \varnothing$ *for each* $j \in \{1, \ldots, n-1\} \setminus \{n-m\}$*, where m is the smallest natural number such that $M_m \in \mathcal{S}^1$.*

As in the proof of Theorem B, we construct the set

$$\mathcal{D}' = \overline{\Delta}_n^{\varepsilon} = \overline{\Delta}_n \cap \{y \geq \varepsilon\} \tag{48}$$

verifying for $\Psi_{a,\theta}$ statement a) of Theorem B, which proves statement a) of Theorem C.

Remark 5.9. *We have seen that each one of the n disjoint pieces $\mathcal{D}_j$ that contain the strange attractor for $\Gamma_{a,\theta}$ for $1 < a < a_1$ splits itself into another n pieces D_i' when $1 < a < a_2 < a_1$, thus giving rise to n^2 disjoint pieces that contain the strange attractor for $\Gamma_{a,\theta}$. When considering the restriction of $\Gamma_{a,\theta}^{n^2}$ on each one of these n^2 pieces, it is necessary to take into account the fold $\mathcal{F}_{\mathcal{L}_{a,\theta}}$. Then, applying Lemma 2.3 we obtain the restriction of $\Gamma_{a,\theta}^{n^2}$ as a 3-fold EBM.*

5.1.2. Case n even. Let $n = 2\nu$. The line $\mathcal{L}_{a,\theta}$ is orthogonal to the ray $\mathcal{S}^{\nu-1}$ limiting the regions $\mathcal{R}^{\nu-2}$ and $\mathcal{R}^{\nu-1}$. See Figure 11. We will distinguish two cases: ν odd, and ν even, according to statements ii) and iii) of Theorem C.

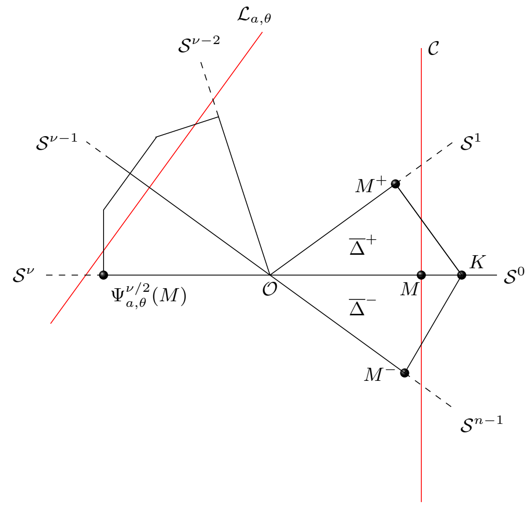


Figure 11: Case $n = 2\nu$ with ν even: There exist two disjoint restrictive domains respectively contained in $\overline{\Delta}^+$ and $\overline{\Delta}^-$

Assume first that ν is odd. Then, we will follow the steps in the proof of Theorem B for the map $\Psi_{a,\theta}$ in (47) with $\theta = 2\theta_1 = 2\pi/\nu$. It is straightforward to check that $\Psi_{a,\theta}$ satisfies the hypotheses and $m = 1$. Let us take the domains

$$\overline{\mathcal{R}}^0 = \mathcal{R}^0 \cup \mathcal{R}^1, \ldots, \overline{\mathcal{R}}^{\frac{\nu-1}{2}} = \mathcal{R}^{\nu-2} \cup \mathcal{R}^{\nu-1}, \ldots, \overline{\mathcal{R}}^{\nu-1} = \mathcal{R}^{n-2} \cup \mathcal{R}^{n-1},$$

and $\overline{\mathcal{R}}^\nu = \overline{\mathcal{R}}^0$. It is easily checked that $\Psi_{a,\theta}(\overline{\mathcal{R}}^j) = \overline{\mathcal{R}}^{j+1}$ for $j = 0, \ldots, \nu - 1$.

Inside $\overline{\mathcal{R}}^0$ let us take the triangle $\overline{\Delta} = \Omega(\mathcal{O}, M_1, K)$, where $M_1 = \Psi_{a,\theta}(M_0)$, with $M_0 = (1, 0)$ and where K is the point of intersection between $y = 0$ and the orthogonal line to $\mathcal{S}^2$ at the point M_1, whose equation is

$$x \cos \frac{2\pi}{\nu} + y \sin \frac{2\pi}{\nu} = \delta = a^n.$$

Then, we can proceed as in the proof of Theorem B to obtain

$$\overline{\Delta}_{(\nu-1)/2} = \Psi^{(\nu-1)/2}_{a,\theta}(\overline{\Delta}).$$

If the line $\mathcal{L}_{a,\theta}$ does not intersect $\overline{\Delta}_{(\nu-1)/2}$, then the process follows as in the proof of Theorem B until obtaining $\overline{\Delta}_\nu \subset \overline{\Delta}$. On the other hand, if the line $\mathcal{L}_{a,\theta}$ intersects $\overline{\Delta}_{(\nu-1)/2}$, then the fold $\mathcal{F}_{\mathcal{L}_{a,\theta}}$ intervenes when it comes to obtaining both $\Psi_{a,\theta}(\overline{\Delta}_{(\nu-1)/2})$ and $\Psi^2_{a,\theta}(\overline{\Delta}_{(\nu-1)/2})$. As we have explained in the case when n is odd, despite the intervention of such fold, it still holds that $\overline{\Delta}_\nu \subset \overline{\Delta}$. Therefore, taking again $\mathcal{D}'$ as in (48) we obtain statement a) of Theorem B for the map $\Psi_{a,\theta}$, equivalently, statement b) of Theorem C.

Remark 5.10. *In this case, each one of the n disjoint pieces $\mathcal{D}_j$ that contain the strange attractor of $\Gamma_{a,\theta}$ for $1 < a < a_1$ splits now into $\nu = n/2$ new pieces D'_i when $1 < a < a_2 < a_1$, thus giving rise to $n^2/2$ disjoint pieces in which the strange attractor of $\Gamma_{a,\theta}$ is contained. When considering the restriction of $\Gamma^{n^2/2}_{a,\theta}$ on each one of these*

$n^2/2$ pieces, it is necessary to take into account the fold along the line $\mathcal{F}_{\mathcal{L}_{a,\theta},\mathcal{O}}$. If this fold intervenes in the dynamics, then again by Lemma 2.3 we know that this restriction $\Gamma_{a,\theta}^{n^2/2}$ is a 3-fold EBM.

Finally, let us suppose that ν is even. In this case, we construct the triangles

$$\Delta^+ = \Omega(\mathcal{O}, M^+, K), \quad \Delta^- = \Omega(\mathcal{O}, M^-, K^-),$$

with $M^\pm = (l\cos\theta_1, \pm l \sin\theta_1)$. The orthogonal lines to the rays $\mathcal{S}^1$ and $\mathcal{S}^{n-1}$ through M^+ and M^-, respectively, intersect $y = 0$ at $K = (l\sec\theta_1, 1)$. Take $l > 1$ such that

$$l\cos\theta_1 < 1, \quad l\sec\theta_1 > 1. \tag{49}$$

The critical line $x = 1$ intersects $\Delta^\pm$ at $M_0 = (1, 0)$ and $H^\pm = (1, \pm(l\csc\theta_1 - \cot\theta_1))$. See Figure 11. It is clear that $\Psi_{a,\theta}^{\nu/2}(M_0) \in \mathcal{S}^\nu$, while $\Psi_{a,\theta}^{\nu/2}(M^-) \in \mathcal{S}^{\nu-1}$ and $\Psi_{a,\theta}^{\nu/2-1}(M^+) \in \mathcal{S}^{\nu-1}$. The respective distances of these points from the origin are $\delta^{\nu/2}$, $l\delta^{\nu/2}$, and $l\delta^{\nu/2-1}$. Arguments as in the proofs of Lemma 4.1 and Lemma 5.7 allows to prove the next result.

Lemma 5.11. *There exists $\delta_M > 1$ such that for all $1 < \delta = a^n < \delta_M$ the following statements hold:*

(i) For $j = 0, 1, \ldots, \nu - 1$, the points $\Psi_{a,\theta}^j(M_0)$, $\Psi_{a,\theta}^j(M^+)$ and $\Psi_{a,\theta}^j(M^-)$ belong to the half-plane $\{x \leq 1\}$.

(ii) $\mathcal{F}_{\mathcal{L}_{a,\theta}}(\Psi_{a,\theta}^j(M^+)) = \Psi_{a,\theta}^j(M^+)$ for all $j \neq \nu/2 - 1$.

(iii) $\mathcal{F}_{\mathcal{L}_{a,\theta}}(\Psi_{a,\theta}^j(M^-)) = \Psi_{a,\theta}^j(M^-)$ for all $j \neq \nu/2$.

Actually, $\mathcal{F}_{\mathcal{L}_{a,\theta}}(\Psi_{a,\theta}^{\nu/2-1}(M^+)) = \Psi_{a,\theta}^{\nu/2-1}(M^+)$ if and only if $l\delta^{\nu/2-1} \leq r(a)$ and $\mathcal{F}_{\mathcal{L}_{a,\theta},\mathcal{O}}(\Psi_{a,\theta}^{\nu/2}(M^-)) = \Psi_{a,\theta}^{\nu/2}(M^-)$ if and only if $l\delta^{\nu/2} \leq r(a)$.

Let us first consider the triangle Δ^+ to analyse the different iterates $\Delta_j^+ = \Psi_{a,\theta}^j(\Delta^+)$ under $\Psi_{a,\theta}$. The segment $\overline{OM_0}$ returns into $\mathcal{S}^0$ after ν iterates. According to Lemma 5.11, in each of these iterates it does not undergo any fold and thus returns with a length of $\delta^\nu > 1$. On the other hand, the segment $\overline{OM^+}$ also returns into $\mathcal{S}^1$ after ν iterates, but it may also be folded by $\mathcal{F}_{\mathcal{L}_{a,\theta},\mathcal{O}}$ in its $\nu/2$th iterate if $l\delta^{\nu/2-1} > r(a)$. Either way, the segment $\overline{OM^+}$ returns into $\mathcal{S}^1$ with a length less than or equal to $r(a)\delta^{\nu/2+1}$. Therefore, in order that $\Delta_\nu^+ = \Psi_{a,\theta}^\nu(\Delta^+)$ is contained in Δ^+, it is enough to take $\delta^\nu < l\sec\theta_1$ and $r(a)\delta^{\nu/2+1} < l$. Since $l > 1$ is fixed and satisfies (49) and $\lim_{a\to1}\delta^\nu = \lim_{a\to1}\delta^{\nu/2+1}r(a) = 1$, we can conclude that there exists $\delta_1 \leq \delta_M$ such that $\Delta_\nu^+ = \Psi_{a,\theta}^\nu(\Delta^+) \subset \Delta^+$ para $1 < \delta = a^n < \delta_1$. Moreover, the restriction of $\Psi_{a,\theta}^\nu$ to Δ_ν^+ is again an EBM of at most two fold. Taking $\mathcal{D}' = \Delta_\nu^+ \cap \{y \geq \varepsilon\}$ as in (48), we conclude that $\mathcal{D}'$ verifies the first three statements of Theorem B, and the restriction of $\Psi_{a,\theta}^\nu$ to $\mathcal{D}'$ belongs to the set $\mathbb{F}_{a,\theta}$.

Let us now consider the triangle Δ^-. The segment $\overline{OM^-}$ returns into $\mathcal{S}^{n-1}$ after ν iterates, but it may also be folded by $\mathcal{F}_{\mathcal{L}_{a,\theta}}$ in its $(\nu/2+1)$th iterate if $l\delta^{\nu/2} > r(a)$. Either way, the segment $\overline{OM^-}$ returns into $\mathcal{S}^{n-1}$ with a length less than or equal to $r(a)\delta^\nu$. Therefore, the same value δ_1 in the previous case is valid to guarantee that

$\Delta_\nu^- = \Psi_{a,\theta}^\nu(\Delta^-) \subset \Delta^-$ for all $1 < \delta = a^n < \delta_1$. Then again, the restriction of $\Psi_{a,\theta}^\nu$ to Δ_ν^- is an EBM of at most two folds. Taking $\mathcal{D}' = \Delta_\nu^- \cap \{y \geq \varepsilon\}$ as in (48), we conclude that $\mathcal{D}'$ verifies the first three statements of Theorem B, and the restriction of $\Psi_{a,\theta}^\nu$ to $\mathcal{D}'$ belongs to the set $\mathbb{F}_{a,\theta}$.

The proof of the third statement of Theorem C is complete.

6. Conclusions and Open Questions

An attractor for a transformation as well as the closure of its basin of attraction (if bounded) are both invariant compact sets. Moreover, attractors are minimal strictly invariant compact sets. Strictly invariant sets for maps defined by a fold along a certain line (such as $\Gamma_{a,\theta}$) are self-similar in some sense and are limited by the successive images of such line. This fact allowed us to prove in [21] the existence of strictly invariant sets $\mathcal{K}_{a,\theta}$ for every $\theta \in (0, \pi)$ and every $a > 1$ sufficiently close to 1. These sets $\mathcal{K}_{a,\theta}$ are polygons, and there exists a nonincreasing sequence $\{a_n(\theta)\}_{n\in\mathbb{N}}$ of values of a at which the number of sides of $\mathcal{K}_{a,\theta}$ increases. This sequence is finite (eventually constant) if and only if θ/π is a rational number. The question of the minimality of $\mathcal{K}_{a,\theta}$ is of utmost importance.

For $\theta = 3\pi/4$ it was proved in [17] that $\mathcal{K}_{a,\theta}$ is a minimal strictly invariant pentagon for all $(1 + \sqrt{2})^{1/4} < a \leq \sqrt{2}$. The expansivity of $\Gamma_{a,\theta}$ for these values of a guarantees then that $\mathcal{K}_{a,\theta}$ is a strange attractor. In [16] and [15] it was numerically shown that for slightly smaller a-values, the pentagon $\mathcal{K}_{a,\theta}$ loses its minimality only to contain another attractor, which is connected but not simply connected (a hole around the origin appears). For even smaller values of a, such attractor splits into eight pieces starting a process of doubling. These processes of splitting and doubling were analytically proved in [18] and [19]. Later in [20] it was proved that each one of these strictly invariant sets contains a disconnected strange attractor whose pieces are contained in those of the strictly invariant set.

The interest in extending the above-mentioned results to any $\theta \neq 3\pi/4$ motivated the following conjectures:

- (i) For every $\theta \in (0, \pi)$ there exists $a_0 = a_0(\theta) > 1$ such that $\Gamma_{a,\theta}$ displays a strictly invariant polygon $\mathcal{K}_{a,\theta}$ for all $a \in (1, a_0)$.
- (ii) The polygon $\mathcal{K}_{a,\theta}$ is minimal (hence a strange attractor) if and only if $(2, 0) \in \mathcal{K}_{a,\theta}$.
- (iii) Let $a_0' > 1$ be the minimum of the values of a for which $(2, 0) \in \mathcal{K}_{a,\theta}$. Then, there exists $a_0'' \in [1, a_0')$ such that for every $a \in [a_0'', a_0')$ there exists a connected neighborhood $\mathcal{U}_{a,\theta}$ of $\mathcal{O}$ such that $\mathcal{K}_{a,\theta} \setminus \mathcal{U}_{a,\theta}$ is strictly invariant and minimal (hence a connected but not simply connected strange attractor). It holds that $a_0'' = 1$ if and only if θ/π is not a rational number.
- (iv) If θ/π is a rational number, a process of splitting and/or doubling of attractors occurs for a sequence of values of a in $(1, a_0'')$.

Conjecture i was positively solved in [21]. From Theorem A we then conclude that there exist strange attractors for any $\theta \in (0, \pi)$ and any $a \in (1, a_0)$.

Conjectures ii and iii remain open, though some numerical experiments (see Figures 12, 13, and 14) seem to support them. A numerical approach to the irrational case is not possible, so that it can only be analytically tackled.

Conjecture iv is actually a complex issue which requires to control a complicated process of renormalization. Given $\theta = 2\pi k/n \in (0, \pi)$ with $k, n \in \mathbb{N}$ and $n > 3$ and $\gcd(k, n) = 1$, we have proved in this paper the existence of two consecutive splittings of the strictly invariant set and therefore of the strange attractor (one splitting for $n = 3$). However, when n is odd or $n = 2\nu$ with ν odd, for the study of a possible third splitting it is necessary to consider a 3-fold EBM. Then, it seems natural to ask whether the splitting process finishes at that point or further renormalizations can be carried out considering other EBMs of more folds. A possible increasing number of folds could make this problem analytically unmanageable. Numerically, some simulations in Figure 12 show that for $n = 7$ each one of the 49 pieces obtained for $a = 1.006$ split again into another 7 pieces as is shown for $a = 1.0006$.

After the proof of Theorem C, we would like to reformulate Conjecture iv in a more precise way: Given the levels of the Sarkovskii ordering for all numbers greater than 3,

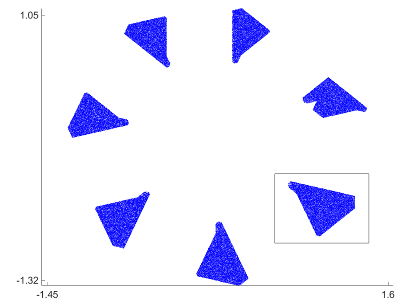
$$S_0\colon 5 \rhd 7 \rhd 9 \rhd \cdots$$
$$S_1\colon 3 \cdot 2 \rhd 5 \cdot 2 \rhd 7 \cdot 2 \rhd \cdots$$
$$S_q\colon 3 \cdot 2^q \rhd 5 \cdot 2^q \rhd 7 \cdot 2^q \rhd \cdots$$
$$S_\infty\colon \cdots \rhd 16 \rhd 8 \rhd 4,$$

we know by Corollary 1.4 that for every $n \in S_\infty$ there exists a sequence of doubling of attractors, and we wonder whether for every $n \in S_q$ there exist $q - 1$ doublings along with a infinite sequence of splitting of attractors.

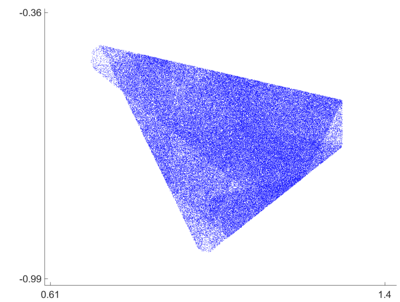
The singular case $n = 3$ should be also studied. See Figure 15.

Acknowledgements

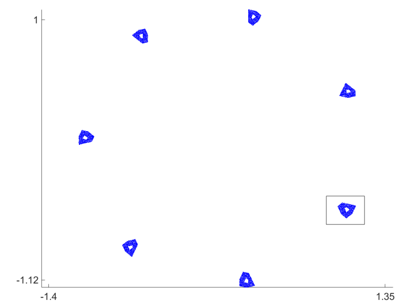
This work has been supported by the project PID2020-113052GB-I00.



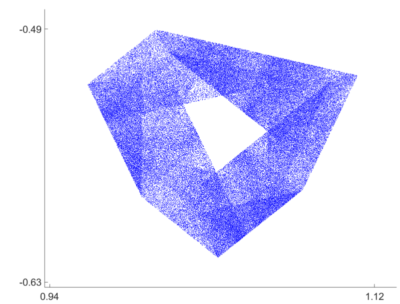
(a) 7-piece attractor for $a = 1.05$



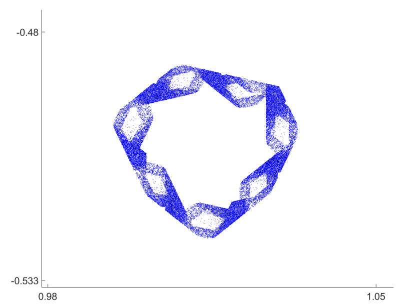
(b) Amplification of a piece in (a)



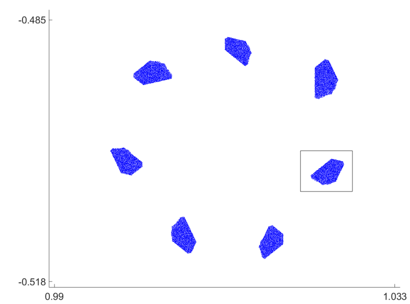
(c) 7-piece attractor for $a = 1.02$



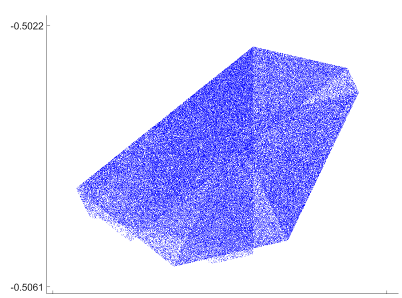
(d) Amplification of a piece in (c)



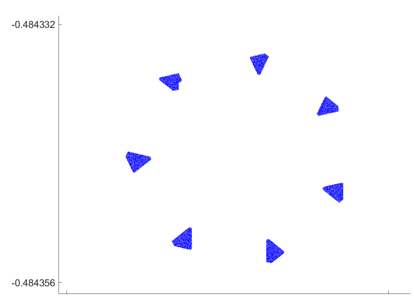
(e) Evolution of (d) before the splitting



(f) Splitting of (d) for $a = 1.006$

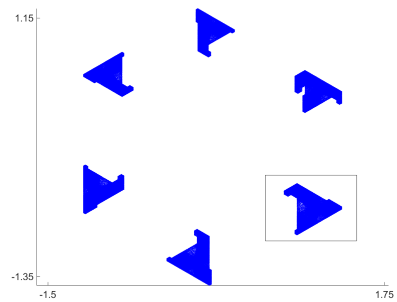


(g) Amplification of a piece in (f)

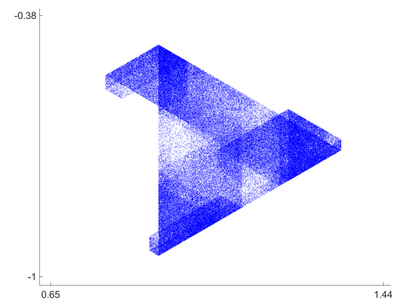


(h) Splitting of (g) for $a = 1.0006$

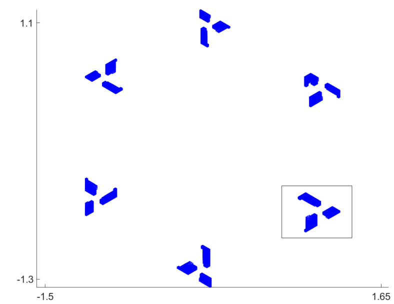
Figure 12: Numerical simulation for $\theta = 2\pi/7$ and decreasing values of a



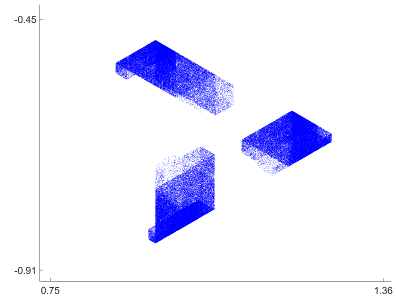
(a) 6-piece attractor for $a = 1.05$



(b) Amplification of a piece in (a)

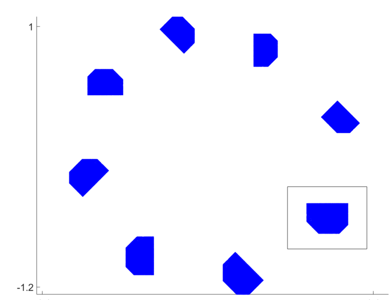


(c) Splitting of (b) for $a = 1.04$

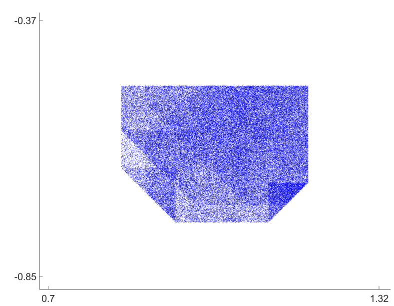


(d) Amplification of a piece in (c)

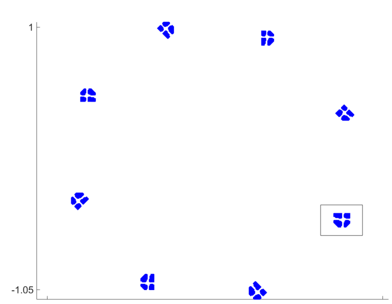
Figure 13: Numerical simulation for $\theta = 2\pi/6$ and decreasing values of a



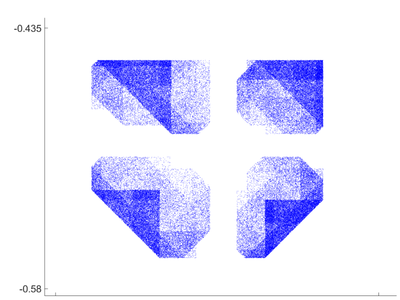
(a) 8-piece attractor for $a = 1.05$



(b) Amplification of a piece in (a)

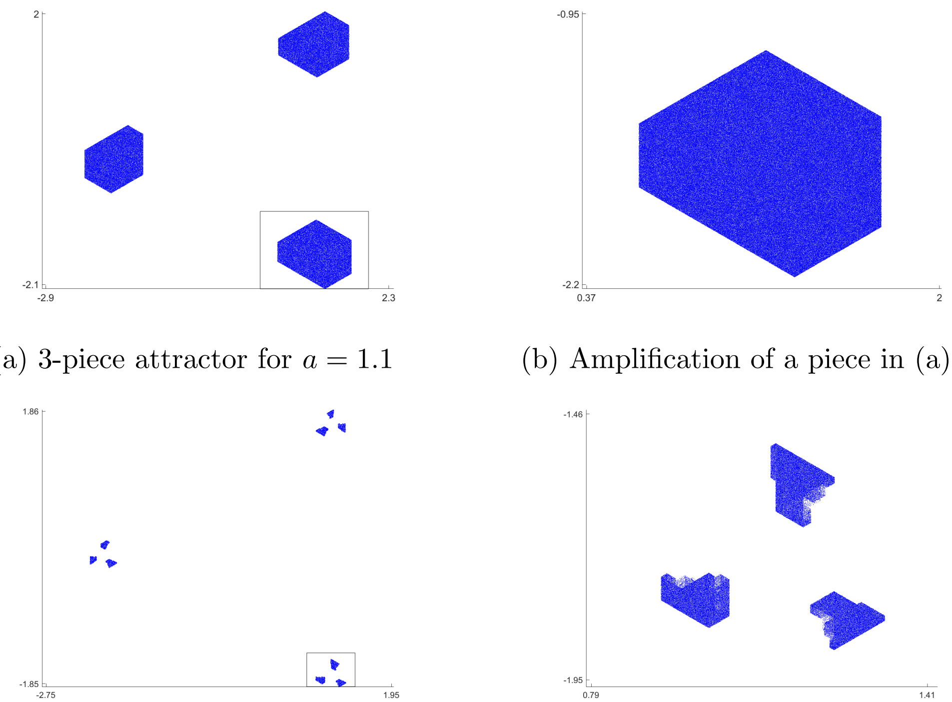


(c) Splitting of (b) for $a = 1.02$



(d) Amplification of a piece in (c)

Figure 14: Numerical simulation for $\theta = 2\pi/8$ and decreasing values of a



(a) 3-piece attractor for $a = 1.1$

(b) Amplification of a piece in (a)

(c) Splitting of (b) for $a = 1.05$

(d) Amplification of a piece in (c)

Figure 15: Numerical simulation for $\theta = 2\pi/3$ and decreasing values of a

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