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Degenerations of Generalized Krichever - Novikov Algebras on Tori

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Abstract. Degenerations of Lie algebras of meromorphic vector fields on elliptic curves (i.e. complex tori) which are holomorphic outside a certain set of points (markings) are studied. By an algebraic geometric degeneration process certain subalgebras of Lie algebras of meromorphic vector fields on $\mathbb{P}^1$ the Riemann sphere are obtained. In case of some natural choices of the markings these subalgebras are explicitly determined. It is shown that the number of markings can change.

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1. Introduction

The Virasoro algebra is of fundamental interest in certain branches of mathematics and physics (conformal field theory etc.). The Virasoro algebra is the graded Lie algebra given as the universal central extension of the algebra of meromorphic vector fields on the Riemann sphere S^2 holomorphic outside the points $z = 0$ and $z = \infty$. The grading is given by the order of the zeros at the point $z = 0$ (a negative order corresponds to a pole). For details, see Section 5.

In [11] Krichever and Novikov introduced a generalization of this graded algebra to higher genus Riemann surfaces. The algebra (without central extension) consists of vector fields meromorphic on the compact Riemann surface X and holomorphic outside two arbitrary generic but fixed points. Of course, this step of the generalization is quite forward. Their important step was to introduce a grading by giving a suitable basis of the algebra by defining them to be the homogeneous elements. With respect to this grading the Lie algebra is almost graded (see Sec. 3).

In the interpretation of X together with the two points as the string world sheet of one incoming and one outgoing string it is quite natural to ask for generalizations of these almost graded Lie algebras for the case of several strings coming in and several strings (not necessarily the same number) going out. Such a generalization was given by me [16-18] (see also [14,15]). For related work of other authors on such generalizations see [6,13,9]. Again the first step is quite forward. The algebra without central extension consists of vector fields which are meromorphic on X outside a fixed finite set of points. After dividing up this set in two nonempty subsets which I call in- resp. out-points a grading can be introduced in such a way that the Lie algebra structure is almost graded. In fact, all of this can be done for the whole Lie algebra of differential operators of degree less or equal 1, which is a semidirect product of the vector field algebra with the abelian algebra of functions. Generalizations of affine Kac-Moody algebras can be introduced (see [17,18]).

If one deforms the complex structure of the Riemann surface together with the set of points a lot of interesting problems appear. For example, it is quite natural to study such deformations in string theory. Here one has to 'integrate' over the whole moduli space of marked Riemann surfaces. This moduli space is not compact. To compactify it one has to introduce 'singular Riemann surfaces' and to study the behaviour of the nonsingular objects when approaching the boundary of the moduli space. After 'resolving the singularity' (see below), one obtains honest Riemann surfaces of lower genera. By this one obtains a relation between disjoint unions of moduli spaces for nonsingular Riemann surfaces. (see [8] for possible physical interpretation).

To perform such a study one has to know how the basis of the vector field algebra or more generally the basis of the moduls of forms of arbitrary weights behave under the deformation of the complex structure of the Riemann surface and under the deformation of the positions of the points where poles are allowed. For this it is quite useful to have explicit expressions of the basis elements. Such were given in [15,17] in terms of rational functions (for genus $g = 0$) , theta functions ($g \geq 1$) and Weierstraß σ function ($g = 1$). Even in these representations it is very hard to see what happens under degeneration to singular 'Riemann surfaces'.

By using the Weierstraß $\wp$ function Deck [3] and Ruffing [12] (see also [5]) studied the case of the torus (i.e. $g = 1$) with 2, 3 and 4 'punctures' in certain symmetric positions. They were able to express the structure constants of the vector field algebra in purely algebraic deformation parameters, and observed that for certain values corresponding to certain degenerations there is a (formal) relation to the Virasoro algebra (see also [4]).

In this letter I want to study the deformation for the $g = 1$ case in detail and give a precise formulation on geometrical grounds of the relation between the involved objects. Having at least a large part of the prospective readership in mind, I have to say a few words on the algebraic geometric language before I can give a rough outline of the result. This will be done in Section 2.
In Section 3 I translate the setting of marked complex tori into the setting of marked elliptic curves (i.e. nonsingular cubic curves).
In Section 4 I study the degenerations of the marked cubic curves with some natural choices of the markings into marked singular cubic curves, together with the basis of the associated vector field algebras.
In Section 5 the objects obtained by degenerations are identified with objects living on the complex projective line (i.e. the Riemann sphere).
The main result can be found in Theorem 1 in this section.

In this letter I will restrict myself to the case of two and three markings. All possible effects occur already here. The studies have been done for forms of arbitrary integer weight λ. Due to lack of space I will concentrate on the vector fields (i.e. $\lambda = -1$). The reader will have no difficulties doing the general case including the differential operator algebra along the outline given in this letter by himself.

2. A few words on the language of algebraic geometry and an outline of the results

The right theory to describe degenerations of curves is the theory of algebraic geometry. Here we can confine ourselves with the language of algebraic varieties. Roughly speaking, a closed variety is the set of zeros of a polynomial. A (complex) affine or projective curve is an closed algebraic variety over $\mathbb{C}$ of dimension 1 either in affine or projective space. I will call them just curves. For the notions above and in the following see [19] for a quick introduction or any other book on algebraic geometry. Singular curves are incorporated in the theory of algebraic curves from the beginning. By well-known theorems due to Chow and Kodaira every compact Riemann surface X can be holomorphically embedded as a complex nonsingular projective curve in projective space in a very strong sense. For example, meromorphic functions (differentials) on X become rational functions (differentials) on the embedded curve [19,p.60]. Recall, a rational function on the embedded curve in $\mathbb{P}^n$ can be given as quotient of homogeneous polynomials of the same degree in $n+1$ variables.

Using the differential equation of the associated Weierstraß $\wp$ function I will give in Section 3 the well-known embedding for a torus into the complex plane $\mathbb{P}^2$ in an explicit manner

By this embedding all tori can be identified as cubic curves (i.e. curves on which the points are the zeros of a homogeneous cubic polynomial) which are nonsingular. Such curves are termed elliptic curves. By allowing the cubic polynomial to degenerate one obtains two nonisomorphic types of irreducible singular cubic curves. The first one is the nodal cubic E_N with one singular point which is a node, i.e. a point where two branches of the curve meet transversly. The other one is the cuspidal cubic E_C with one singularity, which is a cusp, i.e. a point with only one branch of the curve meeting this point with higher multiplicity. In Section 4 I will give exact polynomials describing these curves (see also [19,p.79]).

By an algebraic geometric process (normalization or blow-up) every singular curve X can be desingularized, i.e. there is a nonsingular curve $\widehat{X}$ and a surjective algebraic map $\psi : \widehat{X} \to X$ such that outside the singularities of X ψ is an isomorphism and above the singularities there are only finitely many points of $\widehat{X}$. Moreover, the desingularization curve is up to algebraic isomorphisms uniquely defined by the above features.

A family of curves consists of 2 (not necessarily closed) varieties B and $\mathcal{X}$ and a surjective algebraic map $\phi : \mathcal{X} \to B$ such that the fibres $\phi^{-1}(b)$ for every base point $b \in B$ is an algebraic curve. The different fibres are called the members of the family. B is called the base. Intuitively, a family of curves consists of curves depending on algebraic parameters. If X and X' are different curves, but there is a family $\mathcal{X}$ over an irreducible base which contains both curves as members, we call X a deformation of X' and vice versa.

If X is a nonsingular member of a family of curves over an irreducible base then all other nonsingular members have the same genus g as X. For members of the family which are singular curves the genus g defined as dimension of the global holomorphic differentials does not make sense.[1] But at least we know that the genus of their desingularization will be less then g. Hence, in our situation the desingularization of E_N and E_C is a nonsingular curve of genus 0. This only could be $\mathbb{P}^1$ the complex projective line. We obtain maps

$$\psi_N : \mathbb{P}^1 \to E_N, \qquad \psi_C : \mathbb{P}^1 \to E_C \ . \tag{2-1}$$

For the nodal cubic there are 2 points lying above the singularity, for the cuspidal cubic there is one point lying above the singularity.

If objects are defined in a global way for a family of cubic curves (maybe containing singular ones) they are defined for every member (curve) including the singular ones. In the case I will consider in this letter the meromorphic vector fields etc. can be expressed as rational expressions in the affine coordinate functions X and Y of the complex plane. Considered as functions on the curves they will depend on the algebraic deformation parameters. The objects will make sense also in the limit on the singular cubics. The X and Y coordinate functions of the curve in the limit can be expressed as functions of the affine coordinate function t on $\mathbb{P}^1$, i.e. they can be pullbacked to $\mathbb{P}^1$. By this process they define objects on $\mathbb{P}^1$.

We have to keep in mind that in our family of cubic curves every curve posesses a certain finite set of marked points (sometimes called 'punctures') where our objects are allowed to have poles. If we want to speak of a deformation of marked curves we have to require that the marked points will vary 'smoothly' along the different members. In more precise terms, the maps assigning to every base point of the family the corresponding markings are required to be algebraic maps.

The most natural choice of points on elliptic curves are the $n-$torsion points ($n \in \mathbb{N}$). They are defined as follows. An elliptic curve E comes with an abelian group structure compatible with the algebraic geometric structure. A $n-$torsion point is a point $a \in E$ with $n \cdot a = 0$. It is called a primitive $n-$torsion point if n is the smallest such natural number. In the complex analytic picture tori are realized as

$$T \ = \ \mathbb{C}/L, \qquad L = \langle\ 1\ ,\ \tau\ \rangle, \qquad \tau \in \mathbb{C},\ \operatorname{Im}\tau > 0 \tag{2-2}$$

where L is a lattice. Here the above group structure is nothing else as addition in $\mathbb{C}$. For example the primitive $2-$torsion points are the point $(\mod L)$

$$w_1 = \frac{1}{2}, \qquad w_2 = \frac{1+\tau}{2}, \qquad w_3 = \frac{\tau}{2} \ . \tag{2-3}$$

[1]The correct invariant object for the family is the arithmetic genus $p_a = \dim H^1(X, \mathcal{O}_X)$ of the curves X. Due to Serre duality it coincides for nonsingular curves with the (geometric) genus g.

Such $n-$torsion points can universally be defined for the whole family of elliptic curves. Only such markings will be considered here.

Now I come to an outline of the results. In Section 3 I will define a family of cubic curves depending on 3 Parameters e_1, e_2 and e_3. The curves are nonsingular if all 3 parameters are pairwise distinct. By giving different choices of the marking I obtain different families of marked cubic curves. We can continue the whole situation to all possible values of the parameters e_1, e_2 and e_3. We obtain above the additional points singular cubic curves. In the case that the marking maps stay distinct also in the limit case and none of marked points in the limit case coincide with a nodal singularity (cuspidal singularities are allowed) we obtain from an torus situation with N markings (punctures) a Riemann sphere situation with the same number of markings. If one of the marked points coincide with a nodal singularity we obtain from a N marking situation on the torus an $N+1$ situation on the sphere. Of course, it is also possible that the some markings coincide in the limit. This will give us a transition to situation with less markings. We see that if we want to consider degenerations we can not stay at a fixed number of markings.

Because we have to study forms of weight λ, (i.e. sections in the λ-tensor power of the canonical bundle K_E) we have to examine the behaviour of $(dz)^\lambda$ under the algebraization and the pullback. Here an additional effect at the singularities occur. In the case of the vector fields ($\lambda = -1$) additional zeros are introduced.[2] Hence we obtain only a subalgebra in the degeneration, if the singularity is not a marked point.

By this well defined process of algebraic geometric degenerations we have the deformations of the algebras of vector fields on the elliptic curves (resp. tori) with N markings under explicit control. In the 3 point case I studied the following degenerations were obtained:
(1) the full 3 point vector field algebra on $\mathbb{P}^1$, or (2) some subalgebra which can explicitly given, (3) the full Virasoro-algebra (without central extension) or (4) a explicitly given subalgebra of it.
Which case occurs depends on the markings and the algebraic geometric deformation under consideration. Theorem 1 in Section 5 gives the exact result. Such a list can easily given for every N.

Let me mention here that it is also possible to study deformations of the situation in the sense that the genus (i.e. the topology) does not change but two marked points come together (see [5]).

[2] In the case of the differentials ($\lambda = 1$) additional poles of certain orders are introduced.

3. From marked tori to marked elliptic curves

By definition a torus T is the quotient of $\mathbb{C}$ by a lattice $L = \langle \omega_1, \omega_2 \rangle$, $T = \mathbb{C}/L$, where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. The complex structure of the quotient is the complex structure induced from $\mathbb{C}$. Up to analytic isomorphy T can be given as quotient with respect to a lattice L as given in (2-2). The field of meromorphic functions on T can be identified with the field of doubly-periodic functions on $\mathbb{C}$, which in turn can be generated by the Weierstraß $\wp$ function and its derivative $\wp'$ subject to the relation

$$\wp'(z,\tau)^2 = 4\wp(z,\tau)^3 - g_2(\tau)\wp(z,\tau) - g_3(\tau) \ . \tag{3-1}$$

Here everything depends on the lattice parameter τ. The discriminant $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$ is for every admissible τ different from zero. For further reference we collect the following well-known results. $\wp$ has a pole of order 2 at the lattice points and is an even function. $\wp'$ has a pole of order 3 at the lattice points and is an odd function. It has zeros at the the primitive $2-$torsion points on T.

Introducing $e_i(\tau) = \wp(z_i, \tau)$ we can reformulate (3-1)

$$\wp'(z,\tau)^2 = 4\big(\wp(z,\tau) - e_1(\tau)\big)\big(\wp(z,\tau) - e_2(\tau)\big)\big(\wp(z,\tau) - e_3(\tau)\big) \ . \tag{3-2}$$

The condition $\Delta(\tau) \neq 0$ is equivalent to the fact, that alle $e_i(\tau)$ are distinct.[3] We obtain $e_1(\tau) + e_2(\tau) + e_3(\tau) = 0$.

Now we want to rewrite the situation in terms of algebraic geometry. We embed the torus T given by an arbitrary but fixed τ into $\mathbb{P}^2$ the complex projective plane via

$$\Phi : T \to \mathbb{P}^2, \quad z \mod L \mapsto \begin{cases} (\wp(z) : \wp'(z) : 1), & z \notin L \\ (0:1:0) & z \in L \ . \end{cases} \tag{3-3}$$

If we denote the homogeneous coordinates in $\mathbb{P}^2$ by $(x : y : z)$ the image of Φ is the set of zeros of the homogeneous polynomial

$$Y^2Z - 4(X - e_1(\tau)Z)(X - e_2(\tau)Z)(X - e_3(\tau)Z) \ . \tag{3-4}$$

Hence, it is an algebraic curve. which turns out to be nonsingular. This curve E is called an elliptic curve. Indeed Φ is an analytic isomorphism on its image. Conversely every zero set of a homogeneous cubic polynomial which is a nonsingular curve can be described (after a suitable coordinate transformation) as a zero set of a polynomial of the type (3-4) with $e_1, e_2, e_3 \in \mathbb{C}$, $e_1 + e_2 + e_3 = 0$ and comes from an analytic situation.

[3] for more information see [10,19],etc.

Notice, in (3-4) all e_i are on equal footing. This represents the fact that in the algebraic geometric picture all primitive 2-torsion points are equivalent. If we define

$$B := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0,\ e_i \neq e_j \text{ for } i \neq j\},$$

then B is a open subvariety of the affine (even linear) variety

$$\widehat{B} := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0,\ \}\ .$$

Eq. (3-4) defines a family $\mathcal{X}$ of nonsingular cubic curves over the base B. Obviously (3-4) also makes sense over the whole of $\widehat{B}$. The additional members of the family are the singular cubic curves which will be dealt with in Section 4.

In the following the point $0 \mod L$ will always be a marking and has nothing to do with the singularity. Hence, it is possible and convenient to use affine coordinates in $\mathbb{P}^2$ (obtained by setting the 3. coordinate equal to 1). If we denote $(0:1:0)$ by the symbol ∞ we can rewrite (3-3)

$$z \mod L \mapsto (\wp(z), \wp'(z)), \quad z \notin L, \qquad \text{and} \quad L \mapsto \infty\ . \tag{3-5}$$

The affine part of the elliptic curve is given as the zero set of the polynomial

$$Y^2 - 4(X - e_1)(X - e_2)(X - e_3)\ . \tag{3-6}$$

Under this identification the affine coordinate function X corresponds to $\wp$ and Y corresponds to $\wp'$. The field of meromorphic function on the torus corresponds to the field of rational functions on the curve E

$$\mathbb{C}(X)[Y]/\big(Y^2 - f(X)\big), \quad f(T) = 4(T - e_1)(T - e_2)(T - e_3)\ . \tag{3-7}$$

Every meromorphic function on the torus can be described as a rational function (i.e. as a function which is a quotient of polynomials) in X and Y. In fact, (3-7) shows even that it can be given as a rational function in X plus Y times another rational function in X. A function f is called regular on a subset of the curve E if f can be given as quotient of polynomials such that the the denominator polynomial does not vanish on E. Regular functions correspond to holomorphic functions. Due to this equivalence I will usually call rational (regular) functions meromorphic (holomorphic) functions.

We have to introduce markings, i.e. choose points where poles are allowed. As explained in Section 2 the natural choices are $n-$torsion points. I will consider here the following cases.

(Case A). In the two point case I choose $z_0 = 0$ and $z_1 = 1/2$ in the analytic picture. In the elliptic curve picture this corresponds to choosing the point ∞ and the point with the affine coordinates $(e_1, 0)$. A different choice of a 2−torsion point instead of $z_1 = 1/2$ (as done in [2]) yields an isomorphic algebra.

(Case B). In the three point case I choose $z_0 = 0$, $z_1 = 1/2 + q$, and $z_2 = 1/2 - q$ with the two subcases $q = \tau/4$ and $q = 1/2 + \tau/4$. In both cases z_1 and z_2 are primitive 4−torsion points. In fact, these are the two pairs of solutions to $2 \cdot (z \mod L) = \tau/2 \mod L$. Other choices of 4−torsion points will give the same behaviour of the above situations (studied at different deformations). On the elliptic curve, the markings correpond to the points ∞ , (a, b) , and $(a, -b)$ with

$$a = \wp(1/2 + q), \quad b = \wp'(1/2 + q) = \sqrt{4(a - e_1)(a - e_2)(a - e_3)} \ . \tag{3-8}$$

Both (a, b) and $(a, -b)$ occur as markings. Hence, there is no ambiguity of sign in the complex square root of b . We are interested in varying the algebraic parameters e_1, e_2 and e_3. For this goal we have to express a (and automatically b) in terms of them. Using the addition theorem of the $\wp$ function (see [7]) we obtain

$$a(e_1, e_2, e_3) \ = \ e_3 + \sqrt{e_3 - e_1}\sqrt{e_3 - e_2} \ . \tag{3-9}$$

Here the sign of the product of the square root has to be choosen following the rules given in [7]. The two possibilities occuring correspond to the two choices for q. We obtain two pairs of values (a_1, b_1) , $(a_1, -b_1)$ and (a_2, b_2) , $(a_2, -b_2)$.

The main reason for choosing the above markings is that there coincide with the cases considered by Deck and Ruffing [3-5,12]. Hence, I am able to refer to their results and avoid redoing the calculation. By setting $q = 0$ in (Case B) we obtain formally the 2 point case from the 3 point case. This allows an elegant compact form of notation if we define $(a_0, b_0) = (e_1, 0)$ as we will see immediately.

Proposition 1. *A basis of the vector space of meromorphic (rational) functions on the elliptic curve E holomorphic outside the points ∞ and $(e_1, 0)$ in the 2 point case ($s = 0$), resp. ∞, (a_s, b_s), and $(a_s, -b_s)$ for the 3 point cases ($s = 1, 2$) is given by the elements*

$$A^s_{2k} := (X - a_s)^k, \qquad A^s_{2k+1} := \frac{1}{2}Y(X - a_s))^{k-1} \tag{3-10}$$

where k runs over all integers. ($s = 0, 1, 2$ denotes the different cases.)

Proof. Up to reindexing and rewriting it is just the basis introduced by Deck and Ruffing. For a proof see [5]. In fact, this is quite easy to see directly. Using the information on the functions on E following Eq. (3-7) we see that the denominator polynomial in the rational functions in X could only be powers of $(X - a_s)$. By developing the numerator polynomial in powers of $(X - a_s)$ we obtain the result. $\square$

Using the following facts about the orders at the points of the elliptic curve ($\gamma \in \mathbb{C}$, $a \neq e_1, e_2, e_3$)

$$\begin{aligned} \mathrm{ord}_\infty(X-\gamma) &= -2, \qquad & \mathrm{ord}_\infty(Y) &= -3, \\ \mathrm{ord}_{(e_i,0)}(X-e_i) &= 2, \qquad & \mathrm{ord}_{(e_i,0)}(Y) &= 1, \\ \mathrm{ord}_{(a,b)}(X-a) &= 1, \qquad & \mathrm{ord}_{(a,-b)}(X-a) &= 1\ , \end{aligned} \tag{3-11}$$

we obtain

Proposition 2. *The divisors for the basis elements are given as follows:*
(a) in the two point case:

$$\begin{aligned} (A^0_{2k}) &= -2k[\infty] + 2k[(e_1,0)], \\ (A^0_{2k+1}) &= -(2k+1)[\infty] + (2k-1)[(e_1,0)] + 1[(e_2,0)] + 1[(e_3,0)], \end{aligned}$$

(b) in the three point cases: $(s = 1, 2)$

$$\begin{aligned} (A^s_{2k}) &= -2k[\infty] + k[(a,b)] + k[(a,-b)], \\ (A^s_{2k+1}) &= -(2k+1)[\infty] + (k-1)[(a_s,b_s)] + (k-1)[(a_s,-b_s)] \\ &\quad + 1[(e_1,0)] + 1[(e_2,0)] + 1[(e_3,0)]. \end{aligned}$$

Recall that a divisor of a function (or generally of a section of a line bundle) denotes the points where zeros of this function occur with their muliplicities. A zero of negative multiplicity is a pole.

In the genus 1 case the canonical bundle K and hence all its tensor powers are trivial. If f is a meromorphic function on the torus T then $f(z)(dz)^\lambda$ is a meromorphic section of the bundle $K^{\otimes\lambda} = K^\lambda$. We have to express the analytic differential dz as

$$dz = \frac{dX}{Y}\ . \tag{3-12}$$

Observe that $(X-\gamma)$ for $\gamma \notin \{e_1, e_2, e_3\}$ is a uniformizing variable at $(\gamma, \pm\sqrt{f(\gamma)})$ and Y does not vanish there. For $\gamma = e_1, e_2$ or e_3 the function $(X-\gamma)$ vanishes of second order at $(e_i, 0)$ and hence compensates for the pole of $1/Y$ at this point. For the tensor powers we obtain

$$(dz)^\lambda = Y^{-\lambda}(dX)^\lambda, \qquad \frac{d}{dz} = Y\frac{d}{dX} \tag{3-13}$$

where we used for the special case $\lambda = -1$ the usual derivation notation. Hence a basis of the sections in K^λ with the same regularity condition as in Prop. 1 are given by the elements

$$A^s_m Y^{-\lambda}(dX)^\lambda, \quad m \in \mathbb{Z}\ . \tag{3-14}$$

In the case of vector fields we obtain from Prop. 1

Proposition 3. *A basis of the space of meromorphic (rational) vector fields on the elliptic curve E holomorphic (regular) outside the points ∞ and $(e_1,0)$ in the 2 point case (corresponding to $s=0$), resp. ∞, (a_s,b_s), and $(a_s,-b_s)$ for the two 3 point cases $(s=1,2)$ is given by the elements*

$$V^s_{2k} := (X-a_s)^k Y \frac{d}{dX}, \qquad V^s_{2k+1} := \frac{1}{2} f(X)(X-a_s))^k \frac{d}{dX} \tag{3-15}$$

where k runs over all integers and $f(X)=4(X-e_1)(X-e_2)(X-e_3)$.

By defining

$$\deg(A^0_n Y^{-\lambda}(dX)^\lambda) = n, \tag{3-16}$$
$$\deg(A^s_{2k} Y^{-\lambda}(dX)^\lambda) = \deg(A^s_{2k+1} Y^{-\lambda}(dX)^\lambda) = k \tag{3-17}$$

we introduce a grading in the vector space of all sections.

The vector fields with the above regularity conditions define a Lie algebra under the usual Lie bracket. It is usually called generalized Krichever - Novikov algebra (see [16-18] for details). The sections in K^λ (again with the same regularity conditions) are Lie modules over this algebra with respect to taking the Lie derivative. The function algebra itself operates as multiplication on the space of sections. Putting this two actions together the space of sections are Lie modules over the Lie algebra of differential operators of degree ≤ 1, which is given as semidirect product of the algebra of vector fields with the functions. Because, the situation is completely analogous in the general situation I will only do here the vector field case. To avoid cumbersome notation I suppress the superscript s

Proposition 4. *The Lie algebra structure of the Krichever - Novikov algebra is given by the following structure equations in the generators introduced in Prop. 3, $(k,l\in\mathbb{Z})$*

(a) in the two point case:

$$\begin{aligned}
[V_{2k},V_{2l}] &= (2l-2k)V_{2(k+l)+1} \\
[V_{2k+1},V_{2l+1}] &= ((2l+1)-(2k+1))\{V_{2(l+k+1)+1} + 3\,e_1 V_{2(l+k)+1} \\
&\qquad + (e_1-e_2)(e_1-e_3)\,V_{2(l+k-1)+1}\} \\
[V_{2k+1},V_{2l}] &= (2l-(2k+1))V_{2(l+k+1)} + (2l-(2k+1)+1)\,3\,e_1 V_{2(l+k)} \\
&\qquad + (2l-(2k+1)+2)(e_1-e_2)(e_1-e_3)\,V_{2(l+k-1)}\ ,
\end{aligned} \tag{3-18}$$

(b) in the three point case: (where a denotes a_1 or a_2 depending on the case s)

$$\begin{aligned}
[V_{2k}, V_{2l}] &= (2l-2k)V_{2(k+l)+1} \\
[V_{2k+1}, V_{2l+1}] &= ((2l+1)-(2k+1))\{V_{2(l+k+1)+1} + 3\,a\,V_{2(l+k)+1} \\
&\qquad\qquad + (3a^2-(e_2^2+e_2e_3+e_3^2))\,V_{2(l+k-1)+1} + (1/4)\,b^2 V_{2(l+k-2)+1} \\
[V_{2k+1}, V_{2l}] &= (2l-(2k+1))V_{2(l+k+1)} + (2l-(2k+1)+1)\,3\,a\,V_{2(l+k)} \\
&\quad + (2l-(2k+1)+2)\big(3a^2-(e_2^2+e_2e_3+e_3^2)\big)V_{2(l+k-1)} \\
&\qquad + (2l-(2k+1)+3)(1/4)\,b^2\,V_{2(l+k-2)}.
\end{aligned} \tag{3-19}$$

Proof. This is a reformulation of results obtained by Ruffing and Deck which again could be calculated directly by using informations about the possible pole orders of the Lie bracket. [5,3,12].

If d denotes the sum of the degrees of the generators on the left hand side of (3-18), (3-19) then the degrees of the generators appearing on the right hand side are within the range $d-3$ and $d+1$. Hence, the Lie algebra structure is almost graded with respect to the gradings (3-16),(3-17). The same is true for the Lie module structures of the space of sections. These features are important to construct semi-infinite wedge representations (see [16,17]).

Of course, it would be possible to avoid completely the use of the complex analytic tori and to start with the cubic curves. But to see the relation with the in conformal field theory more familiar analytic picture I decided not to do so. To a certain extend an algebraic formulation has been given in a recent preliminary version of a preprint by Anzaldo-Meneses [1].

4. From marked elliptic curves to marked singular cubic curves

The singular members of the family of cubic curves given by the polynomial (3-4) are exactly the curves lying above points in $\widehat{B}$ where at least two of the e_i coincide. We obtain two different situations.
If only two coincide we get the nodal cubic E_N given by

$$Y^2 = 4(X-e)^2(X+2e)\ . \tag{4-1}$$

Here e denotes the value of the coinciding ones. By $e_1+e_2+e_3=0$ the third one has the value $-2e$.
If all three have the same value, which necessarily equals 0, we get the cuspidal cubic E_C

$$Y^2 = X^3\ . \tag{4-2}$$

The singularity on E_N is the point $(e,0)$, a node. The singularity on E_C is the point $(0,0)$, a cusp. In both cases the point $\infty = (0:1:0)$ lies on the cubic curves but is not a singularity.

As explained in Section 2 in both cases the complex projective line $\mathbb{P}^1$ is the desingularization. The points of $\mathbb{P}^1$ are given by homogeneous coordinates (t,s). We define the following maps $\psi_n, \psi_C : \mathbb{P}^1 \to \mathbb{P}^2$ by

$$\psi_N(t:s) = (\ t^2s - 2es^3\ :\ 2t(t^2-3es)\ :\ s^3\ ), \tag{4-3}$$

$$\psi_C(t:s) = (\ t^2s\ :\ 2t^3\ :\ s^3\ )\ . \tag{4-4}$$

Under these maps the point $\infty = (1:0)$ on $\mathbb{P}^1$ corresponds to (and only to) the point ∞ on both curves E_N and E_C. The maps are given by homogeneous polynomial. Hence, they are obviously algebraic maps. Again it is enough to consider the affine part (i.e. we are setting $s=1$). We use the same symbols for the affine maps. A direct calculation shows the following facts.
(1) Image $\psi_N = E_N$ and Image $\psi_C = E_C$.
(2) ψ_C is $1:1$.
(3) The only points where ψ_N is not $1:1$ are the points $t=\sqrt{3e}$ and $t=-\sqrt{3e}$. They both project onto the singular point $(e,0)$. The point $(-2e,0)$ correspond to $t=0$. Hence,

Proposition 5. *The maps*

$$\psi_N : \mathbb{P}^1 \to E_N \quad \textit{and} \quad \psi_C : \mathbb{P}^1 \to E_C$$

are the unique desingularization of the nodal cubic E_N resp. the cuspidal cubic E_C.

Now we have to consider what happens to the markings. In all cases $t=\infty$ on $\mathbb{P}^1$ corresponds to a point where poles are allowed.

Recall that the other markings are given by $(e_1, 0)$ in the 2 point case, resp. (a, b) and $(a, -b)$ in the 3 point case where a and b are given by (3-8), (3-9).

The situation in the cuspidal case is easy to describe. Because $e_1 = e_2 = e_3 = 0$ all remaining markings go to the singularity (either 2 point case or 3 point case). What remains is $\mathbb{P}^1$ with the markings $t = 0$ and $t = \infty$.

In the case of nodal degenerations and 2 points we have to distinguish 2 different degenerations.
(1) If we have $e = e_1$ then the marking becomes the singular point. Hence on $\mathbb{P}^1$ two points $t_1 = \sqrt{3e}$ and $t_2 = -\sqrt{3e}$ correspond to the second marking. We obtain $\mathbb{P}^1$ with 3 markings.
(2) If $e \neq e_1$ then the marking is an ordinary point. Only the point $t = 0$ correspond to the second marking. We obtain $\mathbb{P}^1$ with 2 markings.

In the case of nodal degenerations and 3 points we have the following possibilities.
(3) If $e = e_1 = e_3$ or $e = e_2 = e_3$ we find $a_s = e$ and $b_s = 0$ for $s = 1, 2$ using (3-8), (3-9). Here the two markings (a_s, b_s), $(a_s, -b_s)$ join at the singularity. This corresponds to the markings $t_1 = \sqrt{3e}$ and $t_2 = -\sqrt{3e}$. We obtain $\mathbb{P}^1$ with 3 markings.
(4) If $e_1 = e_2$ we have to check the sign of the square root in (3-9). For $s = 1$, (i.e. $q = \tau/4$ we obtain $a_1 = e_1 = e$, $(b_1 = 0)$. Hence the same situation as in (3). The sign of the product of the roots can be determined either by following the prescription given in [7] or by observing that the marking $1/2 + \tau/4$ is 'squeezed' between the two approaching points $1/2$ and $1/2 + \tau/2$ under this degeneration.
(5) In the remaining case $e = e_1 = e_2$ and $s = 2$ (i.e. $q = 1/2 + \tau/4$) we have to take the other sign of the product of the roots and obtain $a_2 = -5e$ with two associated values for b_2. The markings (a_2, b_2), $(a_2, -b_2)$ stay distinct and none of them coincides with the singularity. They are given by the t values $\pm\mathbf{i}\sqrt{3e}$. We obtain $\mathbb{P}^1$ with 3 markings.

As shown in Section 3 a section of K_E^λ corresponding to the different situations s for E a nonsingular cubic curve is given by[4]

$$w(X,Y) = \sum_{k\in\mathbb{Z}}{}' \big(\alpha_k (X - a_s)^k + \beta_k Y (X - a_s)^{k-1}\big)\big(\frac{dX}{Y}\big)^\lambda, \qquad \alpha_k, \beta_k \in \mathbb{C} \ . \tag{4-5}$$

On the nonempty open set of nonsingular points these elements make perfectly sense also in the degenerate cases E_N and E_C. By pulling it back via ψ_N, resp. ψ_C on the open set of $\mathbb{P}^1$ mapping to the nonsingular points we obtain a well defined meromorphic (rational) Section of $K_{\mathbb{P}^1}^\lambda$

$$\psi_N^*(w)(t) = w(X(t), Y(t)) \qquad \psi_C^*(w)(t) = w(x(t), y(t)) \ .$$

[4]The $'$ indicates that only finitely many summands occur.

To find their representation functions we have to plug in the expression for X and Y in terms of t and to rewrite the differential in terms of dt. For the power of the differential we obtain in the nodal case

$$\left(\frac{dX}{Y}\right)^{\lambda} = \frac{1}{(t^2-3e)^{\lambda}}(dt)^{\lambda}, \tag{4-6}$$

and

$$\left(\frac{dX}{Y}\right)^{\lambda} = \frac{1}{t^{2\lambda}}(dt)^{\lambda}, \tag{4-7}$$

in the cuspidal case. By this we see very explicitly that additional poles and zeros (depending on the sign of λ) are introduced at the singular points. A important example is the pullback of the constant differential $dz = \dfrac{dX}{Y}$. We obtain

$$\frac{1}{t+\sqrt{3e}}\,\frac{1}{t-\sqrt{3e}}\,dt \qquad \text{resp.} \qquad \frac{1}{t^2}\,dt\ .$$

We get by degeneration to the nodal cubic a meromorphic differential with poles of order 1 at the points $t = \pm\sqrt{3e}$, resp. a pole of order 2 at $t = 0$. These are the Rosenlicht differentials [20].

In this letter we are especially interested in the case $\lambda = -1$, the vector field case. Here only additional zeros at the the points above the singularities are introduced. All Lie derivatives can be calculated on the (Zariski-) open set of points which do not become singular points under the degeneration. Hence, its Lie structure is not inflicted by the degeneration. We obtain

Proposition 6. *Under the geometric process of degeneration and desingularization one obtains a geometrically induced 'deformation' of the algebra of vector fields on the torus into a subalgebra of the vector fields on $\mathbb{P}^1$ consisting of vector fields with poles only at the points t with $(X(t), Y(t))$ is a marked point on the degenerate cubic curve. The degenerate algebra can be obtained by replacing the generators by their pullbacks and setting the structure constants to their limit values.*

By examples, I will show that the number of markings can change (even increase) and that not necessarily the full vector field algebra will occur.

Because it is clear from the description in which situation we are I drop the index s to avoid cumbersome notations. Let me give first the pull back of the generators (3-14).

They are ($k \in \mathbb{Z}$)

$$\psi_N^*(A_{2k}\left(\frac{dX}{Y}\right)^\lambda) = (t^2-3e)^{-\lambda}(t^2-2e-a)^k(dt)^\lambda, \tag{4-8}$$

$$\psi_N^*(A_{2k+1}\left(\frac{dX}{Y}\right)^\lambda) = t\,(t^2-3e)^{1-\lambda}(t^2-2e-1)^{k-1}(dt)^\lambda, \tag{4-9}$$

$$\psi_C^*(A_{2k}\left(\frac{dX}{Y}\right)^\lambda) = t^{-2\lambda}(t^2-a)^k(dt)^\lambda, \tag{4-10}$$

$$\psi_C^*(A_{2k+1}\left(\frac{dX}{Y}\right)^\lambda) = t^{3-2\lambda}(t^2-a)^{k-1}(dt)^\lambda, \tag{4-11}$$

The above formula are valid for arbitrary values of a. Here we only consider the natural choices described above.
In the cuspidal case we see that the limit value of a equals zero. Hence (4-10), (4-11) specializes to (in the case $\lambda = -1$)

$$\psi_C^*(A_{2k}\left(\frac{dX}{Y}\right)^{-1}) = t^{2k+2}\frac{d}{dt}, \qquad \psi_C^*(A_{2k+1}\left(\frac{dX}{Y}\right)^{-1}) = t^{2k+3}\frac{d}{dt}\ . \tag{4-12}$$

By this we see that in the cuspidal degeneration (with the above markings) we always obtain the full (2 point) Virasoro algebra. This coincides with the degeneration of the structure equations (3-18) and (3-19) to

$$[V_n, V_m] = (m-n)V_{n+m+1}, \quad n, m \in \mathbb{Z}\ . \tag{4-13}$$

In the nodal degeneration case, we have to distinguish several subcases. In the 2 marking case ($a = e_1$) we obtain the following picture.
(1) For $e = e_1$ the limit marking is a singularity. For the pullback we obtain

$$\begin{aligned}\psi_N^*(A_{2k}\left(\frac{dX}{Y}\right)^{-1}) &= (t+\sqrt{3e})^{k+1}(t-\sqrt{3e})^{k+1}\frac{d}{dt}, \\ \psi_N^*(A_{2k+1}\left(\frac{dX}{Y}\right)^{-1}) &= t\,(t+\sqrt{3e})^{k+1}(t-\sqrt{3e})^{k+1}\frac{d}{dt}\ .\end{aligned} \tag{4-14}$$

We see explicitly that on the desingularization 3 markings occur. From a 2 point situation we came to a 3 point situation. The limit algebra is defined by the limit of the structure equations (3-18) for $e_1 = e_2 = e$ or $e_1 = e_3 = e$

$$\begin{aligned}[V_{2k}, V_{2l}] &= (2l-2k)V_{2(k+l)+1}, \\ [V_{2k+1}, V_{2l+1}] &= ((2l+1)-(2k+1))\{V_{2(k+l+1)+1} + 3\,e\,V_{2(l+k)+1}\}, \\ [V_{2k+1}, V_{2l}] &= (2l-(2k+1))V_{2(k+l+1)} + (2l-(2k+1)+1)3\,e\,V_{2(l+k)}\ .\end{aligned} \tag{4-15}$$

In the next Section I will identify this and the following algebras in more detail.
(2) The next subcase for the two point situation is $e_1 \neq e$. The limit marking is not a singularity. The pullbacks are

$$
\begin{aligned}
\psi_N^*\big(A_{2k}\left(\frac{dX}{Y}\right)^{-1}\big) &= t^{2k}(t+\sqrt{3e})(t-\sqrt{3e})\frac{d}{dt},\\
\psi_N^*\big(A_{2k+1}\left(\frac{dX}{Y}\right)^{-1}\big) &= t^{2k-1}(t+\sqrt{3e})^2(t-\sqrt{3e})^2\frac{d}{dt}\ .
\end{aligned}
\tag{4-16}
$$

Only at two points poles can occur ($t=0$ and $t=\infty$). Hence, we do not leave the 2 point situation. But we do not obtain the full Virasoro algebra, because the vector fields are forced to have zeros at $t=\pm\sqrt{3e}$. Indeed, there is a difference in the two formulas of (4-16). We get an additional condition to identify the exact subalgebra. This is due to the fact that only those functions on $\mathbb{P}^1$ (given by Laurent polynomials) can be obtained by pullback from E_N if they fullfill $f(\sqrt{3e})=f(-\sqrt{3e})$. For the even functions this is automatically. For the odd function this forces them to be zero at the two points. Hence, only functions generated by t^{2k} and $f^{2k+1}(t^2-3e)$ for $k\in\mathbb{Z}$ could occur. (Remember the second term t^2-3e comes from the differential.)
The structure equations of the algebra (obtained by the usual process) are

$$
\begin{aligned}
[V_{2k},V_{2l}] &= (2l-2k)\,V_{2(k+l)+1},\\
[V_{2k+1},V_{2l+1}] &= ((2l+1)-(2k+1))\{V_{2(k+l+1)+1}-6\,e\,V_{2(l+k)+1}\}+9\,e^2V_{2(l+k-1)+1}\},\\
[V_{2k+1},V_{2l}] &= (2l-(2k+1))V_{2(k+l+1)}+(2l-(2k+1)+1)(-6\,e)V_{2(l+k)}\\
&\quad +(2l-(2k+1)+2)\,9\,e^2V_{2(l+k-1)}.
\end{aligned}
\tag{4-17}
$$

Now we are coming to the three point cases.
(3) For $e=e_1=e_3$ or $e=e_2=e_3$ we obtain for both values of q that the markings (a,b) and $(a,-b)$ move into the singularity $(e,0)$. This yields exactly the same generators and the same deformed algebra as as considered in (1), resp. in (4-14),(4-15). Notice, we obtain the Equations (3-18) formally by setting $a=e_1$ and $b=0$ in (3-19) using the fact $e_1+e_2+e_3=0$.
(4) For $q=\tau/4$ and the remaining case $e=e_1=e_2$ the situation is the same as described under (3).
(5) For $q=1/2+\tau/4$ the remaining case is $e=e_1=e_2$. We had obtained above $a=-5e$ and two different associated b values. The pullbacks of the generators are

$$
\begin{aligned}
\psi_N^*\big(A_{2k}\left(\frac{dX}{Y}\right)^{-1}\big) &= (t+\mathbf{i}\sqrt{3e})^k(t-\mathbf{i}\sqrt{3e})^k(t+\sqrt{3e})(t-\sqrt{3e})\frac{d}{dt}\\
\psi_N^*\big(A_{2k+1}\left(\frac{dX}{Y}\right)^{-1}\big) &= t\,(t+\mathbf{i}\sqrt{3e})^{k-1}(t-\mathbf{i}\sqrt{3e})^{k-1}(t+\sqrt{3e})^2(t-\sqrt{3e})^2\frac{d}{dt}\ .
\end{aligned}
\tag{4-18}
$$

Poles are at $t = \mathbf{i}\sqrt{3e},\ -\mathbf{i}\sqrt{3e}, \infty$. Hence we stay in a three point situation. As in (2) additional zeros at $t = \pm\sqrt{3e}$ are introduced adn we get an additional condition yielding the difference between even and odd elements. We get only a subalgebra of the corresponding 3 point algebra on $\mathbb{P}^1$. The structure equations can be easily be obtained from (3-19). I ommit writing them down here.

5. The involved algebras on $\mathbb{P}^1$ with arbitrary markings

The Virasoro algebra $\mathcal{V}$ (without central extension) can be given as the complex vector space generated by the vector fields

$$L_n \ := \ t^{n+1}\frac{d}{dt}, \qquad n \in \mathbb{Z} \ . \tag{5-1}$$

Its structure equations are

$$[L_n, L_m] = (m-n)L_{n+m}, \quad m, n \in \mathbb{Z} \ . \tag{5-2}$$

By the geometric reasoning we obtain in the cuspidal cases for the 2 or 3 point situation with (4-12)

$$\psi_N^*\big(A_n \left(\frac{dX}{Y}\right)^{-1}\big) = L_{n+1} \ . \tag{5-3}$$

Of course, the formal isomorphism (without giving the geometric explaination) could be seen by the map $\Phi : V_n \mapsto L_{n+1}$ in the degenerate structure equations (4-13).[5]

We are now considering certain subalgebras of the Virasoro algebra. The vector fields vanishing at the points α and $-\alpha$, with $\alpha \neq 0, \infty$ are the vector fields

$$f(t)\,(t^2-\alpha^2)\,\frac{d}{dt} \tag{5-4}$$

where $f(t)$ is a Laurent polynomial in t. Obviously, they define a subalgebra of $\mathcal{V}$ with (vector space) basis

$$t^k\,(t^2-\alpha^2)\frac{d}{dt} \ = \ L_{k+1} - \alpha^2 L_{k-1}, \qquad k \in \mathbb{Z} \ . \tag{5-5}$$

Inside this subalgebra we consider the subspace generated by the vector fields

$$M_{2k} := t^{2k}(t^2-\alpha^2)\frac{d}{dt} = L_{2k+1} - \alpha^2 L_{2k-1}, \tag{5-6}$$

$$M_{2k+1} := t^{2k-1}(t^2-\alpha^2)^2\frac{d}{dt} = L_{2k+2} - 2\alpha^2 L_{2k} + \alpha^4 L_{2k-2} \ . \tag{5-7}$$

[5]Such kind of observations made by Deck and Ruffing where the starting point of this work. Hence I am very much indebted for their calculations.

Either by direct calculation of the Lie bracket, or calculation inside the Virasoro algebra we see this is a subalgebra $\mathcal{W}^\alpha$ of $\mathcal{V}$.
For $\alpha = \sqrt{3e}$ the M_n coincide with the pullbacks (4-16). The map $\Phi : V_n \mapsto M_n,\ n \in \mathbb{Z}$ gives the identification of the geometrically induced degenerated algebra (4-17) with the algebra $\mathcal{W}^\alpha$. In [4] a relation of the 'deformed' algebra in the 2 point case with the Virasoro algebra was found by purely formal algebraic investigations. Indeed, it was tried (without sucess) to identify it with the whole Virasoro algebra. By the above, we see the reason why this could not work.

I am now coming to the algebra $\mathcal{Z}^\alpha$ of vector fields on $\mathbb{P}^1$ holomorphic outside the points $\alpha, -\alpha$ and ∞ ($\alpha \ne 0,\ \infty$). In [16,17] a general method for obtaining a basis was given. If we split the marking into two disjoint sets, the in-points $\{\alpha, -\alpha\}$ and the out-point $\{\infty\}$ the algebra is generated (as vector space) by elements $e_{n,1}$ and $e_{n,2}$ with $n \in \mathbb{Z}$. Their divisors are

$$\begin{aligned}(e_{n,1}) &= (n-1)[\alpha] + n[-\alpha] + (-2n+3)[\infty], \\ (e_{n,2}) &= n[\alpha] + (n-1)[-\alpha] + (-2n+3)[\infty]\ .\end{aligned}$$

These are the generators of degree n. We can symmetrize the situation by taking suitable linear combinations of them to obtain generators H_n and G_n with the divisors

$$\begin{aligned}&(H_n) = (n-1)[\alpha] + (n-1)[-\alpha] + (-2n+4)[\infty], \\ &(G_n) = (n-1)[\alpha] + (n-1)[-\alpha] + 1[0] + (-2n+3)[\infty]\ .\end{aligned} \tag{5-8}$$

These generators are still homogeneous elements of degree n and a basis of the algebra $\mathcal{Z}^\alpha$. They can explicitly be given as

$$H_n(t) := (t-\alpha)^{n-1}(t+\alpha)^{n-1}\frac{d}{dt}, \qquad G_n(t) := t\,(t-\alpha)^{n-1}(t+\alpha)^{n-1}\frac{d}{dt}\ . \tag{5-9}$$

A direct calculation shows

$$\begin{aligned}&[H_n, H_m] = 2(m-n)G_{n+m-2}, \\ &[G_n, G_m] = 2(m-n)(G_{n+m-1} + \alpha^2 G_{n+m-2}), \\ &[G_n, H_m] = (2(m-n)-1)H_{n+m-1} + 2(m-n)\,\alpha^2 H_{n+m-2}\ .\end{aligned} \tag{5-10}$$

For $\alpha = \sqrt{3e}$ the elements (5-9) coincide with the pullbacks (4-14) obtained in the nodal degeneration in Section 4 in the cases (1) and (3). Under the map
$\Phi :\ V_{2k} \mapsto H_{k+2},\quad V_{2k+1} \mapsto G_{k+2}$ we obtain the geometrically induced identification of the degenerate algebra in the case (4-14) with the full full algebra $\mathcal{Z}^\alpha$.

In the remaining case (5) we have to consider again the subalgebra of $\mathcal{Z}^\alpha$ (now for $\alpha = \mathbf{i}\sqrt{3e}$) generated by the vector fields with zeros at $\pm\sqrt{3e}$ and a similar additional condition as in the case $\mathcal{W}^\alpha$. Again this algebra can be identified with the degenerated algebra obtained in Section 4. The results are completely analogous hence I will not write them down here.

We should not forget that it is not only the algebras, but also their grading which is of importance. The only graded Lie algebra (in the strict sense) here is the full Virasoro algebra. All other algebras are only almost graded. Hence it does not make sense to investigate whether the maps Φ are grading preserving. Instead I introduce the following

Definition. *Let T and S be almost graded Lie algebras. A Lie homomorphism $\Phi : S \to T$ respects the almost grading if there are positive natural numbers a, k, l such that for every homogeneous element $s \in S$ we get*

$$a \cdot \deg(s) - k \;\le\; \deg(\Phi(s)) \;\le\; a \cdot \deg(s) + l\ .$$

By inspection of the formalas above and collecting the results we obtain

Theorem 1. *Under the geometrically defined deformation of the 2 or 3 point vector field algebra on the ellitic curve (i.e. torus) with the markings ∞ and $(e_1, 0)$ in the 2 point case, and ∞, (a_s, b_s) and $(a_s, -b_s)$ for the two 3 point cases ($s = 1, 2$) introduced in Section 3 we abtain the following deformed algebras,*
(A) the full Virasoro algebra $\mathcal{V}$ in all cases of the cuspidal degeneration,
(B) the subalgebra $\mathcal{W}^\alpha$, $\alpha = \sqrt{3e}$ of the Virasoro algebra in the case of the nodal degeneration of the 2 point situation if $(e_1, 0)$ does not become the singular point,
(C) the full vector field algebra $\mathcal{Z}^\alpha$, $\alpha = \sqrt{3e}$ of 3 markings in the case of the nodal degeneration either in the 2 point situation if $(e_1, 0)$ becomes a singular point or all 3 point situations with the exception of the case considered in (D),
(D) a subalgebra of $\mathcal{Z}^\alpha$, $\alpha = \mathbf{i}\sqrt{3e}$ (which could be given explictly) for the nodal degeneration in the 3 point situation with $e = e_1 = e_2$ and $q = 1/2 + \tau/4$.

All isomorphisms Φ induced by the degenerations respect the almost gradings of the involved algebras.

6. Further results

As already indicated, what have been done here for the vector field algebra could have been done for the whole Lie algebra of differential operators of degree less or equal one and their Lie modules consisting of sections of K^λ of arbitrary integer weight λ. One obtains again by the well defined geometric deformation certain subalgebras, resp. modules of the corresponding objects on $\mathbb{P}^1$. Note however that for forms of positve weight additional poles are introduced at the points lying above the singularities. Roughly speaking, one obtains by degenerations subalgebras operating on bigger spaces. This has to be considered if one uses this moduls to construct semi-infinite wedge representations (i.e. $b-c$ systems), see [16,17].

In this letter I only considered the algebras without central extensions. As can easily seen the geometric cocycles depending on the partition of the markings defining central extensions [16-18] can be calculated completely outside the singularities. Moreover, they are calculated by calculating residues (for example, at the point ∞), hence in an algebraic manner. They make sense under the described degeneration process and in this way everything makes sense also for the central extensions.

At least in principle the method is not restricted to the genus 1 case. However, additional problems occur at higher genus. In the genus 1 case I used that every elliptic curve can be realized as the set of zeros of one polynomial in the projective plane. The degenerations could be easily studied at the defining polynomial. We obtained that there are exactly two nonisomorphic possible degenerations. In general, curves of higher genus can not be embedded in the complex plane. One has to use projective spaces of higher dimensions and needs more polynomial to define them. Under degeneration of the curves (if one considers only 'stable' degenerations) one approaches the boundary of the compactified moduli space of curves of genus g. This boundary is of positive dimension and consists of different components, which can be identified with the moduli space of lower genus curves (together with their degenerations) [19,p.78]. Even without considering the markings it is not at all clear in which 'direction' one should degenerate. A possibility could be 'maximal degeneration' in the sense that one obtains a (reducible) connected union of curves either isomorphic to $\mathbb{P}^1$ or with desingularization $\mathbb{P}^1$. By desingularization of the whole curve one obtains a disjoint union of many copies of $\mathbb{P}^1$s. Hence, one expects under the limit procedure for the vector fields a subalgebra of the direct sum of the associated vector field algebras on different copies of $\mathbb{P}^1$s.

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