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ABSOLUTELY CONTINUOUS SPECTRUM FOR SPARCE POTENTIALS

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Abstract. In this paper we show that bounded perturbation of the discrete laplacian with a potential that is sparcely supported (a notion made precise in the paper) produces absolutely continuous spectrum in the interval [-2ν, 2ν] for large dimension ν. We note that the potential need not give rise to a compact operator, let alone have decay at infinity.

1. Introduction

The study of the absolutely continuous spectrum is relevent in several models associated with scattering on the one hand and the extended states , as is expected in the Anderson model, etc., In our attemp to understand the mechanism of occurance of the absolutely continuous spectrum in the Anderson model, we come across a situation where a certain non compact perturbation of the discrete laplacian in 4 or more dimensions exhibits absolutely continuous spectrum if the potential is sparcely supported in the lattice. We may consider this as a hierarchial model also for the purposes of understanding the mechanism responsible for producing absolutely continuous spectrum in the higher dimensional Anderson model. We stress here that we only consider a deterministic operator and show the presence of absolutely continuous spectrum.

We also would like to add here that in the Anderson model the diffusive behaviour is supposed to be responsible for the occurance of the absolutely continuous spectrum, while in this work it is scattering that produces the absolutely continuous spectrum. We would like to still present this result with the hope that it might be useful for further work in the area. We recall that in [1] it was shown that if the potential is i.i.d. random variable with distribution having finite variance times a function with a overall decay at infinity as a power $\alpha < -1$ of the distance from the origin, then the perturbation of the laplacian with such potentials has absolutely continuous spectrum in $[-2\nu, 2\nu]$ almost everywhere. In the proof of that theorem the behaviour of the kernel of the free evolution in time is used together with the conditions on the potentials to show the existence of the wave operators and thus the presence of the absolutely continuous spectrum. It was sufficient to make crude estimates to obtain the result there. In this paper we show that it is actually not necessary for the potential to have decay at infinity provided it has some sparce support , in such a way that we can still show the existence of the wave operators.

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But for technical reasons we stick to the deterministic case and consider only single potentials. It is clear that some random potentials of small disorder would have sparce suport but not those assumed here. Therefore we cannot extend our results yet to cover them.

2. The Model

We consider the space $l^2(\mathbb{Z}^\nu)$ and the laplacian Δ defined by

$$(\Delta u)(n) = \sum_{|n-i|=1} u(i) \tag{2.1}$$

and a potential $\mathbf{V}$ coming from a real valued bounded funtion $V : \mathbb{Z}^\nu \to \mathbb{R}$. We then consider the operator H = Δ + V and note that it is self-adjoint on $l^2(\mathbb{Z}^\nu)$. We also note that the operator Δ has spectrum [-2ν , 2ν] which is purely absolutely continuous and H has spectrum in [-2$\nu - \|V\|_\infty$, $2\nu + \|V\|_\infty$]. We make the following assumption on the potential $\mathbf{V}$.

Assumption 2.1 *We assume that if Λ is a ν-dimensional cubic region in $\mathbb{Z}^\nu$ centered at any point in $\mathbb{Z}^\nu$,then the volume $|S|$ of the set $S = \{n \in \Lambda : V(n) \neq 0\}$ satisfies*

$$|S| \leq |\Lambda|^{1-\eta}$$

for $|\Lambda|$ sufficiently large. We assume that η satisfies $\frac{4}{\nu} > \eta > \frac{3}{\nu}$.

Our main theorem of the paper is the following.

Theorem 2.2 *Consider the operator H defined above on $l^2(\mathbb{Z}^\nu)$ and suppose the potential* $\mathbf{V}$ *satisfies the assumption 2.1, then for $\nu \geq 4$ the absolutely continuous spectrum of H contains the interval [-2ν , 2ν].*

At this stage we would like to mention the strategy of proof. It turns out that the time evolution generated by the operator Δ has decay like $t^{\frac{-\nu}{2}}$ in ν dimensions in most of the regions of the lattice. However there is a thin region in the lattice where the decay is slightly lower. Therefore if the potential is nonvanishing in regions which are in small volume compared to the volume of the region under consideration, then the overall decay of the sum fo matrix elements of the time evoluition would have integrable decay , which would then give using standard techniques of scattering theory the necessary result from the existence of the wave operators.

Proof. We notice that to prove the theorem we need to show the existence of the limits $\lim_{t\to\infty}$ exp(itH) exp(-itΔ)f for each f $\in l^2(\mathbb{Z}^\nu)$. This is reduced to showing by density arguments that the sequence

$$F(\phi, f, t) =< \phi, exp(itH)exp(-it\Delta)f > \tag{2.2}$$

is Cauchy in t , uniformly in ϕ and for all elements f of $l^2(\mathbb{Z}^\nu)$ of finite support.

Thus we have to show that

$$\lim_{t,w\to\infty} \sup_{\|\phi\|=1} \{| < \phi, exp(itH)exp(-it\Delta)f > \; - \; < \phi, exp(iwH)exp(-iw\Delta)f > |\}$$
$$= 0 \tag{2.3}$$

By writing the difference on the left hand side as an integral of the differential one has to show that

$$\lim_{t\to\infty} \sup_{\|\phi\|=1} \{ \int_w^t \frac{d}{ds} < \phi, exp(isH)exp(-is\Delta)f >\} = 0 \tag{2.4}$$

Computing the integrand in the above expression and simplifying we get that it is sufficient to show the integrability in **s** of

$$\sup_{\|\phi\|=1} \{| < \phi, exp(isH)V exp(-is\Delta)f > |\} \tag{2.5}$$

Since by the argument above f is a vector of finite support in the above expression , it is sufficient to consider the case when f is a vector with support at a single point in $\mathbb{Z}^\nu$ in particular the element δ_0. We shall not make use of any special properties of the point **0** of $\mathbb{Z}^\nu$, in the following. Therefore we consider,

$$\sup_{\|\phi\|=1} \{| < \phi, exp(isH)V exp(-is\Delta)\delta_0 > |\} \tag{2.6}$$

By introducing a partition of unity in the above expression we find that it is equal to

$$\sup_{\|\phi\|=1} \{| < \phi, exp(isH) \sum_{n\in\mathbb{Z}^\nu} |\delta_n >< \delta_n| V exp(-is\Delta)\delta_0 > |\} \tag{2.7}$$

Since the operator exp(isH) is unitary for each fixed s , the above quantity can be estimated ,using the Cauchy-Schwarz inequality, by the following quantity

$$\sup_{\|\phi\|=1} \{\|\phi\|^2 \sum_{n\in\mathbb{Z}^\nu} |V(n)|^2| < \delta_n, exp(-is\Delta)\delta_0 > |^2\}^{\frac{1}{2}} \tag{2.8}$$

Therefore it is enough to show that the following estimate is valid as s$\to\infty$,

$$\sum_{n\in\mathbb{Z}^\nu} |V(n)|^2| < \delta_n, exp(-is\Delta)\delta_0 > |^2 \approx O(|s|^{-2-}) \tag{2.9}$$

To show this we need to consider the sum split into several regions in $\mathbb{Z}^\nu$ depending upon the parameter **s** and estimate the summand in each of these regions . We shall do this in the following lemmas and complete the proof of the theorem.

Lemma 2.3 *Consider the number β as in Proposition A.3 and consider the region $S_s = \{n \in \mathbb{Z}^\nu : |n_i| > \beta s,\ for some\ i = 1,..,\nu\}$. Then the following estimate is valid.*

$$\sum_{n \in S_s} |V(n)|^2 | < \delta_n, exp(-is\Delta)\delta_0 > |^2 \approx O(|s|^{-2-}) \tag{2.10}$$

as $s \to \infty$.

Proof. To evaluate the sum in (2.10) , we note that $S_s = \cup_i \{n \in \mathbb{Z}^\nu : |n_i| > \beta|s|\} = \cup_i S_s(i)$. Therefore on each of these regions we have the estimate,coming from Propositions A.2 and A.3 ,

$$| < \delta_n, exp(-is\Delta)\delta_0 > |^2 \leq exp(-\alpha(\beta)|n_i|) \prod_{j \neq i} |J_{n_j}(2s)|^2 \tag{2.11}$$

Therefore we have the estimate

$$\sum_{n \in S_s} |V(n)|^2 | < \delta_n, exp(-is\Delta)\delta_0 > |^2$$

$$\leq \sum_{i=1}^{\nu} \sum_{S_s(i)} |V(n)|^2 exp(-\alpha(\beta)|n_i|) \prod_{j \neq i} |J_{n_j}(2s)|^2 \tag{2.12}$$

Now V(n) is a bounded function , $|n_i| > \beta|s|$ and $n_j, j \neq i$ is in $\mathbb{Z}$. Therefore the exponential term in $|n_i|$ gives a decay in s of the required order for the sum since $J_{n_j}(2s)$ is in$l^2(\mathbb{Z})$ with $\sum_{n_j} |J_{n_j}(2s)|^2$ uniformly bounded in s.

Now it remains to show that in the region $\mathbb{Z}^\nu - S_s$ the decay at infinity in s required by equation (2.9) is valid. Infact in the above Lemma we did not make use of the Assumption 2.1 at all. It is in the other region that we need the asssumption as will be seen in the proof of the following lemma.

Lemma 2.4 *Consider the region $\mathbb{Z}^\nu - S_s$, S_s as in Lemma 2.3. Then in this region the following estimate is valid.*

$$\sum_{n \in \mathbb{Z}^\nu - S_s} |V(n)|^2 | < \delta_n, exp(-is\Delta)\delta_0 > |^2 \approx O(|s|^{-2-}) \tag{2.13}$$

as $|s| \to \infty$.

Proof. The region $\mathbb{Z}^\nu - S_s$ is nothing but the region $\{n \in \mathbb{Z}^\nu : |n_i| \leq \beta|s|, \forall i = 1,..,\nu\}$, let us call it S_s^c. We split the region S_s^c further into the following regions.

$$S_s^c = \cup_{k=0}^{\nu} S_s^c(k)$$

and

$$S_s^c(k) = S_s^c \cap \{|n_i - 2s| < \sqrt{s}, i = 1,..,k\} \cap \{|n_i - 2s| \geq \sqrt{s}, i = k+1,..,\nu\} \tag{2.14}$$

where k = 0 is to be understood as the absence of the second factor. We shall show that in each of the above regions we have the estimate calimed in the lemma for the left hand side of equation (2.13). All the constants that occur in the following estimates would be independent of s, but may depend on the dimension ν , the point $\mathbf{0}$ of $\mathbb{Z}^\nu$, on $\|V\|$ and the exponent η of assumption 2.1. Now let us consider first the region $S_s^c(0)$. Then we have, as $s \to \infty$, by Propositions A.2 and A.4 (1),

$$\sum_{n \in S_s^c(0)} |V(n)|^2| < \delta_n, exp(-is\Delta)\delta_0 > |^2 \leq C\ Vol\{n \in S_s^c(0) : V(n) \neq 0\}|s|^{-\nu} \tag{2.15}$$

where Vol (A) denotes the cardinality of the set A. Now by assumption 2.1, the Vol in the above inequality is estimated as

$$Vol(n \in S_s^c(0) : V(n) \neq 0) \leq \{Vol(S_s^c(0))\}^{1-\eta} \leq Const.|s|^{\nu(1-\eta)} \tag{2.16}$$

where the Const. is independent of s. Combining equations (2.15) and (2.16) we see that

$$\sum_{n \in S_s^c(0)} |V(n)|^2| < \delta_n, exp(-is\Delta)\delta_0 > |^2 \leq C\ |s|^{-\nu\eta} \tag{2.17}$$

Now η being bigger than $\frac{3}{\nu}$, by assumption 2.1, the required estimate follows for the sum in this region. As for the other region we shall estimate for a general $\mathbf{k}$. We have by Propositions A.2 and A.4 (1),(2) that

$$\sum_{n \in S_s^c(k)} |V(n)|^2| < \delta_n, exp(-is\Delta)\delta_0 > |^2$$
$$\leq C\ Vol\{n \in S_s^c(k) : V(n) \neq 0\}|s|^{-(\nu-k)}|s|^{-\frac{2k}{3}} \tag{2.18}$$

and the estimate for the Vol in the above inequality for each k is done as follows. We consider a region $S_s^c(k)$ and cover it by disjoint cubic regions of sides $\sqrt{s}$. Then , since in each of such cubic regions the set of points where V does not vanish scales as $(1-\eta)$ of the volume, and since there are about $|s|^{\frac{1}{2}(\nu-k)}$ such cubes covering the region $S_s^c(k)$ we have the estimate that for each k = 1,..,ν,

$$Vol(\{n \in S_s^c(k) : V(n) \neq 0\}) \leq Const.|s|^{(\frac{\nu}{2})(1-\eta)}|s|^{\frac{(\nu-k)}{2}} \tag{2.19}$$

On the otherhand for some values of k the above estimate could turnout to be a gross over estimate of the actual volume of the points where V is non zero. Therefore we also consider the estimate that

$$Vol(\{n \in S_s^c(k) : V(n) \neq 0\}) \leq Vol(\{n \in S_s^c : V(n) \neq 0\}) \leq Const.|s|^{\nu(1-\eta)} \tag{2.20}$$

¿From equations (2.18) - (2.20) , it is clear that the decay as $s \to \infty$ claimed in lemma is valid if , for each k = 1,..,ν we have that

$$min\{\frac{(\nu-k)}{2}+\nu\frac{(1-\eta)}{2},\nu(1-\eta)\}+(-\nu+k-\frac{2k}{3})<-2 \tag{2.21}$$

This inequality simplifies to the following

$$-\eta-max\{\eta,\frac{k}{\nu}\}+\frac{2}{3}\frac{k}{\nu}<-\frac{4}{\nu} \tag{2.22}$$

This equation is satisfied for each k = 1,..,ν if η is chosen so that $\frac{4}{\nu}>\eta>\frac{3}{\nu}$, when $\nu\geq 4$. Thus under the assupmtion 2.1 , the lemma is proved.

We shall give an example of a potential V satisfying assumption 2.1 , below.

Example. Let us consider $\mathbb{Z}^4$ and the set S defined by

$$S=\{n\in\mathbb{Z}^4:\forall i=1,..,4,n_i=k_i^5\ for some\ k_i\in\mathbb{Z}\}$$

Now consider any real valued function V which takes non zero values on S and is zero in $\mathbb{Z}^4-S$. It is not difficult to check that such a V satisfies the assumption 2.1 , with $\eta=\frac{4}{5}$.

One can construct in a similar fashion a host of example of functions in any dimension bigger than 3 . It is also clear that if V is bounded but does not vanish on any point of S , it cannot give rise to a compact oprator in $l^2(\mathbb{Z}^\nu)$. Of course it goes without saying that if the potential consits of two parts one satisfying the assumptions given in assumption 2.1 and the other having a decay at infinity as in [1], the Theorem 2.2 will be valid for the sum.

Appendix

In the appendix we collect a few propositions that are standard and which are used in the paper.

Proposition A.1 *If A and B are two bounded self-adjoint operators on a Hilbert space and suppose the limits* $slim_{t\to\infty}$ *exp(iAs)exp(-iBs)*$E_{ac}(B)$ *exists. Then* $\sigma_{ac}(B)\subset\sigma_{ac}(A)$

The above theorem is standard and the proof can be found in any standard work on scattering theory , for example [2] [3].

Proposition A.2 *Let m,n $\in \mathbb{Z}^\nu$and Δ be as in (2.1). Then we have*

$$< \delta_n, exp(-i\Delta s)\delta_m >= \prod_{i=1}^{\nu} (-i)^{(n-m)_i} J_{(n-m)_i}(2s) \tag{A.1}$$

where $J_k(s)$ is the Bessel function of integral order k.

The expression for the matrix elements of the free evolution in terms of the Bessel functions follows from direct verification , see for example the Appendix of [1].

Proposition A.3 *Consider $J_n(x)$,$n \in \mathbb{Z}$ and $x > 0$. Then $\exists\ \beta > 0$ and $\alpha(\beta) > 0$ such that*

$$|J_n(x)| \leq Cexp(-\alpha(\beta)|n|) \tag{A.2}$$

whenever $|n| > \beta|x|$.

Proof. We have the following relation for the Bessel function of integral order. $J_n(x) = (-1)^n J_{-n}(x)$ for n a positive integer. Therefore we consider only positive n in the following. Now the power series representation of $J_n(x)$, we have that

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}x)^{n+2m}}{m!(n+2m)!}$$

¿From this it follows that

$$|J_n(x)| \leq \frac{\frac{1}{2}|x|^n}{n!}\ exp(\frac{1}{4}\frac{|x|^2}{(n+1)})$$

Now we have the Stirling's approximation for the factorial n, when $n \to \infty$, as

$$n! \approx \frac{1}{\sqrt{2\pi n}}\ exp(-n)n^n$$

Hence, for some constant C ,

$$|J_n(x)| \leq C\ \sqrt{2\pi n}\ exp\{n[\log(\frac{|x|e}{2n}) + \frac{|x|^2}{4n(n+1)}]\}$$

Therefore if $|n| > \frac{e^2|x|}{2}$ we can easily check that

$$|J_n(x)| \leq C\ \sqrt{2\pi n}\ exp\{\ (-1 + \frac{1}{e^4})n\ \} \leq C\ exp(-\alpha(\beta)n)$$

for $\beta = \frac{1}{2}e^2$ and an appropriate α.

Proposition A.4 *Consider $J_n(x)$ for x and n both large. Then we have the following behaviour as $x \to \infty$, in the regions indicated below.*

$$(1) \qquad If\ |x-n| \geq |x|^{\frac{1}{2}},\ then\ |J_n(x)| \leq C|x|^{\frac{-1}{2}}.$$

$$(2) \qquad If\ |x-n| \leq |x|^{\frac{1}{2}},\ then\ |J_n(x)| \approx O(|x|^{\frac{-1}{3}}).$$

In both the cases the error involved would be $|x|^{-1}$.

The above proposition is well known classically and is proved using the stationary phase techniques and the details can be found in Watson [4].

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