# A Remark on the Decay of Superconducting Correlations in One and Two Dimensional Hubbard Models

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**Abstract**

Upper bounds on the decay of various correlation functions are derived for a general class of lattice fermion models with long range hopping matrix. These bounds extend previous results of Koma and Tasaki [1] and rule out the possibility of magnetic ordering and condensation of superconducting electron pairs in one and two dimensions for finite temperatures.

KEY WORDS: Hubbard model, ODLRO, superconducting correlations, gauge symmetry.

The absence of spontaneous breakdown of continuous symmetry in one and two dimensional classical and quantum statistical mechanical systems is a well known phenomenon (see e.g. [2], [8] and references therein). For the Hubbard model, which has a global $SU(2)$ symmetry, Ghosh [3] proved the absence of magnetic ordering at finite temperatures, using the Bogoliubov inequality. More recently, Koma and Tasaki [1] extended the Mc Bryan and Spencer method [4], to a general class of Hubbard models with *finite range hoppings.* Making use of the global $U(1)$ gauge symmetry of any quantum system conserving the particle number, they proved the absence of off diagonal long range order (ODLRO) corresponding to the condensation of superconducting electron pairs (such as Cooper pairs).

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In this letter, we report on a number of upper bounds for off diagonal correlation functions in the case of *long range hopping matrix.* They are based on the Mc Bryan and Spencer bound proved in Ref. [1] to which we apply the estimates of Messager *et al.* [5]. The case of random hopping is also considered.

We consider on the $(d \le 2)$-dimensional lattice $Z^d$, an itinerant electron model with Hamiltonian

$$H = -\sum_{x,y\in Z^d}\sum_{\sigma=\uparrow,\downarrow} t_{xy} c^+_{x,\sigma} c_{y,\sigma} + V(\{n_{x,\sigma}\}) \tag{1}$$

Here $(t_{xy})$ is the hopping matrix, $c^+_{x,\sigma}$ and $c_{x,\sigma}$ are the creation and annihilation operators with Fermi statistics and the interaction $V(\{n_{x,\sigma}\})$ is an arbitrary function of the number operators $n_{x,\sigma} = c^+_{x,\sigma} c_{x,\sigma}$. We introduce the expectation of an arbitrary observable $A$ as $\langle A\rangle = \mathrm{Tr}\, A e^{-\beta(H-\mu N)} / \mathrm{Tr}\, e^{-\beta(H-\mu N)}$ where $\mu$ is the chemical potential and $N = \sum_{x,\sigma} n_{x,\sigma}$ is the total number operator. $\langle A\rangle$ is to be interpreted as the thermodynamic limit $\Lambda \uparrow Z^d$ of the corresponding finite volume expression $\langle A\rangle_\Lambda$ where the sites are restricted to a finite box $\Lambda$. The following theorem sets bounds on the correlations between the superconducting order parameter $\Delta^+_x = c^+_{x,\uparrow} c^+_{x,\downarrow}$, $\Delta_y = c_{y,\downarrow} c_{y,\uparrow}$.

**Theorem 1** (a) *If* $d = 1$ *and* $|t_{uv}| \le \frac{t}{|u-v|^\alpha}$ *with* $\alpha > 2$*, then*

$$\langle \Delta^+_x \Delta_y + h.c.\rangle \le \frac{C_0}{|x-y|^{2(\alpha-1)}} \tag{2}$$

(b) *If* $d = 1$ *and* $|t_{uv}| \le \frac{t}{|u-v|^2 \log^{(p)} |u-v|}$*, then*

$$\langle \Delta^+_x \Delta_y + h.c.\rangle \le \frac{C_1}{\left(\log^{(p)} |x-y|\right)^{\lambda_1(\beta)}} \tag{3}$$

(c) *If* $d = 2$ *and* $|t_{uv}| \le \frac{t}{|u-v|^\alpha}$ *with* $\alpha > 4$*, then*

$$\langle \Delta^+_x \Delta_y + h.c.\rangle \le \frac{C_2}{|x-y|^{\lambda_2(\beta)}} \tag{4}$$

(d) *If* $d = 2$ *and* $|t_{uv}| \leq \frac{t}{|u-v|^4}$, *then*

$$\langle \Delta_x^+ \Delta_y + h.c. \rangle \leq \frac{C_3}{(\log|x-y|)^{\lambda_3(\beta)}} \tag{5}$$

(e) *If* $d = 2$ *and* $|t_{uv}| \leq t\frac{\log^{(2)}|u-v| \cdots \log^{(p)}|u-v|}{|u-v|^4}$, *then*

$$\langle \Delta_x^+ \Delta_y + h.c. \rangle \leq \frac{C_4}{\left(\log^{(p)}|x-y|\right)^{\lambda_4(\beta)}} \tag{6}$$

*where* $t$, $C_0$, $C_1$, $C_2$, $C_3$, $C_4$ *are positive constants and* $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ *are non increasing functions of* $\beta$ *proportional to* $\beta^{-1}$ *for large* $\beta$. *In (3) and (6),* $\log^{(p)}$ *denotes the p-times iterated logarithm.*

*Proof.* Formula (6), (10) and the first inequality of (11) in Ref. [1] give

$$\langle \Delta_x^+ \Delta_y + h.c. \rangle \leq \exp\left\{-2(\varphi_x - \varphi_y) + \beta \sum_{u,v} |t_{uv}|[\cosh(\varphi_u - \varphi_v) - 1]\right\} \tag{7}$$

where $\{\varphi_u\}$ is an arbitray family of real numbers. They are related to the gauge transformation

$$A \longrightarrow e^{-\sum_{u\sigma} \varphi_u c_{u,\sigma}^+ c_{u,\sigma}} A \, e^{\sum_{u\sigma} \varphi_u c_{u,\sigma}^+ c_{u,\sigma}} \tag{8}$$

that plays, in the case under consideration, the role of the complex translation of Ref. [4]. The right hand side of (7) has been estimated by a suitable choice of the variables $\varphi_u$ according to the different hypotheses on $t_{uv}$, to obtain upper bounds on the decay of two-points correlation functions of $SO(N)$-symmetric spins systems [5]. We thus refer to [5] (see Section 2 and Section 3) to conclude the proof. $\diamondsuit$

*Remarks.*

- Koma and Tasaki proved the decay given in Statements (a) and (c) of Theorem 1 in the case of finite range hopping matrix: $|t_{xy}| = 0$ if $|x-y| > R$ where $R$ is some constant. They already mentioned that the case of long range hopping could be treated by their techniques using the result of Ito [6]. However the treatment of the Mc Bryan Spencer bound given by Ito leads to much slower decay than those obtained from the estimates of Messager *et al.* [5] proposed here.

- In dimension $d = 1$, it can also be shown that power law decay holds for small $\beta$ when $|t_{uv}| \leq t|u-v|^{-2}$ and exponential decay holds for all $\beta$ when $t_{uv}$ decays exponentially.

- The bounds of the theorem hold also for the correlations $\langle c^{+}_{x,\sigma} c_{y,\sigma} + h.c. \rangle$, except in case (a) where the decay goes like $|x-y|^{-(\alpha-1)}$.

- The infinite volume limits of the free energy and the pressure exist for the models under consideration if $|t_{uv}| \leq t|u-v|^{-\alpha}$, $\alpha > d$. This follows from standard arguments in the theory of the thermodynamic limit. Here we tacitly assume that it is also the case for the correlation functions.

We now consider the case of random hopping. Obviously the bounds given in Theorem 1 immediatly extend to this case, but we can obtain decaying bounds even if $d < \alpha < 2d$, thanks to methods used for classical spin-glasses [7].

**Theorem 2** *Whenever $t_{uv} = \tilde{t}_{uv}|u-v|^{-\alpha}$, where $\tilde{t}_{uv}$ are bounded i.i.d. random variables with zero mean, then*

(a) *if $d = 1$ and $\alpha > 1$, there is a positive random variable $f(\{\tilde{t}_{uv}\})$ which is finite almost surely such that*

$$\langle \Delta_x^+ \Delta_y + h.c.\rangle(\{\tilde{t}_{uv}\}) \leq f(\{\tilde{t}_{uv}\}) \, \frac{1}{|x-y|^{(2\alpha-1)}} \tag{9}$$

(b) *if $d = 2$ and $\alpha > 2$, then for all $0 < K < \alpha - 1$ and $\gamma < 1$, there exists a sequence of random variables $F_N$ such that for $|x - y| = N$:*

$$-\log |\langle \Delta_x^+ \Delta_y + h.c.\rangle| \geq F_N \tag{10}$$

*with*

$$\lim_{N\to\infty} \frac{F_N}{(\log N)^\gamma} = K \quad \text{in } L^2\text{−sense} \tag{11}$$

*Proof.* ¿From the second inequality of formula (10) in Ref. [1] and the variational principle (or Peierls inequality [8]), we get

$$\begin{aligned} \langle A \rangle &\leq e^{-2(\varphi_x - \varphi_y)} \, \frac{\mathrm{Tr}\, e^{-\beta(H'-\mu N)}}{\mathrm{Tr}\, e^{-\beta(H-\mu N)}} \\ &\leq \exp\left\{ -2(\varphi_x - \varphi_y) + \beta \sum_{u,v} t_{uv}[\cosh(\varphi_u - \varphi_v) - 1] \langle c^+_{u,\sigma} c_{v,\sigma}\rangle' \right\} \end{aligned} \tag{12}$$

where $H'$ is the Hamiltonian with modified hoppings $t'_{uv} = t_{uv}\cosh(\varphi_u - \varphi_v)$ and $\langle\ \rangle'$ is the expectation with respect to this modified hopping. As in Ref. [7], it is sufficient, for $d = 1$, to prove that

$$\mathbb{E}\left\{ |\sum_{uv} X_{uv}| \right\} \equiv \mathbb{E}\left\{ |\sum_{uv} t_{uv}[\cosh(\varphi_u - \varphi_v) - 1]| \right\} < \infty \tag{13}$$

where $\mathbb{E}$ is the expectation with respect to the hoppings $\tilde{t}_{uv}$.

Since, by Schwartz inequality, $\mathbb{E}\{|\sum_{uv} X_{uv}|\} \leq \left(\mathbb{E}\left\{(\sum_{uv} X_{uv})^2\right\}\right)^{1/2}$, one needs to bound a diagonal part $\sum_{uv} \mathbb{E}\{X^2_{uv}\}$ and a non-diagonal part $\sum_{\substack{ij,kl \\ ij\neq kl}} \mathbb{E}\{X_{ij}X_{kl}\}$. For the diagonal part each term is bounded as in Ref. [7] by

$$\frac{\mathbb{E}\left\{\tilde{t}^2_{uv}\right\}}{|u-v|^{2\alpha}} \, [\cosh(\varphi_u - \varphi_v) - 1]^2 \tag{14}$$

because of $|\langle c^+_{u,\sigma} c_{v,\sigma}\rangle'| \leq 1$. Indeed (for finite volume)

$$
\begin{aligned}
|\mathrm{Tr}\, c^+_{u,\sigma} c_{v,\sigma} e^{-\beta(H'-\mu N)}| &\leq \| c^+_{u,\sigma} c_{v,\sigma} e^{-\beta(H'-\mu N)} \|_1 \\
&\leq \| c^+_{u,\sigma} c_{v,\sigma} \|_\infty \cdot \| e^{-\beta(H'-\mu N)} \|_1 \\
&\leq \mathrm{Tr}\, e^{-\beta(H'-\mu N)}
\end{aligned}
$$

where $\| \ . \|_1$ is the trace class norm and $\| \ . \|_\infty$ the operator norm (c.f. Ref. [9]).

For the non diagonal part we use the Dyson expansion

$$
\begin{aligned}
& e^{-\beta(H_0+H_1)} = e^{-\beta H_0} + \qquad (15)\\
& \sum_{n\geq 1} \int_0^\beta ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n e^{-s_n H_0} H_1 e^{-(s_{n-1}-s_n)H_0} H_1 \cdots H_1 e^{-(\beta - s_1)H_0}
\end{aligned}
$$

where

$$
\begin{aligned}
H_0 &= -\sum_{\substack{x,y\\ \neq ij,kl}} \sum_{\sigma=\uparrow,\downarrow} t'_{xy} c^+_{x,\sigma} c_{y,\sigma} + V(\{n_{x,\sigma}\}) \qquad (16)\\
H_1 &= t'_{ij} c^+_{i,\sigma} c_{j,\sigma} + t'_{kl} c^+_{k,\sigma} c_{l,\sigma} \qquad (17)
\end{aligned}
$$

This leads to a convergent series for $\langle c^+_{i,\sigma} c_{j,\sigma}\rangle' \langle c^+_{k,\sigma} c_{l,\sigma}\rangle'$ in powers of $\tilde{t}_{ij}$ and $\tilde{t}_{kl}$. Since $\mathbb{E}\{\tilde{t}_{ij}\tilde{t}^n_{kl}\} = \mathbb{E}\{\tilde{t}_{ij}\}\mathbb{E}\{\tilde{t}^n_{kl}\} = 0$, for any integer $n$, all the first order terms disappear and each term of the non diagonal part can be bounded by

$$
\begin{aligned}
C\, \frac{\mathbb{E}\left\{\tilde{t}^2_{ij}\right\} \mathbb{E}\left\{\tilde{t}^2_{kl}\right\}}{|i-j|^{2\alpha}|k-l|^{2\alpha}} \quad & [\cosh^2(\varphi_i - \varphi_j) - \cosh(\varphi_i - \varphi_j)] \qquad (18)\\
& \times\ [\cosh^2(\varphi_k - \varphi_l) - \cosh(\varphi_k - \varphi_l)]
\end{aligned}
$$

We choose $\varphi_z - \varphi_x = (\alpha - 1)\sum_{j=1}^n j^{-1}$ when $|z-x| = n$ and conclude the proof of the one dimensional case by the estimates of Ref. [5].

For the two dimensional case we use the choice of Picco [10], $\varphi_z - \varphi_x = K\sum_{j=1}^n \left(j[\max\{1, \log j\}]^{1-\gamma}\right)^{-1}$ when $z$ belongs to the square with sides of length $2n+1$ centered at $x$, and apply the estimates of Ref.[5] to (14) and (18), which yields instead of (13)

$$\mathbb{E}\left\{ |\sum_{uv} X_{uv}| \right\} \leq O((\log N)^{2\gamma-1}) + \mathrm{Cte} \tag{19}$$

for any $\gamma < 1$, and conclude the proof as in Van Enter [7]. $\diamondsuit$

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