On the Arnold Stability Criterion

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The Arnold stability criterion suggests that a stationary flow of an ideal incompressible fluid is stable if a certain quadratic form is definite. We show that, in three or more dimensions, this quadratic form is never definite. Typically the form is indefinite, and the spectrum of the associated Hermitian operator ranges from $-\infty$ to ∞. The exceptional case is where the velocity field is harmonic (solenoidal and irrotational) in which case the quadratic form is identically zero.

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1. Background

V.I. Arnold [1,2] was the first to recognize and develop a new set of geometric ideas in the hydrodynamics of an ideal fluid. In particular, his stability criterion [3] has been used in a great number of later publications on the subject of hydrodynamic stability. This criterion states that a velocity field that is a steady-state solution to the hydrodynamic Euler equations is stable if a certain quadratic form, defined on the space of infinitesimal deformations to this velocity field, is definite. Arnold also gave important applications of his stability criterion for 2 dimensional flows. He essentially proved nonlinear stability under the conditions of the criterion in a certain metric, using a method that avoided problems related to the geometry of a foliated family of orbits in the vicinity of a stationary point [4,5]. We refer the reader to reference [6] for a discussion of the applications of the stability criterion to 2D and quasi-2D flows. H.K. Moffatt [7] gave an explicit computation of the quadratic form for ABC flows and found that it is indefinite for these important examples.

In this paper we examine the applicability of the Arnold criterion in 3 or more dimensions. We find that the quadratic form is never definite. In a few special cases it is identically zero. In all other cases it is indefinite, and the spectrum of the associated Hermitian operator is not even bounded from above or below. It should be emphasized

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that this does *not* imply that all velocity fields in 3 or more dimensions are unstable. It merely means that the conditions of Arnold's stability criterion are never met.

2. The stability criterion

We begin by reminding the reader of the geometric setting of the stability criterion [3]. For simplicity, we work at first in 3 dimensions, and later generalize to higher dimensions. Let M be a 3-dimensional smooth (C^∞) orientable Riemannian manifold with a smooth volume form μ, defined by the Riemannian metric. Let $\mathrm{SDiff}_\mu(M)$ denote the group of smooth volume-preserving diffeomorphisms, and let $\mathrm{sdiff}_\mu(M)$ be the corresponding Lie algebra of solenoidal (divergence-free) vector fields. The fundamental questions of developing Lie theory for $\mathrm{SDiff}_\mu(M)$ were partially addressed by D. Ebin and J. Marsden [8]. Surprising phenomena related to the differential geometry of $\mathrm{SDiff}_\mu(M)$ were discovered by Schnirelman [9,10].

$\mathrm{SDiff}_\mu(M)$ is the configuration space of a 3-dimensional ideal fluid (see, however, [9,10]). For any smooth curve $g_t : I \mapsto \mathrm{SDiff}_\mu(M)$, where I is an interval of the real line, we define the Eulerian velocity vector $u(t) \in \mathrm{sdiff}_\mu(M)$ to be

$$u(t) = R_{{g_t^{-1}}_*}(\dot{g}_t), \tag{1}$$

where R_g denotes right-translation by g on the Lie group $\mathrm{SDiff}_\mu(M)$. The Euler-Arnold equations [1,2,3] arise from the variational principle

$$\delta\left(\frac{1}{2}\int dt \int_M \big(u(t,x)\cdot u(t,x)\big)\mu(dx)\right) = 0. \tag{2}$$

This means that the flows of an ideal fluid on M are geodesics of the right-invariant metric $<\cdot,\cdot>$ on $\mathrm{SDiff}_\mu(M)$, defined on the tangent space $T_e\mathrm{SDiff}_\mu(M)$ as

$$< u, v > = \int_M (u\cdot v)\mu(dx). \tag{3}$$

The Euler-Arnold equations then take the form

$$\dot{u} = -(u\cdot\nabla)u - \nabla p; \qquad\qquad \nabla\cdot u = 0, \tag{4}$$

where u is an Eulerian velocity, p is a pressure function defined on M, chosen such that $\nabla\cdot\dot{u} = 0$, and ∇ acts on u via the Levi-Civita connection on M.

Let $u(t) \in \mathrm{sdiff}_\mu(M)$ be a solution to (4). Kelvin's theorem says that $u(t)$ is restricted to a single orbit of the coadjoint representation of $\mathrm{SDiff}_\mu(M)$ on $(\mathrm{sdiff}_\mu(M))^*$, the dual space of $\mathrm{sdiff}_\mu(M)$. We identify $\mathrm{sdiff}_\mu(M)$ with $(\mathrm{sdiff}_\mu(M))^*$ by the metric $<\cdot,\cdot>$.

The tangent space of the coadjoint orbit at a point $u \in \mathrm{sdiff}_\mu(M)$ is spanned by vectors fields of the form $ad^*_\xi = -\xi\times(\nabla\times u) - \nabla p$, where ξ is an arbitrary solenoidal vector field and p is chosen to make the entire expression $-\xi\times(\nabla\times u) - \nabla p$ solenoidal. The second variation of the energy $H(u) = \frac{1}{2} < u, u >$ on the orbit is given by the quadratic form

$$d^2H(\xi) = \frac{1}{2} < -\xi\times(\nabla\times u) - \nabla p, -\xi\times(\nabla\times u) - \nabla p > + \frac{1}{2} < -\xi\times(\nabla\times u) - \nabla p, \nabla\times(\xi\times u) > . \tag{5}$$

The Arnold stability criterion states that a steady-state solution to the Euler-Arnold equations (4) is stable if the quadratic form (5) is positive (or negative) definite, and if the stratification of $(\mathrm{sdiff}_\mu(M))^*$ into orbits of the coadjoint representation is regular in a neighborhood of the steady-state solution u.

The corresponding (unbounded) self-adjoint operator is defined from the expression (5) in $\mathcal{H}$, the space of all solenoidal square-integrable vector fields ξ. It is a simple exercise to see that the null-space of this operator contains the null-space K of the bounded operator $\mathcal{B} : \mathcal{H} \mapsto \mathcal{H}$

$$\mathcal{B}(\xi) = -\xi \times (\nabla \times u) - \nabla p, \tag{6}$$

leading to a self-adjoint operator defined in the factor-space $\mathcal{H}/K$, the formal tangent space to the orbit.

3. The second variation for 3-dimensional fluid flow

Main Theorem *Let M be a 3-dimensional closed manifold. If $\nabla \times u$ is not identically zero, then the quadratic form d^2H is indefinite. Moreover, the spectrum of the corresponding self-adjoint operator is neither bounded from below nor from above.*

Proof: If $\nabla \times u$ is not identically zero, we can pick a point $x_0 \in M$ where u and $\nabla \times u$ are both nonzero. Let $\phi(x)$ be a function on M with $\nabla\phi \cdot u$ and $\nabla\phi \cdot (\nabla \times u)$ both being nonzero at x_0. By continuity, $\nabla\phi \cdot u$ and $\nabla\phi \cdot (\nabla \times u)$ are both nonzero on a small neighborhood N of x_0. We then pick a smooth vector field a_R, everywhere orthogonal to $\nabla\phi$, that vanishes outside of N, and define the real vector field $a_I = (\nabla\phi) \times a_R/|\nabla\phi|$ and the complex vector field $a = a_R + ia_I$.

Next we construct deformations ξ_ϵ for which $d^2H(\xi_\epsilon)$ is arbitrarily positive or negative. Let

$$\xi_\epsilon = \epsilon \nabla \times \left(\left(\frac{1}{|\nabla\phi|} \right) a e^{i\phi/\epsilon} \right) = a e^{i\phi/\epsilon} + O(\epsilon). \tag{7}$$

The quantity $d^2H(\xi_\epsilon)$ can be expanded in a power series in ϵ. To leading order,

$$\begin{aligned} d^2H(\xi_\epsilon) =& \frac{1}{2} < -\xi_\epsilon \times (\nabla \times u) - \nabla p, -\xi_\epsilon \times (\nabla \times u) - \nabla p >_c \\ & + \frac{1}{2} < -\xi_\epsilon \times (\nabla \times u) - \nabla p, \nabla \times (\xi_\epsilon \times u) >_c \\ =& \frac{-i}{2\epsilon} < -\xi_\epsilon \times (\nabla \times u) - \nabla p, \nabla\phi \times (\xi_\epsilon \times u) >_c + O(1) \\ =& \frac{-1}{\epsilon} \int (u \cdot \nabla\phi)(a_R \times a_I \cdot (\nabla \times u))\mu(dx) + O(1) \\ =& \frac{-1}{\epsilon} \int \frac{|a_R|^2}{|\nabla\phi|} (u \cdot \nabla\phi)((\nabla \times u) \cdot \nabla\phi)\mu(dx) + O(1), \end{aligned} \tag{8}$$

where $< \cdot, \cdot >_c$ is the Hermitian inner product, linear in the first argument and anti-linear in the second, that extends the real inner product $< \cdot, \cdot >$. Since $(u \cdot \nabla\phi)(\nabla \times u \cdot \nabla\phi)$

is nonzero on N, the integral over N is nonzero. By choosing the sign of ϵ and picking ϵ sufficiently close to zero, we can make $d^2H(\xi_\epsilon)$ as negative (or positive) as we wish, so d^2H cannot be a definite form.

As $\epsilon \to 0$, the L^2 norm of ξ_ϵ remains bounded, as does the L^2 norm of $-\xi_\epsilon \times (\nabla \times u)$. Since $-\xi_\epsilon \times (\nabla \times u) - \nabla p$ is the L^2-orthogonal projection of $-\xi_\epsilon \times (\nabla \times u)$ onto the solenoidal vector fields, the norm of $-\xi_\epsilon \times (\nabla \times u) - \nabla p$ is also bounded. Since we can get arbitrarily large (positive or negative) values of $d^2H(\xi_\epsilon)$ with a bounded set of ξ_ϵ's, the self-adjoint operator associated to d^2H is bounded neither above nor below. Its spectrum runs, possibly with gaps, from $-\infty$ to ∞. ∎

4. Generalization to higher dimensions

We now briefly describe the generalization of the main theorem to higher dimensions. Let M be an n-dimensional smooth manifold ($n \geq 3$), with a smooth volume form μ that does not necessarily come from the Riemannian metric [11]. For each vector $v \in TM$, we let $\tilde{v}$ be the 1-form obtained by "lowering indices". That is, $\tilde{v}(w) = v \cdot w$.

In n dimensions, the Euler-Arnold equations take the form

$$\dot{u} = -(u \cdot \nabla)u - \nabla p; \qquad \mathrm{div}_\mu u = 0, \tag{9}$$

where $u \in \mathrm{sdiff}_\mu(M)$ and ∇ acts on vectors by the Levi-Civita connection. The condition $\mathrm{div}_\mu u = 0$ can be written invariantly as $d(i_u \mu) = 0$. Alternatively, if μ is $\rho(x)$ times the Riemannian volume form, then this condition reduces to $\nabla \cdot (\rho u) = 0$, or equivalently $d^*(\rho \tilde{u}) = 0$. As before, the pressure p is chosen such that $\mathrm{div}_\mu \dot{u} = 0$. In this more general case, our main theorem reads as follows

Theorem *Let u be a smooth steady-state solution to the equations (9). If $d\tilde{u} = 0$ everywhere, then the quadratic form d^2H is identically zero. Otherwise, the quadratic form d^2H is indefinite. Moreover, the spectrum of the corresponding self-adjoint operator is neither bounded from below nor from above.*

Proof: In this case the quadratic form d^2H is given by

$$d^2H(\xi) = \frac{1}{2} < i_\xi d\tilde{u} - df, i_\xi d\tilde{u} - df > + \frac{1}{2} < i_\xi d\tilde{u} - df, \widetilde{[\xi, u]} >, \tag{10}$$

where the function f is chosen to make $i_\xi d\tilde{u} - df$ correspond to a μ-solenoidal vector field. When $d\tilde{u}$ vanishes, d^2H is identically zero. When $d\tilde{u}$ does not vanish we construct, as before, highly oscillatory circularly polarized vector fields ξ_ϵ that make d^2H arbitrarily positive or negative.

Let $x_0 \in M$ be a point where u and $d\tilde{u}$ are both nonzero. Let $\phi(x)$ be a function on M with $u \cdot \nabla \phi$ and $d\phi \wedge d\tilde{u}$ both nonzero on a neighborhood N of x_0. Pick smooth vector fields a_R and a_I, everywhere orthogonal to $\nabla \phi$, that vanish outside of N, for which $d\tilde{u}(a_R, a_I) \geq 0$, and for which $d\tilde{u}(a_R, a_I) > 0$ in a smaller neighborhood $N' \subset N$. This is possible since $d\phi \wedge d\tilde{u}$ is smooth and nonzero on N. Let $a = a_R + i a_I$, and let

$$\tilde{\xi}_\epsilon = \frac{i\epsilon}{\rho} d^* \left(\frac{\rho d\phi \wedge \tilde{a}}{|d\phi|^2} e^{i\phi/\epsilon} \right) = \tilde{a} e^{i\phi/\epsilon} + O(\epsilon). \tag{11}$$

The leading term in the expansion of $d^2H(\xi_\epsilon)$ is

$$\begin{aligned}&\frac{1}{2\epsilon} < i_{\xi_\epsilon} d\tilde{u} - df, i(u\cdot\nabla\phi)\tilde{a}e^{i\phi/\epsilon} >_c = \frac{-i}{2\epsilon}\int_M (u\cdot\nabla\phi)d\tilde{u}(a,\bar{a})\mu(dx)\\ =&\frac{-1}{\epsilon}\int_M (u\cdot\nabla\phi)d\tilde{u}(a_R,a_I)\mu(dx).\end{aligned} \tag{12}$$

By assumption, $u\cdot\nabla\phi$ is nonzero, and hence of definite sign, on N, while $d\tilde{u}(a_R,a_I)$ is positive on N' and is never negative. The integral over M is therefore nonzero, and we can make $d^2H(\xi_\epsilon)$ arbitrarily large or small by taking ϵ sufficiently close to zero and of appropriate sign. ∎

5. Manifolds with boundary

Finally, we consider the Arnold stability criterion for fluid flow on a manifold with boundary. As before, if u is irrotational, then d^2H is identically zero. Otherwise, we construct the vector fields ξ_ϵ as before, in an arbitrarily small neighborhood of a point. In particular, we can choose ξ_ϵ to vanish near the boundary, so our conclusions about d^2H being indefinite are unaffected by the presence of the boundary. Translating this conclusion into a statement about the spectrum of the self-adjoint operator associated to d^2H is more difficult, however, since it is not immediately clear what the domain of this operator would be. In this section we construct this operator.

For simplicity, we consider M to be a compact smooth domain in $\mathbf{R}^3$, with the standard metric and volume form. We work on the space

$$\mathcal{H} = \left\{u\in L^2(M)^3 | \nabla\cdot u = 0, (u\cdot n)|_{\partial M} = 0\right\}. \tag{13}$$

Let $V = C_0^\infty(M)\cap\mathcal{H}$. V is a dense subspace of $\mathcal{H}$, consisting of smooth solenoidal vector fields with compact support in the interior of M. We define an unbounded self-adjoint operator $\mathcal{A}:\mathcal{H}\to\mathcal{H}$ on the following domain $D(\mathcal{A})\supset V$.

$$D(\mathcal{A}) = \{\xi\in\mathcal{H}| - u\times\nabla\cdot((\nabla\times u)\times\xi - \nabla p) - \nabla g\in\mathcal{H}\}, \tag{14}$$

where the inclusion in $\mathcal{H}$ is in the following sense. The vector $\mathcal{B}(\xi) = (\nabla\times u)\times\xi - \nabla p$ is in $\mathcal{H}$, since multiplication by $\nabla\times u$ and L^2-projection onto solenoidal vectors are bounded operations. The vector $-u\times\nabla\cdot((\nabla\times u)\times\xi-\nabla p)$ must be understood in the distributional sense as an element of $\mathcal{D}'(M^\circ)^3$. The statement that $-u\times\nabla\cdot((\nabla\times u)\times\xi-\nabla p)-\nabla g\in\mathcal{H}$ means that there exists $w\in\mathcal{H}$ such that, for all $v\in V$,

$$< -u\times\nabla\cdot((\nabla\times u)\times\xi - \nabla p) - w, v >= 0, \tag{15}$$

in which case we define $\mathcal{A}(\xi) = w$.

It is easy to check that, for any ξ, $\zeta\in D(\mathcal{A})$, $<\xi,\mathcal{A}(\zeta)> = <\mathcal{A}(\xi),\zeta>$ and that $\frac{1}{2}<\mathcal{A}(\xi),\xi> + \frac{1}{2}<\mathcal{B}(\xi),\mathcal{B}(\xi)> = d^2H(\xi)$. The quadratic forms d^2H and $<\mathcal{A}(\xi),\xi>$ are clearly defined on the factor space $\mathcal{H}/K$, where K is the kernel of the bounded operator

$\mathcal{B}$. Since $\mathcal{B}$ is a bounded operator, unboundedness of the quadratic form $< \mathcal{A}(\xi), \xi >$ both from above and below implies the same statement for d^2H on $\mathcal{H}/K$ and similarly for the self-adjoint operator $\frac{1}{2}(\mathcal{A} + \mathcal{B}^*\mathcal{B})$ mod K.

6. Conclusions

The quadratic form d^2H ((5) or, more generally (10)) contains two terms, a positive-definite first term and an indefinite second term. The second term contains one more derivative of ξ than the first term, and so should be expected to dominate when ξ is highly oscillatory. When ξ is a circularly polarized transverse wave this is in fact the case. At any point where u and $\nabla \times u$ are both nonzero, there exist directions where right (or left) circularly polarized waves give large positive values of d^2H and left (or right) circularly polarized waves give large negative values. In 3 or more dimensions this makes d^2H unbounded and indefinite.

In two dimensions, however, circular polarization does not exist. One can only construct linearly polarized waves, for which the $O(1/\epsilon)$ contribution to d^2H is identically zero. So in two dimensions d^2H is sometimes definite, and the Arnold criterion is a useful measure of stability. In 3 or more dimensions, d^2H is never definite, and the Arnold criterion can never be applied.

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