Coupled Analytic Maps

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Abstract

We consider a lattice of weakly coupled expanding circle maps. We construct, via a cluster expansion of the Perron-Frobenius operator, an invariant measure for these infinite dimensional dynamical systems which exhibits space-time-chaos.

1 Introduction

It is an important problem to determine which parts of the rich theory of finite dimensional dynamical systems (e.g. hyperbolic attractors, SRB measures [11]) can be extended to infinite dimensional ones. The latter are usually given by non-linear partial differential equations of the form $\partial_t u = F(u, \partial u, \partial^2 u, \ldots)$, i.e. the time derivative of $u(x,t)$ is given in terms of $u(x,t)$ and its partial space derivatives. One would like to find natural invariant measures for the flow.

In a bounded spatial domain and F suitably dissipative, such equations tend to have finite dimensional attracting sets [26] and thus fall in into the class of finite dimensional systems. Genuinely infinite dimensional phenomena are expected to occur for dissipative PDE's on unbounded domains [9]. In particular, invariant measures for the flow might have infinite dimensional supports and there might be several of them (corresponding to a "phase transition").

A class of dynamical systems, possibly modelling such PDE's, are obtained by discretizing space and time and considering a recursion

$$u(x, t+1) = F(x, u(\cdot, t)) \tag{1}$$

i.e. $u(x, t+1)$, with x being a site of a lattice, is determined by the values taken by u at time t (usually on the sites in a neighbourhood of x). For a suitable class of F's such dynamical systems are called Coupled Map Lattices [14, 15].

*Supported by EC grants SC1-CT91-0695 and CHRX-CT93-0411
†Supported by NSF grant DMS-9205296

The first rigorous results on such systems are due to Bunimovich and Sinai who studied a one dimensional lattice of weakly coupled maps [5]. They established the existence of an invariant measure with exponential decay of correlations in space-time. Their method was to construct a Markov partition and to show uniqueness of the Gibbs state for the corresponding two-dimensional spin system. This is a natural extension of the method used for a single map or for hyperbolic systems [25, 23]. These results were strengthened by Volevich [27] and extended by Pesin and Sinai to coupled hyperbolic attractors [20] (for a review, see [6]). An extension to lattices of any dimension is announced in [28] (for coupled hyperbolic attractors).

Since the Gibbs measure constructed by Bunimovich and Sinai describes statistical mechanics in two dimensions, the possibility of phase transitions i.e. non-uniqueness of invariant measure is open (for recent results on this, see [1, 2, 4, 7, 19, 22] and references therein). In statistical mechanics, Gibbs measures are often easy to construct in weak coupling (which corresponds to high temperature) and in strong coupling (low temperature) using convergent expansions. The purpose of the present paper is to develop these expansion methods for the dynamical system problems in infinite dimensions.

We consider weakly coupled circle maps and derive a convergent cluster expansion for the Perron-Frobenius operator (transfer matrix in the statistical mechanics terminology). This allows us to prove exponential mixing in space and time for an invariant SRB measure. These results are similar to those of Bunimovich and Sinai, but our method works immediately in any dimension and is simpler. However, for technical reasons we need to restrict ourselves to real analytic maps.

The Perron-Frobenius operator has been a powerful tool to analyze quite general maps, of bounded variation [18] (for reviews see [8, 17]). This approach was used also for coupled maps in [16], but weaker results were obtained there in the infinite volume limit. An open problem still remains to develop expansion methods for coupled maps that are of bounded variation. These are the most natural candidates that might exhibit interesting phase transitions as the coupling is increased.

We have tried to make the paper self-contained for readers having no background in the expansion methods of statistical mechanics. Appendix 2 contains some of the standard combinatorical estimates needed. A reader who is familiar with these methods will find a slightly novel application of them because our expansion is applied directly to the Perron-Frobenius operator.

2 Results

We consider the following infinite dimensional dynamical system. The state space of the system is

$$M = (S^1)^{\mathbf{Z}^d},$$

the direct product of circles over $\mathbf{Z}^d$, i.e., $m \in M$ is given as $m = \{m_i\}_{i \in \mathbf{Z}^d}$, $m_i \in S^1$. M carries the product topology and the Borel σ-algebra inherited from S^1. To describe the dynamics, we consider a map $f : S^1 \to S^1$ and let $\mathcal{F} : M \to M$ be $\mathcal{F} = \times_{i \in \mathbf{Z}^d} f$ i.e.,

$\mathcal{F}(m)_i = f(m_i)$. $\mathcal{F}$ is the uncoupled map. The coupling map $\Phi : M \to M$ is given by

$$\Phi(m)_i = m_i e^{2\pi i\varepsilon \sum_j g_{|i-j|}(m_i, m_j)} \tag{2}$$

where $g_n : S^1 \times S^1 \to \mathbf{R}$ (g will be chosen such that the sum converges) and $\varepsilon > 0$ is a parameter. We define now the coupled map $T : M \to M$ by

$$T = \Phi \circ \mathcal{F}. \tag{3}$$

We assume the following:

A. f is expanding and real analytic.

B. g_n are exponentially decreasing and real analytic.

More precisely, for B we assume that there exists a neighbourhood V of S^1 in $\mathbf{C}$ such that g_n are analytic in $V \times V$ and

$$|g_n(u, v)| \leq Ce^{-\lambda n} \tag{4}$$

for some $C < \infty, \lambda > 0$ and all $u, v \in V$.

Let us denote by σ_i the shifts in $\mathbf{Z}^d$, $i = 1, \ldots, d$. Then our main results are

Theorem. *Suppose f and g_n satisfy A, B. There is an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ there exists a Borel measure μ on M such that*

1. μ is invariant under T and the shifts σ_i. For any finite $\Lambda \subset \mathbf{Z^d}$, the marginal distribution of μ on $(S^1)^\Lambda$ is absolutely continuous with respect to the Lebesgue measure.

2. The $\mathbf{Z}^{d+1}$ action generated by T and $\{\sigma_i\}_{i=1}^d$ is exponentially mixing.

3. $T^n m \to \mu$ weakly as $n \to \infty$, where m is the product of Lebesgue measures on S^1.

Remark 1. For a precise statement of 2, see Proposition 7. Properties 1 and 2 are usually called "space-time chaos".

Remark 2. The result holds for a much more general class of interactions Φ : we could take

$$\Phi(m)_i = m_i e^{2\pi i \sum_{X : i \in X} g_X(m_X)}$$

where $X \subset \mathbf{Z}^d$, $m_X = m|_X$, $|X|$ is finite and $g_X : (S^1)^{|X|} \to \mathbf{R}$ is analytic in $V^{|X|}$ with the bound

$$|g_X(z_X)| \leq \varepsilon^{|X|} e^{-\lambda\tau(X)}$$

where $\tau(X)$ is the length of the shortest tree graph on the set X. Then the theorem holds again for $\varepsilon < \varepsilon_0$ small enough provided $g_{X+j} = g_X, j \in \mathbf{Z}^d$. Without this translation invariance μ is not σ_i-invariant, but the other claims still hold. Similarily, $\mathcal{F}$ may be replaced by $\times_{i \in \mathbf{Z}^d} f_i$ where f_i satisfy A, uniformly in i. Our results also extend to coupled maps of the interval $[0, 1]$, of the type considered by Bunimovich and Sinai [5], provided we take their f and α analytic.

Remark 3. We do not prove, but conjecture, that there is only one invariant measure whose local marginal distributions on the sets $(S^1)^\Lambda, \Lambda \subset \mathbf{Z^d}, |\mathbf{\Lambda}|$ finite are absolutely continuous with respect to Lebesgue measure. This is true in a class of measures with analytic marginals: the proof of 3 extends to the case where m is replaced with a measure satisfying some clustering and the analyticity of the densities of the local marginal distributions. This is similar to the results of Volevich [27, 28]. Thus μ can be considered as a natural extension to the infinite dimensional context of an SRB measure.

3 Decoupling of the Perron-Frobenius operator

Let $\Lambda \subset \mathbf{Z}^d$ be a finite connected set (a set Λ in $\mathbf{Z}^d$ is connected if every point of Λ has a nearest neighbour in Λ, or if Λ consists of a single point). We denote $M_\Lambda = (S^1)^\Lambda$. Let now $f_\Lambda = \times_{i\in\Lambda} f$ and let $\Phi_\Lambda : M_\Lambda \to M_\Lambda$ be given by (2) where the sum is over j in Λ. We first construct a $T_\Lambda = \Phi_\Lambda \circ f_\Lambda$ invariant measure μ_Λ on M_Λ and later μ as a suitable limit of μ_Λ. The measures μ_Λ will be constructed by means of the Perron-Frobenius operator of T_Λ. To describe this, we first collect some straightforward facts about f and its invariant measure.

The Perron-Frobenius operator P for f on $L^1(S^1)$ is defined as usual by

$$\int (g \circ f) h \; dm = \int g P h \; dm$$

for $h \in L^1(S^1), g \in L^\infty(S^1)$ and dm the Lebesgue measure (i.e., dx in the parametrization $m = e^{2\pi i x}$). We wish to consider P on a smaller space, namely H_ρ, the space of bounded holomorphic functions on the annulus $A_\rho = \{1 - \rho < |z| < 1 + \rho\}$. H_ρ is a Banach space in the sup norm. The assumption A for f implies the following spectral properties for P:

Proposition 1. *There are constants $\rho_0 > 0, \gamma > 1$ such that*

a) $P : H_\rho \to H_{\gamma\rho}$ is continuous for all $\rho \le \rho_0$.

b) $P : H_\rho \to H_\rho$ can be written as

$$P = Q + R \tag{5}$$

with Q a 1-dimensional projection operator:

$$Qg = \left(\int_{S^1} g \, dm \right) h \equiv \ell(g) h \tag{6}$$

with $h \in H_{\rho_0}, h > 0$ on S^1, $\int_{S^1} h \, dm = 1$ and

$$||| \, R^n \, ||| \le C \mu^n, \;\; QR = RQ = 0 \tag{7}$$

for some $\mu < 1$, $C < \infty$ and all n; we use $||| \cdot |||$ to denote operator norms.

Remark 1. a) is a consequence of the expansiveness of f and shows that the operator P improves the domain of analyticity, while b) means that there is a unique absolutely continuous invariant measure for f, with density in H_{ρ_0}, and the rest of the spectrum of

P in H_{ρ_0} is strictly inside the unit disc. Since all this is rather standard (see [8, 17, 18]), we defer the proof to Appendix 1.

Remark 2. Throughout the paper C will denote a generic constant, which may change from place to place, even in the same equation.

To describe the Perron-Frobenius operator $\mathcal{P}_\Lambda$ for the coupled map T_Λ, we introduce some notation. We denote $\mathcal{H}_\rho^\Lambda = \otimes_{i\in\Lambda} H_\rho$, $P_\Lambda = \otimes_{i\in\Lambda} P_i$ where P_i acts on the i-th variable. Also, we denote by dm_Λ the Lebesgue measure on M_Λ. Then

$$\mathcal{P}_\Lambda = \Phi_\Lambda^* P_\Lambda \tag{8}$$

where

$$(\Phi_\Lambda^* G)(m) = \det D\Phi_\Lambda^{-1}(m) G(\Phi_\Lambda^{-1}(m)) \tag{9}$$

and thus

$$\int (G \circ T_\Lambda) H \; dm_\Lambda = \int G \mathcal{P}_\Lambda H \; dm_\Lambda \tag{10}$$

for $G \in L^\infty(M_\Lambda), H \in L^1(M_\Lambda)$. We have of course to show that (9) is well defined. Actually, the strategy of our proof will be to first derive a "cluster expansion" [12, 13] for Φ_Λ^* in terms of localized operators with good bounds on norms. Then we shall construct the invariant measure in Section 4 by studying the limit $n \to \infty$ of $\mathcal{P}_\Lambda^n$; a cluster expansion for $\mathcal{P}_\Lambda^n$ will be obtained by combining (5) for P_Λ and (11) below for Φ_Λ^*.

Proposition 2. *Let Φ_Λ given by (2)with $i, j \in \Lambda$, and let g_n satisfy* **B**. *The following holds uniformly in Λ : There exists $\rho_1 > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, Φ_Λ^* maps $\mathcal{H}_{\rho_1}^\Lambda$ into $\mathcal{H}_{\rho_1-\delta}^\Lambda$ where we may take $\delta \to 0$ as $\varepsilon \to 0$. Moreover*

$$\Phi_\Lambda^* = \sum_{\mathcal{Y}} \otimes_{Y\in\mathcal{Y}} \delta_Y \otimes 1_{\Lambda\setminus\mathcal{Y}} \tag{11}$$

where $\mathcal{Y}$ runs through sets of disjoint subsets of Λ, $\Lambda \setminus \mathcal{Y} = \Lambda \setminus \cup Y$ and 1_Z denotes the imbedding of $\mathcal{H}_{\rho_1}^Z$ into $\mathcal{H}_{\rho_1-\delta}^Z$. The operators $\delta_Y : \mathcal{H}_{\rho_1}^Y \to \mathcal{H}_{\rho_1-\delta}^Y$ are bounded, with

$$||| \; \delta_Y \; ||| \leq \eta^{|Y|} \sum_\tau e^{-\frac{\lambda}{2}|\tau|} \equiv \eta^{|Y|} T(Y, \lambda) \tag{12}$$

where the sum runs over tree graphs on Y, $|\tau|$ is the length of τ and $\eta \to 0$ as $\varepsilon \to 0$.

Proof. Let us introduce decoupling parameters $s = \{s_{ij}\}$, $i, j \in \Lambda$, $i < j$ in some linear order of Λ, for our map Φ_Λ^*:

$$(\Phi_\Lambda^* G)(m) = \det D\Phi_{\Lambda s}^{-1}(m) G(\Phi_{\Lambda s}^{-1}(m))|_{s=1} \tag{13}$$

where

$$\Phi_{\Lambda s}(m)_i = m_i e^{2\pi i \varepsilon \sum_{j\in\Lambda} s_{ij} g_{|i-j|}(m_i, m_j)} \tag{14}$$

and we let $s_{ii} \equiv 1$, $s_{ij} = s_{ji}$. Our assumption **B** for g_n implies that there exists a $\rho_1 > 0$ such that all g_n's are holomorphic in $A_{\rho_1} \times A_{\rho_1}$ (A_ρ is the annulus) and (4) holds there, together with

$$|\partial g_n(u,v)| \leq Ce^{-\lambda n}. \tag{15}$$

where $\partial = \partial/\partial u$ or $\partial/\partial v$. Consider now s complex, in the polydisc $D_\Lambda = \{s_{ij} | i < j,\ |s_{ij}| < re^{\frac{\lambda}{2}|i-j|}\} \subset \mathbf{C}^{\frac{1}{2}(|\Lambda|^2 - |\Lambda|)}$ (we shall take r large for ε small). Then, we have the following

Lemma. *There exists $\varepsilon_0(\rho_1, r, \lambda)$ such that, for $\varepsilon < \varepsilon_0(\rho_1, r, \lambda)$ $\Phi^{-1}_{\Lambda s}$ is a holomorphic family (for $s \in D_\Lambda$) of holomorphic diffeomorphisms from $A^\Lambda_{\rho_1 - \delta}$ into $A^\Lambda_{\rho_1}$ where $\delta \to 0$ as $\varepsilon \to 0$ (and A^Λ_ρ denotes the polyannulus $\{z_i \in A_\rho,\ i \in \Lambda\}$).*
Moreover, the bound

$$\| \det D\Phi^{-1}_{\Lambda s} \| \leq \exp(C\varepsilon r(1 + \lambda^{-d})|\Lambda|) \tag{16}$$

holds uniformly in Λ. $\| \cdot \|$ is the norm in $\mathcal{H}^\Lambda_{\rho_1 - \delta}$.

Proof. (4) and (15) imply

$$|D\Phi_{\Lambda,s}(z)_{ij} - \delta_{ij}| \leq C\varepsilon re^{-\frac{\lambda}{2}|i-j|} \tag{17}$$

$$|\Phi_{\Lambda,s}(z)_i - z_i| \leq C(1 + \lambda^{-d})\varepsilon r \tag{18}$$

for $z \in A^\Lambda_{\rho_1}$ and $s \in D_\Lambda$. From these inequalities it follows easily that $\Phi_{\Lambda,s}$ is a diffeomorphism from $A^\Lambda_{\rho_1}$ onto a set containing $A^\Lambda_{\rho_1 - \delta}$ provided $C(1 + \lambda^{-d})\varepsilon r < \delta$. The inverse of $\Phi_{\Lambda,s}$ then satisfies bounds (17), (18) too (with different constants) and then (16) follows from Hadamard's inequality [10]. □

Thus, $\Phi^*_{\Lambda s} : \mathcal{H}^\Lambda_{\rho_1} \to \mathcal{H}^\Lambda_{\rho_1 - \delta}$ is a holomorphic family of operators, for $s \in D_\Lambda$.

The cluster expansion for $\Phi^\star_\Lambda$ is the following repeated application of the fundamental theorem of calculus :

$$\Phi^*_\Lambda = \Phi^*_{\Lambda s}|_{s=1} = \sum_\Gamma \int_{[0,1]^{|\Gamma|}} ds_\Gamma \partial^\Gamma \Phi^*_{\Lambda s}|_{s_{\Gamma^c} = 0} \equiv \sum_\Gamma \delta_{\Lambda\Gamma} \tag{19}$$

where the notation is as follows. Γ is a subset of $\Lambda \times \Lambda$ and each pair $(i,j) \in \Gamma$ is such that $i < j$. $ds_\Gamma \equiv \prod_{(i,j)\in\Gamma} ds_{ij}$, $\partial^\Gamma = \prod_{(i,j)\in\Gamma} \frac{\partial}{\partial s_{ij}}$, $s_{\Gamma^c} = \{s_{ij}\}_{(i,j)\in\Gamma^c}$ and $|\Gamma|$ is the cardinality of Γ. (19) then follows by writing $1 = \prod_{i<j}(I_{ij} + E_{ij})$ with $I_{ij} = \int_{[01]} ds_{ij} \frac{\partial}{\partial s_{ij}}$ and E_{ij} is the evaluation map at $s_{ij} = 0$ (for more details, see [12] and Proposition 18.2.2 in [13]).

Now observe that $\delta_{\Lambda\Gamma}$ factorizes as follows. For a set $A \subset \Lambda \times \Lambda$, let $\overline{A}$ denote the union of its projections on the two factors. We say $p_1, p_2 \in \Gamma$ are connected to each other if $\overline{p}_1 \cap \overline{p}_2 \neq \emptyset$. Let Γ_α be the maximal connected components of Γ with respect to this relation. Then we have

$$\delta_{\Lambda\Gamma} = \otimes_\alpha\, \delta_{\overline{\Gamma}_\alpha \Gamma_\alpha} \otimes 1_{\Lambda \backslash \overline{\Gamma}} \tag{20}$$

where $\delta_{\overline{\Gamma}_\alpha \Gamma_\alpha} : \mathcal{H}_{\rho_1}^{\overline{\Gamma}_\alpha} \to \mathcal{H}_{\rho_1-\delta}^{\overline{\Gamma}_\alpha}$ is given by the integral in (19) (with Λ replaced by $\overline{\Gamma}_\alpha$ and Γ by Γ_α).

We now estimate the norm of $\delta_{\overline{\Gamma}_\alpha \Gamma_\alpha}$. Using (16) (which holds for arbitrary Λ) and a Cauchy estimate in (19) (a circle of radius $\frac{r}{2} e^{\frac{\lambda}{2}|i-j|}$ around any s_{ij} in the integral (19) is inside D_Λ), we get

$$|||\delta_{\overline{\Gamma}_\alpha \Gamma_\alpha}||| \leq \left(\frac{r}{2}\right)^{-|\Gamma_\alpha|} e^{C\varepsilon r(1+\lambda^{-d})|\overline{\Gamma}_\alpha|} e^{-\frac{\lambda}{2}\sum_{(i,j)\in\Gamma_\alpha}|i-j|}. \tag{21}$$

We arrive at the final formula (11) by setting

$$\delta_Y = \sum_\Gamma \delta_{Y\Gamma} \tag{22}$$

with $Y = \overline{\Gamma}$ and Γ connected. Given any such Γ which we may view as a connected graph on the points of Y, pick a connected tree graph τ with $\overline{\tau} = Y$ and estimate the sum over Γ in (22) by

$$|||\delta_Y||| \leq (\sum_{j\in\mathbf{Z}^d} e^{-\frac{1}{2}\lambda|j|})^{|Y|} \tilde{\eta}^{|Y|-1} \sum_\tau e^{-\frac{\lambda}{2}|\tau|} \tag{23}$$

where $\tilde{\eta} = r^{-1}\exp[C\varepsilon r(1+\lambda^{-d})]$ (see Appendix 2 for more details). Since we may take $r \to \infty$ as $\varepsilon \to 0$, see (16-18), the claim (12) follows. □

Remark. Propositions 1 and 2 imply that there is an $\varepsilon_0 > 0$ and $\rho > 0$ such that $\mathcal{P}_\Lambda$ maps $\mathcal{H}_\rho^\Lambda$ into itself for all Λ and all $\varepsilon \leq \varepsilon_0$: the domain of analyticity shrinks by an amount δ when we apply Φ_Λ^*, but it is expanded when we apply P_Λ (by Proposition 1,a). So, we may choose ε small enough so that $\gamma\rho - \delta > \rho$. We will fix this ρ now once and for all.

4 Space time expansion for the invariant measure

The T_Λ-invariant measure μ_Λ will be constructed by studying $\mathcal{P}_\Lambda^n$ as $n \to \infty$. This will yield spectral information on $\mathcal{P}_\Lambda$ uniformly in Λ. From (8), we have

$$\mathcal{P}_\Lambda^n = (\Phi_\Lambda^* P_\Lambda)^n, \tag{24}$$

into which we insert (11) for Φ_Λ^* and, for P_Λ, we use (5):

$$P_\Lambda = \otimes_{i\in\Lambda} (Q_i + R_i) = \sum_{I\subset\Lambda} \otimes_{i\in I} R_i \otimes_{j\in\Lambda\setminus I} Q_j \equiv \sum_{I\subset\Lambda} R_I \otimes Q_{\Lambda\setminus I}, \tag{25}$$

and we get

$$\mathcal{P}_\Lambda^n = \sum_{\{I_t\}} \sum_{\{\mathcal{Y}_t\}} \prod_{t=1}^n (\otimes_{Y\in\mathcal{Y}_t} \delta_Y \otimes 1_{\Lambda\setminus\mathcal{Y}_t})(R_{I_t} \otimes Q_{\Lambda\setminus I_t}) \tag{26}$$

where the product of operators is ordered, with $t = 1$ on the right.

To understand the structure of the terms in (26), let us first consider a simple example. Consider the term in (26) with $\mathcal{Y}_t = \emptyset$ for $t \neq m < n$ and $\mathcal{Y}_m = \{Y\}$, where Y is some finite subset of $\mathbf{Z}^d$. Let also $I_t = \{i\}$ for $1 \leq t \leq m$ and $I_t = \{j\}$ for $m+1 \leq t \leq n$, where $i, j \in Y$. The contribution to (26) from this term is given by

$$\Big((R_j^{n-m} \otimes Q_{Y\setminus j})\delta_Y(R_i^m \otimes Q_{Y\setminus i})\Big) \otimes Q_{\Lambda\setminus Y} \tag{27}$$

where we used $Q^2 = Q$ repeatedly. Let us write Q in (6) as

$$Q = h\hat{\otimes}\ell \tag{28}$$

where we identify bounded operators $H_\rho \to H_\rho$ with elements of H_ρ tensored with its dual H_ρ^* and use $\hat{}$ to distinguish this tensoring from the one used for spaces indexed by $\mathbf{Z}^d$. Using the shorthand

$$\ell_Y \equiv \otimes_{i\in Y}\, \ell,\ h_Y = \otimes_{i\in Y}\, h$$

and inserting (28) into (27), the latter becomes

$$W \otimes (h_{\Lambda\setminus j}\hat{\otimes}\ell_{\Lambda\setminus i})$$

where $W : \mathcal{H}_\rho^{\{i\}} \to \mathcal{H}_\rho^{\{j\}}$ is given by (let $g \in \mathcal{H}_\rho^{\{i\}}$)

$$Wg = \ell_{Y\setminus j}\left(R_j^{n-m}\delta_Y(h_{Y\setminus i} \otimes R_i^m g)\right).$$

To make these remarks systematic, it is useful to introduce a "space-time" lattice $\mathbf{Z}^{d+1}$, where the extra dimension corresponds to the "time" t in the product in (26), and to establish a correspondence between the terms in (26) and geometrical objects defined on this lattice.

We let $\mathcal{S}$ denote the set of all finite "spacelike" subsets of $\mathbf{Z}^{d+1}$, i.e. $Z \in \mathcal{S}$ is of the form $Z = Y \times \{t\}$ for some $Y \subset \mathbf{Z}^d$ and $t \in \mathbf{Z}$. Also, let $\mathcal{B}$ denote the set of "timelike bonds" of $\mathbf{Z}^{d+1}$, i.e. $b \in \mathcal{B}$ is of the form $b = \{(i,t),(i,t+1)\} \equiv b_i(t)$ for some $i \in \mathbf{Z}^d$ and integer t. The correspondence with terms in (26) is defined as follows: to each $Y \in \mathcal{Y}_t, t = 1, \cdots, n$, we associate $Z = Y \times \{t\}$ and to each I_t we associate the set of bonds $\{b_i(t-1), i \in I_t\}$.

A *polymer* γ is then defined as a connected finite subset of $\mathcal{S} \cup \mathcal{B}$, i.e. the elements of γ are spacelike subsets and timelike bonds of $\mathbf{Z}^{d+1}$. We define two bonds $b, b' \in \gamma$ to be connected, if $b \cap b' \neq \emptyset$ or if there exists a $Z \in \gamma$ such that b and b' intersect Z. γ is then defined to be *connected* if the set of bonds $b \in \gamma$ is connected with respect to this relation, or if γ consists of a single element belonging to $\mathcal{S}$. Denote by $\overline{\gamma}$ the support of γ, i.e. the subset of $\mathbf{Z}^{d+1}$ that is the union of the elements of γ. We say that two polymers are *disjoint*, if $\overline{\gamma}_1 \cap \overline{\gamma}_2 = \emptyset$. Thus, in the example above we have $\gamma = \{Y \times \{m\}, b_i(0), \ldots, b_i(m-1), b_j(m), \ldots, b_j(n-1)\}$.

Let γ be a polymer. Denote by π_s and π_t the projections on $\mathbf{Z}^{d+1} = \mathbf{Z}^d \times \mathbf{Z}$ to the first and the second factor. Then $\pi_t(\overline{\gamma})$ is connected, i.e. it is an interval denoted $[t_-, t_+]$. Let

$\gamma_\pm = \pi_s(\overline{\gamma} \cap \pi_t^{-1}(t_\pm))$ The *weight* of the polymer γ, $W(\gamma)$, is a bounded linear operator $W(\gamma) : \mathcal{H}_\rho^{\gamma_-} \to \mathcal{H}_\rho^{\gamma_+}$ or equivalently $W(\gamma) \in \mathcal{H}_\rho^{\gamma_+} \hat{\otimes} (\mathcal{H}_\rho^{\gamma_-})^*$. W in the example above is a weight with $\gamma_+ = j$ and $\gamma_- = i$.

We may now return to (26). Let us in general denote by $\Lambda_{t_1 t_2}$ the set $\Lambda_{t_1 t_2} = \Lambda \times \{t_1, t_1 + 1, \ldots, t_2\} \subset \mathbf{Z}^{d+1}$ where $-\infty \leq t_1 < t_2 \leq \infty$ and $\Lambda \times \{t\} \equiv \Lambda_t$. γ is said to be in $\Lambda_{t_1 t_2}$ if $\overline{\gamma} \subset \Lambda_{t_1 t_2}$, and each $Z \in \gamma$ has $t_1 + 1 \leq \pi_t(Z) \leq t_2$. In (26) we will encounter a family of polymers Γ in Λ_{0n} and we need to make a distinction whether their support intersects the boundary of Λ_{0n} i.e. Λ_0 or Λ_n. We denote the family of those intersecting neither by Γ_v (v stands for "vacuum polymers"), those intersecting Λ_0 but not Λ_n by Γ_0, the ones intersecting Λ_n but not Λ_0 by Γ_n and the ones intersecting both by Γ_{0n} (the example in (27) belongs to Γ_{0n}). Finally, let $\partial\gamma$ be the set of $i \in \overline{\gamma}$ such that i belongs to exactly one $b \in \gamma$ and to no $Z \in \gamma$.

The convergence of the polymer expansion is due to two reasons: thanks to (12), each δ_Y brings a small factor, which decays with the size of Y or the distance between the points of Y. On the other hand, bonds are associated to R factors and long strings of such factors are suppressed by (7). Note however that μ need not be small, only less than one. Thus, this expansion is similar to the one of a lattice of weakly coupled one dimensional systems, but where, within each system, the couplings are not necessarily small.

We have now

Proposition 3. (26)*can be written as*

$$\mathcal{P}_\Lambda^n = \sum_\Gamma \prod_{\gamma \in \Gamma_v} \langle W(\gamma) \rangle \otimes_{\gamma \in \Gamma_{0n}} W(\gamma)$$
$$\otimes \Big((h_{\Lambda_+} \otimes_{\gamma \in \Gamma_n} W(\gamma) h) \hat{\otimes} (\ell_{\Lambda_-} \otimes_{\gamma \in \Gamma_0} \ell W(\gamma)) \Big) \tag{29}$$

where the sum is over sets Γ (possibly empty) of mutually disjoint polymers γ with $\partial\gamma \subset \Lambda_0 \cup \Lambda_n$. The following notation was used: $\ell W(\gamma) = \ell_{\gamma^+} W(\gamma) \in (\mathcal{H}_\rho^{\gamma_-})^*$, $W(\gamma)h = W(\gamma) h_{\gamma_-} \in \mathcal{H}_\rho^{\gamma_+}$, $\langle W(\gamma) \rangle = \ell_{\gamma^+} W(\gamma) h_{\gamma_-} \in \mathbf{R}$ and $\Lambda_+ = \Lambda \backslash \cup_{\gamma \in \Gamma_n \cup \Gamma_{0n}} \gamma_+$ and $\Lambda_- = \Lambda \backslash \cup_{\gamma \in \Gamma_0 \cup \Gamma_{0n}} \gamma_-$.
The weights satisfy

$$||| \, W(\gamma) \, ||| \leq \mu^B \prod_{Z \in \gamma} (C\eta)^{|Z|} T(Z, \lambda) \tag{30}$$

where B is the number of bonds in γ and $||| \cdot |||$ is the norm in $\mathcal{H}_\rho^{\gamma_+} \hat{\otimes} (\mathcal{H}_\rho^{\gamma_-})^*$.

Proof. Consider a given term in (26). $\{I_t\}_{t=1}^n$ determines a set of bonds $\mathcal{B}_0 = \{b_i(t) \mid i \in I_{t+1} \ , \ t \in [0, n-1]\}$ and $\{\mathcal{Y}_t\}_{t=1}^n$ a subset of $\mathcal{S}$, $\mathcal{S}_0 = \{Y \times \{t\} \ , \ Y \in \mathcal{Y}_t\}$. Decompose $\mathcal{B}_0 \cup \mathcal{S}_0 = \cup_{\gamma \in \Gamma} \gamma$ where Γ is a set of mutually disjoint polymers. Since $QR = RQ = 0$, we see moreover that $\partial\gamma \subset \Lambda_0 \cup \Lambda_n$. Thus the sum in (26) can be written as

$$\mathcal{P}_\Lambda^n = \sum_\Gamma O_\Gamma(\Lambda) \tag{31}$$

where Γ runs through such sets and $O_\Gamma(\Lambda)$ is the product in (26) corresponding to Γ. Note also that we may, since $Q^2 = Q$, replace each Q in $O_\Gamma(\Lambda)$ which is on a bond that is disjoint from $\overline{\Gamma}$ and $\Lambda_0 \cup \Lambda_n$ by the identity operator. For the remaining Q's use the identity

$$QKQ = \ell(Kh)Q, \tag{32}$$

where $K : \mathcal{H}_\rho \to \mathcal{H}_\rho$, to factorize $O_\Gamma(\Lambda)$: We apply (32) (extended to tensor products) to each product $Q_i \delta_Y Q_i$ in (26) where $i \in Y$. The result is the summand in (29) where the weights $W(\gamma) : \mathcal{H}_\rho^{\gamma_-} \to \mathcal{H}_\rho^{\gamma_+}$ are given by (here $F \in \mathcal{H}_\rho^{\gamma_-}$)

$$W(\gamma)F = \ell_{Y\setminus\gamma_+}(O_\gamma(Y)(F \otimes h_{Y\setminus\gamma_-})). \tag{33}$$

Y in (33) is $\pi_s(\overline{\gamma})$ and $O_\gamma(Y)$ is given by the product in (26) with Λ replaced with Y.

To bound $W(\gamma)$, we use (12) and (7), the bounds $\|\ell\| \le 1$, $\|h\| \le C$ (the norm of linear functionals is also denoted by $\|\cdot\|$) and the fact that on bonds b of $Y \times \pi_t(\overline{\gamma})$ that are disjoint from $\overline{\gamma}$, we have identity operators. $\square$

Equation (29) is an example of polymer expansion in statistical mechanics and it is well known (see e.g. [3, 21, 24]) that the bounds (30) will enable us to prove exponential falloff of correlations (i.e. mixing) and construct the $\Lambda \to \mathbf{Z}^d$ limit (in the standard treatments the weights of polymers are scalars and not operators as here, but, as we will see, the combinatorical part of the proof is as in the standard case). We refer the reader not familiar with the combinatorical methods needed in this analysis to the references cited above and to Appendix 2 and just spell out the main steps here.

First we wish to cancel in (29) the contribution from the polymers $\gamma \in \Gamma_v$. Let us call these the vacuum polymers. These are "freely floating" in Λ_{0n-1} unlike the others that are attached to the boundary $\Lambda_0 \cup \Lambda_n$, but they tend to cancel each other. To see this, note that, by (10) with $G = 1$, $\ell_\Lambda \mathcal{P}_\Lambda^k = \ell_\Lambda$ for all k, so, since $\ell(h) = 1$, and $Rh = RQh = 0$ (which implies $W(\gamma)h = 0$ for $\gamma \in \Gamma_0 \cup \Gamma_{0n}$), we get

$$1 = \ell_\Lambda(\mathcal{P}_\Lambda^{n-1} h_\Lambda) = \sum_\Gamma \prod_\gamma \langle W(\gamma)\rangle \tag{34}$$

where Γ is a set of disjoint vacuum polymers in Λ_{0n-1} (note that the vacuum polymers in (29) lie in Λ_{0n-1} too). The cancellation we are after is accomplished by using (34) to write

$$\mathcal{P}_\Lambda^n = (\sum_\Gamma \prod_\gamma \langle W(\gamma)\rangle)^{-1} \mathcal{P}_\Lambda^n \tag{35}$$

and substituting (29). The standard combinatorics (see Appendix 2) now yields

Proposition 4. (35) *can be written as*

$$\mathcal{P}_\Lambda^n = \sum_\Gamma \otimes_{\gamma\in\Gamma_{0n}} V(\gamma) \otimes \Big((h_{\Lambda_+} \otimes_{\gamma\in\Gamma_n} V(\gamma)h)\hat{\otimes}(\ell_{\Lambda_-} \otimes_{\gamma\in\Gamma_0} \ell V(\gamma))\Big) \tag{36}$$

where the sum is over sets Γ (possibly empty) of disjoint polymers γ in Λ_{0n} with $\overline{\gamma} \cap (\Lambda_0 \cup \Lambda_n) \neq \emptyset$ and $\partial\gamma \subset \Lambda_0 \cup \Lambda_n$. The weights satisfy

$$\|| V(\gamma) \||\le \mu^B \prod_{Z\in\gamma} (C\eta)^{|Z|} T(Z, \lambda/2). \tag{37}$$

Notice now that, as $n \to \infty$, the $\gamma \in \Gamma_{0n}$ in (36) give exponentially small contributions due to (37), and (36) will factorize. Let us define a function $\tilde{\Omega}_\Lambda \in \mathcal{H}_\rho^\Lambda$

$$\tilde{\Omega}_\Lambda = \sum_\Gamma h_{\Lambda \backslash \Gamma_+} \otimes_{\gamma \in \Gamma} V(\gamma) h \tag{38}$$

where Γ is a set of finite, mutually disjoint polymers γ in $\Lambda_{-\infty 0}$ with $\overline{\gamma} \cap \Lambda_0 \neq \emptyset$ and $\partial\gamma \subset \Lambda_0$. Similarily, define a linear functional $\tilde{\mathcal{L}}_\Lambda : \mathcal{H}_\rho^\Lambda \to \mathbf{R}$ by

$$\tilde{\mathcal{L}}_\Lambda = \sum_\Gamma \ell_{\Lambda \backslash \Gamma_-} \otimes_{\gamma \in \Gamma} \ell V(\gamma) \tag{39}$$

where γ are now in $\Lambda_{0\infty}$, with $\emptyset \neq \partial\gamma \subset \Lambda_0$. Put

$$\Omega_\Lambda = \frac{\tilde{\Omega}_\Lambda}{\ell_\Lambda(\tilde{\Omega}_\Lambda)}, \quad \mathcal{L}_\Lambda = \ell_\Lambda(\tilde{\Omega}_\Lambda)\tilde{\mathcal{L}}_\Lambda. \tag{40}$$

Then we have

Proposition 5. *(Spectral decomposition of P_Λ). There exist $\varepsilon_0 > 0$, $\mu_1 < 1$, $c < \infty$, independent of Λ, such that, for $\varepsilon < \varepsilon_0$, Ω_Λ and $\mathcal{L}_\Lambda$ have the following properties*

$$\|\Omega_\Lambda\| \leq e^{c|\Lambda|}, \quad \mathcal{L}_\Lambda = \ell_\Lambda \tag{41}$$

$$\mathcal{P}_\Lambda \Omega_\Lambda = \Omega_\Lambda \tag{42}$$

$$|||\ \mathcal{P}_\Lambda^n - \Omega_\Lambda \hat{\otimes} \mathcal{L}_\Lambda\ ||| \leq \mu_1^n\ e^{c|\Lambda|} \tag{43}$$

for all $\Lambda \subset \mathbf{Z}^d$ and $n \in \mathbf{N}$.

Remark. Thus the spectrum of $\mathcal{P}_\Lambda$ consists of the eigenvalue 1 with multiplicity 1 and the rest is in the disc $\{|z| \leq \mu_1\}$, uniformly in Λ.

Proof. We have

$$\ell_\Lambda(\tilde{\Omega}_\Lambda) = \sum_\Gamma \prod_\gamma \langle V(\gamma) \rangle \tag{44}$$

and (using $\|h\| \leq C$)

$$\| \tilde{\Omega}_\Lambda \| \leq C^{|\Lambda|} \sum_\Gamma \prod_\gamma \| V(\gamma) \| .$$

Since $\| \ell \| \leq 1$, $|\langle V(\gamma) \rangle|$ satisfies (37). Now standard estimates (see Appendix 2) give

$$e^{-C|\Lambda|} \leq \ell_\Lambda(\tilde{\Omega}_\Lambda) \leq e^{C|\Lambda|}, \ \ \| \tilde{\Omega}_\Lambda \| \leq e^{C|\Lambda|} \tag{45}$$

which yields the estimate in (41).

Consider next $\mathcal{P}_\Lambda^n - \tilde{\Omega}_\Lambda \hat{\otimes}\ \tilde{\mathcal{L}}_\Lambda$. Using (36),(38) and (39), this is given again by (36), but with different constraints for the set Γ: either there exists a γ such that $\overline{\gamma} \cap \Lambda_0 \neq \emptyset, \overline{\gamma} \cap \Lambda_n \neq \emptyset$ or these exists a pair $\{\gamma, \gamma'\}$ such that $\overline{\gamma} \cap \overline{\gamma}' \neq \emptyset$ and $\overline{\gamma} \cup \overline{\gamma}'$ intersects both Λ_0 and Λ_n. The first set of terms come from $\mathcal{P}_\Lambda^n$ and are not canceled by the corresponding terms in $\Omega_\Lambda \hat{\otimes} \mathcal{L}_\Lambda$ and the second set are the uncanceled terms coming from (38,39). In both cases there is an overall μ^n factor and using (37) and again standard estimates for the combinatorics (see Appendix 2) we get

$$ ||| \, \mathcal{P}^n_\Lambda - \Omega_\Lambda \hat{\otimes} \mathcal{L}_\Lambda \, ||| \leq \mu^n (1+C\eta)^n \, e^{c|\Lambda|}. \tag{46} $$

Since $\eta \to 0$ as $\varepsilon \to 0$, (43) follows with $\mu_1 = \mu(1+C\eta)$. To show $\mathcal{L}_\Lambda = \ell_\Lambda$ recall that, from (10), $\ell_\Lambda \mathcal{P}^n_\Lambda = \ell_\Lambda$ for all n. This together with (43) gives

$$ \ell_\Lambda = \ell_\Lambda(\Omega_\Lambda \hat{\otimes} \mathcal{L}_\Lambda) = \ell_\Lambda(\Omega_\Lambda)\mathcal{L}_\Lambda = \mathcal{L}_\Lambda. $$

To prove (42), use (43) to get

$$ \mathcal{P}^n_\Lambda \Omega_\Lambda \longrightarrow \ell_\Lambda(\Omega_\Lambda)\Omega_\Lambda = \Omega_\Lambda $$

as $n \to \infty$, and then use the continuity of $\mathcal{P}_\Lambda$. □

We will pass shortly to the $\Lambda = \mathbf{Z}^d$ limit but before that we need a more refined mixing condition than (43):

Proposition 6. *There exists $\mu_1 < 1, c < \infty$ such that, if $d\mu_\Lambda = \Omega_\Lambda dm_\Lambda$ denotes the T_Λ invariant measure constructed in Proposition 5, then, for any $X, Y \subset \Lambda$ and $F \in \mathcal{H}^X_\rho$, $G \in \mathcal{H}^Y_\rho$,*

$$ |\int (F \circ T^n_\Lambda) G \, d\mu_\Lambda - \int F \, d\mu_\Lambda \int G \, d\mu_\Lambda| \leq \mu_1^n \, e^{c \min(|X|,|Y|)} e^{-\frac{\lambda}{4} d(X,Y)} \|F\| \, \|G\| \tag{47} $$

Proof. We want to compare

$$ \int (F \circ T^n_\Lambda) G \, d\mu_\Lambda = \ell_\Lambda \left(F \mathcal{P}^n_\Lambda (G \Omega_\Lambda) \right) \tag{48} $$

with

$$ \ell_\Lambda(F\Omega_\Lambda)\ell_\Lambda(G\Omega_\Lambda). $$

To do this, we expand them. Let us denote the terms in the sum (38) by $\widetilde{\Omega}_\Lambda(\Gamma)$ and the ones in (39) by $\widetilde{\mathcal{L}}_\Lambda(\Gamma)$. Then we get, see (40,41),

$$ \ell_\Lambda(F\Omega_\Lambda) = \widetilde{\mathcal{L}}_\Lambda(F\widetilde{\Omega}_\Lambda) = \sum_{\Gamma\Gamma'} \widetilde{\mathcal{L}}_\Lambda(\Gamma) F \widetilde{\Omega}_\Lambda(\Gamma') \tag{49} $$

where Γ is a set in $\Lambda_{0\infty}$, Γ' in $\Lambda_{-\infty 0}$ and their members satisfy $\overline{\gamma}' \cap \Lambda_0 \neq \emptyset$ and $\emptyset \neq \partial\gamma \subset \Lambda_0$.

We want next to group together all γ, γ' that "connect" to X. For this, consider the projections $\pi_s(\overline{\gamma}) \equiv Y_\gamma$ of $\overline{\gamma}$ to Λ and define $Y_{\gamma'}$ similarily. Define any two sets in $\{X, Y_\gamma, Y_{\gamma'}\}$ to be connected if they intersect. Let Y be the connected component including X under this relation and Γ_Y, Γ'_Y the sets of γ and γ' contributing to Y. We may then rewrite (49) as

$$ \ell_\Lambda(F\Omega_\Lambda) = \sum_{Y: X \subset Y} \ell_{\Lambda\backslash Y}(\Omega_{\Lambda\backslash Y}) \sum_{\Gamma_Y, \Gamma'_Y} \widetilde{\mathcal{L}}_Y(\Gamma_Y) F \widetilde{\Omega}_Y(\Gamma'_Y) \tag{50} $$

Since $\ell_Z(\Omega_Z) = 1$ for all Z, (50) equals

$$\ell_\Lambda(F\Omega_\Lambda) = \sum_\Gamma V_F(\Gamma) \tag{51}$$

where Γ is a set of polymers γ in $\Lambda_{-\infty 0}$ or in $\Lambda_{0\infty}$, intersecting Λ_0 and with projections on Λ connected to X in the above sense, and $V_F(\Gamma)$ is the corresponding term in (51). $\ell_\Lambda(G\Omega_\Lambda)$ has a similar expansion where the Γ's are connected to Y, so consider (48). We proceed as above, using (36):

$$\ell_\Lambda\left(F\mathcal{P}^n_\Lambda(G\Omega_\Lambda)\right) = \sum_{\overline{\Gamma}_1\cap\overline{\Gamma}_2=\emptyset} V_F(\Gamma_1)V_G(\Gamma_2) + \sum_\Gamma V_{F,G}(\Gamma) \tag{52}$$

where $\overline{\Gamma} = \bigcup_{\gamma\in\Gamma}\overline{\gamma}$, the first sum comes from the $\Gamma_{0n} = \emptyset$ terms in (36) and, in the second one, we sum over sets Γ with the following properties: $\Gamma = \Gamma_{-\infty 0}\cup\Gamma_0\cup\Gamma_n\cup\Gamma_{0n}\cup\Gamma_{n\infty}$, with by now an obvious notation, and $\Gamma_{0n}\neq\emptyset$; moreover, the projections $\pi_s(\gamma)$ are now connected to X and Y. $V_{F,G}$ is given by

$$V_{F,G}(\Gamma) = \tilde{\mathcal{L}}_Y(\Gamma_{n\infty})F(\otimes_{\gamma\in\Gamma_{0n}}V(\gamma)\otimes(\tilde{\Omega}_Y(\Gamma_n)\hat{\otimes}\tilde{\mathcal{L}}_Y(\Gamma_0)))G\tilde{\Omega}_Y(\Gamma_{-\infty 0}). \tag{53}$$

The bound (47) follows now from (51) and (53): the left hand side of (47) has an expansion like (52) but with $\overline{\Gamma}_1\cap\overline{\Gamma}_2\neq\emptyset$ in the first sum and then each term in both sums contributes $\mu^n\ e^{-\frac{\lambda}{4}d(X,Y)}$ (as in the proof of (43)). The combinatorics is controlled by $e^{c\min(|X|,|Y|)}$ because each Γ must be connected to X and Y (see Appendix 2). $\square$

5 The $\Lambda\longrightarrow\mathbf{Z}^d$ limit: Proof of the Theorem

We have now all the ingredients for the proof of the Theorem. First we describe the invariant measure μ. To do this, rewrite $\tilde{\Omega}_\Lambda$ in (38) as

$$\tilde{\Omega}_\Lambda = \sum_\Gamma \otimes_\gamma \tilde{V}(\gamma)h_\Lambda$$

with $\tilde{V}(\gamma) = \frac{V(\gamma)h_{\gamma_-}}{h_{\gamma_+}}$. Since $h > 0$ on S^1 (Proposition 1), $h\neq 0$ on S_ρ for ρ small enough and thus $\|\tilde{V}\|$ satisfies (37) too. Thus we may exponentiate (see Appendix 2)

$$\tilde{\Omega}_\Lambda = \exp(\sum_\gamma U(\gamma))h_\Lambda, \tag{54}$$

U satisfies

$$\|U(\gamma)\| \leq \mu^{|B|}\prod(C\eta)^{|Y|}T(Y,\frac{\lambda}{3}) \tag{55}$$

Define, for $Y\subset\Lambda$, $H_Y\in\mathcal{H}^Y_\rho$ by

$$H_Y = \sum_{\gamma_+=Y} U(\gamma)$$

where $\gamma \subset \mathbf{Z}^d_{-\infty 0}$ and let $H_{Y\Lambda}$ be given by the same sum with $\gamma \subset \Lambda_{-\infty 0}$. Then

$$\|H_Y\|, \ \|H_{Y\Lambda}\| \leq (C\eta)^{|Y|} T(Y, \frac{\lambda}{4}) \tag{56}$$

and

$$\|H_Y - H_{Y\Lambda}\| \leq (C\eta)^{|Y|} e^{-\frac{\lambda}{8} d(Y, \Lambda^c)} T(Y, \frac{\lambda}{4}) \tag{57}$$

where d denotes the distance. We have

$$d\mu_\Lambda = \frac{\exp(\sum_{Y \subset \Lambda} H_{Y\Lambda}) h_\Lambda dm_\Lambda}{\int \exp(\sum_{Y \subset \Lambda} H_{Y\Lambda}) h_\Lambda \ dm_\Lambda}$$

It is easy to see that there is a unique Gibbs measure corresponding to the Hamiltonian H, for η small. We take μ to be this measure, i.e., the unique Borel probability measure on M with conditional probability densities given by

$$\mu(dm_\Lambda | m_{\Lambda^c}) = \frac{\exp[-\sum_{Y \cap \Lambda \neq \emptyset} H_Y(m)]}{\int \ exp[-\sum_{Y \cap \Lambda \neq \emptyset} H_Y(m)]} h_\Lambda dm_\Lambda \tag{58}$$

for any $\Lambda \subset \mathbf{Z}^d$ finite [23, 24].

The σ_i-invariance follows since the unique Gibbs measure satisfying (58) is translation invariant. For the T-invariance of μ it suffices to show

$$\int F \circ T \ d\mu = \int F \ d\mu$$

for all $F \in \mathcal{H}^X_\rho$, all X finite. Let $T^{(\Lambda)} = T_\Lambda \ \otimes \ f_{\Lambda^c}$ and $d\mu^{(\Lambda)} = d\mu_\Lambda \ \otimes \ h_{\Lambda^c} dm_{\Lambda^c}$; then, $\mu^{(\Lambda)}$ is $T^{(\Lambda)}$ invariant. Moreover $F \circ T$ is continuous and $F \circ T^{(\Lambda)} \to F \circ T$ in the sup norm. (56) and (57) imply that $\mu^{(\Lambda)} \to \mu$ weakly (here $\Lambda \to \mathbf{Z}^d$ is taken in the sense of the net of finite subsets of $\mathbf{Z}^d$, see [23, 24]). Hence,

$$\int F \circ T \ d\mu = \lim \int F \circ T^{(\Lambda)} d\mu^{(\Lambda)} = \int F \ d\mu.$$

Mixing follows from (47) which carries over to the limit:

Proposition 7. *Let τ denote the $\mathbf{Z}^{d+1}$ action generated by T and the σ_i. There exist $\alpha > 0$, $c < \infty$, such that for all finite $X \subset \mathbf{Z}^d$, all F, $G \in \mathcal{H}^X_\rho$, and all $\mathbf{n} \in \mathbf{Z}^{d+1}$*

$$|\int (F \circ \tau^{\mathbf{n}}) G \, d\mu - \int F \, d\mu \int G \, d\mu| \leq e^{-\alpha |\mathbf{n}|} e^{c|X|} \|F\| \ \|G\|.$$

Finally, to prove part 3 of the Theorem, it is enough, by a density argument, to show, for all finite $X \subset \mathbf{Z}^d$, and all $F \in \mathcal{H}^X_\rho$ that

$$\lim_{n \to \infty} \int F \circ T^n dm = \int F d\mu$$

But,

$$\int F \circ T^n dm = \lim_{\Lambda \to \mathbf{Z}^d} \int F \circ T^n_\Lambda dm = \lim_{\Lambda \to \mathbf{Z}^d} \ell_\Lambda (F \mathcal{P}^n_\Lambda 1)$$

and

$$\int F d\mu = \lim_{\Lambda \to \infty} \ell_\Lambda (F \Omega_\Lambda)$$

So, it is enough to show

$$|\ell_\Lambda (F \mathcal{P}^n 1) - \ell_\Lambda (F \Omega_\Lambda)| \leq \mu_1^n \, e^{c|X|} \|F\|$$

with μ_1, c, independent of Λ. But this follows from (47) with $G = h_\Lambda^{-1}$. □

Acknowledgments

We would like to thank L.A. Bunimovich, E. Järvenpää, J. Losson and Y.G. Sinai for interesting discussions. This work was supported by NSF grant DMS-9205296 and by EC grants SC1-CT91-0695 and CHRX-CT93-0411.

Appendix 1: Proof of Proposition 1

We work in terms of the lift of f to the covering space $\mathbf{R}$ of S^1, and denote it by f again. Thus, our assumptions are: There exists a $\rho_0 > 0$, $\gamma > 1$ such that

(a) f is holomorphic and bounded on $S_{\rho_0} = \{z \in \mathbf{C} | |Imz| < \rho_0\}$ with $f(\mathbf{R}) = \mathbf{R}$,

(b) $f(z+1) = f(z) + k \quad z \in S_{\rho_0}, \quad k \in \mathbf{N},\ k \geq 2$,

(c) $f'(x) \geq \gamma \quad x \in \mathbf{R}$.

Hence, by changing ρ_0, γ if necessary, we may assume $f(S_\rho) \supset S_{\gamma\rho} \quad \rho \leq \rho_0$ and $\psi \equiv f^{-1}$ is holomorphic and bounded on $S_{\gamma\rho_0}$ and $|f'(z)| \geq \gamma$, $|\psi'(z)| \leq 1/\gamma$ on S_{ρ_0} and $S_{\gamma\rho_0}$ respectively. Let H_ρ be the space of periodic bounded holomorphic functions on S_ρ, of period one (which can be identified with H_ρ of Sect.2). Since P is given by

$$(Pg)(z) = \sum_{j=0}^{k-1} \frac{g\left(\psi(z+j)\right)}{f'\left(\psi(z+j)\right)}$$

(59)

we get from the above remarks that $P : H_\rho \to H_{\gamma\rho}$ for $\rho \le \rho_0$, which is the claim a) of Proposition 1.

The proof of the spectral decomposition b) goes via finite rank approximations. Let $\psi_q = f^{-q}$ and let $P_{qn} : H_\rho \to H_\rho$ be the finite rank operator

$$(P_{qn}g)(z) = \sum_{j=0}^{k^q-1} [(f^q)'(\psi_q(z+j))]^{-1} \sum_{m=0}^{n-1} \frac{1}{m!} g^{(m)}(\psi_q(j))(\psi_q(z+j)-\psi_q(j))^m.$$

By Taylor's theorem and Cauchy's estimates, we have, for $z \in S_\rho$, $Rez \in [0,1]$,

$$\begin{aligned} |P^q g(z) - P_{qn}g(z)| &\le \frac{\gamma^{-q}}{n!} \sum_j |\psi_q(z+j)-\psi_q(j)|^n \|g^{(n)}\|_{\rho/2} \\ &\le \gamma^{-q(n+1)} k^q C^n \rho^{-n} \|g\|_\rho \end{aligned}$$

where we assume $\psi_q(S_\rho) \subset S_{\rho/2}$, which holds for q large enough and we used $\|\psi_q'\|_\rho \le \gamma^{-q}$ in the second inequality. Since g is periodic, we may restrict ourselves to $Rez \in [0,1]$.

Take now first q large enough such that $C\rho^{-1}\gamma^{-q} \le \frac{1}{2}$ and then n large enough such that $k^q 2^{-n} < \frac{1}{2}$. Then

$$||| \; P^q - P_{qn} \; ||| < \frac{1}{2}.$$

From this we conclude, since P_{qn} is of finite rank, that the spectrum of P^q outside of the disc of radius $\frac{1}{2}$ consists of a finite number of eigenvalues with finite multiplicities. Therefore the same holds for P outside of a disc strictly inside the unit disc. On the other hand it is well known that in a space of L^1-functions of bounded variation, the spectrum of P consists of 1 and a subset of $\{z||z| \le \mu < 1\}$ for a map like the one we are considering. Eigenvalue 1 comes with multiplicity 1 and the eigenvector h is strictly positive [8]. Since eigenvectors in H_ρ are also in L^1, we only need to prove that h is in H_ρ. If this wasn't true, the spectral radius of P in H_ρ would be less than 1 and we would have $P^n 1 \to 0$ in H_ρ, hence in L^1, which is impossible. $\square$

Appendix 2: Combinatorics

We collect here some details on the combinatorical estimates used in the paper.

1. **Proof of (23):**

Inserting (21) into (22), we get

$$||| \; \delta_Y \; ||| \le \tilde{\eta}^{|Y|-1} \sum_{\substack{\Gamma: \\ \overline{\Gamma}=Y}} e^{-\frac{\lambda}{2}\sum_{(i,j)\in\Gamma}[i-j|} \tag{60}$$

where we used $|\overline{\Gamma}| - 1 = |Y| - 1 \leq |\Gamma|$.
Now, associate to each Γ in (60) a tree graph $\tau = \tau(\Gamma)$ with $\overline{\tau} = Y$, and write

$$\sum_{\Gamma} = \sum_{\tau} \sum_{\substack{\Gamma: \\ \tau(\Gamma)=\tau}}$$

and

$$\sum_{(i,j)\in\Gamma} |i-j| \geq |\tau| + \sum_{(i,j)\in\Gamma\setminus\tau} |i-j| \tag{61}$$

Finally, the sum over Γ with $\tau(\Gamma) = \tau$ is bounded by the sum over all choices of lines connecting points of Y, which yields:

$$\sum_{\substack{\Gamma: \\ \tau(\Gamma)=\tau}} e^{-\frac{\lambda}{2}\sum_{(i,j)\in\Gamma\setminus\tau}|i-j|} \leq \prod_{i\in Y}(1 + \sum_{\substack{j\neq i \\ j\in Y}} e^{-\frac{\lambda}{2}|i-j|}) \leq (\sum_{j\in\mathbf{Z^d}} e^{-\frac{\lambda}{2}|j|})^{|Y|} \tag{62}$$

and this proves (23). □

2. **Proof of Proposition 4:**

Let us first consider the denominator in (35). The basic result of the polymer expansion formalism [3, 21, 24] is that (34) can be written as

$$\sum_{\Gamma}\prod_{\gamma}\langle W(\gamma)\rangle = \exp\sum_{\gamma}\tilde{U}(\gamma) \tag{63}$$

whith

$$\tilde{U}(\gamma) = \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\substack{(\gamma_1,\cdots,\gamma_m) \\ \cup_1^m\overline{\gamma_i}=\overline{\gamma}}}\prod_{i=1}^{m}\langle W(\gamma_i)\rangle\sum_{G}\prod_{(i,j)\in G}\chi_{ij} \tag{64}$$

where we sum over sequences of polymers in Λ_{0n-1} (not necessarily disjoint or even distinct) and the sum $\sum_G$ is over all connected graphs with vertices $\{1, \cdots, m\}$ and $\chi_{ij} = 0$ if $\overline{\gamma_i}\cap\overline{\gamma_j} = \emptyset, \chi_{ij} = -1$ if $\overline{\gamma_i}\cap\overline{\gamma_j} \neq \emptyset$, so that this sum vanishes unless γ is connected. $\tilde{U}(\gamma)$ satisfies the bound:

$$|\tilde{U}(\gamma)| \leq \mu^B \prod_{Z\in\gamma}(C\eta)^{|Z|}T(Z, 2\lambda/3). \tag{65}$$

Formulas (63, 64) follow from the polymer formalism, provided we have the bound:

$$\sum_{\substack{\gamma: \\ \overline{\gamma}\ni x}} |\langle W(\gamma)\rangle| \leq C\eta \tag{66}$$

for any $x \in \mathbf{Z}^{d+1}$. To prove (66), we use (30) which holds also for $|\langle W(\gamma)\rangle|$, since $\|\ell\| \leq 1, \|h\| \leq C$. Next, note that

$$\sum_{Z\ni x}(C\eta)^{|Z|}T(Z,\lambda) \leq \sum_{n\geq 0}\frac{(C\eta)^{n+1}}{n!}\sum_{(z_1,\cdots,z_n)}\sum_{\tau} e^{-\frac{\lambda}{2}|\tau|} \tag{67}$$

where $(z_1, \cdots, z_n)$ is a sequence of mutually distinct $z_i \in \mathbf{Z}^d$, and the last sum is over tree graphs on $\{x, z_1, \cdots, z_n\}$. Now, for each fixed tree, the sum over $z_1, \cdots, z_n$ is bounded by $(C\lambda^{-d})^n$: we start by summing over vertices with incidence number one, remove those vertices, get a new tree and iterate (i.e., we "roll back" the tree). The number of tree graphs is bounded by $c^n n!$ so that (67) is bounded by $C\eta$.

¿From this we obtain (66) by repeating the argument in (62), so as to reduce the sum in (66) to a sum over trees whose vertices are now sets $Z_i \in S$, and the edges carry powers of μ. Then the sum over the trees is done as in (67), using the fact that the edges are now one-dimensional (only in the "time" direction) and that $\sum_{n=1}^{\infty} \mu^n = \frac{\mu}{1-\mu} < \infty$. Of course we need to choose η, and therefore ε, so that $\frac{\eta}{1-\mu}$ is small enough. This finishes the proof of (66), hence of (63,64).

To prove (65), we use (64) and sum first over the graphs G corresponding to a given tree as in the proof of (62). Then one has to control the fact that, since γ in (64) can be written in many ways as a union of γ_i's, each Z in γ can also be decomposed in many ways (each time-like bond in γ can also occur several times in the bonds of the γ_i's, but this brings extra powers of μ, and we use $\sum_{n=1}^{\infty} \mu^n < \infty$). Let us write $\eta = \hat{\eta}(\eta/\hat{\eta})$, and $T(Z, \lambda) \le T(Z, \frac{\lambda}{3}) T(Z, \frac{2\lambda}{3})$. For any term in (64), we have

$$\prod_{i=1}^{n} \prod_{Z \in \gamma_i} T(Z, \frac{2\lambda}{3}) \le \prod_{Z \in \gamma} T(Z, \frac{2\lambda}{3})$$

and the factor $\frac{1}{\hat{\eta}}$ can be absorbed into the constant C in (65). Then, the sum over all ways of decomposing γ in (64) is bounded by

$$\prod_{Z \in \gamma} \Big(\sum_{n=1}^{\infty} \sum_{\substack{(Z_i)_{i=1}^n \\ \cup Z_i = Z}} \prod_{i=1}^{n} \hat{\eta}^{|Z_i|} T(Z_i, \frac{\lambda}{3}) \Big) \le \prod_{Z \in \gamma} \prod_{Y \subset Z} \Big(1 + \sum_{n=1}^{\infty} (\hat{\eta}^{|Y|} T(Y, \frac{\lambda}{3}))^n \Big) \tag{68}$$

This in turn is bounded by

$$\prod_{Z \in \gamma} \exp\left(\sum_{Y \subset Z} \sum_{n=1}^{\infty} (\hat{\eta}^{|Y|} T(Y, \frac{\lambda}{3}))^n \right) \le \prod_{Z \in \gamma} \exp(C\hat{\eta}|Z|) \tag{69}$$

using the fact that the sum (67), and therefore each term in that sum, is bounded by $(C\eta)$; this holds for any $\lambda > 0$, for η small. So we may replace λ by $\frac{\lambda}{3}$ and η by $\hat{\eta}$ and obtain (69). This establishes (65).

Now, turning to the numerator in (35), write

$$\mathcal{P}_{\Lambda}^{n} = \sum_{\Gamma} \varphi(\Gamma) \sum_{\Gamma_v} \prod_{\gamma \in \Gamma_v} \langle W(\gamma) \rangle \tag{70}$$

where the sum over Γ has the same constraints as in Proposition 4, $\varphi(\Gamma)$ denotes the product in (29) (with $\Gamma_v = \emptyset$) and the sum over Γ_v runs over sets of disjoint vaccum polymers in Λ_{0n-1} so that $\overline{\gamma}\cap\overline{\gamma}' = \emptyset, \forall\gamma\in\Gamma, \forall\gamma'\in\Gamma_v$. Applying the polymer formalism to this last sum, we get

$$\sum_{\Gamma_v}\prod_{\gamma\in\Gamma_v}\langle W(\gamma)\rangle = \exp(\sum_{\substack{\gamma:\\ \overline{\gamma}\cap\overline{\Gamma}=\emptyset}}\tilde{U}(\gamma)), \tag{71}$$

and, using (63), (70) becomes

$$\mathcal{P}^n_\Lambda = \sum_{\Gamma}\varphi(\Gamma)\exp(-\sum_{\substack{\gamma:\\ \overline{\gamma}\cap\overline{\Gamma}\neq\emptyset}}\tilde{U}(\gamma)) \tag{72}$$

with $\overline{\Gamma} = \bigcup_{\gamma\in\Gamma}\overline{\gamma}$. Now, we expand the exponential in (72), and combine each of the terms with Γ. Concretely, write (72) as

$$\sum_{\Gamma}\varphi(\Gamma)\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{(\gamma_i)_1^n}\prod_{i=1}^{n}(-\tilde{U}(\gamma_i))$$

where $\overline{\gamma_i}\cap\overline{\Gamma}\neq\emptyset, \forall i$. This can be written as

$$\sum_{\Gamma}\varphi(\Gamma)\sum_{(n_\gamma)}\prod_{\gamma}\frac{(-\tilde{U}(\gamma))^{n_\gamma}}{n_\gamma!} \tag{73}$$

where the product runs over all polymers (the number of polymers is finite for n and Λ finite) and $n_\gamma\in\mathbf{N}$. Now, decompose

$$\Gamma\cup\{\gamma|n_\gamma\neq 0\} = \cup\gamma_i'$$

into mutually disjoint polymers and define $V(\gamma')$, for a polymer γ', to be given by the sum (73), with the constraint

$$\Gamma\cup\{\gamma|n_\gamma\neq 0\} = \gamma' \tag{74}$$

Since $\tilde{U}(\gamma)$ is a number, this does not change the type of operators being considered. Since the constraints on the sum over Γ in (73) are the same as in (36), we have established (36).

The bound (37) follows from (30) applied to the factors in $\varphi(\Gamma)$ and to $\tilde{U}(\gamma)$. The sum over all possible decompositions of γ' in (74) can be controlled by using (68,69), and by going from $2\lambda/3$ to $\lambda/2$. □

3. **Proof of Propositions 5 and 6**

The inequalities (45) on $\ell_\Lambda(\tilde{\Omega}_\Lambda)$ follow from (44) and the bounds (37) on $|||\ V(\gamma)\ |||$, which hold also for $|\langle V(\gamma)\rangle|$: the polymer formalism allows us to write (44) as $\exp(\sum_\gamma\hat{V}(\gamma))$, which is similar to (63,64) but with $\overline{\gamma}\cap(\Lambda_0\cup\Lambda_n)\neq\emptyset$. Then, a bound like (65) implies (45).

To prove (46), consider the terms where $\Gamma \ni \gamma$, with $\overline{\gamma} \cap \Lambda_0 \neq \emptyset, \overline{\gamma} \cap \Lambda_n \neq \emptyset$. Using (36,37), we bound them by

$$\exp(c|\Lambda|) \sum_{\gamma} \mu^B \prod_{Z \in \gamma} (C\eta)^{|Z|} T(Z, \lambda/2)$$

where $e^{c|\Lambda|}$ controls the sum over $\Gamma \backslash \gamma$ (as in (45)). Next, we extract a factor μ^n from μ^B and we control the sum over γ by $(1+c\eta)^n$ as follows: To each γ, associate a tree by choosing, for each time $t = 1, \cdots, n$, a set Z_t (possibly empty) and a time-like line joining successive non empty Z's. The sum over the rest of γ is handled as in (62). The choice of the lines fixes the "origins", x_t of Z_t. So, we have $|Z_t|$ choices for each line and $|\Lambda|$ choices for the intersection of γ with Λ_0. These latter factors can be absorbed in $e^{c|\Lambda|}$ or $C^{|Z|}$ and $(1+c\eta)^n$ is then an upper bound on

$$\prod_{t=1}^{n} (1 + \sum_{\substack{Z \subset \Lambda_t \\ x_t \in Z}} (C\eta)^{|Z|} T(Z, \frac{\lambda}{2}))$$

for any choice of $\{x_t\}$ (as in the bound on (67)). The other terms contributing to the LHS of (46) are bounded in a similar way.

The bound on (53) leading to (47) is also similar. We get, however, $e^{c\min(|X|,|Y|)}$ instead of $e^{c|\Lambda|}$ (which is crucial) because here all contributing polymers are "connected" to X and Y. The connection is defined through the projection of $\overline{\gamma}$ on Λ, but it is easy to see that we may again reduce the estimates to a sum over tree graphs and that for two polymers γ, γ' such that $\pi_s(\overline{\gamma}) \cap \pi_s(\overline{\gamma}') \neq \emptyset$ and $\overline{\gamma} \cap \Lambda_0 \neq \emptyset, \overline{\gamma}' \cap \Lambda_0 \neq \emptyset$, we have that

$$d(\overline{\gamma}', \overline{\gamma}) \leq \sum_{Z \in \gamma'} [Z|. \tag{75}$$

So, for γ fixed, the sum over γ' with $\pi_s(\overline{\gamma}) \cap \pi_s(\overline{\gamma}') \neq \emptyset$ can be bounded by

$$\sum_{x} e^{-d(x,\overline{\gamma})} \sum_{\overline{\gamma}' \ni x} \mu^B \prod_{Z \in \gamma'} (C\eta)^{|Z|} T(Z, \lambda/2)$$

where we used (75) to absorb a factor $e^{d(x,\overline{\gamma})}$ into $\prod_{Z \in \gamma'} C^{|Z|}$.
Finally, in (55, 56, 57), we perform resummations, using (68,69), which explains why we have smaller fractions of λ.

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