Invariant Measures for a Two Species Asymmetric Process

P. A. Ferrari, L. R. G. Fontes
Y. Kohayakawa

Universidade de São Paulo

Summary. We consider a process of two classes of particles jumping on a one dimensional lattice. The marginal system of the first class of particles is the one dimensional totally asymmetric simple exclusion process. When classes are disregarded the process is also the totally asymmetric simple exclusion process. The existence of a unique invariant measure with product marginals with density ρ and λ for the first and first plus second class particles, respectively, was shown by Ferrari, Kipnis and Saada (1991). Recently Derrida, Janowsky, Lebowitz and Speer (1993) and Speer (1994) have computed this invariant measure for finite boxes and performed the infinite volume limit. Based on this computation we give a complete description of the measure and derive some of its properties. In particular we show that the invariant measure for the simple exclusion process as seen from a second class particle with asymptotic densities ρ and λ is equivalent to the product measure with densities ρ to the left of the origin and λ to the right of the origin.

Keywords and phrases. Two species process. Asymmetric simple exclusion. Second class particles.

AMS 1991 Classification. 60K35, 82C22, 82C24, 82C41.

Short title: Two species asymmetric process.

1. Introduction

The simplest way of defining the two species system is by using the basic coupling of the totally asymmetric simple exclusion process (SEP). We define the *simple exclusion process* $\eta_t \in \{0,1\}$ $(t \geq 0)$ as follows. At each site $x \in$ we attach a random clock that rings according to a Poisson process of parameter 1. The clocks are mutually independent. When the clock of an occupied site x rings, if $x+1$ is empty, the particle at x jumps to $x+1$. If $x+1$ is occupied, nothing happens. Thus, in this process, the particles are generally drifting to the right. If one considers two initial configurations η^1 and $\eta^2 \in \{0,1\}$ such that $\eta^1(x) \leq \eta^2(x)$ for all x and uses the same clocks for both realizations, then one has a coupled process (η_t^1, η_t^2) with the property that $\eta_t^1(x) \leq \eta_t^2(x)$ for all x and all $t \geq 0$ (cf. Liggett (1976, 1985)). The two species process (σ_t, ξ_t) $(t \geq 0)$ is defined by putting $\sigma_t(x) = \eta_t^1(x)$ and $\xi_t(x) = \eta_t^2(x) - \eta_t^1(x)$. The σ particles are the *first class* particles and the ξ particles are the *second class* particles. The reason for these terms is that, when a clock rings for a first class particle at site x and a second class particle is at site $x+1$, the particles interchange positions, whereas if a second class particle is at x and a first class particle is at $x+1$, they do *not* move. It is easy to see that the two species process is Markovian.

In this paper we are concerned with the invariant measures for the two species process. Since the marginal processes σ_t and $\sigma_t+\xi_t$ are simple exclusion processes, the corresponding marginal measures of any invariant measure for the two species process must be invariant for the SEP. Now, the invariant measures for the SEP are convex combinations of the product measures ν_ρ with density $\rho \in [0,1]$ and the blocking measures concentrated on the configuration $\ldots 000111 \ldots$ and its translates. Let us say that a distribution of (σ, ξ) has *good marginals* if, for some $\rho \leq \lambda$, its σ marginal is ν_ρ and its $\sigma + \xi$ marginal is ν_λ. It is easy to construct a product measure π_2 for (σ, ξ) with good marginals. Ferrari, Kipnis and Saada (1991) proved that, for the two species process (σ_t, ξ_t), there exists a unique invariant measure μ_2 with good marginals, and that the process started with the product measure π_2 converges to μ_2 as $t \to \infty$. Derrida, Janowsky, Lebowitz and Speer (1993) have recently computed the invariant measure μ_2 in finite boxes and, performing the infinite volume limit, they have investigated μ_2. In a sequel, Speer (1994) makes this approach rigorous.

One important fact discovered by Derrida, Janowsky, Lebowitz and Speer (1993) is that, under the invariant measure μ_2, the distribution to the right of a second class particle is *independent* of the distribution to its left. This suggests studying the process 'as seen from a second class particle'. To make this precise, assume that at time $t = 0$ there is a second class particle at the origin and let X_t be its position at time t. The process *as seen from this second class particle* is $(\tau_{X_t}\sigma_t, \tau_{X_t}\xi_t)$, where τ_x denotes translation by x. Thus, $(\tau_{X_t}\sigma_t(x), \tau_{X_t}\xi_t(x)) = (\sigma_t(x + X_t), \xi_t(x + X_t))$ for all $x \in$. Now assume that the initial distribution μ of the two species process is translation invariant, and that it has a positive density of second class particles. Then, at time t, the process as seen from a second class particle, started with the measure μ conditioned to having a second class particle at the origin, has the same distribution as the two species process started with the unconditioned measure μ, but itself conditioned to having a second class particle at the origin at time t. This means that, when the density of second class particles is positive, the invariant measures for the two species process have a corresponding invariant measure for the process as seen from a second class particle (see Ferrari, Kipnis and Saada (1991)). But in fact the process as seen from a second class particle is richer: it has invariant measures with only a finite number of second class particles with no corresponding measure in the two species process.

Our main contribution is a complete description of the invariant measures for the two species process as seen from a fixed second class particle. This description is based on computations in Derrida, Janowsky, Lebowitz and Speer (1993). We consider two densities $0 < \rho \leq \lambda < 1$ and construct a measure $\mu_2' = \mu_2'(\rho, \lambda)$ that is invariant for our process (see Theorem 1). The parameters correspond to the asymptotic densities (as $x \to \pm\infty$) of the first and the first plus second class particles. The cases $\rho = 0$ or $\lambda = 1$ are easier and were considered before. In the particular case in which $\rho = 0$ or $\lambda = 1$, either the σ marginal or the $\sigma + \xi$ marginal is trivial. Moreover, in this case, the other marginal is trivial if $\rho = \lambda$ and it is the SEP if $\rho < \lambda$. In the latter case, the process corresponds to the SEP as seen from a tagged particle, as studied by Ferrari (1986) and De Masi, Kipnis, Presutti and Saada (1990). An important general property of the measure μ_2' is 'translation invariance', in the sense that it is the same seen from *any* second class particle. When $\rho < \lambda$ this 'translation invariance' implies that there exists a unique translation invariant μ_2 such that μ_2' is μ_2 conditioned to having a second class particle at the origin. When $\lambda = \rho$, the average distance between two successive second class particles is infinite. This implies that there is no translation invariant measure μ_2 such that μ_2' is μ_2 conditioned to having a second class particle at the origin.

Recall that our particles are drifting to $+\infty$. The definition of the two species process is such that a first class particle can overtake a second class one, but the other way around is prohibited. Let us start the process with a second class particle at the origin, and, along the evolution of the process, we refer to this particle as the *0-th second class particle*. We consider the second class particles ordered from left to right, so that we may speak of the i-th second class particle for any $i \in$. If one identifies the two classes of particles starting from the i-th second class particle ($i > 0$), one has an operator Φ_i acting on the configurations (σ, ξ), which commutes with the semigroup corresponding to the evolution. Similarly, for any fixed $j < 0$, one can identify *holes*, i.e. empty sites, with second class particles starting from, and to the left of, the j-th second class particle to obtain an operator Ψ_j that also commutes with the semigroup. Hence, applying any (or both) of these operators to the 'translation invariant' stationary measure μ_2', we obtain another invariant measure (see Theorem 2). Incidentally, these new measures are clearly not 'translation invariant'. Now, identifying first and second class particles to the right of the particle at the origin and holes and second class particles to the left of it, we obtain the invariant measure for the process as seen from a single, isolated second class particle. When $\rho < \lambda$ this corresponds to a shock in the SEP (Ferrari, Kipnis and Saada (1991), Ferrari (1992)). When $\rho = \lambda$ there is a reminiscence of the shock, as the density to the right of the second class particle is bigger than the density to the left of it and the approach to the asymptotic density λ, which equals ρ here, is slow (Derrida, Janowsky, Lebowitz and Speer (1993)). If one makes the identification for all but two second class particles, one gets that the distance d between the two second class particles is, following the terminology of Derrida, Janowsky, Lebowitz and Speer (1993), a 'bounded state' even when $\lambda = \rho$. Indeed, the distribution of d is the same as the distribution of the distance between two successive second class particles under μ_2'. It turns out that this distance has the same distribution as the hitting time of 1 for a nearest neighbor random walk with jumps in $\{-1, 0, 1\}$ with probabilities $\rho(1-\lambda)$, $1-\lambda(1-\rho)-\rho(1-\lambda)$ and $\lambda(1-\rho)$ respectively (see Lemma 2.5).

Our approach relies on the work of Derrida, Janowsky, Lebowitz and Speer (1993) and Speer (1994) but we work directly in the infinite volume. In Section 3 we describe completely the measure μ_2' and show that it is invariant for the process as seen from a fixed second class particle. Derrida, Janowsky, Lebowitz and Speer (1993) state the following remarkable property of the measure μ_2': the distribution of first class particles to the right of the tagged second class particle is the product measure ν_ρ with density ρ while the distribution of empty sites to the left of the tagged second class particle is the product measure $\nu_{1-\lambda}$ with density $1-\lambda$. Speer (1994) proves this statement and here we give an alternative proof of this fact by showing that one may construct μ_2' as follows. We first put a second class particle at the origin and distribute the first class particles to the right of the origin according to the product measure ν_ρ. Then we give a recipe for deciding where to put the second class particles among the unnoccupied sites. To the left of the second class particle at the origin the positions of the empty sites are chosen according to the product measure $\nu_{1-\lambda}$ and a similar recipe is used to decide where to put the second class particles. (See Proposition 1.)

When $\lambda > \rho$ there exists a unique translation invariant measure μ_2 with the property that it coincides with μ_2' when it is conditioned to having a second class particle at the origin. As explained above, the invariance of μ_2' for the process as seen from the second class particle implies that the measure μ_2 is invariant for the two species process. Using the property that the first class particles to the right of the tagged second class particle are distributed according to a product measure, we show that μ_2 has good marginals (cf. Theorem 3). This already followed from the infinite volume limit of Derrida, Janowsky, Lebowitz and Speer (1993) but in a somewhat indirect way. We also show that it is possible to construct a coupling $\tilde{\mu}$ with marginals μ_2' and μ_2 in such a way that the number of sites where the two marginals differ is a random variable with a finite exponential moment. (See Theorem 4.)

Let $\nu_{\rho,\lambda}$ be the product measure with density ρ to the left of the origin and density λ to the right of the origin. Using the results of Ferrari, Kipnis and Saada (1991), Ferrari (1992) proved that the SEP as seen from a second class particle starting with the product measure $\nu_{\rho,\lambda}$ presents a shock: uniformly in time the asymptotic densities are ρ and λ to the left and right of the origin, respectively. Indeed the process with a unique second class particle, started at the origin, with initial product distribution $\nu_{\rho,\lambda}$ can be coupled to the two species process with initial product distribution π_2 (with marginals ν_ρ and ν_λ) in such a way that at all times the single second class particle of the first process has the same position as the tagged second class particle in the second process. As mentioned above, this can be done by identifying first and second class particles to the right of the origin and empty sites and second class particles to the left of the origin. Applying the results for the two species process to the shock in the SEP, Derrida, Janowsky, Lebowitz and Speer (1993) have computed the rate of convergence of the density of the shock to the asymptotic densities ρ and λ. We make a further step proving that the invariant measure μ' for the process as seen from a single second class particle has the following property. One may construct a coupling between μ' and $\nu_{\rho,\lambda}$ in such a way the the number of sites where the two marginals differ is a random variable with a finite exponential moment. This implies in particular that μ' is equivalent to $\nu_{\rho,\lambda}$. (See Theorem 5 and its corollary.)

Let us now mention some related results. Speer (1994) have described the set of all invariant measures for the two species process and has shown that the invariant measure μ_2 is not Gibbsian. Ferrari and Fontes (1993) have computed the asymptotic variance of the position of the second class particle for the process with initial distribution μ_2', and they have studied the density fluctuation fields for the exclusion process with a shock initial condition.

This article is organised as follows. In the next section we prove three basic lemmas (Lemmas 2.1, 2.2, and 2.3) that are used in later sections. In Section 3 we give our construction of the measure μ_2', and we prove Theorem 1, which asserts that μ_2' is invariant for the process $(\tau_{X_t}\sigma_t, \tau_{X_t}\xi_t)$. Also in this section are Theorem 2, concerning other invariant measures constructed from μ_2' with the aid of the operators Φ_i and Ψ_j, and Proposition 1. In Section 4 we deal with the invariant measure μ_2 for the process (σ_t, ξ_t), and prove that it has good marginals (cf. Theorem 3). In that section we also prove Theorem 4, concerning the coupling $\tilde{\mu}$ between μ_2 and μ_2' mentioned above. The last section is devoted to proving Theorem 5, on the coupling between μ' and $\nu_{\rho,\lambda}$.

2. A distribution on the set of finite configurations

Let $\mathbf{Y}$ be the space of *finite configurations* of 0s and 1s, i.e.

$$\mathbf{Y} = \bigcup_{n\geq 0} \{0,1\}^n = \{\emptyset, 0, 1, 00, 01, 10, 11, 000, \ldots\}.$$

Usually, we think of a sequence ζ in $\mathbf{Y}$ of length n as indexed by $\{1,\ldots,n\}$. In this section we define and study a certain probability distribution p on the space $\mathbf{Y}$. This distribution will be used in the next section to construct the invariant measure for the system as seen from a second class particle.

Let $\zeta \in \mathbf{Y}$ be given. We write $N(\zeta)$ for the length of ζ, and $K(\zeta)$ for the number of 1s in ζ. Formally, we have $N(\zeta) = n$ if and only if $\zeta \in \{0,1\}^n$, and $K(\zeta) = \sum_{x=1}^{N(\zeta)} \zeta(x)$. An important definition that we shall need is the following. For $\zeta \in \mathbf{Y}$, let $M(\zeta)$ be the number of *distinct configurations that can be obtained from ζ by shifting ones to the right, including ζ itself*. Thus, for example, we have $M(100) = 3$, $M(0011) = 1$, and $M(1010) = 5$.

We may now define the distribution p on $\mathbf{Y}$. In fact, we shall define a distribution $p = p_{\rho,\lambda}$ for each $0 < \rho \leq \lambda < 1$. Let ρ and λ as above be fixed. Given $\zeta \in \mathbf{Y}$, we put

$$p(\zeta) = p_{\rho,\lambda}(\zeta) = \lambda(1-\rho)M(\zeta)(\lambda\rho)^{K(\zeta)}\big((1-\lambda)(1-\rho)\big)^{N(\zeta)-K(\zeta)}. \tag{2.1}$$

We show in Lemma 2.1(i) below that p does indeed define a probability distribution over $\mathbf{Y}$. It is with the aid of $p = p_{\rho,\lambda}$ that we shall construct the invariant measure for the two species process as seen from a second class particle, when the asymptotic densities of the first class particles and the first plus second class particles are respectively ρ and λ.

The rest of this section is devoted to proving that p gives a probability measure over $\mathbf{Y}$ and to the study of some simple properties of the space $(\mathbf{Y}, p)$ and of the function $M(\zeta)$ ($\zeta \in \mathbf{Y}$). In particular, we shall consider the random variable $N = N(\zeta)$, that is, the random length of a sequence ζ drawn from $\mathbf{Y}$ according to p. The main results of this section are given in Lemmas 2.1, 2.2 and 2.3, which we now state.

Lemma 2.1. *Let ρ, $\lambda \in (0,1)$ be fixed. Then (i) $\sum_{\zeta\in\mathbf{Y}} p(\zeta) = 1$ if and only if $\rho \leq \lambda$. Assuming that $\rho \leq \lambda$ and, writing $= _{\rho,\lambda}$ for the expectation in $(\mathbf{Y}, p)$, we have: (ii) if $\rho < \lambda$ then $(N+1) = 1/(\lambda-\rho)$, and (iii) if $\rho = \lambda$ then $(N+1) = \infty$. Finally, (iv) if $\rho < \lambda$, then N has a finite exponential moment. In other words, there exists $\theta > 0$ such that*

$$e^{\theta N} < \infty. \tag{2.2}$$

The distribution of the random variable N is given in Lemma 2.5 below. The generating function of N is given in the following lemma.

Lemma 2.2. *Let $\rho \le \lambda$. The generating function of N is given by*

$$s^N = \frac{1}{2as^2}\left\{1 - cs - \sqrt{(1-cs)^2 - 4abs^2}\right\}, \tag{2.3}$$

where $a = \rho(1-\lambda)$, $b = \lambda(1-\rho)$ and $c = 1-a-b$. The closed disc $|s| \le \left\{1 - (\sqrt{b} - \sqrt{a})^2\right\}^{-1}$ is its domain of convergence.

Our next lemma, Lemma 2.3, is inspired by Derrida, Janowsky, Lebowitz and Speer (1993).

Lemma 2.3. *For all ζ, $\gamma \in \mathbf{Y}$, we have $M(\zeta 1 0 \gamma) = M(\zeta 1 \gamma) + M(\zeta 0 \gamma)$.*

We now define a random walk on the integers that will be important in the sequel. Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ be independent 0–1 random variables with $(X_i) = \lambda$ and $(Y_i) = \rho$ $(i \ge 1)$. Put $Z_i = X_i - Y_i$ $(i \ge 1)$, and let $\widetilde{Z}_n = \sum_{1 \le i \le n} Z_i$ $(n \ge 0)$. Note that then $(\widetilde{Z}_n)_0^\infty$ is a random walk on , and let $T = \inf\{n \ge 0 : \widetilde{Z}_n = 1\}$ be the hitting time of the event $\{\widetilde{Z}_n = 1\}$.

The rest of this section is devoted to prove the lemmas above. The other sections of this note may be read independently from what follows. Our first auxiliary lemma is the following

Lemma 2.4. *For all integers n, $k \ge 0$, we have*

$$\sum M(\zeta) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k}, \tag{2.4}$$

where the sum ranges over all $\zeta \in \mathbf{Y}$ with $N(\zeta) = n$ and $K(\zeta) = k$.

We defer the proof of Lemma 2.4 until later, and pass on to a result that is crucial in the proof of Lemma 2.1(i).

For a finite configuration $\zeta \in \mathbf{Y}$, recall that $N(\zeta)$ denotes its length and $K(\zeta)$ its number of 1s. For integers n and k, set $p_{n,k} = \sum_\zeta p(\zeta)$, where the sum ranges over all $\zeta \in \mathbf{Y}$ with $N(\zeta) = n$ and $K(\zeta) = k$. Note that once we know that p is a probability measure on $\mathbf{Y}$, the quantity $p_{n,k}$ is simply the probability that a random configuration $\zeta \in \mathbf{Y}$ has length n and k elements equal to 1. In particular, the lemma below in this case simply states that $P\{N = n\} = P\{T = n+1\}$.

Lemma 2.5. *Let λ, $\rho \in (0,1)$ be fixed. Then, for any $n \geq 0$, we have*

$$\sum_k p_{n,k} = \{T = n+1\}. \tag{2.5}$$

Proof. By Lemma 2.4, we have

$$\begin{aligned} p_{n,k} &= \frac{1}{n+1}\binom{n+1}{k+1}\lambda^{k+1}(1-\lambda)^{n-k}\binom{n+1}{k}\rho^k(1-\rho)^{n+1-k} \\ &= \frac{1}{n+1}(W_1 = k+1)(W_2 = k), \end{aligned}$$

where W_1 and W_2 are two independent binomial random variables with parameters $n+1$ and λ and $n+1$ and ρ, respectively. Now, summing over all k, for $n \geq 0$ we have

$$\sum_k p_{n,k} = \frac{1}{n+1}\{W_1 - W_2 = 1\} = \frac{1}{n+1}\{\widetilde{Z}_{n+1} = 1\}, \tag{2.6}$$

where $(\widetilde{Z}_n)_0^\infty$ is the random walk introduced above. On the other hand, recalling that T is the hitting time of 1 for that walk, we have

$$\{T = n+1\} = \frac{1}{n+1}\{\widetilde{Z}_{n+1} = 1\} \tag{2.7}$$

for all integers $n \geq 0$. Identity (2.7) is Exercise (IV.12) of Spitzer (1976), but for completeness we give a combinatorial proof for it in Lemma 2.6 below. Lemma 2.5 follows from (2.6) and (2.7). $\square$

Remark 2.1. The Local Central Limit Theorem (or direct calculations) and (2.7) imply that, when $0 < \rho = \lambda < 1$, we have $(T = n) = (c + o(1))n^{-3/2}$ as $n \to \infty$, where $c = c(\rho) > 0$ depends only on ρ. For the case in which $\rho < \lambda$, see Remark 2.2.

Let us now prove (2.7). The proof below is entirely combinatorial and more elementary than the one suggested in Spitzer (1976), which is based on Lagrange's inversion formula.

Lemma 2.6. *Let $(V_i)_1^\infty$ be a family of i.i.d. $\{\pm 1, 0\}$-random variables and let $\widetilde{V}_n = \sum_{1 \leq i \leq n} V_i$ $(n \geq 0)$ be the associated random walk on . Let $T = \inf\{n : \widetilde{V}_n = 1\}$ be the hitting time of $\{\widetilde{V}_n = 1\}$. Then $\{T = n\} = n^{-1}\{W_n = 1\}$ for all integers $n \geq 1$.*

Proof. We deduce this result from a lemma of Raney (1960) (see also Example 4 in Section 7.5 of Graham, Knuth, and Patashnik (1989)): *if $\mathbf{x} = (x_1, \ldots, x_n)$ is a sequence of integers with $\sum_{1 \leq i \leq n} x_i = 1$, then there is a unique cyclic permutation of $\mathbf{x}$,* say $(x_j, x_{j+1}, \ldots, x_n, x_1, \ldots, x_{j-1})$, *all of whose proper initial partial sums are non-positive, i.e. such that* x_j, $x_j + x_{j+1}, \ldots, x_j + \cdots + x_{j-2} \leq 0$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a $\{\pm 1, 0\}$-sequence with $\sum_{1 \leq i \leq n} x_i = 1$, and let $E_{\mathbf{x}}$ be the event that $(V_1, \ldots, V_n)$ is a cyclic permutation of $\mathbf{x}$. It is simple to check, and in fact it follows from Raney's lemma, that all the n cyclic permutations of $\mathbf{x}$ are distinct. Also, clearly, the probability that $(V_i)_1^n$ is any of these n permutations is $(1/n)(E_{\mathbf{x}})$. Now, by Raney's lemma, exactly one of these permutations corresponds to the event $\{T = n\}$, and hence Lemma 2.6 follows. $\square$

We may now prove Lemma 2.1, the first main result of this section.

Proof of Lemma 2.1. (i) We need to prove that $\sum_{n,k} p_{n,k} = 1$ if and only if $\rho \leq \lambda$. In view of (2.5), we have $\sum_{n,k} p_{n,k} = \sum_n (T = n+1) = (T < \infty)$, where T is the hitting time of 1 for the walk $(\widetilde{Z}_n)_1^\infty$ defined just after Lemma 2.3. It now suffices to notice that $T < \infty$ almost surely if and only if the walk $(\widetilde{Z}_n)_0^\infty$ has non-negative drift. This proves (i).

We assume from now on that $\rho \leq \lambda$, and rewrite (2.5) as $\{N = n\} = \{T = n + 1\}$ $(n \in \ )$. Let us now prove (ii). Suppose that $\rho < \lambda$. Then, again considering the random walk $\widetilde{Z}_n = \sum_{1 \leq i \leq n} Z_i$ and the hitting time T, by Wald's identity we obtain $1 = (\widetilde{Z}_T) = (Z_i)(T) = (\lambda - \rho)(T)$, and hence $(N + 1) = (T) = 1/(\lambda - \rho)$, as required.

To see (iii), note that for $\lambda = \rho$ the expected hitting time T is infinite. Finally, to prove (iv), we prove that $\{N = n\}$ decays exponentially with n. By (2.5) and (2.7), we have $\{N = n\} = \{T = n + 1\} = (n + 1)^{-1}\{\widetilde{Z}_{n+1} = 1\} \leq \{\widetilde{Z}_{n+1} = 1\}$ for all integers $n \geq 0$. But then it suffices to notice that this last probability is exponentially small, since $(Z_i) = \lambda - \rho > 0$. Indeed, if n is large enough with respect to $\lambda - \rho$, we have that

$$(\widetilde{Z}_{n+1} = 1) \leq \exp\left\{-\frac{(\lambda - \rho)^2}{5(\lambda - \rho + 1)} n\right\} \tag{2.8}$$

by Hoeffding's inequality. (See Hoeffding (1963) or McDiarmid (1989).) $\square$

Proof of Lemma 2.2. The result is obtained by a standard application of Wald's identity to the stopping time T (see Breiman (1968)) and standard analytic continuation arguments. $\square$

Remark 2.2. It follows from Lemma 2.2 that $\{N = n\}$ decays a little faster than it is suggested in (2.8) in the proof of Lemma 2.1(iv). The rate of exponential decay of the distribution of N when $\lambda > \rho$ is given by

$$\limsup_{n \to \infty} \{N = n\}^{1/n} = 1 - \left\{\sqrt{\lambda(1 - \rho)} - \sqrt{\rho(1 - \lambda)}\right\}^2.$$

We now turn to the proof of Lemma 2.3. Let $\zeta \in \mathbf{Y}$ be given. Write $\mathcal{M}(\zeta)$ for the set of configurations that can be obtained from ζ by translating ones to the right. Thus $M(\zeta)$ is simply the cardinality $|\mathcal{M}(\zeta)|$ of $\mathcal{M}(\zeta)$. If $\eta \in \mathbf{Y}$ then $\eta\zeta$ will denote the sequence in $\mathbf{Y}$ obtained by the concatenation of η and ζ. Finally, if $\mathbf{X} \subset \mathbf{Y}$, we let $\mathbf{X}\zeta = \{\eta\zeta : \eta \in \mathbf{X}\}$.

Proof of Lemma 2.3. We fix $\zeta \in \mathbf{Y}$, and use induction on $N(\gamma)$. If $N(\gamma) = 0$, that is, if γ is the empty sequence, then it suffices to notice that $\mathcal{M}(\zeta 10) = \mathcal{M}(\zeta 1)0 \cup \mathcal{M}(\zeta 0)1$, where the union is clearly disjoint. Thus the result follows in this case. Assume now that $N(\gamma) \geq 1$, and that the result holds for smaller values of $N(\gamma)$. We now analyse two cases.

Case 1. The sequence γ does not contain the segment 10.

In this case we clearly have that $\gamma = 0^k 1^\ell$ for some $k,\ \ell \geq 0$. If $\ell \geq 1$, using the fact that $\mathcal{M}(\eta 1) = \mathcal{M}(\eta)1$ for any $\eta \in \mathbf{Y}$ and the induction hypothesis, we are home. Thus we may assume that $\gamma = 0^k$ for some $k \geq 1$. Now note that

$$\mathcal{M}(\zeta 100^k) = \mathcal{M}(\zeta)10^{k+1} \cup \mathcal{M}(\zeta 0)10^k \cup \cdots \cup \mathcal{M}(\zeta 0^{k+1})1, \tag{2.9}$$

where clearly the sets on the right-hand side are pairwise disjoint. Similarly, we have

$$\mathcal{M}(\zeta 10^k) = \mathcal{M}(\zeta)10^k \cup \mathcal{M}(\zeta 0)10^{k-1} \cup \cdots \cup \mathcal{M}(\zeta 0^k)1, \tag{2.10}$$

with all the unions disjoint. We now observe that, by (2.10), the elements in all but the last set on the right-hand side of (2.9) are in natural one-to-one correspondence with the elements in $\mathcal{M}(\zeta 10^k)$. Moreover, since the elements in the last set on the right-hand side of (2.9) correspond to the elements in $\mathcal{M}(\zeta 0^{k+1}) = \mathcal{M}(\zeta 0\gamma)$ in an obvious way, we have that

$$M(\zeta 10\gamma) = |\mathcal{M}(\zeta 10^{k+1})| = |\mathcal{M}(\zeta 10^k)| + |\mathcal{M}(\zeta 0^{k+1})| = M(\zeta 1\gamma) + M(\zeta 0\gamma),$$

as required.

Case 2. The sequence γ contains the segment 10.

In this case let us write $\gamma = \gamma_1 10 \gamma_2$. Using the induction hypothesis, we have that

$$\begin{aligned} M(\zeta 10\gamma) = M(\zeta 10\gamma_1 10\gamma_2) &= M(\zeta 10\gamma_1 1\gamma_2) + M(\zeta 10\gamma_1 0\gamma_2) \\ &= M(\zeta 1\gamma_1 1\gamma_2) + M(\zeta 0\gamma_1 1\gamma_2) + M(\zeta 1\gamma_1 0\gamma_2) + M(\zeta 0\gamma_1 0\gamma_2) \\ &= M(\zeta 1\gamma_1 10\gamma_2) + M(\zeta 0\gamma_1 10\gamma_2) = M(\zeta 1\gamma) + M(\zeta 0\gamma), \end{aligned}$$

completing the induction step, and hence the proof. $\square$

To close this section, we need to prove Lemma 2.4. To this end, we consider a function $R(\zeta)$ ($\zeta \in \mathbf{Y}$), implicit in Derrida, Janowsky, Lebowitz and Speer (1993), which will turn out to give an alternative combinatorial description of the quantity $M(\zeta)$. It is using this description that we shall prove Lemma 2.4.

Let $W = (W_i)_1^n$ be a $\{\pm 1, 0\}$-sequence and $L = (L_i)_1^n$ a 0–1 sequence. We say that (W, L) is a *labelled closed walk* of length n if (i) W is a *closed walk on* ${}_+$ *starting at* 0, that is, if all initial partial sums $\sum_{1\le i\le j} W_i$ $(0 \le j \le n)$ are non-negative and $\sum_{1\le i\le n} W_i = 0$, and (ii) L is such that, for all $1 \le i \le n$, if $W_i = 1$ then $L_i = 1$, if $W_i = -1$ then $L_i = 0$, and if $W_i = 0$ then $L_i \in \{0, 1\}$. For brevity, we refer to a closed walk on ${}_+$ starting at 0 simply as a *closed walk*. Given $\zeta \in \mathbf{Y}$, let $\mathcal{R}(\zeta)$ be the set of all labelled closed walks (W, L) with $L = \zeta$, and put $R(\zeta) = |\mathcal{R}(\zeta)|$.

Proof of Lemma 2.4. We start by proving the following claim.

Claim. *For all ζ, $\gamma \in \mathbf{Y}$, we have $R(\zeta 10\gamma) = R(\zeta 1\gamma) + R(\zeta 0\gamma)$.*

Proof of the Claim. Let ζ, $\gamma \in \mathbf{Y}$ be fixed. Suppose $W = (W_i)_1^n$ is a closed walk for which $(W, \zeta 10\gamma)$ is a labelled closed walk. Assume $W = W^{(1)} w_1 w_2 W^{(1)}$, where $W^{(1)}$ and $W^{(2)}$ are $\{\pm 1, 0\}$-sequences of length $N(\zeta)$ and $N(\gamma)$ respectively, and w_1, $w_2 \in \{\pm 1, 0\}$. We put

$$\varphi(W, \zeta 10\gamma) = \begin{cases} (W^{(1)} 0 W^{(2)}, \zeta 1\gamma) & \text{if } (w_1, w_2) = (0, 0) \\ (W^{(1)} 0 W^{(2)}, \zeta 0\gamma) & \text{if } (w_1, w_2) = (1, -1) \\ (W^{(1)} 1 W^{(2)}, \zeta 1\gamma) & \text{if } (w_1, w_2) = (1, 0) \\ (W^{(1)} (-1) W^{(2)}, \zeta 0\gamma) & \text{if } (w_1, w_2) = (0, -1). \end{cases}$$

Then it is straightforward to check that φ defines a bijection between $\mathcal{R}(\zeta 10\gamma)$ and $\mathcal{R}(\zeta 1\gamma) \cup \mathcal{R}(\zeta 0\gamma)$, proving the claim. $\square$

Putting together the claim above and Lemma 2.3, we deduce that $R(\zeta) = M(\zeta)$ for all $\zeta \in \mathbf{Y}$, since $R(0^k 1^\ell) = M(0^k 1^\ell) = 1$ for all k, $\ell \ge 0$. We are now ready to start the proof of Lemma 2.4 proper. The calculations below, which are included for completeness, appear in the Appendix of Derrida, Janowsky, Lebowitz and Speer (1993) in a slightly different form.

Let a and $b \ge 0$ be integers. For convenience, let us say that a 0–1 sequence L is of *type* (a, b) if L has a elements equal to 1 and b elements equal to 0. Let $r_{a,b}$ be the number of labelled closed walks (W, L) with L of type (a, b). Thus $r_{a,b} = \sum_\zeta R(\zeta) = \sum_\zeta M(\zeta)$, where the sum ranges over all ζ with $N(\zeta) = a + b$ and $K(\zeta) = a$. Moreover, if W_0 is a given closed walk, let $r_{W_0,a,b}$ be the number of labelled closed walks (W, L) with $W = W_0$ and L a sequence of type (a, b). Clearly $r_{a,b} = \sum_W r_{W,a,b}$, where the sum ranges over all closed walks W of length $a + b$.

The easiest way of handling the numbers $r_{a,b}$ and $r_{W,a,b}$ is by using generating functions. In the sequel, we shall consider bivariate formal power series with formal variables x and y. Let $n \geq 0$ be an integer. We put $\psi_n(x, y) = \sum_{a,b} r_{a,b} x^a y^b$, where the sum ranges over all pairs (a, b) with a, $b \geq 0$ and $a + b = n$. Moreover, for a closed walk W, we put $\psi_W(x, y) = \sum_{a,b\geq 0} r_{W,a,b} x^a y^b$. Then clearly $\psi_n(x, y) = \sum_W \psi_W(x, y)$, where the sum is over all closed walks W of length n.

Now, if a closed walk $W = (W_i)_1^n$ has $2q$ non-zero entries, it is immediate that we have $\psi_W(x, y) = (x + y)^{n-2q} x^q y^q$. Now note that the number of closed walks $W = (W_i)_1^n$ of length n with $2q$ non-zero entries is $(q + 1)^{-1}\binom{2q}{q}\binom{n}{2q}$. Indeed, to each such walk W, associate the walk $W' = (W_i')_1^{n+1}$ with $W_i' = W_i$ for $1 \leq i \leq n$ and $W_{n+1}' = -1$. Then all proper partial initial sums of W' are non-negative and $\sum_{1\leq i\leq n+1} W_i' = -1$. The number of such sequences W' is $(n+1)^{-1}\binom{2q+1}{q}\binom{n+1}{2q+1}$: choose where to have the ± 1 in W' randomly, and then Raney's lemma (cf. the proof of Lemma 2.6) tells us that a fraction of $1/(n+1)$ of such choices will do for W'. Thus the number of closed walks of length n and $2q$ non-zero entries is

$$\frac{1}{n+1}\binom{2q+1}{q}\binom{n+1}{2q+1} = \frac{1}{2q+1}\binom{2q+1}{q}\binom{n}{2q} = \frac{1}{q+1}\binom{2q}{q}\binom{n}{2q},$$

as claimed. (Here and in the sequel, the reader is referred to Chapter 5 of Graham, Knuth, and Patashnik (1989) for identities involving binomial coefficients.) Therefore

$$\begin{aligned}
\psi_n(x, y) &= \sum_q \frac{1}{q+1}\binom{2q}{q}\binom{n}{2q}(x + y)^{n-2q} x^q y^q \\
&= \sum_{q,j} \frac{1}{q+1}\binom{2q}{q}\binom{n}{2q}\binom{n-2q}{j} x^{q+j} y^{n-(q+j)} \\
&= \sum_{q,k} \frac{1}{q+1}\binom{2q}{q}\binom{n}{2q}\binom{n-2q}{k-q} x^k y^{n-k} \\
&= \sum_{q,k} \frac{1}{q+1}\binom{n}{k}\binom{k}{q}\binom{n-k}{q} x^k y^{n-k} \\
&= \sum_k \frac{1}{k+1}\binom{n}{k} x^k y^{n-k} \sum_q \binom{k+1}{q+1}\binom{n-k}{q} \\
&= \sum_k \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k} x^k y^{n-k}.
\end{aligned}$$

Therefore we have that $r_{k,n-k} = (k+1)^{-1}\binom{n}{k}\binom{n+1}{k}$, and hence Lemma 2.4 follows. $\square$

We close this section with the following remark. Recall that $M(\zeta)$ appears in the definition of the probability measure $p = p_{\rho,\lambda}$ on $\mathbf{Y}$, and that we shall use p to construct invariant measures for our two species asymmetric processes. We feel that the definition of $M(\zeta)$ makes it natural that this quantity should be involved in our construction. The alternative description of $M(\zeta)$ as a certain number of labelled walks on $_+$, given in the proof of Lemma 2.4 above, allows us to perform some calculations, and in particular to prove Lemma 2.4.

3. Invariant measures for the process as seen from a second class particle

In the sequel, we shall always have $\rho \le \lambda$. Given $0 < \rho \le \lambda < 1$ we construct here a 'translation invariant' measure μ_2', in the sense this measure is invariant under translations that leave a second class particle at the origin. The parameter ρ corresponds to the asymptotic density of the first class particles, and λ corresponds to the asymptotic density of all the particles, with classes disregarded. In Proposition 1 we show that under μ_2' the distribution of first class particles to the right of the origin and the distribution of empty sites to the left of it are product measures with densities ρ and $1-\lambda$ respectively. Another important and nice property of the measure μ_2' is that the distribution of the distance between two successive second class particles is the same as the distribution of the hitting time of 1 for the random walk $\tilde{Z}_n$ introduced after Lemma 2.3. This observation and Proposition 1 give an alternative way of computing the decay of densities found by Derrida, Janowsky, Lebowitz and Speer (see Remark 3.2 below).

In Theorem 1 we show that μ_2' is invariant for the process. We then construct other invariant measures for the process as seen from a second class particle randomly drawing a configuration according to μ_2' and identifying first and second class particles to the right of the origin and empty sites and second class particles to the left of it. In particular, we get the shocks when $\lambda > \rho$: the invariant measure as seen from a single, isolated second class particle. We may also obtain an invariant measure with only two second class particles. If $\lambda = \rho$ the distance between these two particles is a non-degenerate random variable with an infinite first moment. In this case the corresponding random walk $\tilde{Z}_n$ is *symmetric* and the hitting time of 1 is finite with probability one but has an infinite mean.

Let $\{\zeta_i\}_{i\in} \subset \mathbf{Y}$ be a doubly infinite i.i.d. sequence of finite configurations with distribution $(\zeta_i = \zeta) = p(\zeta)$, where $p(\zeta)$ is given in (2.1). A configuration (σ, ξ) with distribution μ_2' is obtained by displaying the ζ_i on the integers separated by second class particles. More rigorously, for $i \ge 0$, let $N_i = N(\zeta_i) + 1$ and $S_i = \sum_{j=0}^{i-1} N_j$. Let $I(x) = i$ if and only if

$S_i \le x < S_{i+1}$ $(i \in)$. Set $\sigma(0) = 0$, $\xi(0) = 1$ and for $x > 0$ put

$$\sigma(x) = \begin{cases} \zeta_{I(x)}(x - S_{I(x)}) & \text{if } S_{I(x)} < x < S_{I(x)+1} \\ 0 & \text{if } x = S_{I(x)}, \end{cases}$$

$$\xi(x) = \begin{cases} 1 & \text{if } x = S_{I(x)} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\sigma(x)$ and $\xi(x)$ for $x < 0$ analogously. The resulting distribution of (σ, ξ) is the measure μ_2' that we seek.

Theorem 1. *Let $0 < \rho \le \lambda < 1$. The measure μ_2' is invariant for $(\tau_{X_t}\sigma_t, \tau_{X_t}\xi_t)$, the process as seen from a second class particle.*

Before proving the theorem above, we construct other invariant measures using μ_2' and identification operators. Let $\xi = \{x_i\}_{i\in}$ be the set of occupied sites of a configuration ξ of second class particles with the origin occupied, where $x_0 = 0$ and $x_i < x_{i+1}$ for all integers i. Let Φ_i and Ψ_i $(i \in)$ be operators on configurations (σ, ξ) defined by setting, for all $x \in$,

$$\Phi_i(\sigma(x), \xi(x)) = \begin{cases} (\sigma(x) + \xi(x), 0) & \text{if } x \ge x_i \\ (\sigma(x), \xi(x)) & \text{otherwise,} \end{cases}$$

$$\Psi_i(\sigma(x), \xi(x)) = \begin{cases} (\sigma(x), 0) & \text{if } x \le x_i \\ (\sigma(x), \xi(x)) & \text{otherwise.} \end{cases}$$

In words, Φ_i identifies first and second class particles to the right of the i-th second class particle and Ψ_i identifies empty sites and second class particles to the left of the i-th second class particle. The next lemma, which is a straightforward generalization of an observation in Ferrari, Kipnis and Saada (1991), says that the identification operators commute with the process as seen from a second class particle. The reason is that, owing to the nearest neighbour interaction rules, namely, the total asymmetry of the jumps and the exclusion interaction, the second class particles to the right of a given second class particle behave as though they were first class particles. For the same reason, second class particles to the left of a given second class particle are just like empty sites.

Lemma 3.1. *For any $i > 0$ and $j < 0$, the operators Φ_i and Ψ_j commute with the generator L_2' of the process as seen from the second class particle:*

$$\Phi_i L_2' = L_2' \Phi_i, \qquad \Psi_j L_2' = L_2' \Psi_j.$$

An immediate corollary of Lemma 3.1 is the following. Let $\Phi_\infty = \Psi_{-\infty} = I$, the identity operator.

Theorem 2. *For any $0 < \rho \leq \lambda < 1$ and any $0 < i \leq \infty$ and $-\infty \leq j < 0$ the measures*

$$\mu'_{2,i,j} = \Phi_i \Psi_j \mu'_2$$

are invariant for the process as seen from a second class particle.

Remark 3.1. For all $0 < \rho \leq \lambda < 1$, the measure $\mu'_{2,1,-1}$ is the measure as seen from an isolated second class particle. Moreover, the measure $\mu'_{2,2,-1}$ is a measure with only two second class particles.

Proof of Theorem 1. By a standard construction of the process (Liggett (1985), Andjel (1982)), it is sufficient to verify the equality

$$\int L'_2 f(\sigma,\xi) d\mu'_2(\sigma,\xi) = 0 \tag{3.1}$$

for $f = 1_\Gamma\{(\sigma,\xi)_\Lambda\}$, where Λ is a finite subset of containing the origin, $(\sigma,\xi)_\Lambda$ is the projection of the configuration (σ,ξ) in Λ and Γ is an arbitrary configuration of the form

$$2\gamma_{-k}2\gamma_{-k+1}2\ldots2\gamma_{-1}2\gamma_1 2\ldots 2\gamma_k 2,$$

with arbitrary k and arbitrary $\gamma_i \in \mathbf{Y}$ $(i = -k, -k+1, \ldots, -1, 1, \ldots, k)$. Put $l_- = -\sum_{i=-k}^{-1}(N(\gamma_i)+1)$ and $l_+ = \sum_{i=1}^{k}(N(\gamma_i)+1)$. Then, here, we have $\Lambda = \{x \in \ : l_- \leq x \leq l_+\}$. Moreover $1\,.\,\{\,\cdot\,\}$ is the usual indicator function and

$$\begin{aligned}
L'_2 f(\sigma,\xi) = \sum_{x\neq 0} &\{\sigma(x)[1-\sigma(x+1)][f(\sigma^{x,x+1},\xi^{x,x+1}) - f(\sigma,\xi)] \\
&\quad + \xi(x)[1-\sigma(x+1)][1-\xi(x+1)][f(\sigma,\xi^{x,x+1}) - f(\sigma,\xi)]\} \\
&+ \sigma(-1)[f(\tau_{-1}\sigma^{-1,0},\tau_{-1}\xi^{-1,0}) - f(\sigma,\xi)] \\
&+ [1-\sigma(1)][1-\xi(1)][f(\tau_1\sigma^{0,1},\tau_1\xi^{0,1}) - f(\sigma,\xi)].
\end{aligned}$$

Let μ''_2, μ'''_2 and μ''''_2 be the projections of μ'_2 on Λ, $\{l_- - 1\}\cup\Lambda$ and $\Lambda\cup\{l_+ + 1\}$, respectively. Then verifying (3.1) amounts to verifying the equality of the following two expressions:

$$\mu'''_2(1\Gamma) + \sum_{x\in\Lambda_1} \mu''_2(\Gamma) + \mu''''_2(\Gamma 0) \tag{3.2}$$

and

$$\mu'''_2(20\Gamma_-) + \sum_{x\in\Lambda_2} \mu''_2(\Gamma_{x,x+1}) + \mu''''_2(\Gamma^- 12), \tag{3.3}$$

where

$$\Lambda_1 = \{x \in \Lambda : \Gamma(x) = 1, \Gamma(x+1) = 0 \text{ or } 2\} \cup \{x \in \Lambda^- : \Gamma(x) = 2, \Gamma(x+1) = 0\},$$

$$\Lambda_2 = \{x \in \Lambda : \Gamma(x) = 0 \text{ or } 2,\ \Gamma(x+1) = 1\} \cup \{x \in \Lambda : \Gamma(x) = 0, \Gamma(x+1) = 2\},$$

$$\Lambda^- = \Lambda \backslash \{l_+\},$$

$$\Gamma_- = \gamma_{-k} 2 \gamma_{-k+1} 2 \ldots 2 \gamma_{-1} 2 \gamma_1 2 \ldots 2 \gamma_k 2, \quad \Gamma^- = 2 \gamma_{-k} 2 \gamma_{-k+1} 2 \ldots 2 \gamma_{-1} 2 \gamma_1 2 \ldots 2 \gamma_k,$$

$$\Gamma_{x,x+1} = \{\Gamma(y), y \in \Lambda, y < x\} \Gamma(x+1) \Gamma(x) \{\Gamma(y), y \in \Lambda, y > x+1\}.$$

We first show $\mu_2'''(1\Gamma) = \mu_2''''(\Gamma^- 12)$. Notice that

$$\mu_2'''(1\Gamma) = \mu_2'(\sigma(-1) = 1) \mu_2''(\Gamma) = \rho \lambda \mu_2''(\Gamma).$$

On the other hand

$$\mu_2''''(\Gamma^- 12) = \prod_{i=-k}^{k-1} p(\gamma_i) \times p(\gamma_k 1) = \prod_{i=-k}^{k-1} p(\gamma_i) \times p(\gamma_k) \rho \lambda = \rho \lambda \mu_2''(\Gamma),$$

where p is the probability measure given by (2.1). Similarly, $\mu_2''''(\Gamma 0) = \mu_2'''(20\Gamma_-)$, so we only need to show that the two central sums in (3.2) and (3.3) are equal. The first thing to notice is that $\mu_2''(\Gamma)$ factors in the following way.

$$\mu_2''(\Gamma) = m(\Gamma) \times \lambda(1-\rho)(\lambda\rho)^{k(\Gamma)}((1-\lambda)(1-\rho))^{n(\Gamma)-k(\Gamma)}, \tag{3.4}$$

where $m(\Gamma) = \prod_{i=-k}^{k} M(\gamma_i)$, $k(\Gamma) = \sum_{i=-k}^{k} K(\gamma_i)$ and $n(\Gamma) = \sum_{i=-k}^{k} N(\gamma_i)$. The measure $\mu_2''(\Gamma_{x,x+1})$ factors in a similar way when $x \in \Lambda^- \backslash \{\sum_{i=-k}^{-1}(N(\gamma_i)+1)\}$. If moreover $x \in \Lambda_2$, then $k(\Gamma) = k(\Gamma_{x,x+1})$ and $n(\Gamma) = n(\Gamma_{x,x+1})$. So the last factor in the product (3.4) for $\mu_2''(\Gamma)$ equals the corresponding one in $\mu_2''(\Gamma_{x,x+1})$.

When $l_- \in \Lambda_2$, we have

$$\mu_2''(\Gamma_{x,x+1}) = m(\Gamma_{x,x+1}) \times \lambda(1-\rho)(\lambda\rho)^{k(\Gamma_{x,x+1})}((1-\lambda)(1-\rho))^{n(\Gamma_{x,x+1})-k(\Gamma_{x,x+1})}$$
$$\times \mu_2'(\sigma(-1) = 1).$$

Since in this case $k(\Gamma_{x,x+1}) = k(\Gamma) - 1$, $n(\Gamma_{x,x+1}) = n(\Gamma) - 1$ and $\mu_2'(\sigma(-1) = 1) = \lambda\rho$, the product of the last two terms in the above expression equals the last term in (3.4). A similar thing happens when $l_+ \in \Lambda_2$. Hence the factors dependent on λ and ρ in the terms of both central sums in (3.2) and (3.3) are the same and so it is sufficient to verify

$$\sum_{x \in \Lambda_1} m(\Gamma) = \sum_{x \in \Lambda_2} m(\Gamma_{x,x+1}). \tag{3.5}$$

This is proven in exactly the same way as (3.5) in Derrida, Janowsky, Lebowitz and Speer (indeed, m here is the same object as w in that paper), by observing the following properties of m, which are inherited from M:

$$m(\Gamma'10\Gamma'') = m(\Gamma'1\Gamma'') + m(\Gamma'0\Gamma''),$$

$$m(\Gamma'12\Gamma'') = m(\Gamma'2\Gamma'') \quad \text{and} \quad m(\Gamma'20\Gamma'') = m(\Gamma'2\Gamma'').$$

The conclusion is that both sides in (3.5) equal

$$\sum_{j:y_j=1,0} m(y_1^{k_1} \dots y_j^{k_j-1} \dots y_m^{k_m}),$$

where $y_i^{k_i}$ $(i = 1, \dots, m)$ are the maximal blocks of y_i $(y_i \in \{0, 1, 2\})$ constituting Γ, that is, $\Gamma = y_1^{k_1} \dots y_j^{k_j} \dots y_m^{k_m}$. $\square$

The next result was announced by Derrida, Janowsky, Lebowitz and Speer (1993) and proved by Speer (1994) through direct computations. Our proof is based on constructing the measure μ_2' by first displaying the first class particles to the right of the origin according to a product measure and then specifying the positions of the second class particles. The same is done to the left of the origin with the empty sites.

Proposition 1. *Under μ_2', the distribution of first class particles to the right of the origin is the product measure ν_ρ of parameter ρ. Similarly, the distribution of holes to the left of the origin is the product measure $\nu_{1-\lambda}$ with parameter $1 - \lambda$.*

Proof. Let a configuration $\eta \in \{0, 1\}$ be given. For $x \geq 1$, let $\eta|_x \in \mathbf{Y}$ be the finite configuration $\eta(1)\eta(2)\dots\eta(x-1)$ of length $x - 1$ determined by η. For all $x \geq 1$, put $M(\eta, x) = M(\eta|_x)$ and similarly put $K(\eta, x) = K(\eta|_x)$. For $x \geq 1$, let

$$p(x \mid \eta) = \begin{cases} 0 & \text{if } \eta(x) = 1 \\ \lambda M(\eta, x) \lambda^{K(\eta,x)} (1-\lambda)^{x-K(\eta,x)-1} & \text{if } \eta(x) = 0. \end{cases} \tag{3.6}$$

Notice that $p(x \mid \eta)$ depends on η only through sites $1, \dots, x$. To prove our result it suffices to prove that

$$\sum_{x>0} p(x \mid \eta) = 1 \tag{3.7}$$

ν_ρ almost surely, which is proven in Lemma 3.2 below. The reason is that we can interpret $p(x \mid \eta)$ as the probability of the leftmost second class particle to the right of the origin be at site x given that the first class particles are at the sites occupied by η. To see this compute for instance the probability that the configuration $\zeta = 11010$ appears between the second class particle at the origin and the next second class particle (at site 6). According to our construction, first ditribute three first class particles and two holes at sites $\{1, \dots, 5\}$ with probability $\rho^3(1-\rho)^2$. Then put a hole at site 6 with probability $1-\rho$. Finally, the conditional probability of putting a second class particle at site 6 given the configuration $110100\dots$ is

$$p(6 \mid 110100\dots) = \lambda M(11010)\lambda^3(1-\lambda)^2.$$

The resulting distribution is exactly the one given by (2.1). This argument can be applied to an arbitrary configuration but the the notation is too heavy, and hence we omit the details. $\square$

Lemma 3.2. *Let $p(x \mid \eta)$ be defined as in (3.6). Then for all $\rho \in [0, \lambda]$*

$$\sum_{x>0} p(x \mid \eta) = 1 \tag{3.8}$$

ν_ρ almost surely. Furthermore (3.8) holds for all configurations $\eta \in \{0,1\}$ with a finite number of particles.

Proof. We first prove the identity (3.8) for configurations $\eta \in \{0,1\}$ with a finite number of particles. Observe that if $\eta(x) = 0$ for all $x > 0$, then

$$\sum_{x>0} p(x \mid \eta) = \lambda \sum_{x>0} (1-\lambda)^{x-1} = 1.$$

Assume that the identity holds for any configuration with n particles. Let η be a configuration with $n+1$ particles, whose rightmost particle is located at $z > 0$. Let η^z be the configuration η modified only at site z. Hence η^z has n particles. From Lemma 2.3, for $x \geq z+2$, we have

$$M(\eta, x) = M(\eta^z, x-1) + M(\eta, x-1). \tag{3.9}$$

Divide the sum in (3.8) in two parts:

$$\sum_{x>0} p(x \mid \eta) = \sum_{x=1}^{z+1} p(x \mid \eta) + \sum_{x=z+2}^{\infty} p(x \mid \eta). \tag{3.10}$$

Apply identity (3.9) to all terms of the second sum of the right-hand side of (3.10) to obtain

$$\begin{aligned}\sum_{x=z+2}^{\infty} p(x \mid \eta) = &\sum_{x=z+2}^{\infty} (1-\eta(x))M(\eta^z, x-1)\lambda^{K(\eta,x)+1}(1-\lambda)^{x-1-K(\eta,x)} \\ &+ \sum_{x=z+2}^{\infty} (1-\eta(x))M(\eta, x-1)\lambda^{K(\eta,x)+1}(1-\lambda)^{x-1-K(\eta,x)}.\end{aligned}$$

Since for $x \geq z+2$, $K(\eta, x) = K(\eta, x-1) = K(\eta^z, x-1)+1$ and $1-\eta(x) = 1-\eta(x-1) = 1$, we obtain

$$= \lambda \sum_{x=z+2}^{\infty} (1-\eta(x-1))(\eta^z, x-1)\lambda^{K(\eta^z, x-1)+1}(1-\lambda)^{(x-1)-1-K(\eta^z, x-1)}$$
$$+ (1-\lambda) \sum_{x=z+2}^{\infty} (1-\eta(x-1))M(\eta, x-1)\lambda^{K(\eta, x-1)+1}(1-\lambda)^{(x-1)-1-K(\eta, x-1)}.$$

Hence

$$\sum_{x=z+2}^{\infty} p(x \mid \eta) = \lambda \sum_{x=z+1}^{\infty} p(x \mid \eta^z) + (1-\lambda) \sum_{x=z+1}^{\infty} p(x \mid \eta). \tag{3.11}$$

Observe that for $x \leq z$, $M(\eta, x) = M(\eta^z, x)$, $K(\eta, x) = K(\eta^z, x)$ while $1-\eta(x) = 1-\eta^z(x)$ for $x < z$ and $1-\eta(z) = 0$. Hence multiplying by $(1-\lambda)+\lambda$ the first z terms of the first sum in the right hand side of (3.10),

$$\begin{aligned}\sum_{x=1}^{z+1} p(x \mid \eta) = (1-\lambda) &\sum_{x=1}^{z} p(x \mid \eta) \\ &+ \lambda \sum_{x=1}^{z-1} (1-\eta^z(x))M(\eta^z, x)\lambda^{K(\eta^z, x)+1}(1-\lambda)^{x-1-K(\eta^z, x)} \\ &+ (1-\eta(z+1))M(\eta, z+1)\lambda^{K(\eta, z+1)+1}(1-\lambda)^{(z+1)-K(\eta, z)-1}.\end{aligned} \tag{3.12}$$

Now $M(\eta, z+1) = M(\eta^z, z)$, $K(\eta, z+1) = K(\eta^z, z)+1$ and $1-\eta(z+1) = 1-\eta^z(z) = 1$. Hence the last line equals $\lambda p(\eta^z \mid z)$ and the second plus the third line equal $\lambda \sum_{x=1}^{z} p(x \mid \eta^z)$. So, putting together (3.11) and (3.12) we get

$$\sum_{x>0} p(x \mid \eta) = (1-\lambda) \sum_{x>0} p(x \mid \eta) + \lambda \sum_{x>0} p(x \mid \eta^z). \tag{3.13}$$

Since the second sum in the right-hand side of (3.13) above is one by inductive hypothesis, this complete the induction step. Thus the result holds for finite η. Since the sum of the first n terms in (3.8) depends only on $\eta(1), \ldots, \eta(n)$, the validity of (3.8) for finite η implies that for any η and $n \geq 1$, we have $\sum_{x=1}^{n} p(x \mid \eta) \leq 1$, which in turns implies that for any η

$$\sum_{x=1}^{\infty} p(x \mid \eta) = c(\eta) \leq 1. \tag{3.14}$$

Assume that there exists a set $\mathbf{X_0}$ with positive ν_ρ probability such that if $\eta \in \mathbf{X_0}$, then $c(\eta) \leq c < 1$. This and (3.14) imply that

$$1 > \int d\nu_\rho(\eta) \sum_{x=1}^{\infty} p(x \mid \eta) = (N < \infty) = 1, \qquad \text{for } \rho \leq \lambda,$$

where N is the random variable whose distribution is given by (2.1) and (2.3). The contradiction above proves that (3.8) holds ν_ρ almost surely for any $\rho \in [0, \lambda]$. $\square$

Remark 3.2. Using the generating function of N given in (2.3), one can estimate precisely the rate of convergence of the densities of the particles computed by Derrida, Janowsky, Lebowitz and Speer (1993). Let $(\widetilde{Z}_n)_1^\infty$ be the random walk on defined after Lemma 2.3. We say that there is a *record* at time $n \geq 1$ if $\widetilde{Z}_n > \widetilde{Z}_j$ for all $0 \leq j < n$. Hence the probability that under μ_2' a second class particle is present at site $n \geq 1$ is the probability that the random walk $(\widetilde{Z}_n)_1^\infty$ establishes a record at time n. Put $u_0 = 1$, and for $n \geq 1$ let $u_n = (\lambda - \rho) + \rho(1-\lambda)(T \geq n)$, where T is the time of the first record or, in other words, the hitting time of 1. Now (2.3), the relation between N and T and the renewal equation (see Feller (1968)) imply that u_n is precisely the probability that a record is established at time n. From the rate of convergence of the distribution of T (see remarks to the proofs of Lemmas 2.2 and 2.5) it is clear that u_n goes exponentially fast to $\lambda - \rho$ when $\lambda > \rho$ and like $n^{-1/2}$ when $\lambda = \rho$. Since the density of first class particles to the right of the origin is a product measure with constant density ρ, this gives also the asymptotic density of holes to the right of the origin. Analogous arguments work to the left of the origin by observing that the density of holes is $1 - \lambda$.

4. The invariant measure for the translation invariant process

In this section we assume $0 < \rho < \lambda < 1$. Let μ_2 be the unique translation invariant measure satisfying $\mu_2(\,\cdot\,|\,\xi(0) = 1) = \mu_2'(\,\cdot\,)$. As mentioned in the introduction, μ_2 must be invariant for the two species process. We show next that the measure μ_2 has good marginals.

Theorem 3. *The σ marginal of μ_2 is ν_ρ while the $\sigma + \xi$ marginal of μ_2 is ν_λ.*

Proof. To construct the measure μ_2' we started by assigning the positions of the second class particles and then we gave the distribution of the first class particles, given the position of the second class particles. The positions of the second class particles form a (discrete time) renewal process with finite interarrival time, being the first renewal at time 0. When $\rho < \lambda$ the average distance between two renewals is $(\lambda - \rho)^{-1} < \infty$. Hence we can use the key renewal theorem to construct μ_2 in the following way:

$$\mu_2 = \lim_{x \to -\infty} \mu_2' \tau_x = \lim_{x \to +\infty} \mu_2' \tau_x \tag{4.1}$$

where τ_x is the translation by x operator. To show the theorem take a cylinder function $f(\sigma, \xi)$ depending only on σ. Take a negative z such that the support of f is contained in

(z, ∞). By Proposition 1,

$$\mu_2' \tau_z f = \nu_\rho f.$$

This and (4.1) implies that $\mu_2 f = \nu_\rho f$. To show that the $\sigma + \xi$ marginal is ν_λ apply the same reasoning for a positive z and show that $\mu_2 f = \nu_\lambda$ if f depends only on $\sigma + \xi$. $\square$

Our next result exploits the embedded renewal process in both μ_2 and μ_2'.

Theorem 4. *It is possible to construct a coupling $\tilde{\mu}_2$ with marginals μ_2 and μ_2' such that if $(\sigma, \xi, \sigma', \xi')$ has distribution $\tilde{\mu}_2$ and $H(\sigma, \xi, \sigma', \xi') = \sum_x |\sigma(x) - \sigma'(x)| + |\xi(x) - \xi'(x)|$ is the number of sites where (σ, ξ) is different from (σ', ξ') then, under $\tilde{\mu}_2$, the random variable H has a finite exponential moment. In other words, there exists $\theta > 0$ such that*

$$\int d\tilde{\mu}_2 e^{\theta H} < \infty. \tag{4.2}$$

Proof. Let (σ, ξ) be a realization of the translation invariant point process related to the point process with distribution μ_2' and T_i the stationary process related to ξ. Thus T_i denotes the position of the i-th ξ particle, where $T_0 \leq 0$ is the position of the rightmost ξ particle to the left of the origin. Let $(\bar{\sigma}', \bar{\xi}')$ be a realization of the process with distribution μ_2' and let S_i be the renewal process associated to $\bar{\xi}'$, with $S_0 = 0$ and S_i denoting the position of the i-th $\bar{\xi}'$ particle. The random variables $T_i - T_{i-1}$ are independent and have the same distribution as $S_i - S_{i-1}$ for $i \neq 1$, while (T_0, T_1) has the limiting distribution

$$P(T_1 > u, -T_0 > v) = \lim_{t \to \infty} P(S_{I(t)+1} > u, S_{I(t)} > v).$$

Similarly, for all cylinder f, we have $\mu_2 f = \lim_{x \to \infty} \mu_2' \tau_x f$. Now we construct a coupling $(\sigma, \xi, \sigma', \xi')$ with the property that the two first marginals have distribution μ_2 and the two last marginals have distribution μ_2'. If $T_0 = 0$, put $\xi'(x) \equiv \xi(x)$ and $\sigma'(x) \equiv \sigma(x)$. If $T_0 \neq 0$, let $J^+ = \min\{i > 0 : S_i \in \xi\}$ and $J^- = \max\{i \leq 0 : S_i \in \xi\}$, and let

$$(\sigma'(x), \xi'(x)) = \begin{cases} (\sigma(x), \xi(x)) & \text{if } x \geq J^+ \text{ or } x \leq J^- \\ (\bar{\sigma}'(x), \bar{\xi}'(x)) & \text{if } J^- < x < J^+. \end{cases}$$

It is clear that the resulting distribution of $(\sigma, \xi, \sigma', \xi')$ has marginals μ_2 and μ_2'. To show (4.2) notice that under $\tilde{\mu}_2$, we have $H \leq J^+ - J^-$. Since T_0 and T_0' have a finite exponential moment, it follows from Lindvall (1992), pp. 30–31, that both J^+ and $|J^-|$ have a finite exponential moment. $\square$

5. Shocks in the simple exclusion process

If we take the process as seen from a second class particle and identify particles of both classes to the right of the origin and second class particles with holes to the left of it, we get η'_t, the simple exclusion process as seen from an isolated second class particle. Rigorously, Lemma 3.1 says that the process $\eta'_t := \tau_{X_t}\eta_t = \Phi_1\Psi_{-1}\tau_{X_t}(\sigma_t + \xi_t)$ is the SEP as seen from a second class particle. We consider the shock measure constructed in the remark after Theorem 2 of Section 3. Let $\mu' = \Phi_1\Psi_{-1}\mu'_2$. For $0 < \rho \le \lambda < 1$, it follows from Theorem 2 that μ' is invariant for the process η'_t. Notice that X_t can be seen as either a tagged second class particle for the (σ_t, ξ_t) process or as an isolated second class particle for the η_t process. Our next result implies in particular that, for $0 < \rho < \lambda < 1$, the measure μ' is equivalent to $\nu_{\rho,\lambda}$ the product measure with densities ρ and λ to the left and right of the origin respectively.

Theorem 5. *If $0 < \rho < \lambda < 1$, it is possible to construct jointly the invariant measure μ and the product measure $\nu_{\rho,\lambda}$ in such a way that the number of sites where the configurations differ has a finite exponential moment.*

Proof. First we construct a configuration with distribution $\nu_{\rho,\lambda}$ using two independent configurations with distribution μ_2. Let (σ^+, ξ^+) and (σ^-, ξ^-) be two independent realizations of μ_2 and (σ', ξ') a realization of μ'_2 independent of the other two. Define $\eta \in \{0, 1\}$ by letting

$$\eta(x) = \begin{cases} \sigma^+(x) + \xi^+(x) & \text{if } x \ge 0 \\ \sigma^-(x) & \text{if } x < 0 \end{cases}$$

for all $x \in$. Then, by the marginal properties of μ_2 given by Theorem 3, it is easy to see that η constructed above has distribution $\nu_{\rho,\lambda}$. Here it is important that we take independent realizations of μ_2 to the right and left of the origin. Now couple (σ^+, ξ^+) with (σ', ξ') as in the proof of Theorem 4, letting J_+ be the leftmost positive site where (σ^+, ξ^+) is different from (σ', ξ'). Similarly, couple to the left of the origin letting J^- be the rightmost negative site where (σ^-, ξ^-) differs from (σ', ξ'). By the same argument as before J^+ and J^- have a finite exponential moment. Hence, setting $\eta' = \Phi_1\Psi_{-1}(\sigma', \xi')$, we get

$$\sum_x |\eta'(x) - \eta(x)| \le J^+ - J^-.$$

The result now follows again from Lindvall (1992). □

An immediate consequence of the result above is the following.

Corollary. *The measures μ' and $\nu_{\rho,\lambda}$ are equivalent, i.e. one is absolutely continuous with respect to the other.*

Proof. Since under both measures all non-empty cylinder sets have positive probability, and non-empty sets of measure zero depend on infinitely many coordinates, the corollary follows from Theorem 4. □

Acknowledgments. We thank E. Speer for insightful comments and very helpful discussions. The final version of this paper was written while P.A.F. was a participant of the programme *Random Spatial Processes* held at the Isaac Newton Institute for Mathematical Sciences, University of Cambridge, whose warm hospitality is gratefully acknowledged.

This paper is partially supported by FAPESP "Projeto Temático" Grant number 90/3918-5, CNPq and the SERC under Grant GR G59981. Y.K. was partially supported by FAPESP under grant 93/0603-1, and by CNPq under grant 300334/93-1.

References

E.D. Andjel (1982). Invariant measures for the zero range process. *Ann. Probab.* **10** 525-547.

L. Breiman (1968). *Probability*, Addison-Wesley.

A. De Masi, C. Kipnis, E. Presutti, E. Saada (1990). Microscopic structure at the shock in the asymmetric simple exclusion. *Stochastics* **27** 151-165.

B. Derrida, S. Janowsky, J.L. Lebowitz, E. Speer (1993). Exact solution of the totally asymmetric simple exclusion process: shock profiles. To appear in *J. Stat. Phys.*

W. Feller (1968). *An Introduction to Probability Theory and its Applications*, John Wiley & Sons, Inc.

P.A. Ferrari (1986). The simple exclusion process as seen from a tagged particle. *Ann. Probab.* **14** 1277–1290.

P.A. Ferrari (1992). Shock fluctuations in asymmetric simple exclusion. *Probab. Theor. Related Fields.* **91** 81–101.

P.A. Ferrari, L.R.G. Fontes (1993). Shock fluctuations in the asymmetric simple exclusion process. *Probab. Theor. Related Fields.*

P.A. Ferrari, C. Kipnis, E. Saada (1991). Microscopic structure of travelling waves for the asymmetric simple exclusion process. *Ann. Probab.* **19** 226–244.

R.L. Graham, D.E. Knuth, O. Patashnik (1989). *Concrete Mathematics*, Addison-Wesley.

W. Hoeffding (1963). Probability inequalities for sums of bounded random variables, *Jour. Amer. Statistical Assoc.* **58** 13–30.

T.M. Liggett (1976). Coupling the simple exclusion process. *Ann. Probab.* **4** 339–356.

T.M. Liggett (1985). *Interacting Particle Systems.* Springer-Verlag, Berlin.

T. Lindvall (1992). *Lectures on the Coupling Method.* John Wiley & Sons, Inc.

C.J.H. McDiarmid (1989). On the method of bounded differences, in *Surveys in Combinatorics 1989* (J. Siemons, ed.), London Mathematical Society Lecture Notes Series 141, Cambridge University Press, Cambridge 1989, pp. 148–188.

G.N. Raney (1960). Functional composition patterns and power series reversion, *Transactions of the AMS* **94** 441–451.

E.R. Speer (1994). The two species asymmetric simple exclusion process, in *Micro, Meso, and Macroscopic Approaches in Physics* (M. Fannes, C. Maes, A. Verbeure, eds), Procedings of the NATO AR Workshop "On Three Levels", Leuven, July 1993.

F. Spitzer (1976). *Principles of Random Walk*, 2nd edition, Springer-Verlag.

Instituto de Matemática e Estatística — Universidade de São Paulo
Cx. Postal 20570 — 01452-990 São Paulo SP — Brasil
`<pablo@ime.usp.br>`, `<lrenato@ime.usp.br>`, `<yoshi@ime.usp.br>`