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The smooth cohomology of N=2 supersymmetric Landau-Ginzburg field theories

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Abstract. We compute the smooth cohomology (both unrestricted and compactly supported) of the supercharge of an ultraviolet cutoff $N = 2$ supersymmetric Landau - Ginzburg field theory.

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I. Introduction

I.A. The two-dimensional $N = 2$ supersymmetric Landau - Ginzburg models (a.k.a. as Wess - Zumino models) were introduced in the physics literature in the seventies. For early results, see [CG] and references therein. Constructive field theory aspects of these models were a subject of many investigations, see [JL] and [J] for references. The Landau - Ginzburg models provide useful examples to study complex physical and mathematical phenomena of supersymmetric quantum field theory which are much harder to control in the four dimensional world. Recent revival of interest in the Landau - Ginzburg models stems largely from the fact that they seem to play a role in various "compactification" scenarios of string theory (see e.g. [CV] and references therein).

Supersymmetric quantum field theories provide non-trivial examples of infinite dimensional non-commutative geometries [C]. In particular, supersymmetric field theories with $N = 2$ supersymmetries lead naturally to structures which can be regarded as examples of non-commutative Kähler geometry. For the Landau - Ginzburg models, the underlying infinite dimensional geometry is flat. What makes them non-trivial is the non-linear self-interaction term in the Hamiltonian. One of the fundamental difficulties in studying the mathematical structures associated with this model is of technical character: to show that the Hamiltonian is well defined on a dense domain, and that its heat kernel is trace class. This requires a detailed analysis of a suitably regularized form of the Hamiltonian.

I.B. In this paper, we choose a particular regularization, namely the sharp ultraviolet cutoff M. This amounts to suppressing all the modes with $|p| > M$ in the Fourier expansion of the field operators. The regularized Hamiltonian has then the following form. Let δ_M be the following regularized delta function,

$$\delta_M(\sigma) = \frac{\sin(M + 1/2)\sigma}{\sin \sigma/2}. \tag{I.1}$$

By γ_μ we denote the two-dimensional Dirac matrices,

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{I.2}$$

and define the chiral projections,

$$\Lambda_\pm = \frac{1}{2}\,(I \pm \gamma_0\gamma_1). \tag{I.3}$$

Finally, define the following operator

$$D_V(\sigma) = i\gamma_1\partial_\sigma + \Lambda_+ P_M \nabla^2 V(\phi(\sigma)) + \Lambda_- P_M \overline{\nabla^2 V(\phi(\sigma))}, \tag{I.4}$$

where the projection operator P_M is defined by $(P_M f)(\sigma) = \int_{S^1} \delta_M(\sigma - \tau) f(\tau)\, d\tau$. Then

$$\begin{aligned} \overline{\Box}_V = & \int_{S^1} \left(\overline{\pi}(\sigma)\pi(\sigma) + \partial_\sigma \overline{\phi}(\sigma) \partial_\sigma \phi(\sigma)\right) d\sigma + \int_{S^1} \overline{\psi}(\sigma) D_V(\sigma) \psi(\sigma)\, d\sigma \\ & + \int_{S^1 \times S^1} \overline{\nabla V(\phi(\sigma))}\, \delta_M(\sigma - \tau)\, \nabla V(\phi(\tau))\, d\sigma\, d\tau, \end{aligned} \tag{I.5}$$

where $\pi(\sigma)$ is the canonical momentum operator, and where

$$\psi(\sigma) = \begin{pmatrix} \psi_1(\sigma) \\ \psi_2(\sigma) \end{pmatrix}, \qquad \overline{\psi}(\sigma) = \psi^*(\sigma)\, \gamma_0 = \begin{pmatrix} \psi_2^*(\sigma) & \psi_1^*(\sigma) \end{pmatrix}. \tag{I.6}$$

Throughout this paper, we work with the regularized theory only. All our results are, however, uniform in the cutoff M, and we believe that they hold true for a suitably defined limit $M \to \infty$.

I.C. A particular feature of the above regularized Hamiltonian is that it is a square of a fermionic operator, the supercharge. Specifically, with

$$\begin{aligned} \overline{Q}_V = \frac{1}{\sqrt{2}} \int_{S^1} \big(& - i\psi_1(\sigma)(\pi(\sigma) - \partial_\sigma \phi(\sigma)) + \psi_2(\sigma)(\overline{\pi}(\sigma) + \partial_\sigma \overline{\phi}(\sigma)) \\ & + i\psi_1(\sigma) \nabla V(\phi(\sigma)) + \psi_2(\sigma) \overline{\nabla V(\phi(\sigma))} \big) d\sigma + \text{herm. conj.}, \end{aligned} \tag{I.7}$$

we have $\overline{\Box}_V = \overline{Q}_V^2$. This property indicates that $\overline{Q}_V$ is a Dirac type operator, and the natural problem is to study the properties of its kernel and in particular its index. Physically, the kernel of $\overline{Q}_V$ consists of the (zero energy) ground states of the Hamiltonian. It is easy to see that $\overline{Q}_V$ has the structure $\overline{Q}_V = \overline{\partial}_V + \overline{\partial}_V^*$, where $\overline{\partial}_V$ is a coboundary operator, and where $\overline{\partial}_V^*$ denotes its hermitian adjoint.

In this paper we are concerned with the cohomology groups of the operator $\overline{\partial}_V$ which we believe are related the space of harmonic forms of $\overline{\Box}_V$. We explicitly compute the smooth and compactly supported cohomology groups of this operator, and show that they depend only on the singularity structure of the superpotential. We establish a vanishing theorem for the smooth cohomology associated with the operator $\overline{\partial}_V$: only cohomology groups of definite parity are non-trivial. Furthermore, we show that the cohomology groups are independent of the ultraviolet cutoff M. We believe that similar results hold for the space of harmonic forms of $\overline{\Box}_V$, and present partial arguments to support this hypothesis.

I.D. The paper is organized as follows. In Section II we define the smooth cohomology complex which corresponds to the Landau - Ginzburg model and state our main results. These results are proved in Sections III and IV. In Section V we define the square integrable cohomology corresponding to the Landau - Ginzburg model, formulate a technical conjecture, and study its consequences.

II. The Landau - Ginzburg complex

II.A. In this section we define the equivariant cohomology of a perturbed $\overline{\partial}$ operator on the space $E_M = \mathbb{C}^n \times \mathbb{C}^{2nM}$, where $n \geq 1$ and $M \geq 0$ are integers. We will regard E_M as a vector bundle over $\mathbb{C}^n$ with fiber $\mathbb{C}^{2nM}$. We represent a point $z \in E_M$ as $z = (z_{-M}, \ldots, z_{-1}, z_0, z_1, \ldots, z_M)$, where $z_p \in \mathbb{C}^n$. In the language of quantum field theory, z_0 represents the zero modes while the z_p's with $1 \leq |p| \leq M$ represent the excited modes. The integer M is the ultraviolet cutoff.

The relevant complex is defined as follows. We let $\bigwedge^{p,q}(E_M)$ denote the space of smooth (p,q)-forms on E_M and let $\overline{\partial} : \bigwedge^{p,q}(E_M) \to \bigwedge^{p,q+1}(E_M)$ and $\partial : \bigwedge^{p,q}(E_M) \to \bigwedge^{p+1,q}(E_M)$ denote the usual Dolbeault coboundary operators.

Let S^1 be the circle of circumference 1. There is a natural S^1-action on E_M,

$$z_p \to e^{-2\pi i p\sigma} z_p, \qquad \sigma \in S^1, \qquad |p| \leq M, \tag{II.1}$$

which is generated by the following holomorphic vector field:

$$K(z) = -2\pi i \sum_{|p| \leq M} p z_p \nabla_p, \tag{II.2}$$

where $\nabla_p = \partial / \partial z_p$. K acts on forms by interior multiplication; we denote the corresponding operator mapping $\bigwedge^{p,q}(E_M)$ into $\bigwedge^{p-1,q}(E_M)$ by $i(K)$. Clearly, $i(K)^2 = 0$ and $\overline{\partial}\, i(K) + i(K)\, \overline{\partial} = 0$. As a consequence, the operator

$$\overline{\partial}_0 = \overline{\partial} + i(K) \tag{II.3}$$

satisfies $\overline{\partial}_0^2 = 0$. It is an S^1-equivariant version of the Dolbeault operator.

II.B. We will be concerned with the cohomology of a perturbation of $\overline{\partial}_0$ by a holomorphic 1-form. Let $\bigwedge(E_M) = \bigoplus_{p,q} \bigwedge^{p,q}(E_M)$ be the Grassmann algebra over E_M. We define the usual $\mathbb{Z}_2$-grading on $\bigwedge(E_M)$,

$$\bigwedge(E_M) = \bigwedge(E_M)^0 \oplus \bigwedge(E_M)^1, \tag{II.4}$$

where

$$\bigwedge(E_M)^0 = \bigoplus_{p+q\, even} \bigwedge^{p,q}(E_M), \qquad \bigwedge(E_M)^1 = \bigoplus_{p+q\, odd} \bigwedge^{p,q}(E_M). \tag{II.5}$$

For $\sigma \in S^1$ we define the following field operators on $\bigwedge(E_M)$:

$$
\begin{aligned}
\overline{b}^*(\sigma) &= \sum_{|p|\le M} e^{2\pi i p\sigma} d\overline{z}_p \wedge, \\
b^*(\sigma) &= \sum_{|p|\le M} e^{2\pi i p\sigma} dz_p \wedge, \\
\overline{b}(\sigma) &= \sum_{|p|\le M} e^{-2\pi i p\sigma} i(\overline{\nabla}_p), \\
b(\sigma) &= \sum_{|p|\le M} e^{-2\pi i p\sigma} i(\nabla_p), \\
\phi(\sigma) &= \sum_{|p|\le M} e^{-2\pi i p\sigma} z_p, \\
\overline{\phi}(\sigma) &= \sum_{|p|\le M} e^{2\pi i p\sigma} \overline{z}_p, \\
K(\sigma) &= -2\pi i \sum_{|p|\le M} p\, e^{-2\pi i p\sigma} z_p = \partial_\sigma \phi(\sigma), \\
\overline{K}(\sigma) &= 2\pi i \sum_{|p|\le M} p\, e^{2\pi i p\sigma} \overline{z}_p = \partial_\sigma \overline{\phi}(\sigma), \\
\overline{\pi}(\sigma) &= -i \sum_{|p|\le M} e^{-2\pi i p\sigma} \overline{\partial}_p, \\
\pi(\sigma) &= -i \sum_{|p|\le M} e^{2\pi i p\sigma} \partial_p
\end{aligned}
\tag{II.6}
$$

(these are actually vectors of operators; we will put the indices whenever necessary). In terms of these operators,

$$
\overline{\partial}_0 = i \int_{S^1} \overline{b}^*(\sigma)\, \overline{\pi}(\sigma)\, d\sigma + \int_{S^1} b(\sigma)\, \partial_\sigma \phi(\sigma)\, d\sigma. \tag{II.7}
$$

Let now $V : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic polynomial (the superpotential). We consider the following function on E_M:

$$
h(z) = \int_{S^1} V(\phi(\sigma))\, d\sigma = \int_{S^1} V\Big(\sum_{|p|\le M} e^{-2\pi i p\sigma} z_p \Big)\, d\sigma. \tag{II.8}
$$

Clearly, h is holomorphic on E_M. Its differential dh is a holomorphic 1-form on E_M and acts on $\bigwedge(E_M)$ by exterior multiplication. We denote the corresponding operator mapping $\bigwedge^{p,q}(E_M)$ into $\bigwedge^{p+1,q}(E_M)$ by δ_V. In terms of the field operators (II.6),

$$
\delta_V = \int_{S^1} b^*(\sigma)\, \nabla V(\phi(\sigma))\, d\sigma. \tag{II.9}
$$

Clearly, $\delta_V^2 = 0$, and $\overline{\partial}\,\delta_V + \delta_V\,\overline{\partial} = 0$. Furthermore,

$$
\begin{aligned}
i(K)\,\delta_V + \delta_V\,i(K) &= \int_{S^1} \nabla V(\phi(\sigma))\,\partial_\sigma\phi(\sigma)\,d\sigma \\
&= \int_{S^1} \frac{d}{d\sigma} V(\phi(\sigma))\,d\sigma = 0.
\end{aligned}
$$

As a consequence, the operator

$$\overline{\partial}_V = \overline{\partial} + i(K) + \delta_V = \overline{\partial}_0 + \delta_V \tag{II.10}$$

satisfies $\overline{\partial}_V^2 = 0$. Note also that $\overline{\partial}_V$ is odd with respect to the $\mathbb{Z}_2$-grading defined in (II.4). We thus have a complex

$$\ldots \xrightarrow{\overline{\partial}_V} \bigwedge(E_M)^0 \xrightarrow{\overline{\partial}_V} \bigwedge(E_M)^1 \xrightarrow{\overline{\partial}_V} \bigwedge(E_M)^0 \xrightarrow{\overline{\partial}_V} \ldots . \tag{II.11}$$

Let $H^*_{\overline{\partial}_V}(E_M)$ denote the cohomology of this complex. For future reference, we observe that $H^*_{\overline{\partial}_V}(E_M)$ arises as the total cohomology of the following double complex:

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\ldots \arrow[r, "\Delta_V"] & \bigwedge^{2k+1}(E_M)^0 \arrow[u, "\overline{\partial}"] \arrow[r, "\Delta_V"] & \bigwedge^{2k+1}(E_M)^1 \arrow[u, "\overline{\partial}"] \arrow[r, "\Delta_V"] & \ldots \\
\ldots \arrow[r, "\Delta_V"] & \bigwedge^{2k}(E_M)^0 \arrow[u, "\overline{\partial}"] \arrow[r, "\Delta_V"] & \bigwedge^{2k}(E_M)^1 \arrow[u, "\overline{\partial}"] \arrow[r, "\Delta_V"] & \ldots \\
 & \vdots \arrow[u, "\overline{\partial}"] & \vdots \arrow[u, "\overline{\partial}"] &
\end{tikzcd}
\tag{II.12}
$$

where

$$\bigwedge^q(E_M)^0 = \bigoplus_{p\,even} \bigwedge^{p,q}(E_M), \qquad \bigwedge^q(E_M)^1 = \bigoplus_{p\,odd} \bigwedge^{p,q}(E_M), \tag{II.13}$$

and where

$$\Delta_V = i(K) + \delta_V. \tag{II.14}$$

Finally, let $\Omega^k(\mathbb{C}^n)$ denote the space of holomorphic k-forms on $\mathbb{C}^n$ and let $K_V^*(\mathbb{C}^n)$ be the cohomology of the following Koszul complex:

$$\ldots \xrightarrow{dV} \Omega^k(\mathbb{C}^n) \xrightarrow{dV} \Omega^{k+1}(\mathbb{C}^n) \xrightarrow{dV} \ldots . \tag{II.15}$$

II.C. Our first main result is contained in the following theorem.

Theorem II.1. *We have the following isomorphisms:*

$$\begin{aligned} H^0_{\overline{\partial}_V}(E_M) &\simeq \bigoplus_j K_V^{2j}(\mathbb{C}^n), \\ H^1_{\overline{\partial}_V}(E_M) &\simeq \bigoplus_j K_V^{2j+1}(\mathbb{C}^n). \end{aligned} \tag{II.16}$$

We prove this theorem in Section III.

The complex (II.11) is precisely the complex which was studied in [CGP] and [KL1] in connection with $N = 2$ supersymmetric quantum mechanics (which corresponds to $M = 0$ in our notation). The above theorem asserts that the topological content of the regularized $N = 2$ supersymmetric Landau - Ginzburg theory is identical to that of the zero mode limit of the theory. Furthermore, it is well known (see e.g. [GH]) that if the critical set of V, $cr(V)$, is finite, then

$$K_V^k(\mathbb{C}^n) \simeq \begin{cases} 0, & \text{if } k < n; \\ \Omega^n(\mathbb{C}^n)/dV \wedge \Omega^{n-1}(\mathbb{C}^n), & \text{if } k = n. \end{cases} \tag{II.17}$$

We thus obtain the following corollary.

Corollary II.2. *Let $cr(V)$ be finite. Then*

$$H^0_{\overline{\partial}_V}(E_M) \simeq \begin{cases} \Omega^n(\mathbb{C}^n)/dV \wedge \Omega^{n-1}(\mathbb{C}^n), & \textit{if } n \textit{ is even;} \\ 0, & \textit{if } n \textit{ is odd;} \end{cases} \tag{II.18}$$

and

$$H^1_{\overline{\partial}_V}(E_M) \simeq \begin{cases} 0, & \textit{if } n \textit{ is even;} \\ \Omega^n(\mathbb{C}^n)/dV \wedge \Omega^{n-1}(\mathbb{C}^n), & \textit{if } n \textit{ is odd.} \end{cases} \tag{II.19}$$

The space $\Omega^n(\mathbb{C}^n)/dV \wedge \Omega^{n-1}(\mathbb{C}^n)$ has dimension equal to $\#cr(V)$, the number of critical points of V. We thus obtain the following result.

Corollary II.3. *Let $cr(V)$ be finite. Then*

$$\begin{aligned} \dim H^0_{\overline{\partial}_V}(E_M) + \dim H^1_{\overline{\partial}_V}(E_M) &= \#cr(V), \\ \dim H^0_{\overline{\partial}_V}(E_M) - \dim H^1_{\overline{\partial}_V}(E_M) &= (-1)^n \#cr(V). \end{aligned} \tag{II.20}$$

II.D. Our second main result concerns the compactly supported cohomology of $\overline{\partial}_V$. We let $H^*_{\overline{\partial}_V,comp}(E_M)$ denote the compactly supported cohomology of $\overline{\partial}_V$. In other words, $H^*_{\overline{\partial}_V,comp}(E_M)$ arises as the cohomology of the complex (II.11) with $\bigwedge^{p,q}(E_M)$ replaced by the corresponding space $\bigwedge^{p,q}_{comp}(E_M)$ of smooth forms with compact supports.

Theorem II.4. *Let $cr(V)$ be finite. Then there is an isomorphism*

$$i: H^*_{\overline{\partial}_V,comp}(E_M) \xrightarrow{\sim} H^*_{\overline{\partial}_V}(E_M). \tag{II.21}$$

We prove this theorem in Section IV.

III. The smooth cohomology

III.A. Our proof of Theorem II.1 is based on two lemmas which we first formulate and prove. Let $\Omega^k(E_M)$ denote the space of holomorphic k-forms on E_M and let

$$\Omega(E_M)^0 = \bigoplus_{k\,even} \Omega^k(E_M), \qquad \Omega(E_M)^1 = \bigoplus_{k\,odd} \Omega^k(E_M). \tag{III.1}$$

Consider the following complex

$$\ldots \xrightarrow{\Delta_V} \Omega(E_M)^0 \xrightarrow{\Delta_V} \Omega(E_M)^1 \xrightarrow{\Delta_V} \Omega(E_M)^0 \xrightarrow{\Delta_V} \ldots, \tag{III.2}$$

and let $H^*_{\Delta_V}(E_M)$ denote its cohomology groups.

Lemma III.1. *The complex (II.11) is quasi-isomorphic to the complex (III.2).*

Proof. We use the technique of spectral sequences (see e.g. [McC]). We observe that the first filtration associated with the double complex (II.12) is bounded. Therefore, $'E_r^{*,*}$ converges. By Dolbeault's lemma,

$$'E_1^{p,q} \simeq \begin{cases} 0, & \text{if } q > 0; \\ \Omega(E_M)^0, & \text{if } q = 0,\ p \text{ even}; \\ \Omega(E_M)^1, & \text{if } q = 0,\ p \text{ odd}; \end{cases} \tag{III.3}$$

and the claim follows. □

We let $s: \mathbb{C}^n \to E_M$ denote the natural holomorphic embedding,

$$\mathbb{C}^n \ni z_0 \longrightarrow s(z_0) = (0, \ldots, 0, z_0, 0, \ldots, 0) \in E_M, \tag{III.4}$$

and consider the spaces

$$\tilde{\Omega}^k(E_M) = \{\omega \in \Omega^k(E_M): \quad s^\sharp\omega = 0\}, \tag{III.5}$$

and the corresponding spaces $\tilde{\Omega}(E_M)^*$ defined in analogy with (III.1). In the above expression, $s^\sharp\omega$ denotes the pullback of ω under s. We note that if $s^\sharp\omega = 0$, then also $s^\sharp\,\Delta_V\omega = 0$, and so the following complex is defined:

$$\ldots \xrightarrow{\Delta_V} \tilde{\Omega}(E_M)^0 \xrightarrow{\Delta_V} \tilde{\Omega}(E_M)^1 \xrightarrow{\Delta_V} \tilde{\Omega}(E_M)^0 \xrightarrow{\Delta_V} \ldots, \tag{III.6}$$

We let $\tilde{H}^*_{\Delta_V}(E_M)$ denote its cohomology groups.

Lemma III.2. *The complex (III.6) has a trivial cohomology.*

Proof. We observe that $\tilde{H}^*_{\Delta_V}(E_M)$ arises as the total cohomology of the following double complex:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle i(K)} & & \uparrow{\scriptstyle i(K)} & & \\
\dots & \xrightarrow{\delta_V} & \tilde{\Omega}^{k-1}(E_M) & \xrightarrow{\delta_V} & \tilde{\Omega}^{k}(E_M) & \xrightarrow{\delta_V} & \dots \\
 & & \uparrow{\scriptstyle i(K)} & & \uparrow{\scriptstyle i(K)} & & \\
\dots & \xrightarrow{\delta_V} & \tilde{\Omega}^{k}(E_M) & \xrightarrow{\delta_V} & \tilde{\Omega}^{k+1}(E_M) & \xrightarrow{\delta_V} & \dots \\
 & & \uparrow{\scriptstyle i(K)} & & \uparrow{\scriptstyle i(K)} & & \\
 & & \vdots & & \vdots & &
\end{array}
\tag{III.7}
$$

We claim that $\tilde{H}^*_{\Delta_V}(E_M) \simeq 0$. To prove this, consider a column in (III.7):

$$
0 \longrightarrow \tilde{\Omega}^{(2M+1)n}(E_M) \xrightarrow{\delta_V} \dots \xrightarrow{\delta_V} \tilde{\Omega}^{1}(E_M) \xrightarrow{\delta_V} \tilde{\Omega}^{0}(E_M) \longrightarrow 0. \tag{III.8}
$$

We will construct a homotopy operator for (III.8). We write $z_p = (z_{p1}, \dots, z_{pM})$, $z_{p\alpha} \in \mathbb{C}$, $|p| \leq M$, and represent a form $\theta \in \tilde{\Omega}^k(E_M)$ as

$$
\theta(z) = \sum_m \sum_{\alpha_1,\dots,\alpha_m} \omega_{\alpha_1\dots\alpha_m}(z)\, dz_{0\alpha_1} \wedge \dots \wedge dz_{0\alpha_m}, \tag{III.9}
$$

where the forms $\omega_{\alpha_1\dots\alpha_m}(z)$ do not involve factors of $dz_{0\alpha}$, $1 \leq \alpha \leq n$. Let now $\omega(z) = f(z) dz_{p_1\alpha_{p_1}} \wedge \dots \wedge dz_{p_k\alpha_{p_k}}$, $|p_1| \geq 1, \dots, |p_k| \geq 1$, be a homogeneous component in $\omega_{\alpha_1\dots\alpha_m}(z)$. We set

$$
(J\omega)(z) = \frac{i}{2\pi} \sum_{|p|\geq 1,\alpha} \frac{1}{p} \int_0^1 t^k \frac{\partial}{\partial z_{p\alpha}} f(z_0, tz')\, dt\, dz_{p\alpha} \wedge dz_{p_1\alpha_{p_1}} \wedge \dots \wedge dz_{p_k\alpha_{p_k}}, \tag{III.10}
$$

where $z' = (z_{-M}, \dots, z_{-1}, z_1, \dots, z_M) \in \mathbb{C}^{2nM}$. This defines an operator $J : \tilde{\Omega}^k(E_M) \to \tilde{\Omega}^{k+1}(E_M)$. An elementary (if slightly tedious) computation shows that

$$
(i(K)\, J + J\, i(K))\, \omega(z) = \int_0^1 \frac{d}{dt} (t^k f(z_0, tz'))\, dt\, dz_{p_1\alpha_{p_1}} \wedge \dots \wedge dz_{p_k\alpha_{p_k}}. \tag{III.11}
$$

We claim that $\int_0^1 \frac{d}{dt} (t^k f(z_0, tz'))\, dt = f(z)$. Indeed, this is clear if $k > 0$. If $k = 0$, then $\omega(z) = f(z)$, and $f(z_0, 0) = s^\sharp \omega(z) = 0$ which implies our claim. Consequently, (III.11) equals $\omega(z)$, which means that $i(K)\,J + J\,i(K) = \mathrm{Id}$. We have thus shown that J is a homotopy operator for the complex (III.8).

It is now easy to show that the total cohomology of (III.7) is zero. Indeed, the first filtration associated with (III.7) is bounded, and thus $'E_r^{*,*}$ converges. Since, by the above argument, the rows of (III.7) have zero cohomologies, it follows that $'E_1^{*,*} \simeq 0$, and the lemma is proven. □

III.B. We are now ready to prove Theorem II.1.
Proof of Theorem II.1. Observe that

$$\Omega(E_M)^*/\tilde{\Omega}(E_M)^* \simeq \Omega(\mathbb{C}^n)^*, \tag{III.12}$$

which yields the following short exact sequence of complexes

$$0 \longrightarrow \tilde{\Omega}(E_M)^* \longrightarrow \Omega(E_M)^* \longrightarrow \Omega(\mathbb{C}^n)^* \longrightarrow 0. \tag{III.13}$$

¿From the associated long sequence of cohomology groups

$$\begin{aligned} \ldots \longrightarrow \tilde{H}^0_{\Delta_V}(E_M) \longrightarrow H^0_{\Delta_V}(E_M) \longrightarrow \bigoplus_j K_V^{2j}(\mathbb{C}^n) \\ \longrightarrow \tilde{H}^1_{\Delta_V}(E_M) \longrightarrow H^1_{\Delta_V}(E_M) \longrightarrow \bigoplus_j K_V^{2j+1}(\mathbb{C}^n) \longrightarrow \ldots, \end{aligned}$$

and Lemma III.2 it follows that $H_{\delta_V}{}^*(E_M) \simeq \bigoplus_j K_V^{2j+*}(\mathbb{C}^n)$. But by Lemma III.1 $H_{\Delta_V}^*(E_M) \simeq H_{\overline{\partial}_V}{}^*(E_M)$ and the theorem is proven. □

IV. The compactly supported cohomology

IV.A. The proof of Theorem II.4 is based on the following lemma.

Lemma IV.1. *Let $U \subset E_M$ be an open set such that $U \cap s(cr(V)) = \emptyset$ ($cr(V)$ is not assumed to be finite). Then the cohomology of $\overline{\partial}_V$ restricted to U is trivial.*
Proof. Consider the double complex (II.12) with E_M replaced by U. The second filtration associated with (II.12) is bounded, and thus $''E_r^{*,*}$ converges. We claim that $''E_1^{*,*} \simeq 0$. Indeed, let L be the following operator:

$$L = W(z)^{-1} \int_{S^1} (b^*(\sigma)\, \overline{\partial_\sigma \phi(\sigma)} + b(\sigma)\, \overline{\nabla V(\phi(\sigma))})\, d\sigma, \tag{IV.1}$$

where

$$W(z) = \int_{S^1} |\partial_\sigma \phi(\sigma)|^2\, d\sigma + \int_{S^1\times S^1} \overline{\nabla V(\phi(\sigma))}\, \delta_M(\sigma-\tau) \nabla V(\phi(\tau))\, d\sigma d\tau, \tag{IV.2}$$

where the regularized delta function δ_M is given by (I.1). Since $W(z) \neq 0$, for $z \notin s(cr(V))$, L is well defined. It is easy to see that $L\,\Delta_V + \Delta_V L = \mathrm{Id}$, i.e. L is a homotopy operator for Δ_V. This proves that $''E_1^{*,*} \simeq 0$. □

IV.B. We can now prove Theorem II.4.

Proof of Theorem II.4. Since the coboundary operator $\overline{\partial}_V$ is local, the complex of compactly supported smooth forms is a subcomplex of the complex of smooth forms. Let i denote the natural injection. We claim i induces an isomorphism of cohomologies. Indeed: *(i) i is a monomorphism.* To prove this, we consider a compactly supported form ω such that $\omega = \overline{\partial}_V \eta$, $\eta \in \bigwedge(E_M)$. Let D_1 and D_2 be open balls such that $\operatorname{supp} \omega \cup s(cr(V)) \subset D_1 \subset \overline{D}_1 \subset D_2$, and let U be the complement of $\overline{D}_1$. Then $\overline{\partial}_V \eta = 0$ on U. As a consequence of Lemma IV.1, there is a smooth form ζ on U such that $\eta = \overline{\partial}_V \zeta$ on U. We choose a smooth function χ on E_M such that

$$\chi = \begin{cases} 0, & \text{on } \overline{D}_1; \\ 1, & \text{on } D_2^c, \end{cases} \tag{IV.3}$$

and set $\eta' = \eta - \overline{\partial}_V(\chi\zeta)$. Then η' is compactly supported, cohomologous to η, and $\omega = \overline{\partial}_V \eta'$ on E_M. This shows that ω is trivial in compactly supported cohomology.

(ii) i induces an epimorphism. Let ω be smooth and let $\overline{\partial}_V \omega = 0$. Let D_1 and D_2 be two open balls in E_M such that $s(cr(V)) \subset D_1 \subset \overline{D}_1 \subset D_2$, and let U be the complement of $\overline{D}_1$. As a consequence of Lemma IV.1, there is a smooth form ψ on U such that $\omega = \overline{\partial}_V \psi$ on U. We choose χ as in (IV.3) and set $\omega' = \omega - \overline{\partial}_V(\chi\psi)$. The ω' is compactly supported, closed, and cohomologous to ω. As a consequence, the cohomology class of ω contains a compactly supported representative. □

V. The L^2-cohomology

V.A. We now turn to the analytic part of this study, namely the square integrable cohomology of the operator $\overline{\partial}_V$ and its relation to the previously studied smooth cohomologies. The content of this section has a largely conjectural character, as the proofs of some crucial technical results are still missing.

Let $\star : \bigwedge^{p,q}(E_M) \to \bigwedge^{(2M+1)n-q,(2M+1)n-p}(E_M)$ be the Hodge star operator, and let $(\omega, \eta) = \int_{E_M} \star\omega \wedge \eta$ be the usual inner product defined on $\bigwedge^{p,q}_{comp}(E_M)$. We let $\bigwedge^{p,q}_2(E_M)$ denote the completion of $\bigwedge^{p,q}_{comp}(E_M)$ in the norm induced by this inner product. By $\bigwedge_2(E_M)$ we denote the direct sum of the above Hilbert spaces, and by $\bigwedge_2(E_M)_0$ and $\bigwedge_2(E_M)_1$ we denote its even and odd subspaces, respectively. The operators (II.6) act on $\bigwedge_2(E_M)$. Note that the fermionic operators are bounded while the bosonic operators are unbounded operators, defined on the dense invariant domain $\mathcal{D}_0 = \bigwedge_{comp}(E_M)$ (say). Note also that $\overline{b}^*(\sigma), b^*(\sigma), \overline{\phi}(\sigma), \overline{K}(\sigma)$ and $\overline{\pi}(\sigma)$ are the respective adjoints of $\overline{b}(\sigma), b(\sigma), \phi(\sigma), K(\sigma)$, and $\pi(\sigma)$. We will use the following notation

$$\overline{b}^*_p = d\overline{z}_p \wedge, \quad \overline{b}_p = i(\overline{\nabla}_p), \quad b^*_p = dz_p \wedge, \quad b_p = i(\nabla_p). \tag{V.1}$$

V.B. Consider now the Dirac type operator

$$\overline{Q}_V = \overline{\partial}_V + \overline{\partial}_V^*, \tag{V.2}$$

defined on $\mathcal{D}_0$. Clearly, $\overline{Q}_V$ is symmetric and odd with respect to the $\mathbb{Z}_2$-grading on $\bigwedge_2(E_M)$. Its square, $\overline{\Box}_V = \overline{Q}_V^2 \geq 0$, is given by

$$\begin{aligned}\overline{\Box}_V = &\int_{S^1} \big(\overline{\pi}(\sigma)\pi(\sigma) + \partial_\sigma\overline{\phi}(\sigma)\partial_\sigma\phi(\sigma)\big)\, d\sigma + \int_{S^1} \big(b^*(\sigma)\partial_\sigma\overline{b}^*(\sigma) - \overline{b}(\sigma)\partial_\sigma b(\sigma)\big)\, d\sigma \\ &+ \int_{S^1} \big(\overline{b}^*(\sigma)\overline{\nabla^2 V(\phi(\sigma))}b(\sigma) + b^*(\sigma)\nabla^2 V(\phi(\sigma))\overline{b}(\sigma)\big)\, d\sigma \\ &+ \int_{S^1\times S^1} \overline{\nabla V(\phi(\sigma))}\,\delta_M(\sigma-\tau)\,\nabla V(\phi(\tau))\, d\sigma\, d\tau,\end{aligned} \tag{V.3}$$

where δ_M is given by (I.1). The Friedrichs extension of $\overline{\Box}_V$ defines a positive self-adjoint operator which we denote by the same symbol. The above Laplace operator is precisely the Hamiltonian of the regularized $N = 2$ supersymmetric Landau - Ginzburg quantum field theory. We define X to be the generator of the circle action (II.1) on $\bigwedge_2(E_M)$. Explicitly,

$$X = \int_{S^1} \big(\partial_\sigma\overline{\phi}(\sigma)\overline{\pi}(\sigma) - \pi(\sigma)\partial_\sigma\phi(\sigma)\big)\, d\sigma + \int_{S^1} \big(b^*(\sigma)i\partial_\sigma b(\sigma) - i\partial_\sigma\overline{b}(\sigma)\overline{b}^*(\sigma)\big)\, d\sigma, \tag{V.4}$$

or, in terms of the Fourier modes,

$$X = \sum_{|p|\leq M} p(\overline{b}_p^*b_p + b_p^*b_p) + \sum_{|p|\leq M} p(z_p\nabla_p + \overline{z}_p\overline{\nabla}_p). \tag{V.5}$$

In the physics language, X is the momentum operator.

To make contact with the usual field theoretical expressions for the Hamiltonian (V.3) and the momentum operator (V.4), we introduce the Dirac field operators,

$$\psi_1(\sigma) = \frac{1}{\sqrt{2}}\,(\overline{b}(\sigma) - ib(\sigma)^*), \qquad \psi_2(\sigma) = \frac{1}{\sqrt{2}}\,(b(\sigma) + i\overline{b}(\sigma)^*). \tag{V.6}$$

Then $\overline{\Box}_V$ is given by (I.5), and

$$X = \int_{S^1} (\partial_\sigma\overline{\phi}(\sigma)\overline{\pi}(\sigma) - \pi(\sigma)\partial_\sigma\phi(\sigma))\, d\sigma + \int_{S^1} \overline{\psi}(\sigma)i\gamma_0\partial_\sigma\psi(\sigma)\, d\sigma. \tag{V.7}$$

V.C. Below we formulate a conjecture concerning the analytic properties of $\overline{\Box}_V$ which will be relevant for our purposes. The particular properties whose validity we conjecture to hold are motivated by the results obtained in [KL1], [AO], and [BI] for the case of supersymmetric quantum mechanics, $M = 0$. To formulate the conjecture, we need to introduce a technical assumption on the superpotential V. We say that V is elliptic if, for each multiindex α, there exist positive constants ϵ_α and C_α such that for all z,

$$|\partial^\alpha V(z)| \geq \epsilon_\alpha |z|^{d-|\alpha|} - C_\alpha, \tag{V.8}$$

where d denotes the algebraic degree of V. In other words, we exclude superpotentials V which have flat directions.

Conjecture V.1. *Let V be an elliptic superpotential. Then:*
(i) for all $t \geq 0$,

$$\mathrm{Tr}(\exp\{-\,t\,\overline{\Box}_V\}) < 0. \tag{V.9}$$

(ii) Every eigenvector ω of $\overline{\Box}_V$ is smooth. Furthermore, there exist constants $a > 0$ and C such that

$$|\omega(z)| \leq C \exp\{-a|z|\}. \tag{V.10}$$

V.D. We now consider the square integrable cohomology $H^*_{\overline{\partial}_V,2}(E_M)$, defined as the cohomology of the complex (II.11) with $\bigwedge^{p,q}(E_M)$ replaced by the corresponding space $\bigwedge_2^{p,q}(E_M)$ of square integrable forms, and with the operator $\overline{\partial}_V$ defined as the closure of the corresponding operator with domain $\mathcal{D}_0$. Let $\mathrm{Harm}_V(E_M) = \{\omega \in \bigwedge_2(E_M) : \overline{\Box}_V\,\omega = 0\}$ denote the space of harmonic forms of $\overline{\Box}_V$, and let $\mathrm{Harm}_V(E_M)^*$ denote the corresponding even and odd subspaces. The following corollary to Conjecture V.1 is a version of Hodge's theorem.

Corollary V.2. *Let V be elliptic and assume that Conjecture V.1 is true. Then:*
(i) $\dim \mathrm{Harm}_V(E_M) < \infty$*;*
(ii) we have the decomposition

$$\bigwedge\nolimits_2(E_M)^* = \mathrm{Harm}_V(E_M)^* \oplus \overline{\partial}_V\big(\overline{\partial}_V^* G_V \bigwedge\nolimits_2(E_M)^*\big) \oplus \overline{\partial}_V^*\big(\overline{\partial}_V G_V \bigwedge\nolimits_2(E_M)^*\big), \tag{V.11}$$

where G_V is a self-adjoint compact operator;
(iii) there is a canonical isomorphism

$$H^*_{\overline{\partial}_V,2}(E_M) \simeq \mathrm{Harm}_V(E_M)^*. \tag{V.12}$$

Proof. The proof follows the standard arguments (see e.g. [GH]). Part (i) follows immediately from part (i) of Conjecture V.1. To prove part (ii), we set

$$G_V = \begin{cases} 0, & \text{on } \mathrm{Harm}_V(E_M)^*; \\ \overline{\Box}_V^{-1}, & \text{on the orthogonal complement of } \mathrm{Harm}_V(E_M)^*. \end{cases}$$

Part (iii) is a consequence of part (ii). $\Box$

V.E. The next corollary to Conjecture V.1 is a vanishing theorem for $\overline{Q}_V$ (a special case of the vanishing theorem was first conjectured in [JL]). It states that the kernel of $\overline{Q}_V$ consists of forms which are either purely bosonic or purely fermionic, depending on the superpotential.

Corollary V.3. *Assume that Conjecture V.1 is true. Then there is an isomorphism,*

$$\mathrm{Harm}_V(E_M)^* \simeq H^*_{\overline{\partial}_V}(E_M). \tag{V.13}$$

In particular, the kernel of $\overline{Q}_V$ consists of elements of definite parity.

Proof. Since $\overline{\Box}_V$ is elliptic, the harmonic forms of $\overline{\Box}_V$ are smooth, and so we have a homomorphism $i : \mathrm{Harm}_V(E_M)^* \to H^*_{\overline{\partial}_V}(E_M)$. We assert that i is an isomorphism.

(i) i is a monomorphism. Let ω be harmonic, and let $[\omega]$ be its image in $H^*_{\overline{\partial}_V}(E_M)$. Assume that $[\omega] = 0$, i.e. $\omega = \overline{\partial}_V \eta$. We claim that this implies $\omega = 0$. For the proof, we need the following result [H].

Lemma V.4. *There exists an operator $J : \bigwedge^{p,q}(\mathbb{C}^D) \to \bigwedge^{p,q-1}(\mathbb{C}^D)$, $q \geq 1$, such that J maps polynomially bounded forms into polynomially bounded forms and*

$$J\overline{\partial} + \overline{\partial} J = I. \tag{V.14}$$

We verify easily the identity $(\overline{\partial} + \Delta_V)(I + \Delta_V J) = (I + \Delta_V J)\overline{\partial}$, which implies that

$$\overline{\partial}(I + \Delta_V J)^{-1} = (I + \Delta_V J)^{-1}(\overline{\partial} + \Delta_V), \tag{V.15}$$

where the inverse is defined by a formal power series (note that this formal power series terminates, so no convergence questions arise). This, in turn, implies that

$$\overline{\partial}_V = \overline{\partial}_V J(I + \Delta_V J)^{-1}\overline{\partial}_V. \tag{V.16}$$

Applying this identity to η yields

$$\omega = \overline{\partial}_V J(I+\Delta_V J)^{-1}\omega,$$

and so $\omega = \overline{\partial}_V \eta'$, with $\eta' = J(I+\Delta_V J)^{-1}\omega$. Since ω is bounded, and J maps polynomially bounded forms into polynomially bounded forms, this implies that η' is polynomially bounded, and our claim follows.
(ii) i is an epimorphism. By Theorem II.4, every smooth cohomology class $[\omega]$ has a compactly supported representative ω_0. In particular, ω_0 is square integrable, and so by Corollary V.2 it is cohomologous to a harmonic form. $\square$

The above corollary can be also rephrased as the following index theorem.

Corollary V.5. *Assume that Conjecture V.1 is true. Then the index of the Dirac operator $\overline{Q}_V$ is given by*

$$\operatorname{Ind}(\overline{Q}_V) = (-1)^n \# cr(V). \tag{V.17}$$

V.F. We end this section by describing the $N=2$ supersymmetry structure of the Landau-Ginzburg theory and the underlying Kähler geometry. We define a second coboundary operator,

$$\partial_V = i\int_{S^1} b^*(\sigma)\pi(\sigma)\, d\sigma - \int_{S^1} \overline{b}(\sigma)\partial_\sigma\overline{\phi}(\sigma)\, d\sigma + \int_{S^1} \overline{b}^*(\sigma)\overline{\nabla V(\phi(\sigma))}\, d\sigma. \tag{V.18}$$

and the corresponding Dirac type operator,

$$Q_V = \partial_V + \partial_V^*. \tag{V.19}$$

We verify that the square of Q_V is equal to the square of $\overline{Q}_V$, and that $\overline{Q}_V$ and Q_V anticommute. Let us record these facts in the form of the following algebra. As operators on $\mathcal{D}_0$,

$$\{\overline{Q}_V, \overline{Q}_V\} = 2\,\square_V, \qquad \{Q_V, Q_V\} = 2\,\square_V, \qquad \{Q_V, \overline{Q}_V\} = 0. \tag{V.20}$$

where $\{..\}$ denotes the anticommutator. In fact, (V.20) is a consequence of an algebra satisfied by the coboundary operators and their adjoints:

$$\begin{gathered}
\{\overline{\partial}_V, \overline{\partial}_V\} = 0, \quad \{\overline{\partial}_V, \partial_V\} = -iX, \quad \{\overline{\partial}_V, \overline{\partial}_V^*\} = \overline{\square}_V, \quad \{\overline{\partial}_V, \partial_V^*\} = 0, \\
\{\partial_V, \partial_V\} = 0, \quad \{\partial_V, \partial_V^*\} = \overline{\square}_V, \quad \{\partial_V, \overline{\partial}_V^*\} = 0, \quad \{\partial_V^*, \partial_V^*\} = 0, \\
\{\partial_V^*, \overline{\partial}_V^*\} = iX, \quad \{\overline{\partial}_V^*, \overline{\partial}_V^*\} = 0.
\end{gathered} \tag{V.21}$$

These relations are reminiscent of the algebraic relations arising in ordinary Kähler geometry. We also note that the Hilbert space $\bigwedge_2(E_M)$ carries a representation of $sl(2)$. Namely, we define the operators

$$\begin{aligned}
h &= \int_{S^1} (\overline{b}(\sigma)\overline{b}^*(\sigma) - b^*(\sigma)b(\sigma))\, d\sigma = i\int_{S^1} \big(\overline{\psi}_1(\sigma)\overline{\psi}_2(\sigma) - \psi_1(\sigma)\psi_2(\sigma)\big)\, d\sigma, \\
L &= \int_{S^1} b^*(\sigma)\overline{b}(\sigma)^*\, d\sigma = \int_{S^1} \overline{\psi}_1(\sigma)\psi_1(\sigma)\, d\sigma, \\
\Lambda &= \int_{S^1} \overline{b}(\sigma)b(\sigma)\, d\sigma = \int_{S^1} \overline{\psi}_2(\sigma)\psi_2(\sigma)\, d\sigma,
\end{aligned} \tag{V.22}$$

and verify that they satisfy the following set of relations:

$$[h,\, L] = -2L, \qquad [h,\, \Lambda] = 2\Lambda, \qquad [\Lambda,\, L] = h. \tag{V.23}$$

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