# Discrete random electromagnetic Laplacians

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**Abstract**

We consider discrete random magnetic Laplacians in the plane and discrete random electromagnetic Laplacians in higher dimensions. The existence of these objects relies on a theorem of Feldman-Moore which was generalized by Lind to the nonabelian case. For example, it allows to realize ergodic Schrödinger operators with stationary independent magnetic fields on discrete two dimensional lattices including also nonperiodic situations like Penrose lattices. The theorem is generalized here to higher dimensions. The Laplacians obtained from the electromagnetic vector potential are elements of a von Neumann algebra constructed from the underlying dynamical system respectively from the ergodic equivalence relation. They generalize Harper operators which correspond to constant magnetic fields. For independent identically distributed magnetic fields and special Anderson models, we compute the density of states using a random walk expansion.

**Mathematics subject classification:**

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**Keywords:**

Ergodic Schrödinger operators, Harper operator, Magnetic operators on graphs and tilings, Group cohomology of ergodic $\mathbf{Z}^d$ actions, ergodic equivalence relations.

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# 1 Introduction

We consider ergodic discrete Schrödinger operators which we call *discrete random electromagnetic Laplacians.* An example in two dimensions is the bounded ergodic selfadjoint operator $L = A + A^*$ on $l^2(\mathbf{Z}^2, \mathbf{C}^N)$, where

$$(Au)_n = A_1(n)u_{n+e_1} + A_2(n)u_{n+e_2}$$

and $A_i(n) \in U(N)$ have the property that the magnetic fields

$$B(n) = A_2(n)^* A_1(n+e_2)^* A_2(n+e_1) A_1(n)$$

on the different plaquettes are independent identically distributed $U(N)$-valued random variables with law $\mu$.

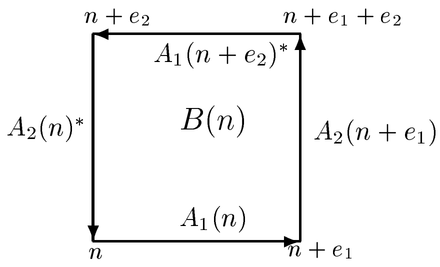


Fig. 1. Magnetic field $B = dA$.

The existence of such operators is nontrivial and relies on cohomological results of Feldman and Moore in ergodic theory. Already the case of independent identically distributed magnetic fields shows that such a result is needed: independent identically distributed random variables $A_i(n)$ lead in general to correlations between the magnetic field variables $B(n)$: if for example $A_i(n)$ take randomly the two values 1 and $e^{i\pi/4}$, then two adjacent plaquettes $P_n, P_m$ can not have magnetic fields $B(n) = 1$ and $B(m) = -1$. In other words, vector potentials $A$, which give independent identically distributed magnetic fields $B$ are not independent in general.

Working in a probabilistic or ergodic setup gives nontrivial constraints which manifest themselves in nontrivial cohomology groups. Eilenberg-McLane's result about the triviality of the two dimensional group cohomology of a free group action and an extension of Dye's result saying that a $\mathbf{Z}^d$-action is orbit equivalent to a $\mathbf{Z}$-action, leads to the result of Feldman-Moore, which makes it possible to speak about some of the objects, we are dealing with. An extension of the Feldman-Moore result due to Lind [31] allows to replace the abelian group $U(1)$ by a nonabelian group like $U(N)$. Lind's proof does not use cohomology and relies on the $\mathbf{Z}^2$-Rohlin lemma in ergodic theory and his result implies the existence of stationary discrete Yang-Mills Laplacians on $l^2(\mathbf{Z}^2, \mathbf{C}^N)$. DePauw has simplified in [11] a part of Feldman-Moore's theorem in dimension $d = 2$ and we generalize the result to $d \geq 2$, getting like this the existence of electromagnetic Laplacians. We will provide in Section 8 a self-contained proof, assuming only that all amenable ergodic group actions are orbit equivalent [8].
Feldman-Moore's result allows also to construct random electromagnetic Laplacians on more general graphs like for example the graph of a Penrose tiling. To get this result, the language of countable ergodic equivalence relations is needed. Since a countable ergodic equivalence relation is generated by a discrete group of automorphisms [14], Schrödinger operators on tilings are also elements of a hyperfinite factor and belong so from the measure theoretical point of view to the class of ergodic Schrödinger operators.

The motivations to study electromagnetic operators come from different directions. We list some of these relations without intending to discuss most of them further in this article.

1) From the physical point of view, discrete magnetic Laplacians in two dimensions are tight binding approximations describing an electron in the plane exposed to an ergodic magnetic field. This generalizes the Harper operator, the case of a constant magnetic field. A nonabelian constant $U(N)$-valued field gives a nonabelian brother of the Harper operator. Independent identically distributed magnetic fields lead to new Anderson type models for which a spectral analysis [6, 10, 36] has still to be done.
2) To every Laplacian $L = A + A^*$ is attached a *lattice gauge field* . The Wilson-Hamiltonian $H_\Lambda(A) = \sum_{x \in \Lambda} \mathrm{tr}(L^4(x))$ defines a statistical mechanical model leading to ergodic translation invariant equilibrium measures $\mu_\beta$ on $(U(N) \times U(N))^{\mathbf{Z}^d}$ which then define ergodic Laplacians $L_\beta$ depending on the inverse temperature $\beta$. Since statistical mechanics was also developed on aperiodic tilings ([16] or [15]), equilibrium measures lead to natural magnetic Laplacians on such tilings.
3) Magnetic or Yang-Mills Laplacians are selfadjoint operators of the form $L = \sum_{i=1}^d A_i + A_i^*$, where the $A_i$ are unitary. If the underlying dynamical system is ergodic, they are elements of a hyperfinite factor $(\mathcal{X}, \mathrm{tr})$. Even so every selfadjoint operator of norm $< 2d$ is of the form $\sum_{i=1}^d A_i + A_i^*$ with unitary $A_i$, random magnetic Laplacians are in a class of operators, for which the fields $F_{ij} = A_j^{-1} A_i^{-1} A_j A_i$ determine all the spectral properties. An example is the discrete magnetic Laplacian [38], where $d = 2$ and $B = F_{12} = e^{2\pi i \alpha}$ is constant. In the view of noncommutative geometry (see [7, 4]), we consider a class of noncommutative tori. A natural problem is to determine the spectral properties and spectral types of $L$ in representations of $\mathcal{X}$.
4) A $\mathbf{Z}^d$-action on an arbitrary abelian group $\mathcal{U}$ defines a complex $d : \mathcal{C}^n \to \mathcal{C}^{n+1}$ with an exterior derivative $d$ given by

$$dA = \sum_{i,|I|=n} [\tau_i, A_I]\tau_I = \sum_{i,|I|=n} A_I^{-1} A_I(T_i)\tau_{iI} \ .$$

This leads to geometric cohomology groups $\mathcal{H}^n_{geom}(\mathcal{U})$. With any Hodge involution $* : \mathcal{C}^p \to \mathcal{C}^{d-p}$, one has an electromagnetic formalism on a group (and it is kind of remarkable that one does not need the structure of a ring or algebra to do that). A vector potential $A \in \mathcal{C}^1$ defines an electromagnetic field $F = dA$. Maxwell equations $dF = 0$, $d^*F = *d*F = j$ define a current $j \in \mathcal{C}^1$. If $\mathcal{U}$ is no more abelian, there is still a first cohomology set $\{A \mid dA_{ij} = A_j^{-1} A_i(T_j)^{-1} A_j(T_i) A_i = 1\}/\{A \mid A_i = (dC)_i = C_i^{-1} C(T_i)\ \}$ which is the moduli space of zero curvature fields modulo fields which can be gauged to the identity. At least in two dimensions, there exists also a nonabelian current. A unitary representation of $\mathcal{U}$ in the unitary group of a von Neumann algebra, $A = \sum_{i=1}^d A_i \tau_i \in \mathcal{C}^1$ defines an electromagnetic operator $L = A + A^*$ having spectral properties depending only on the electromagnetic field $F$. Especially interesting is $\mathcal{U} = \mathcal{L}(X, U)$, where $\mathcal{U}$ is the set of measureable maps from $X$ to $U$ and the $\mathbf{Z}^d$ action is given by Borel automorphisms on $X$. A generalisation of Dye's theorem about orbit equivalence implies that $\mathcal{H}^1(U)$ is independent the $\mathbf{Z}^d$ action (a remark of [39]). We will see here that $\mathcal{H}^1(U)$ is the moduli space of zero curvature vector potentials modulo gradient vector potentials and, in the case $d = 2$, the group $\mathcal{H}^1(U)$ measures also how many currents $j$ exist modulo currents which are of the form $d^*F$. For $d = 3$, $\mathcal{H}^1(U)$ is the space of fields $F$ with no current $d^*F = 0$ modulo fields which satisfy the discrete Bogomolny equation $F^* = d\phi$.
5) In the partition function of *one matrix models* [34] appears a van der Monde determinant which comes from the integration over all unitarily conjugated matrices. The potential theoretical energy $I(L) = -\int \int \log|E - E'| \, dk(E) \, dk(E')$ of the density of states $dk$ of an ergodic Schrödinger operator allows to define a natural infinite-dimensional van der Monde determinant $e^{-I(L)}$. A matrix model with partition function $Z = \int e^{-I(L)} \, d\mu(L)$ is defined by specifying a measure $\mu$ on a class of ergodic operators and it is quite natural to integrate over a class of magnetic or nonabelian Yang-Mills Laplacians.

6) The *flux-phase problem* on a finite planar graph is the variational problem to minimize $-\mathrm{tr}(|L|)$ or maximize $\det(|L|)$ among all magnetic Laplacians on a graph. In an ergodic setup, one could ask, what ergodic magnetic fields $B$ defining a 2-dimensional magnetic or Yang-Mills Laplacian minimize $\mathrm{tr}(|L|)$ or maximize $\det|L| = \exp(\mathrm{tr}\log|L|) = \exp(\int \log|E|\; dk(E))$. According to the flux-phase conjecture [28] proven recently for planar bipartite periodic graphs [29], one would expect that a constant magnetic field $B \equiv -1$ gives the extremum also in the ergodic case.

We outline now the results of this paper. After the definition of electromagnetic Laplacians and the observation that Feldman-Moore-Lind's results imply that ergodic magnetic ($U(1)$-valued) and Yang-Mills ($U(N)$-valued) Laplacians exist, we compute moments of the density of states for independent identically distributed magnetic Laplacians in two dimensions. We show for example that if the law of the magnetic field $B$ is the Haar measure $\mu_{Haar}$ on $U(1)$, the density of states is determined by a random walk in $\mathbf{Z}^2$ having global geometrical constraints: the $n$'th moment of the density of states, $\mathrm{tr}(L^n)$, is the number of closed paths in $\mathbf{Z}^2$ which have length $n$ and give zero winding number to every plaquette. We prove also that random magnetic fields with law $\mu_{Haar}$ can be generated by taking $\mu_{Haar}$-distributed vector potentials, so that in this special case, Feldman-Moore's existence theorem is not needed. We show then that all the spectral properties of the operators in the abelian as well as nonabelian case depend only on the field $F = dA$ and not on the specific realization of the vector potential $A$. The explicit calculation of the moments of the density of states for independent identically distributed fields will lead to an Aubry duality for the deformed operators $L_\lambda = A_1 + A_1^* + \lambda(A_2 + A_2^*)$: the density of states of $L_\lambda$ is related to the density of states of $L_{1/\lambda}$ in the same way as for the Harper case [2] (which is here a special case).
We reprove then a result of Jitomirskaja and Mandelstam [42] stating that a change of the field on a finite set of cells gives a compact perturbation of the operator. A special case is the magnetic Aharonov Bohm operator with magnetic flux $B \in U(1)$ different from 1 only in one cell. This result stays true for aperiodic lattices like the Penrose lattice or in higher dimensions. We show then the existence of electromagnetic Laplacians in any dimension by generalizing Feldman-Moore's theorem to higher dimensions using an idea of dePauw [11] to use geometric cohomology. The translation of geometric cohomology to algebraic group cohomology in higher dimensions (in spirit analogue to the equivalence of simplicial cohomology with de Rham cohomology in differential topology) is a purely algebraic relation, even so our proof uses a topological deformation argument.
We consider then shortly the electromagnetic formalism. To every electromagnetic Laplacian $L = A + A^*$ is attached a field $F = dA$ and so a current $j = d^*F$ which is divergence-free $d^*j = 0$. Zero divergence stays true in the nonabelian case at least if $d = 2$. In two dimensions, not every current is given by a field. The equivalence classes of currents $j$, modulo currents of the form $d^*F$ is the cohomology group $\mathcal{H}^1(U)$ and has so at least countable infinite cardinality, a fact, which we will comment in an appendix. However, in dimensions $d > 2$, as a result of the triviality of higher dimensional cohomology groups, every 1-form $j$ is of the form $j = d^*F$.
We mention then some generalizations like magnetic Laplacians on other graphs or aperiodic tilings. We can prove for example that to any law $\mu$ on $U(1)$, one can realize a measurable vector potential on a *Penrose lattice* such that the magnetic fields in the plaquettes (=Robinson triangles) are independent identically distributed $U(1)$-valued random variables with law $\mu$. Since a Penrose graph is not a Cayley graph of a group, the more abstract set-up of countable ergodic equivalence relations developed in [14] is needed. An electromagnetic Laplacian on a tiling is an element of the hyperfinite factor attached to the tiling.

# 2 Random magnetic Laplacians

We consider first the two dimensional case. Let $(X, \mathcal{F}, m)$ be a probability space. Two commuting measure-preserving invertible transformations $T_1, T_2$ on $X$ define a $\mathbf{Z}^2$-dynamical system. Let $U$ be a Polish (=complete separable metrizable) group. A *2-form* $B\tau_{12}$ is defined by a measurable map $B \in \mathcal{U} = \mathcal{L}(X, U) = \{B \mid X \to U, \text{measurable}\}$. Two measurable circle-valued maps $A_1, A_2 \in \mathcal{U}$ define a *1-form* or *vector potential* $A = A_1\tau_1 + A_2\tau_2$. Define the *curvature* of $A$ as the 2-form $dA\ \tau_{12}$ by

$$dA(x) = A_2^{-1}(x)A_1^{-1}(T_2x)A_2(T_1x)A_1(x) \ .$$

Not every 2-form $B$ can be written as $B = dA$ with a 1-form $A$. For example, if $T_1$ is the identity map and $T_2 = T$ is ergodic, then not every measurable map $B \in \mathcal{U} = \mathcal{L}(X, U)$ can be written as $B = A^{-1}A(T)$ with $A \in \mathcal{U}$ since the *cohomology group*

$$\mathcal{H}^1(U) = \mathcal{U}/\{A \in \mathcal{U} \mid B = A^{-1}A(T)\}$$

of cocycles modulo coboundaries is nontrivial. We know that this group contains at least a countably infinite set (see the Appendix). The following result of Feldman and Moore [14] was extended by Lind [31] to nonabelian groups. A dynamical system given by a group $T^g$ of automorphisms on $(X, \mathcal{F}, m)$ is called *free*, if $m(\{T^g(x) = x\}) > 0$ implies $g = 0$.

**Theorem 2.1 (Feldman-Moore-Lind)** *Assume the $\mathbf{Z}^2$-dynamical system is free. Let $U$ be a (not necessarily abelian) Polish group. For any magnetic field distribution $B\tau_{12}$ with $B \in \mathcal{U}$, there is a vector potential $A = A_1\tau_1 + A_2\tau_2$, which satisfies $dA = B$.*

Example. A magnetic field $B$ taking values in $\{1, -1\}$ is determined by the measureable set $Y = B^{-1}(-1)$. Feldman-Moore's result implies that there exist two measurable sets $Z_1, Z_2$ such that

$$Y = Z_1 + T_1(Z_1) + Z_2 + T_2(Z_2) \ ,$$

where $+$ is the symmetric difference, (the addition in the group $\mathcal{F}$).

Assume now that $U$ is a subgroup of the unitary group $U(N)$ of $n \times n$ matrices. Given a 1-form $A = A_1\tau_1 + A_2\tau_2$, we define a discrete selfadjoint random Schrödinger operator $L = A + A^*$ as follows: for almost all $x \in X$, consider the operator $L(x)$ on $l^2(\mathbf{Z}^2, \mathbf{C}^N)$ given by $(L(x)u) = (A(x) + A(x)^*)u$, where

$$(A(x)u)_n = A_1(x)u_{n+e_1} + A_2(x)u_{n+e_2}$$

and $e_1 = (1, 0), e_2 = (0, 1)$ are the basis vectors in $\mathbf{Z}^2$. We call $L = A + A^*$ a *discrete random magnetic Laplacian*. Such operators are discrete versions of the continuous operators $L = (\nabla - iA)^2$ which have been studied already (see [10]). We call them in the following just *random magnetic Laplacians* if $U = U(1)$ or *random Yang-Mills Laplacians* if $U = U(N)$. Associated to $L$ is a one-parameter family of operators $L = A_1 + A_1^* + \lambda(A_2 + A_2^*), \lambda \in \mathbf{R}$, but we will here mainly concentrate on the case $\lambda = 1$. The *field* of a random Laplacian $L = A + A^*$ is defined for $d \geq 2$ as $F_{ij} = dA_{ij} = A_j^{-1}A_i(T_j)^{-1}A_j(T_i)A_i$. If $U = U(1)$, we speak of a *magnetic field* $B = A_2^{-1}A_1(T_2)^{-1}A_2(T_1)A_1$ having *magnetic flux* $\arg(B)$. The *phases* of $L$ are the functions $\arg(A_i)$.

The operator $L$ is not uniquely defined by $B$. Indeed, given a $0-$form $C \in \mathcal{U}$, (which we call also a *gauge field* ). The gauge transformed operator $CLC^{-1}$ is also a discrete random Laplacian with the same magnetic field $B$ but the gauge potential $A$ has changed to $CAC^*\tau = \sum_i C_iA_iC_i(T_i)^*\tau_i$. The choice of the gauge is not the only source of non-uniqueness. The nontrivial non-uniqueness is measured by the moduli space of flat fields $\{(A_1, A_2) \mid dA = 0\}/\{A = dC\}$ which is for abelian $U$ isomorphic to $\mathcal{H}^1(U)$ (see Section 8).

The above definitions generalize readily to the higher dimensional case. Take $d$ automorphisms $T_1, \ldots, T_d$ on the probability space $(X, \mathcal{F}, m)$ A 1-form $A = \sum_{i=1}^d A_i\tau_i$ is given by $d$ functions

$A_i \in \mathcal{U} = \mathcal{L}(X, U(N))$ and defines a field

$$dA = F = \sum_{i<j} F_{ij} \tau_{ij} ,$$

where $F_{ij} = A_i A_j(T_i) A_j(T_i)^{-1} A_j^{-1}$. This gives random Laplacians $L = A + A^*$ acting on $l^2(\mathbf{Z}^d, \mathbf{C}^N)$

$$(L(x)u) = \sum_{i=1}^{d} A_i(x)u + A_i^* u .$$

As will be shown in Section 8, the condition $dF = 0$ implies the existence of a vector potential $A$ such that $dA = F$ if $U$ is abelian and $\mathbf{Z}^d$ acts freely. A random operator $L = A + A^*$ given by $F$ is then called an *electromagnetic Laplacian with field $F$* .

Examples.
1) A *nonabelian Harper operator* is obtained when the field $B$ takes a constant value in $U(N)$. The "nonabelian Hofstadter butterfly" is then the compact set $\{(B, E) \mid E \in \sigma(A + A^*), dA = B\} \subset U(N) \times \mathbf{R}$. One-parameter families $t \mapsto L^{(t)}$ of such operators are obtained by choosing an element $\alpha$ in the Lie algebra $su(N)$ and to take $L^{(t)}$, an operator with constant field $B = \exp t\alpha$. As in the $U(1)$ case, the density of states will be the same as of a one-dimensional almost periodic operator acting on $l^2(\mathbf{Z}, \mathbf{C}^N)$ defined by

$$(L^{(t)}u)_n = u_{n+1} + u_{n-1} + (\exp(nt\alpha) + \exp(-nt\alpha))u_n .$$

$L$ as well as the deformed operators $L_\lambda$ with potential $\lambda(\exp(n\alpha) + \exp(-n\alpha))$ are "almost periodic operators on the strip" and share many properties of one-dimensional operators (see [25]). Last [26] has shown that in the Harper $U = U(1)$ case and $\lambda = 1$, the spectrum has zero Lebesgue measure if the flux is irrational and satisfies a Diophantine condition. For reviews see [3, 27, 18].
2) An example of a *quasi periodic magnetic Laplacian* is defined by the dynamical system $(U(1), T_1 : \theta \mapsto \theta + \alpha, T_2 : \theta \mapsto \theta + \beta, d\theta)$ and the magnetic field $B(n)(\theta) = \exp(i(\theta + n_1\alpha + n_2\beta)) \in U(1)$. The spectrum of the Laplacian is determined by $\alpha, \beta$. Nonabelian versions are defined similarly. Note however that the vector potential $(A_1, A_2)$ is only measurable and we can not expect it to be almost periodic.
3) An example of a *limit periodic magnetic field* is $B(n) = \sum_{j=0}^{\infty} b_j \cos(2\pi(n_1 2^{-j} + n_2 3^{-j}))$ with $\sum_j |b_j| < \infty$. As in the previous example, the operator $L$ is not limit periodic and only the physically relevant field $B$ is.
4) Given a law $\mu$ on $U(1)$ and independent identically distributed random variables $B(n)$, $n \in \mathbf{Z}^2$ with law $\mu$, which are independent identically distributed magnetic fields. A special case is the Harper operator with constant magnetic field. Again, the vector potential is in general not given by independent identically distributed random variables. We will come back to this case in more detail in Section 4.
5) An example of an aperiodic strictly ergodic field taking only *finitely many values* is $B(n) = 1 - 2 \cdot 1_{[0,\gamma)}(\theta + n_1\alpha + n_2\beta) \in \{-1, 1\}$, where $\gamma, \alpha, \beta$ are rationally independent.
6) In all these examples one gets one-parameter families of deformed operators $L = A_1 + A_1^* + \lambda(A_2 + A_2^*)$ with $\lambda \in \mathbf{R}$. This is especially interesting in the stationary independent magnetic field case because then, some spectral properties of $L$ depend only on $\lambda$ and the field $B = dA$ as we will see below.

## 3 The density of states and the spectrum

If $U$ is a subgroup of $U(N)$, the Laplacian $L$ is an element of a von Neumann algebra $\mathcal{X}$ which is the crossed product of $\mathcal{A} = L^\infty(X, M(N, \mathbf{C}))$ with the $\mathbf{Z}^d$-action generated by automorphisms

$f \mapsto f(T^n)$, where $f(T^n)(x) = f(T^n x)$. The algebra $\mathcal{X}$ is obtained by completing the algebra of all polynomials in the variables $\tau_1, \dots, \tau_d$ with coefficients in $\mathcal{A}$,

$$K = \sum_{n \in F \subset \mathbf{Z}^d} K_n \tau^n, \; (KL)_n = \sum_{l+m=n} K_l L_m(T^l) \tau^n$$

with respect to the norm $|||K||| = | \, ||K(x)|| \, |_\infty$ . Here, $K(x)$ is the bounded linear operator on $l^2(\mathbf{Z}^d, \mathbf{C}^N)$ defined by $(K(x)u)(n) = \sum_m K_m(x)u(n+m)$ and $||\cdot||$ is the operator norm on $\mathcal{B}(l^2(\mathbf{Z}^d))$ and $|\cdot|_\infty$ the essential supremum norm. The involution in $\mathcal{X}$ is $(\sum_n K_n \tau^n)^* = \sum_n K_n^*(T^{-n})\tau^{-n}$ and the *trace* is $\mathrm{tr}(K) = \int_X \mathrm{Tr}(L_0(x)) \, d\mu(x)$, where Tr denotes the usual trace on the finite dimensional matrix algebra $M(N, \mathbf{C})$. This construction of Murray and von Neumann works in the same way, if $\mathbf{Z}^d$ is replaced by a countable amenable group.

For any selfadjoint $L \in \mathcal{X}$, and $f \in C_b(\mathbf{R})$, we can form $f(L) \in \mathcal{X}$. The function $f \mapsto \mathrm{tr}(f(L))$ defines a measure on $\mathbf{R}$, the *density of states* of $L$. If the $\mathbf{Z}^2$-action is ergodic, then the spectrum of $L(x)$ is constant almost everywhere and coincides with the support of the density of states. We have: (see also [23] and Proposition 3.2 below)

**Proposition 3.1** *If $U$ is abelian and $d \geq 2$, the density of states and so the spectrum of a random magnetic Laplacian $L = A + A^*$ depends only on the electromagnetic field $B = dA$.*

*Proof.* It is enough to show that the moments $\mathrm{tr}(L^n) = \int E^n \, dk(E)$ are determined by $B$. But

$$\mathrm{tr}(L^n) = \sum_{\gamma \in \Gamma_n} \prod_P \mathrm{Tr}(B(n)^{n(\gamma,P)}) \; ,$$

where $\Gamma_n$ is the set of closed oriented paths starting at $0 \in \mathbf{Z}^2$ having length $n$ and $n(\gamma, P)$ is the winding number of $\gamma$ with respect to the plaquette $P$. $\square$

Remark. A similar result for hopping matrices on arbitrary finite graphs has been proven by Lieb and Loss ([30] Lemma 2.1): the knowledge of the fluxes through each closed circuit determines the spectrum of the matrices.

Knowing the magnetic field, $L$ is not uniquely determined and we have in general also not an *explicit* example of $L$. However, all the information about the spectrum can be computed knowing $B$ only. To see this, we consider a deterministic setup, where measurability is discarded. Let $\mathcal{U} = U(N)^{\mathbf{Z}^2}$ be the set of possible fields $n \mapsto B(n)$. For every $B \in \mathcal{U}$, we can find a bounded selfadjoint operator $L = A + A^*$ on $l^2(\mathbf{Z}^2, \mathbf{C}^N)$ which has the field $B = dA$. The special gauge $A_2(n,m) = A_1(0,m) = 1$ for $n, m \in \mathbf{Z}$, determines $A$ and makes $L = L_B$ unique. For $N = 1$, a canonical gauge can also be defined in higher dimensions $d \geq 2$: let $E_i \subset \mathbf{Z}^d$ be the vector space spanned by $(e_1, e_2, \dots, e_i)$. Put $A_i = 1$ on $E_1$, then on all lines orthogonal to $E_1$ in $E_2$ and inductively on all lines orthogonal to $E_j$ in $E_{j+1}$. This determines $A$ as can be seen by induction: first construct $A$ on $E_2$ then on edges orthogonal to $E_2$ in $E_3$ etc. always using $dA = B$. The condition $dB = 0$ assures that the definition is consistent. The map $B \mapsto L_B$ is continuous if $\mathcal{U}$ has the product topology and $\mathcal{B}(l^2(\mathbf{Z}))$ has the strong operator topology. However, a change of the magnetic field of one single plaquette changes $L_B$ globally. The Aharonov-Bohm operator (to which we come in Section 7) shows that one can not do better in general.

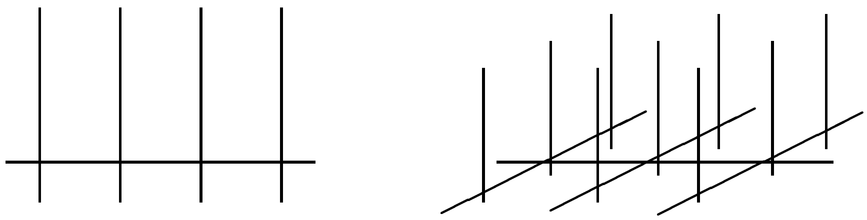

Fig. 2. Gauge fixing in two and more dimensions. Outlined are the bonds $e_i(g)$ on which $A_i(g) = 1$.

**Proposition 3.2** *Assume $N \geq 1, d = 2$ or $N = 1, d \geq 2$. Given $A = (A_1, A_2), \tilde{A} = (\tilde{A}_1, \tilde{A}_2) \in \mathcal{U}^2$ with $d\tilde{A} = dA = B$, there exists an unitary operator $U$ on $l^2(\mathbf{Z}^2, \mathbf{C}^N)$ such that $U^*LU = \tilde{L}$, where $L = A + A^*$ and $\tilde{L} = \tilde{A} + \tilde{A}^*$. In the ergodic case, the density of states, the spectral types and the spectrum depend only on the field $B$.*

*Proof.* Take the diagonal operator $U_{n,m} = G(n)\delta_{n,m}$, where $\overline{G}(n)A_i(n)G(n - e_i) = \tilde{A}_i$. This function $n \mapsto G(n)$ exists since any vector potential can be gauged to the canonical gauge.
In the ergodic case, when the density of states of $L$ and $\tilde{L}$ exist, we have $L^n(x)_{00} = \tilde{L}^n(x)_{00}$ (we do not need the measurability of the conjugating operator to get this). It follows that $\mathrm{tr}(L^n) = \int_X \mathrm{Tr}(L^n(x)_{00})\ dm(x) = \int_X \mathrm{Tr}(\tilde{L}^n(x)_{00})\ dm(x) = \mathrm{tr}(\tilde{L}^n)$.
Similarly, $(\phi, L^n(x), \phi) = (\phi, \tilde{L}^n(x), \phi)$ for every $\phi \in l^2(\mathbf{Z}^d, \mathbf{C}^N)$, so that also the spectral measures are the same. $\square$

Remark. If $B$ and $\tilde{B}$ are both measurable, cohomological constraints prevent that the map $x \mapsto U(x)$ is measurable in general. Examples, where $T_1 = \mathrm{Id}$ show this. In other words, the operators $L(x)$ and $\tilde{L}(x)$ are conjugated in $\mathcal{B}(l^2(\mathbf{Z}^d, \mathbf{C}^N))$ but if they are in $\mathcal{X}$, the conjugation is in general not possible inside $\mathcal{X}$.

**Corollary 3.3** *Given a sequence of operators $L^{(n)} = A^{(n)} + (A^{(n)})^*$ and an operator $L = A + A^*$ defined over the same $\mathbf{Z}^d$-action. Assume the fields $B^{(n)} = dA^{(n)}$ converge to $B = dA$ in $L^\infty(X, U)^d$. If there exists an interval $I \subset \mathbf{R}$ such that $\sigma(L^{(n)}) \cap I = \emptyset, \forall n \in \mathbf{N}$ then also $\sigma(L) \cap I = \emptyset$.*

*Proof.* In the canonical gauge, the operators converge pointwise in the strong operator topology and so in the resolvent sense and the claim follows from general principles ([37] Theorem VIII.24). $\square$

## 4 Independent identically distributed fields

We concentrate in this paragraph to the case of magnetic Laplacians for which the magnetic fields $\{B(n)\}_{n\in\mathbf{Z}^2}$ are independent identically distributed $U(1)$-valued random variables with law $\mu$. Denote by $\hat{\mu}_n = \int_{U(1)} z^n\ d\mu$ the $n$-th moment of $\mu$. The sequence $\hat{\mu}_n$ is the Fourier transform of the measure $\mu$. Denote by $\Gamma_n$ the set of oriented closed paths in $\mathbf{Z}^2$ of length $n$. Let $n(\gamma, P)$ be the *winding number* of the path $\gamma$ with respect to the plaquette $P$. The moments of the density of states can be computed by a random walk expansion (compare [9]):

**Proposition 4.1** *Let $\mu$ be a measure on $U(1)$. The $n$-th moment of the density of states $\mathrm{tr}(L^n)$ of an independent identically distributed magnetic Laplacian with law $\mu$ is*

$$\sum_{\gamma\in\Gamma_n} \prod_P (\int_{U(1)} z^{n(\gamma,P)}\ d\mu(z)) = \sum_{\gamma\in\Gamma_n} \prod_P \hat{\mu}_{n(\gamma,P)} \ .$$

*Proof.* Given a 1-form $A$ satisfying $dA = B$. Write $\int_\gamma A$ for the product of the $A_i$ along the path $\gamma$. The path $\gamma$ encloses a region $\Omega(\gamma)$ which is a collection $\{P\}$ of plaquettes. Use the discrete Green formula $\int_\gamma A = \prod_P B^{n(\gamma,P)}$ to compute

$$\begin{aligned} \mathrm{tr}(L^n) &= \sum_{\gamma\in\Gamma_n} \int_X \int_\gamma A(x)\ dm(x) = \sum_{\gamma\in\Gamma_n} \int_X \prod_P B_P(x)^{n(\gamma,P)}\ dm(x) \\ &= \sum_{\gamma\in\Gamma_n} \prod_P \int_X B_P(x)^{n(\gamma,P)}\ dm(x) = \sum_{\gamma\in\Gamma_n} \prod_P (\int_{U(1)} z^{n(\gamma,P)}\ d\mu(z)) \ . \end{aligned}$$

(We used that the expectation of a product of independent random variables is the product of the expectation and that $X^n$ is independent of $Y^n$ if $X$ is independent of $Y$.) $\square$

Remarks.
1) The random walk expansion breaks down if $U(1)$ is replaced by a nonabelian group. It is then still true that $\mathrm{tr}(L^n) = \sum_{\gamma \in \Gamma_n} \int_X \int_\gamma A(x)\ dm(x)$ , where $\int_\gamma A(x)$ is an ordered product but the Green formula $\int_\gamma A = \prod_P B^{n(\gamma,P)}$ does no more make sense. Furthermore, it is also no more true in general that $E[AB] = E[A] \cdot E[B]$ for nonabelian independent random variables.
2) The moments of the spectral measures of the unit vectors $\delta_k, k \in \mathbf{Z}^2$ can also be computed with a random walk expansion $(\delta_k, L^n(x)\delta_k) = \sum_{\gamma \in \Gamma_n} \prod_\gamma A(T^k x) = \sum_{\gamma \in \Gamma_n} \prod_P B_P(T^k x)^{n(\gamma,P)}$.
3) It follows from Proposition 4.1 that the density of states $dk$ depends continuously on the law $\mu$.

Also for the deformed operators $L_\lambda = A_1 + A_1^* + \lambda(A_2 + A_2^*)$, there is a similar formula for the density of states. Let $y(\gamma) \in 2\mathbf{N}$ be the number of steps, a path makes in the $y$-direction.

**Corollary 4.2 (Aubry-Duality)** *The moments of the density of states of $L_\lambda$ depends only on $\mu$ and $\lambda$:*

$$\mathrm{tr}(L_\lambda^n) = \sum_{\gamma \in \Gamma_n} \lambda^{y(\gamma)} \prod_P \hat{\mu}_{n(\gamma,P)} \ .$$

*and we have the duality*

$$dk(\mu, \lambda, E) = dk(\mu, 1/\lambda, E/\lambda) \ .$$

*Proof.* The random walk expansion is proven in the same way in Proposition 4.1. From this formula, it follows that $L_\lambda^{(1)} = A_1 + A_1^* + \lambda(A_2 + A_2^*)$ and $L_\lambda^{(2)} = \lambda(A_1 + A_1^*) + A_2 + A_2^*$ have the same density of states. The Aubry-Duality follows from $L_\lambda^{(1)}/\lambda = L_{1/\lambda}^{(2)}$. $\square$

Remark. If $\mu$ is the Haar measure and $\{A_1(n), A_2(n)\}_{n \in \mathbf{Z}^2}$ are independent Haar distributed random variables, the duality is stronger: the obvious symmetry $A_1 \leftrightarrow A_2$ implies that the operators $L_\lambda^{(1)} = A_1 + A_1^* + \lambda(A_2 + A_2^*)$ and $L_\lambda^{(2)} = \lambda(A_1 + A_1^*) + A_2 + A_2^*$ are then even isospectral so that also $L_\lambda$ and $\lambda L_{1/\lambda}$ are isospectral. Opposit to Corollary 4.2, this relation is depending on the choice of $A$ and an other choice would in general distroy this stronger symmetry.

The formula for the moments of the density of states becomes especially simple when $\mu$ is the Haar measure:

**Corollary 4.3** *Let $\mu$ be the Haar measure on $U(1)$. Then $\mathrm{tr}(L^n)$ is the number of closed paths of length $n$ beginning at $0 \in \mathbf{Z}^2$ for which every plaquette has zero winding number.*

*Proof.* In this case, $\hat{\mu}_k = \delta_{k,0}, \forall k \in \mathbf{Z}$ and so $\prod_P (\int_{U(1)} z^{n(\gamma,P)}\ d\mu(z)) = 0$, if there exists a plaquette $P$ which has positive winding number $n(\gamma, P)$. If the winding number is zero for all $P$, then $\prod_P (\int_{U(1)} 1\ d\mu(z)) = 1$. $\square$

Remarks.
1) The number of closed paths $\gamma$ of length $n$ in $\mathbf{Z}^2$ for which $n(\gamma, P) = 0$ for all plaquettes $P$ is in general strictly larger than the number of closed contractible paths of length $n$. A path can visit different plaquettes at different times without being contractible.

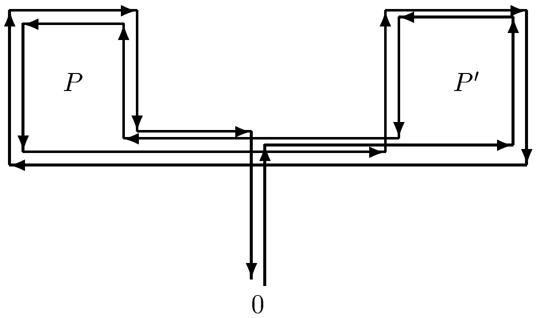


Fig. 3. A non contractible path giving zero winding number to all plaquettes.

The additional paths are so numerous that the radius of convergence for the Green function changes. Random walks with the stronger topological constraint of being contractible have been investigated in [24].
2) For the deformed operator $L = A_1 + A_1^* + \lambda(A_2 + A_2^*)$ with independent Haar distributed functions $A_i$, we get

$$\mathrm{tr}(L^n) = \sum_{\gamma \in \Gamma_n^{(0)}} \lambda^{y(\gamma)} ,$$

where $\Gamma_n^{(0)}$ is the set of paths in $\Gamma_n$ which give zero winding number to every plaquette and $y(\gamma)$ is the number of steps of $\gamma$ in the $y$ direction.

**Corollary 4.4** *Let $\mu$ be the Haar measure on a finite cyclic subgroup $\mathbf{Z}_p$ of $U(1)$. Then $\mathrm{tr}(L^n)$ is the number of closed paths of length $n$ beginning at $0 \in \mathbf{Z}^2$ for which the winding numbers satisfy $n(\gamma, P) = 0 \pmod p$ for all plaquettes $P$.*

*Proof.* $\prod_P(\int_{U(1)} z^{n(\gamma,P)} \, d\mu(z)) = 0$, if there exists a plaquette $P$ which has a winding number $n(\gamma, P)$ which is not zero modulo $p$. $\square$

Remarks.
1) The independence of the magnetic fields is essential in Proposition 4.1 and its Corollaries. Independent vector potentials would not be enough in general.
2) The random walk expansion in Proposition 4.1 shows that $\mathrm{tr}(L^n)$ is a polynomial in the infinite set of variables $\hat{\mu}_k, \hat{\mu}_k^{-1}$ with integer coefficients.

The random walk expansion can also be applied to the almost Mathieu-Harper case, where $\mu$ is a point measure so that the magnetic field is constant:

**Corollary 4.5** *If $\mu$ is the Dirac measure on $e^{2\pi i\alpha} \in U(1)$, then*

$$\mathrm{tr}(L^n) = \sum_{\gamma \in \Gamma_n} \prod_P e^{2\pi i \alpha n(\gamma,P)} .$$

*(Especially, if $\mu$ is the Dirac measure on $1 \in U(1)$, then $\mathrm{tr}(L^n) = ((2n)!)^2 (n!)^{-2}$ is the number of closed paths of length $n$ beginning at $0 \in \mathbf{Z}^2$.) For the Harper magnetic Laplacian $L = A_1 + A_1^* + \lambda(A_2 + A_2^*)$*

$$\mathrm{tr}(L^n) = \sum_{\gamma \in \Gamma_n} \lambda^{y(\gamma)} \prod_P e^{2\pi i \alpha n(\gamma,P)} ,$$

*where $y(\gamma) \in 2\mathbf{N}$ is the number of steps of $\gamma$ in the $y$-direction.*

Remarks.
1) Random walk expansions would work on any planar graph generalizing $\mathbf{Z}^2$ (see also Section 10). A solvable case is the Bethe lattice $\mathbf{B}_{2h}$ with degree $2h$, where Kesten determined the number of closed paths of length $n$ [19]. His calculation leads to $\mathrm{tr}(L-E)^{-1} = (2h-1)/(E(h-1) + h\sqrt{E^2-4(2h-1)})$. The imaginary part of this divided by $\pi$ is the density of states which has support on $[-2\sqrt{2h-1}, 2\sqrt{2h-1}]$. (For an new calculation see [1]). Since there are no closed loops on the Bethe lattice, there is no magnetic field and the density of states of any magnetic Laplacian on the Bethe lattice is the same.
2) Corollary 4.5 shows that $\mathrm{tr}(L^4) = 8 + 4(\hat{\mu}_0 + \hat{\mu}_0^{-1})$ , so that discrete random magnetic Laplacians with different values of $\mathrm{Re}(\hat{\mu}_0)$ have different density of states. Especially, almost Mathieu operators with different $\cos(\alpha)$ can not be isospectral (see also [23]).
3) By changing the orientation of the plaquettes, it becomes obvious that the laws $\mu$ and $\overline{\mu}(A) = \mu(\overline{A})$ give isospectral Laplacians.

In some cases, the theorem of Feldman-Moore is not needed in order to construct the Laplacian.

**Proposition 4.6** *Let $U$ be a compact subgroup of $U(1)$. If the law $\mu$ of the independent identically distributed $U$-valued random variables $\{A_i(n)\}_{n\in\mathbf{Z},i=1,2}$ of the vector potential $A_1\tau_1 + A_2\tau_2$ is the Haar measure on $U$, then $B(n) = dA(n)$ are independent identically distributed $U$-valued random variables with law $\mu$.*

*Proof.* Given measurable subsets $Y_n \subset U, n \in \mathbf{Z}^2$ of positive measure, define $Z_n = B(n)^{-1}(Y_n) \subset X = U^{(\mathbf{Z}^2)}$. Let $m = \mu^{(\mathbf{Z}^2)}$ be the product measure on $X$. The claim is that

$$m(\bigcap_{n\in F} Z_n) = m(Z_k) \cdot m(\bigcap_{n\in F\setminus\{k\}} Z_n) \tag{1}$$

for any finite set $F \subset \mathbf{Z}^2$ and that the law of $B(n)$ is the Haar measure $\mu$.

(i) $m(Z_n) = \mu(Y_n)$, for all $n \in \mathbf{Z}^2$.
Proof. A product of Haar distributed $U$-valued random variables is again Haar distributed because it must be $U$-invariant. It follows that the law of $B(n) = A_2^*(n)A_1^*(n+1)A_2(n+1)A_1(n)$ is the Haar measure $\mu$ and therefore $m(Z_n) = \mu(Y_n)$.

(ii) For any finite set $F$ of sets $\{Y_n\}_{n\in F} \subset U$ with $m(Y_n) > 0$, one has $m(\bigcap_{n\in F} B(n)^{-1}(Y_n)) > 0$.
Proof. We can realize one element in $\bigcap_{n\in F} B(n)^{-1}(Y_n)$ using the canonical gauge. There exists then an open neighborhood of this point in $U^{(\mathbf{Z}^2)}$ which is in $\bigcap_{n\in F} B(n)^{-1}(Y_n)$. An open set has positive measure.

(iii) For $k \in F$, the measure $\tilde{\mu}(Y_k) = m(B(k)^{-1}(Y_k) \mid \bigcap_{n\in F\setminus\{k\}} Z_n)$ is equal to $\mu(Y_k) = m(Z_k) = m(B(k)^{-1}(Y_k))$.
Proof. By the uniqueness of the Haar measure, we have only to show that $\tilde{\mu}$ is translational invariant. By multiplying $A_1(k + l \cdot e_i), l = 1, \dots, |F|$ with some constant $C = e^{2\pi\alpha} \in U$, we change the field $B(k) \mapsto B(k)C$ without affecting $\{B(n)\}_{n\in F\setminus\{k\}}$. Therefore $\tilde{\mu}(Y_k) = \tilde{\mu}(Y_k + \alpha)$ and $\tilde{\mu} = \mu$.

Proof of the claim. By (ii), Eq. 1 can be written as

$$m(Z_k \mid \bigcap_{n\in F\setminus\{k\}} Z_n) = m(Z_k) .$$

The left hand side of this is by $(iii)$ equal to $\tilde{\mu}(Y_k) = \mu(Y_k)$ and the right hand side is by $(i)$ also equal to $\mu(Y_k)$. $\square$

Remarks.
1) There are other possibilities to get independent magnetic Laplacians if $\mu$ is the Haar measure: define $A_2(n) = 1$ for all $n \in \mathbf{Z}^2$ and a family $\{A_1(n)\}_{n\in\mathbf{Z}^2}$ of independent Haar distributed random variables. An argument similar as in the proof of Proposition 4.6 shows that $\{dA(n) = B(n)\}_{n\in\mathbf{Z}^2}$ are independent Haar distributed.
2) We do not know if a generalisation of Proposition 4.6 holds also when $U$ is nonabelian.
3) In dimensions $d > 2$, there is no hope to get a result analogue to Proposition 4.6, since there are then more plaquettes than bonds so that a single bond influences several plaquettes and prevents independent identically distributed fields.
4) One of the open questions here is whether one has (some) pure point spectrum almost everywhere in the case of magnetic Laplacians with Haar distributed magnetic vector potentials.
5) Proposition 4.6 shows that for those specific operators, there is more symmetry as in the Mathieu case. The Aubry-Duality goes deeper: the operators $L_\lambda$ and $L_{1/\lambda}$ have the same spectral type because a multiplication of $L_\lambda$ with $1/\lambda$ gives $L_{1/\lambda}$.

# 5 Electromagnetic Laplacians

We turn now to independent identically distributed magnetic Laplacians in higher dimensions and restrict the discussion for simplicity to the case $d = 3$. As indicated already, we can not realize independent identically distributed electromagnetic fields $F$ by a vector potential, since such fields do not satisfy $dF = 0$. Consider now time-dependent magnetic fields in the plane together with an electric field changing in time. Given a vector potential $A = (A_1, A_2, A_3) \in \mathcal{C}^1$, we think of $A_1$ as the electrostatic potential and of $(A_2, A_3)$ as the magnetic vector potential. Then $dA = F$ is a three dimensional field. $E_1 = F_{12}$ and $E_2 = F_{13}$ are the coordinates of an electric vector field in the plane and $B = F_{23}$ is a magnetic field in the plane. For fixed $k \in \mathbf{Z}$, denote by $L^{(k)}$ the magnetic Laplacian in the plane, given by the vector potential $(n,m) \mapsto (A_2(k,n,m), A_3(k,n,m))$. The operator $L^{(k)}$ is a 2-dimensional magnetic Laplacian at time $k$.
The existence theorem proven in Section 8 shows that a field $F$ satisfying $dF = 0$ defines an electromagnetic Laplacian $L$. By giving the electric fields $E_1, E_2$ and the magnetic field $B^{(k_0)}$ at some time $k_0$, the Maxwell equation $dF = 0$ determines the whole field $F$.

**Proposition 5.1** *Let $F$ be determined by the electric fields and the magnetic field at some time $k_0$. Assume, the electric fields $\{E_1(n), E_2(n)\}_{n\in\mathbf{Z}^3}$ are independent identically distributed random variables with the same distribution $\mu$ which is not a Haar distribution of a subgroup of $U(1)$. Let $B(k_0, n), n \in \mathbf{Z}^2$ be any set of random variables. Then the distribution of the magnetic field of the two dimensional operators $L^{(k)}$ converges in law to the uniform Haar distribution of $U(1)$ for $|k| \to \infty$.*

*Proof.* The Maxwell equation $dF = 0$ (which follows from $F = dA$), implies that

$$B^{(k+1)}(n)^* = B^{(k)}(n)E_1^{(k)}(n+e_1)E_1^{(k)}(n)^*E_2^{(k)}(n)E_2^{(k)}(n+e_1)^* \ .$$

The proof of Proposition 4.6 shows that the random variables

$$\{C(n) = E_2^{(k_0)}(n+e_2)^*E_2^{(k_0)}(n)E_1^{(k_0)}(n)^*E_1^{(k_0)}(n+e_1)\}_{n\in\mathbf{Z}^2}$$

are all independent so that also $\{B^{(k_0\pm l)}(n)\}_{n\in\mathbf{Z}^2}$ is obtained from $\{B^{(k)}(n)\}_{n\in\mathbf{Z}^2}$ by multiplying it with independent identically distributed random variables. The claim follows now from the central limit theorem for independent identically distributed $U(1)$-valued random variables [33]. (On compact topological groups, the Haar measure plays the role of the Gaussian measure in $\mathbf{R}$.)
□

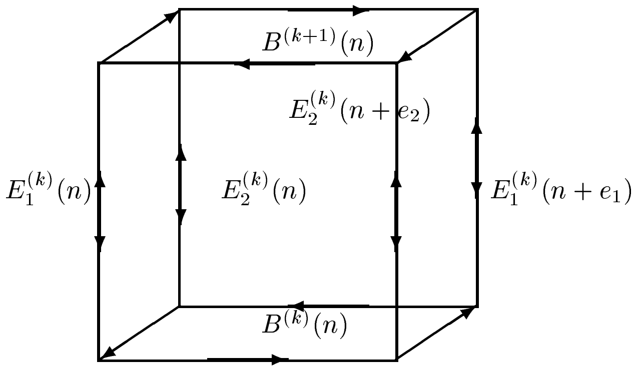


Fig. 4. The Maxwell equation $dF = 0$ determines the magnetic field $B^{(k+1)}(n)^* = B^{(k)}(n)E_1^{(k)}(n+e_1)E_1^{(k)}(n)^* E_2^{(k)}(n)E_2^{(k)}(n+e_1)^*$ at time $(k+1)$ from the magnetic field $B^{(k)}$ and the electric field $(E_1^{(k)}, E_2^{(k)})$ at time $k$.

Proposition 5.1 has the following interpretation: a time-dependent random electric field (which might be arbitrarily small but which is not taking values in a subgroup of $U(1)$) turns an initially arbitrary magnetic field for time $|k| \to \infty$ into an independent identically distributed Haar distributed magnetic field.

# 6 One dimensional operators

Take an electromagnetic Laplacian $L = A + A^*$ in $d$ dimensions, where the electromagnetic field $dA = F$ has only electric components $F_{1k}(n) = E_k(n)$ which are constant in space ($\mathbf{Z}^d = \mathbf{Z} \oplus \mathbf{Z}^{d-1} =$ space $\oplus$ time ) and depend therefore only on the first (=time) coordinate $n = n_1$. The restriction of $L$ to the invariant Hilbert space of functions which are constant in space gives a one-dimensional operator $(Hu)_n = u_{n+1} + u_{n-1} + V(n)u_n$, where

$$V(n) = \sum_{k=1}^{d} E_k(n) + E_k(n)^* = \sum_{k=1}^{d} 2\cos(\arg(E_k(n))) \; .$$

Every one-dimensional operator can be written like this, the dimension depending on the norm. Since $dF = 0$, Feldman-Moore's existence theorem shows that if $V$ is an ergodic potential, then the equation $F = dA$ can be solved with a *measurable* vector potential $A$ which leads to an ergodic electromagnetic Laplacian. The one-dimensional potential $\sum_{k=1}^{d} 2\cos(\arg(E_k(n)))$ is ergodic, if $T_1$ was.

Some Anderson models can be treated as random magnetic Laplacians and allow a combinatorial calculation of the density of states: given independent identically distributed random variables $V(n)$ $n \in \mathbf{Z}^d$ with law $\mu$, define the $\mathcal{B}(l^2(\mathbf{Z}^d))$-valued random variable $(Lu)_n = \sum_{|m-n|=1} u_m + V(n)u_n$ which is an *Anderson model*. By adding to each vertex of $\mathbf{Z}^d$ an oriented loop, one obtains a new lattice $\mathbf{L}^d$. Denote by $\Gamma_n$ the set of paths $\gamma$ in $\mathbf{L}^d$ which have length $n$. (Each loop has length 1 and we distinguish paths which pass in different directions through the loop).

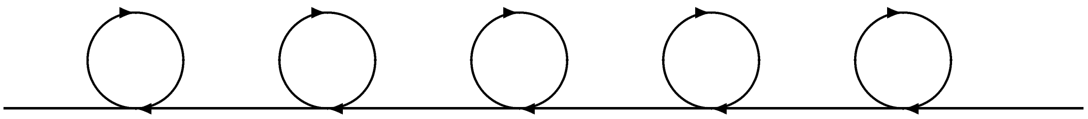

Fig. 5. The graph $\mathbf{L}$ in the case $d = 1$. At each vertex is attached an oriented loop.

**Corollary 6.1** *a) Given the discrete d-dimensional Anderson Schrödinger operator with independent identically distributed potential $V(n) = 2\cos(\alpha(n))$, where $\alpha(n)$ are uniformly distributed in $[0, 2\pi]$. The $n$'th moment of the density of states is the number of closed paths of length $n$ in $\mathbf{L}^d$, for which every loop has vanishing winding number.*
*b) If $V(n) = \pm 2$, where $V(n)$ are uniformly distributed in $\{0, 2\}$, the $n$'th moment of the density of states is the number of closed paths of length $n$ in $\mathbf{L}^d$ for which every loop has an even winding number.*

*Proof.* Write $L = A + A^*$ as a $(d+1)$-dimensional magnetic Laplacian, where $A_i = 1, i = 1, \dots, d$ and $A_{d+1}(n) = \exp(i\alpha(n))$ are independent identically distributed $U(1)$-valued random variables with uniform Haar distribution $\mu$. This is equivalent to taking real-valued random variables $\alpha(n)$ with uniform distribution on $[0, 1]$ and to form the independent identically distributed potential $V(n) = 2\cos(2\pi\alpha(n))$ which has an absolutely continuous law $4(2\pi)^{-1}\sqrt{1 - x^2}$. As before, we compute with the random walk expansion

$$\mathrm{tr}(L^n) = \sum_{\gamma \in \Gamma_n} \prod_P \hat{\mu}_{n(\gamma, P)} \; .$$

Since all nonzero moments of $\nu$ are zero, $\mathrm{tr}(L^n)$ is the number of closed paths in the lattice $\mathbf{L}^d$ which give in case a) zero and in case b) zero (mod 2) winding number to every loop. $\square$

Remark. Relations between two and one-dimensional operators are prototyped by the Harper-Mathieu case $A_1 = \tau, A_2 = e^{2\pi i\alpha}$ which give the one-dimensional operator $\tau + \tau^* + 2\cos(2\pi\alpha)$. For more examples with constant magnetic field see [32]. Other not constant magnetic fields can be obtained as follows: let $A_1\tau_1 = \tau$ be the unitary Koopman operator for a transformation $T$ on a probability space $\Omega$ and let $A_2(x) = e^{2\pi f(x)}$, where $f$ is a $su(N)$-valued random variable. Then $UV = VUe^{2\pi i(f(Tx) - f(x))}$ and we get a 1-dimensional operator $L = \tau + \tau^* + 2\cos(f(x))$ on $l^2(\mathbf{Z}, \mathbf{C}^N)$.

# 7 Deterministic Laplacians

It is illustrative to see what deterministic perturbations of the magnetic field does on the operator. We denote by $L_F$ the $d$-dimensional Laplacian with field $F$ in the special gauge.

**Proposition 7.1 (Jitomirskaja-Mandelstam [42])** *Assume $U$ is abelian. A change of $F \in U^{\mathbf{Z}^d}$ on a finite set of plaquettes is a compact perturbation of the operator $L_F$.*

*Proof.* Assume first $d = 2$. If $B$ is multiplied by $C \in U^{\mathbf{Z}^2}$ such that $C_n \neq 1$ only for finitely many $n$ and $\prod_n C_n = 1$, we call $\tilde{B} = BC$ a zero flux perturbation of 1. It is enough to show the claim for a perturbation of the field $B$ of one single plaquette. By construction, if $\tilde{B}$ is a zero flux perturbation of $B$, then $L_{\tilde{B}}$ is a finite rank perturbation of $L_B$.
Let $L = L_B$ be the original operator and let $\tilde{L} = L_{\tilde{B}}$ be the operator belonging to $\tilde{B}$ satisfying $\tilde{B}(n) = B(n)$ for all $n \in \mathbf{Z}^2$ except one $n_0$, where $\tilde{B}(n_0) = B(n_0)C$ with $C = e^{i\alpha} \in U$. Define for each $k \in \mathbf{N}$ a zero flux perturbation $B_k$ of $B$ by changing $\tilde{B}$ on $k^2$ plaquettes in a box of size $k \times k$ to $\tilde{B}_k C_k^{-1}$ with $C_k = e^{-i\alpha/n^2}$. Then, $L_{B_k} \to L_{\tilde{B}}$ in norm so that $L_{\tilde{B}}$ is a limit of finite rank operators $L_{B_k}$.
For general $d$, we can compose any perturbation by perturbations lying in two dimensional planes for which the previous argument applies. $\square$

Remarks.
1) The Aharonov-Bohm operator (the field $B(n)$ is different from 1 exactly on one plaquette) shows that one has never a finite rank perturbation $L_B \mapsto L_{\tilde{B}}$, if $B\tilde{B}^{-1}$ has compact support and nonzero

flux. It would be interesting to know if the Aharonov-Bohm operator is a trace class perturbation of the free operator.
2) A similar argument shows that the result is also true for some aperiodic tilings like the Penrose tiling.
3) Beside the abelian or nonabelian Ahoronov-Bohm operators (for which a complete spectral analysis is not yet done), other deterministic operators would be interesting to study. An example is a discrete version of the Iwatsuka operator $L$ in $d = 2$ (see [10]), where the magnetic field $B$ is translational invariant in one direction and asymptotically constant in the other direction. Then, $L$ is a direct product of one dimensional operators $(Lu)_n = u_{n+1} + u_{n-1} + \cos(n\alpha(n))u(n)$, where $\alpha(n) \to \alpha^{\pm}$ for constants $\alpha^{\pm}$. If $\alpha^-$ or $\alpha^+$ is rational, then also $L$ has some absolutely continuous spectrum. If both $\alpha^{\pm}$ are irrational, Last's results [26] allows to prove that $L$ has no absolutely continuous spectrum. This is in contrary to the continuous case, where the corresponding operator has purely absolutely continuous spectrum.

## 8 Existence of electromagnetic Laplacians

We consider three cohomological constructions for a pair $(G,\mathcal{U})$, where $G = \mathbf{Z}^d$ is a group of Borel automorphisms acting on the abelian group $\mathcal{U} = \mathcal{L}(X,U) = \{X \to U, \text{ measurable}\}$ and where $U$ is an abelian Polish group. Let $T_1, \dots, T_d$ be $d$ commuting automorphism on the probability space $(X, \mathcal{F}, m)$ which generate the $\mathbf{Z}^d$ action. Write $T^g = \prod_{i=1}^d T_i^{g_i}$ if $g = (g_1, g_2, \dots, g_d)$.

**I) Algebraic group cohomology (Eilenberg-McLane)** (see [13])
The group $G = \mathbf{Z}^d$ acting on $X$ induces a $G$-action on the abelian group $\mathcal{U} = \mathcal{L}(X,U)$ of all measureable maps from $X$ to $U$. Define for $0 \leq p \leq d$ the set $\mathcal{C}^p$ of maps $a : G^{p+1} \to \mathcal{U}$ satisfying $T^g a(g_0, g_1, \dots, g_p) = a(g + g_0, g + g_1, \dots, g + g_p)$. Define the map $d_n : \mathcal{C}^n \to \mathcal{C}^{n+1}$

$$(d_n a)(g_0, \dots, g_{n+1}) = \sum_{j=0}^{n+1} (-1)^j a(g_0, \dots, \hat{g}_j, \dots, g_n) ,$$

where the entry $\hat{g}_j$ has been deleted. Elements in the kernel of $d_n$ are the *algebraic cocycles of degree* $n$, elements in the image of $d_{n-1}$ are the *algebraic coboundaries of degree* $n$. Since $d_{n+1} \circ d_n = 0$, this gives the *algebraic group cohomology groups* $\mathcal{H}_{alg}(G,\mathcal{U}) = \ker(d_n)/\mathrm{im}(d_{n-1})$.

**II) Orbit cohomology (Feldman-Moore)** (see [14])
Define $\mathcal{R}^0 = X$ and $\mathcal{R}^n = \{(x_0, \dots, x_n) \in X^{n+1} \mid \exists\, g_i \in G,\ x_i = T^{g_i} x_0\}$. Define the set $\mathcal{C}^p(\mathcal{R}_G, U)$ of all measurable maps $a : \mathcal{R}^p \to U$. and take the map $d_n : \mathcal{C}^n \to \mathcal{C}^{n+1}$

$$(d_n a)(x_0, \dots, x_{n+1}) = \sum_{j=0}^{n+1} (-1)^j a(x_0, \dots, \hat{x}_j, \dots, x_n) .$$

$\ker(d_n)$ consist of *orbit cocycles of degree* $n$, while $\mathrm{im}(d_{n-1})$ are *orbit coboundaries of degree* $n$. Since $d_{n+1} \circ d_n = 0$, one gets the *orbit cohomology groups* $\mathcal{H}^n_{orb}(\mathcal{R}, U) = \ker(d_n)/\mathrm{im}(d_{n-1})$. This cohomology is defined more generally for hyperfinite equivalence relations (see [14]).

**III) Geometric group cohomology** (see [22], [11])
Define $I = \{1, \dots, d\}$ and let $\mathcal{I}_p$ be the set of sets $J = \{j_1 < j_2 < \dots < j_p\} \subset I$. Let $\mathcal{C}^p$ be the set of maps $A : \mathcal{I}_p \to \mathcal{U}$ which becomes a group by pointwise addition. Extend this map to the set of $p$-tuples $J = (j_1, j_2 \dots, j_p)$ with $j_k \in I$ by requiring $A_{\pi(J)} = \mathrm{sign}(\pi) A_J$ for any permutation $\pi$ of $J$. We write $A = \sum_J A_J \tau_J$. Define $d_n : \mathcal{C}^n \to \mathcal{C}^{n+1}$ by

$$d_n A = \sum_{i,J} (A_J(T_i) - A_J) \tau_{iJ} .$$

(we write at other places in this article the group also multiplicatively). The kernel of $d_n$ contains *cocycles of degree* $n$, whereas the image of $d_{n-1}$ consists of *coboundaries of degree* $n$. Since $d_{n+1} \circ d_n = 0$, one can form the *geometric cohomology groups* $\mathcal{H}^n_{geom}(G,\mathcal{U}) = \ker(d_n)/\mathrm{im}(d_{n-1})$.

Our aim is to show that all these three cohomologies are the same if the group $G$ acts *freely*, that is if $m(\{x \in X \mid T^n x = x\}) = 0$ for all $n \in G \setminus \{0\}$.

**Proposition 8.1 (Feldman-Moore)** *If $G$ acts freely, then $\mathcal{H}^p_{orb}(\mathcal{R}_G, U) \cong \mathcal{H}^p_{alg}(G,\mathcal{U})$ for all $p = 0, \dots, d$.*

*Proof.* The map $\phi : \mathcal{C}^p(G,\mathcal{U}) \to \mathcal{C}^p(\mathcal{R}_G, U)$, $a \mapsto \overline{a}$

$$\overline{a}(x_0, \dots, x_p) = a(0, g_1, \dots, g_p)(x_0), x_i = T^{g_i} x_0$$

is well defined if the action of $G$ is free. It is invertible since

$$a(g_0, \dots, g_p)(x) = \overline{a}(T^{g_0}(x), T^{g_1}(x), \dots, T^{g_p}(x))$$

and $\phi$ commutes with $d$. □

**Proposition 8.2 (Eilenberg-McLane)** $\mathcal{H}^2_{alg}(\mathbf{Z},\mathcal{U}) \cong \{0\}$.

*Proof.* (De Pauw)
(i) Every algebraic cocycle $a \in \mathcal{C}^2(G,U)$ is cohomologous to an alternating cocycle: $a(g_{\pi(0)}, \dots, g_{\pi(n)}) = (-1)^{\mathrm{sign}\pi} a(g_0, \dots, g_n)$.
Proof. Define the cocycle $b(g,h) = 0$ if $g \neq h$ and $b(g,g) = a(g,g,g)$. A first normalization $a_1 = a - db$ gives $a_1(g,g,h) = a_1(g,h,h) = 0$. Define the cocycle $b_1(g,h) = 0$ if $h \geq g$ in the lexicographic ordering on $G = \mathbf{Z}^d$ and $b_1(g,h) = a_1(g,h,g)$ if $h < g$. The second normalization $a_2 = a_1 - db_1$ gives an alternating cocycle $a_2$.
(ii) Given a 2-cocycle $a$. By (i) one can assume that it is alternating. The function $b(k,l) = -\sum_{i=k}^{l-1} a(k,i,i+1)$ is a degree two cochain since

$$\begin{aligned} T^m b(k,l) &= -T^m \sum_{i=k}^{l-1} a(k,i,i+1) = -\sum_{i=k}^{l-1} a(m+k, m+i, m+i+1) \\ &= -\sum_{i=k+m}^{l+m-1} a(k,i,i+1) = b(m+k, m+l) \; . \end{aligned}$$

Since $0 = da(k,l,i,i+1) = a(l,i,i+1) - a(k,i,i+1) + a(k,l,i+1) - a(k,l,i)$, we have

$$\begin{aligned} db(k,l,m) &= b(l,m) - b(k,m) + b(k,l) \\ &= -\sum_{i=l}^{m-1} a(l,i,i+1) + \sum_{i=k}^{m-1} a(k,i,i+1) - \sum_{i=k}^{l-1} a(k,i,i+1) \\ &= \sum_{i=l}^{m-1} -a(l,i,i+1) + a(k,i,i+1) = \sum_{i=l}^{m-1} a(k,k,i+1) - a(k,l,i) \\ &= a(k,l,m) - a(k,l,l) = a(k,l,m) \; . \end{aligned}$$

□

**Theorem 8.3 (Dye,Krieger,Connes,Feldman,Weiss)** *$\mathcal{H}^p_{orb}(\mathcal{R}_{\mathbf{Z}^d}, U) \cong \mathcal{H}^p_{orb}(\mathcal{R}_{\mathbf{Z}}, U)$ for all $n \geq 1$.*

*Proof.* Dye-Krieger-Connes-Feldman-Weiss theorem [8] assures the existence of a measureable invertible map $\psi : X^{n+1} \to X^{n+1}$ mapping $\mathcal{R}_G^n$ bijectively onto $\mathcal{R}_{\mathbf{Z}}^n$. Given a cochain $a \in \mathcal{C}^n(\mathcal{R}_G, U)$, then $\overline{a}(x) = a(\psi(x))$ is in $\mathcal{C}^n(\mathcal{R}_{\mathbf{Z}}, U)$. Because $\psi$ commutes with $d$, cocycles are mapped bijectively onto cocycles and coboundaries onto coboundaries. $\square$

**Corollary 8.4** $\mathcal{H}_{alg}^2(\mathbf{Z}^d,\mathcal{U}) \cong \{0\}$.

*Proof.* By Propositions 8.2,8.3

$$\mathcal{H}_{alg}^2(\mathbf{Z}^d,\mathcal{U}) \cong \mathcal{H}_{orb}^2(\mathcal{R}_{\mathbf{Z}^d}, U) \cong \mathcal{H}_{orb}^2(\mathcal{R}_{\mathbf{Z}}, U) \cong \mathcal{H}_{alg}^2(\mathbf{Z},\mathcal{U}) \cong \{0\} .$$

$\square$

**Theorem 8.5** *If $G = \mathbf{Z}^d$ acts freely on $\mathcal{U}$, then $\mathcal{H}_{geom}^p(\mathbf{Z}^d,\mathcal{U}) \cong \mathcal{H}_{alg}^p(\mathbf{Z}^d,\mathcal{U})$ for $0 \leq p \leq d$.*

*Proof.* For $J \subset \{1, \dots, d\}$ with $|J| = p$, the cube $S_J$ in $G = \mathbf{Z}^d$ spanned by $\{e_j\}_{j \in J}$ is a disjoint union of simplices $\Delta_i = \Delta(g_0^{(i)}, g_1^{(i)}, \dots, g_p^{(i)})$. For a simplex $\Delta(g_0, \dots, g_p)$ and an algebraic cocycle $a \in \mathcal{C}^p$, write $\int_{\Delta(g_0,\dots,g_p)} a = a(g_0, \dots, g_p)$. If $S$ is a finite union of disjoint simplices $\Delta_i$, we set

$$\int_S a = \sum_i \int_{\Delta_i} a .$$

Define a map $\phi : \mathcal{C}_{alg}^p \to \mathcal{C}_{geom}^p$ by

$$\phi(a)_J = A_J = \int_{S_J} a .$$

(i) $\phi$ commutes with $d$.
Let $\sigma_{iJ,K}$ be the signature of the permutation $iJ \mapsto K$.

$$\begin{aligned} d\phi(a)_K = (dA)_K &= \sum_{i \cup J = K} (A_J(T_i) - A_J)\tau_{iJ} = \sum_{i \cup J = K} \sigma_{iJ,K}(A_J(T_i) - A_J)\tau_K \\ &= \sum_{i \cup J = K} \sigma_{iJ,K} \int_{S_J} (a(T_i) - a)\tau_K = \int_{\delta S_K} a = \int_{S_K} da = \phi(da)_K . \end{aligned}$$

(ii) $\phi$ maps cocycles into cocycles and coboundaries into coboundaries.
Proof. From $(i)$ we know that $da = 0$ implies $dA = d\alpha = \phi(da) = \phi(0) = 0$. Also $a = dc$ implies $A = \phi(a) = \phi(dc) = d\phi(c) =: dC$.

For the next step (iii), we need some definitions: a $p$-dimensional simplex and a $(d-p)$-dimensional simplex are *transversal*, if their intersection is a point which is in the interior of both simplices. There are only countably many simplices $\Delta(g_0, \dots, g_p), g_i \in \mathbf{Z}^d$. To each $p$-dimensional oriented cube $S_J(g) = \{g + e_i \cdot [0,1] \mid i \in J\}$ in $\mathbf{Z}^d$ is attached a $(d-p)$-dimensional oriented dual cube $S_{J^*}(g) = \{g + e_i \cdot [0,1] \mid i \in J^* = \{1, \dots, d\} \setminus J\}$. One has $S_K(g) \cap S_{J^*}(g) = \{g\}$ and their product is a positively oriented $d$-dimensional cube. Denote by $W_p$ the union of all $p$-dimensional cubes in $\mathbf{Z}^d$ and call it the *$p$-dimensional web* in $\mathbf{Z}^d$. A $p$-dimensional simplex $\Delta$ is transversal to a $(p-d)$-dimensional web $W_{d-p}$, if $\Delta$ intersects every cube in $W_{d-p}$ transversely. To every transversal intersection is attached a sign, defined by the orientation of their product .

(iii) For each $1 \leq p \leq d$, there exists a $\mathbf{Z}^d-$ periodic homeomorphic deformation $\tilde{W}_p$ of the $(d-p)$-dimensional web $W_p$ in $\mathbf{R}^d$, such that every simplex $\Delta$ has only transversal intersections with the $(d-p)$-dimensional deformed dual cubes $\tilde{S} \subset \tilde{W}_p$.
Proof. The set $\mathcal{W}$ of $\mathbf{Z}^d$-periodic homeomorphisms of $\mathbf{R}^d$ is a complete metric space with metric

$d(f,g) = \sup_{x \in \mathbf{R}^d/\mathbf{Z}^d} |f(x) - g(x)|$. The set $\mathcal{W}(\Delta)$ of homeomorphisms in $\mathcal{W}$ for which a given simplex $\Delta$ intersects every cube in $\tilde{W}_{d-p}$ transversely, is an open dense set. By Bair's theorem, the intersection of all $\mathcal{W}(\Delta)$, where $\Delta$ runs over the countable set of simplices is nonempty and dense.

(iv) If $A$ is a geometric $p$-cocycle, there exists an algebraic $p$-cocycle $a$ such that $\phi(a) = A$.
Proof. Given a geometric cocycle $A$, define the algebraic cocycle

$$a(g_0, \dots, g_p) = \sum_{g: \tilde{S}_J(g) \cap \Delta(g_0, \dots, g_p) \neq \emptyset} A_J(g) \; .$$

If $S_J(g)$ is a $p$-dimensional cube, it is a finite union of $p$-simplices $\Delta_i$. Only one of these simplices has an intersection with the deformed dual cube $\tilde{S}_{J^*}(g)$ since $S_J(g) \cap S_{J^*}(g)$ and so $S_J(g) \cap \tilde{S}_{J^*}(g)$ is a point. This point is by construction of the homeomorphisms in (iii) in the interior of exactly one of the simplices. Therefore

$$\phi(a)_J(g) = \int_{S_J(g)} a = A_J(g) \; .$$

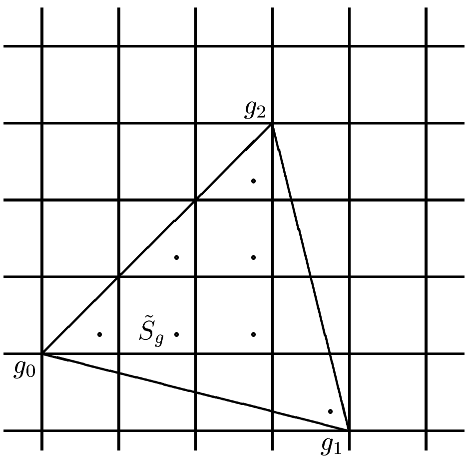


Fig. 6a). Reconstruction of the algebraic cocycle $a$ from the geometric cocycle $A$ in the case $d = p = 2$. The deformed web $\tilde{W}_0$ is zero dimensional and is disjoint from any line connecting two points in $\mathbf{Z}^2$. The value of $a(g_0, g_1, g_2)$ is the flux through the triangle $\Delta(g_0, g_1, g_2)$ and given by the sum of the fluxes $A(g)$ through plaquettes $P(g)$ of all the $g \in \mathbf{Z}^2$ for which $\tilde{S}(g)$ is inside the simplex $\Delta(g_0, g_1, g_2)$.

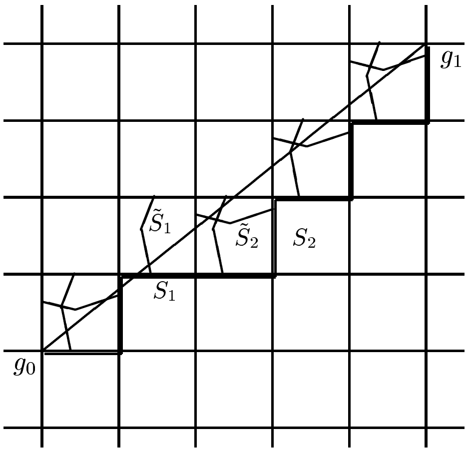

Fig. 6b). Reconstruction of the algebraic cocycle $a$ from the geometric cocycle $A$ in the case $d = 2$, $p = 1$. The deformed web is one dimensional and shown is only the part of this web which intersects the line $\Delta(g_0, g_1)$. The value of $a(g_0, g_1)$ is the flux through the line $\Delta(g_0, g_1)$ and given by the sum of the fluxes $A(g)$ through the edges $S_i(g)$ of all the $g \in \mathbf{Z}^2, i = 1, 2$ for which $\tilde{S}_i(g)$ intersects $\Delta(g_0, g_1)$.

(v) $\mathcal{H}^p_{geom}(\mathbf{Z}^d,\mathcal{U}) \cong \mathcal{H}^p_{alg}(\mathbf{Z}^d,\mathcal{U})$.
Proof. The group homomorphism on cocycles induces the surjective map $\phi : \mathcal{H}^p_{alg}(\mathbf{Z}^d,\mathcal{U}) \to \mathcal{H}^p_{geom}(\mathbf{Z}^d,\mathcal{U})$. If $\phi$ were not injective, it would have a kernel which means that some algebraic not-coboundary would be mapped into a coboundary and this would contradict $(ii)$. $\square$

Remark. DePauw [11] has shown the relation of geometric and algebraic cohomology in the case $d = 2$ and $U = \mathbf{R}$, where the construction of the inverse of $\phi$ is more direct. Our construction of the deformed web is necessary since we do not assume that we can divide in the group $U$. Let us illustrate why the case $U = \mathbf{R}$ would also be easier in higher dimensions. Take $d = 3$ and $p = 2$. We can then construct the algebraic cocycle $a$ from a geometric cocycle $A$ by taking a smooth divergence free vector field in $\mathbf{R}^3$ which has the flux $A_P$ through each plaquette $P$. The flux of this vector field through a triangle $\Delta(g_0, g_1, g_2)$ is then $a(g_0, g_1, g_2)$.

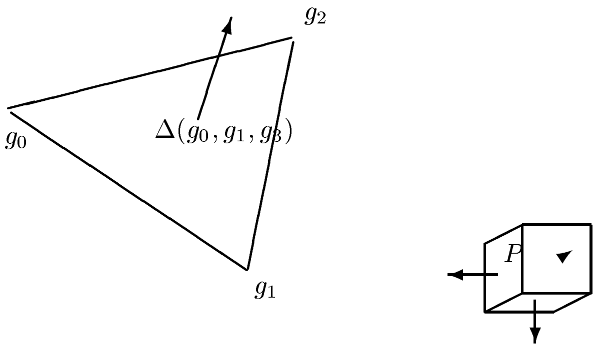


Fig. 7. The problem of constructing an algebraic 2-cocycle given a geometric 2−cocycle in dimension $d = 3$ is the task to determine the flux through a simplex $\Delta(g_0, g_1, g_2)$, if the flux through all plaquettes $P$ is known.

**Corollary 8.6 (Existence of electromagnetic Laplacians)** *Assume the $\mathbf{Z}^d$ action is free. Given a $U(1)$-valued 2-form $F = \sum_{i<j} F_{ij}\tau_{ij}$ such that $dF = 0$. Then there exists a 1-form $A$ satisfying $dA = F$.*

*Proof.* This follows from the fact that the two dimensional geometric cohomology is trivial. $\square$

To see that the higher cohomology groups $\mathcal{H}^p$ with $p \geq 2$ are trivial, one reduces higher dimensional cohomology groups to $\mathcal{H}^2$ with the help of a reduction theorem of Eilenberg-McLane for algebraic cohomology [13, 14]. To any algebraic cochain $a$ is assigned a *non homogeneous cochain* $h$ by the formulas

$$\begin{aligned} a(g_0, g_1, \dots, g_n) &= g_0 h(g_1 - g_0, \dots, g_n - g_{n-1}) \\ h(g_1, \dots, g_n) &= a(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdot \dots \cdot g_n) \,. \end{aligned}$$

The coboundary operator is in the non homogeneous approach given by

$$\begin{aligned} dh(g_1, \dots g_{n+1}) &= g_1 h(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i h(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} h(g_1, \dots, g_n) \,. \end{aligned}$$

A non-homogeneous cocycle is called *normalized* if $h(g_1, \dots, g_n) = 1$ if some $g_i = 1$. Denote by $\mathcal{C}(\mathbf{Z}^d, U)$ the set of all normalized cocycles $h$ from $\mathbf{Z}^d$ to $U$, where the action of $\mathbf{Z}^d$ on $\mathcal{C}(\mathbf{Z}^d, U)$ is given by $(xh)(y) = h(x + y) - yh(x)$.

**Theorem 8.7 (Reduction theorem)** $\mathcal{H}^{n+1}_{alg}(\mathbf{Z}^d, U) \cong \mathcal{H}^n_{alg}(\mathbf{Z}^d, \mathcal{C}(\mathbf{Z}^d, U))\ , n > 0.$

*Proof.* (Eilenberg-McLane [13]) Define the homomorphism $\phi : \mathcal{C}^n(\mathbf{Z}^d, \mathcal{C}(\mathbf{Z}^d, U)) \to \mathcal{C}^{n+1}(\mathbf{Z}^d, U)$ by

$$\phi h(g_1, \dots, g_{n+1}) = (-1)^n h(g_2, \dots, g_{n+1})(g_1)$$

which is an isomorphisms of groups by the homogeneity assumption. A computation shows that $\phi$ commutes with $d$. $\square$

**Corollary 8.8** *If the $\mathbf{Z}^d$ action is free, then $\mathcal{H}^p_{geom}(\mathbf{Z}^d, \mathcal{U}) \cong 0$ for $p \geq 2$.*

# 9 Currents

Given a $\mathbf{Z}^d$-action on the group $\mathcal{U} = \mathcal{L}(X, U)$. A geometric 1-form $A = \sum_i A_i \tau_i \in \mathcal{C}^1$ defines an electromagnetic field $F = dA \in \mathcal{C}^2$ and so a *current* $j = d^* F = *d * F \in \mathcal{C}^1$, where $*$ is the natural Hodge involution $* : \mathcal{C}^n \to \mathcal{C}^{d-n}, A_I \mapsto A_{I^*}$. A current is defined even if the group $\mathcal{U}$ is nonabelian:

**Proposition 9.1** *Assume $N \geq 1, d = 2$ or $N = 1, d \geq 2$. Every current $d^* F = j$ is divergence free: $d^* j = 0$.*

*Proof.* If $d = 2$, the Hodge involution for 1-forms is given by $A_1 \tau_1 + A_2 \tau_2 = (A_1, A_2) \mapsto (A_2, A_1^{-1})$. The divergence of $j$ is given by

$$d^* j = *d * (j_1, j_2) = (*d)(j_2, j_1^*) = j_1 j_2 (T_2)^* j_1^* (T_1) j_2\ .$$

If we plug in $j = (j_1, j_2) = d^* F = (F^* F(T_2), F(T_1)^* F)$, we get

$$d^* j = F^* F(T_2) F(T_2)^* F(T_1 T_2) F(T_2 T_1)^* F(T_1) F(T_1)^* F = 1\ .$$

In the abelian case, $d^* j = 0$ follows in any dimension from $d^* d^* = 0$ $\square$

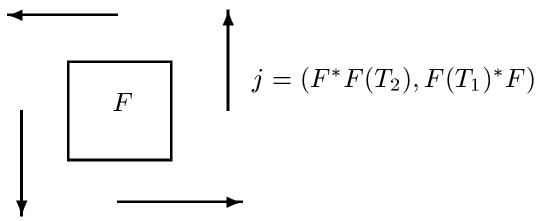


Fig. 8. Illustration of the current of a (not necessarily abelian) Aharonov-Bohm field, where the field $F$ is constant different from 1 on one plaquette only.

One can ask if a current $j$ which is divergence free $d^* j = 0$ does come from a field $F$ satisfying $d^* F = j$. The answer is no in two dimensions and yes in dimensions three or higher. There are at least countably many equivalence classes of currents:

**Proposition 9.2** *Assume $d = 2$ and let $U$ be a Polish group. The moduli space of all divergence free currents $j$ modulo currents $j$ coming from fields $j = d^* F$ is isomorphic to the first cohomology group $\mathcal{H}^1(U)$. For $d \geq 3$ and abelian $U$, every divergence free current $j$ is of the form $d^* F$.*

*Proof.* Assume first $d=2$. $j$ is a cocycle if $j_2 j_1^*(T_1) = j_1^* j_2(T_2)$ and a coboundary if there exists a solution $F$ of $j_1 = F(T_2)F^*, j_2 = FF(T_1)^*$. If $j$ is a cocycle, then the Hodge dual $\tilde{j} = *j$ satisfies a zero curvature equation. Also, $d^*F = j$ if and only if $\tilde{j}$ is a gradient $d(*F) = \tilde{j}$. The moduli space of zero curvature fields modulo gradient fields is $\mathcal{H}^1(U)$.
Assume now $d \geq 3$ and $U$ abelian. Given the 1-form $j$, define the $(d-1)$-form $\tilde{j} = *j$. Since $\mathcal{H}^{d-1}(U)$ is trivial for $d \geq 3$, there exists a $(d-2)$-form $\tilde{F}$ satisfying $d\tilde{F} = \tilde{j}$. Let $F = *\tilde{F}$. Then $d^*F = j$. □

Question. We do not know, if we can realize in dimensions $d \geq 3$ every current $j$ can be written as $d^*F$ with a field $F$ satisfying additionally $dF = 0$. If this were true and $F$ were unique, we would get an interesting class of higher dimensional operators $L = A + A^*$ by taking independent identically distributed random variables $j$, where $d^*dA = j$.

# 10 Magnetic Laplacians on tilings and other lattices

**Magnetic Laplacians on the triangular lattice.**
The triangular lattice is the Cayley graph of the group $G = \mathbf{Z}^2$ with the three generators $e_1, e_2, e_1 + e_2$. A situation with two different fluxes has been considered in [5] (see also [4]). A magnetic field is a cocycle which assigns to each triangle $\Delta(g_1, g_2, g_3), g_i \in \mathbf{Z}^2$ a group element in $U$. This cocycle is determined by the value of $B_d(n)$ on $\Delta(n, n+e_1, n+e_2)$ and $B_u(n)$ on $\Delta(n+e_1, n+e_1+e_2, n+e_2)$ for each $n \in \mathbf{Z}^2$. The two measurable maps $B_d, B_u \in L^\infty(X, U)$ and an ergodic $\mathbf{Z}^2$ action determine so the magnetic field.

**Proposition 10.1** *Every stationary $U(N)$-valued field $B$ on a triangular lattice in $\mathbf{Z}^2$ is given by a vector potential $A$ so that $B = dA$. The spectral properties of $L$ depend only on $B$. If $\{B(n)\}_{n \in \mathbf{Z}^2}$ are independent identically distributed random variables with Haar distribution on $U = U(1)$, then* $\mathrm{tr}(L^n)$ *is the number of closed paths in the triangular lattice which give zero winding number to all triangles.*

*Proof.* In order to get the vector potential $A$, we form $B(x) = B_u(x)B_d(x)$ which is the field on the quadratic plaquette $P(x)$. Feldman-Moore-Lind's theorem gives the existence of the first two coordinates $(A_1, A_2)$ of the vector potential. We define then $A_3$ through $A_3 A_2(T_1) A_1 = B_d$.

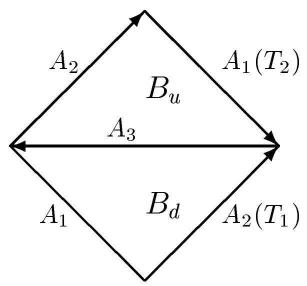


Fig. 9. Construction of the vector potential from $B_u$ and $B_d$.

In the abelian case, a second proof is obtained directly from the algebraic group cohomology for the group $G = \mathbf{Z}^2$ acting on $\mathcal{U} = \mathcal{L}(X, U)$: the magnetic field $B$ with law $\mu$ is an algebraic 2-cocycle. Since the second cohomology group is trivial, it is of the form $dA$, where $A$ is a 1-form.
For abelian $U$, the random walk expansion is done in the same way as for the square lattice by putting $A_2$ identically zero.
In order to see that all the spectral properties depend only on the field $B$, we take the same special gauge as in the square lattice case. □

Remarks.
1) Discrete magnetic Laplacians on more general graphs with uniform magnetic field with values in $U(1)$ have been considered by Sunada [40].
2) If the graph $G$ is the Cayley graph of an infinite abelian group with finitely many generators and $U \subset U(1)$, the magnetic Laplacians are elements in a hyperfinite von Neumann algebra $\mathcal{X}$. The second group cohomology vanishes and every algebraic cocycle $B$ is of the form $B = dA$.

**Magnetic Laplacians on aperiodic tilings**.
Aperiodic tilings in $\mathbf{R}^2$ define a plane graph and one can ask if it is possible to assign to the edges of the graph $U(1)$ random variables in such a way that the magnetic fields in the pieces of the tiling are independent identically distributed $U(1)$-valued random variables. For simplicity, we consider only the case of the *Penrose tiling* with plaquettes built by Robinson triangles. The case, when the plaquettes are Penrose rhombs can be reduced to that by multiplying the field values of the triangles building the rhomb.

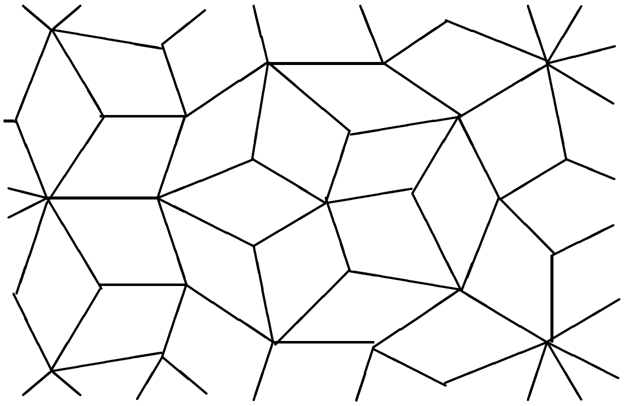

Fig. 10. Part of the Penrose lattice with Penrose rhombs each consisting each of two Penrose triangles.

Define a countable ergodic equivalence relation $\mathcal{R}$ on the compact metric space $X$ of tilings

$$\mathcal{R} = \{(x,y) \in X^2 \mid x = y + v, \text{ where } v \text{ is a difference of vertices of x}\} .$$

A measurable map from $\mathcal{R}^3 = \{(x_1,x_2,x_3) \in X^3 \mid (x_i,x_j) \in \mathcal{R}\}$ to $U$ is an algebraic cocycle and defines a magnetic field distribution $B$ on the Robinson triangles of the tilings.

**Proposition 10.2** *Given a measurable $U = U(1)$-valued field distribution $B$ on the Penrose lattice. There exists a vector potential $A$ such that $dA = B$.*

*Proof.* The equivalence relation $\mathcal{R}$ is defined by a countable group action (see [14] Theorem 1) and defines a hyperfinite von Neumann algebra. The hyperfiniteness follows from the fact that $\mathcal{R}$ is a subset of a countable equivalence relation $\mathcal{R}_G$, where $G \subset \mathbf{R}^2$ is the abelian countable group generated by

$$\{v \in \mathbf{R}^2 \mid v \text{ occurs as a difference of two sites in a tiling }\} .$$

Since $\mathcal{R}_G$ is hyperfinite, also $\mathcal{R}$ is hyperfinite ([14] Proposition 4.2.c). By Proposition 8.3, $\mathcal{H}^2(\mathcal{R},U)$ is isomorphic to to $\mathcal{H}^2(\mathcal{R}_{\mathbf{Z}},U)$ which is trivial. The 2-cocycle $B = B(g_1,g_2,g_3)$ defines the magnetic flux through a triangle spanned by three sites $(g_1,g_2,g_3)$ in the lattice. The triviality of $\mathcal{H}^2(\mathcal{R},U)$ implies that $B = dA$ for some 1-cycle $A$, which is a measurable map from $\mathcal{R}$ to $U$. $\square$

**Corollary 10.3** *Given an independent identically distributed $U(1)$-valued field $B$ on the Robinson triangles of a Penrose tiling. There exists a measurable vector potential $A$ on the edges of the Penrose graph such that $dA = B$.*

Remarks.
1) For more general tilings, where all pieces of the tiling are composed of the same number $k$ of triangles (which is the case in the Penrose tiling where each Penrose rhomb is a union of two Robinson triangles) we can also realize independent identically distributed magnetic field configurations, where the law $\mu = \nu \oplus \cdots \oplus \nu$ is the $k$-th convolution of a measure $\nu$. This is for example the case if $\mu$ is the Haar measure on a closed subgroup of $U(1)$.
2) The existence of the density of states of a magnetic Laplacian $L$ on the tiling follows from the fact $L$ in a finite type von Neumann algebra. Hof [17] has given a direct proof of the existence and proven that the density of states and spectrum is constant on the space of tilings.
3) For independent identically distributed magnetic fields with Haar measure of $U(1)$, we get that $\mathrm{tr}(L^n)$ is the number of closed paths in the tiling graph such that the winding number is zero for each tile. The computation of the density of states is already nontrivial for the free Laplacian with zero magnetic field. There are some numerical results about the random walk on Penrose lattice [41].

## Appendix: The first cohomology group

The first cohomology group $\mathcal{H}^1(U)$ is the only nontrivial cohomology group. It appears as the moduli space of zero curvature fields modulo fields which can be gauged to zero, or as the moduli space of two dimensional currents modulo currents induced by fields. We have therefore some interest in this group. However, we do not know even the cardinality of $\mathcal{H}^1(U)$! We report in this appendix a result of Derrien which sheds some light on this group.

Assume $U$ is abelian. Because the first cohomology group $\mathcal{H}^1(T, U)$ is the same for all aperiodic ergodic abstract dynamical systems $(X, T, m)$, it is enough to investigate it for one specific aperiodic system. For the irrational rotation on the circle $X = \mathbf{R}/\mathbf{Z}$, $T : x \mapsto x + \alpha$ leaving invariant the Lebesgue measure $m$, there is the following corollary of a result of [12].

**Proposition 10.4** *There exists a countable subgroup $K(U)$ of $\mathcal{H}^1(U)$ which is isomorphic to the subgroup of $U^{\mathbf{Z}}$ for which only finitely many elements are different from* 1.

*Proof.* Let $P$ be a partition of $X = \mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ into finitely many intervals $P_i = [t_{i+1} - t_i)$ with rational $t_i$ and let $\phi$ be a measurable map $X \to U$ which is constant on the intervals of $P$ but not constant. There exists a bijective relation between $U^{\mathbf{Q} \cap \mathbf{T}}$ for which only finitely many elements are different from 1 and the set of so defined cocycles: given $\phi$ define $\tilde{\phi} \in U^{\mathbf{Q} \cap \mathbf{T}}$ by $\tilde{\phi}_t = \phi_{t+0} - \phi_{t-0}$ which is different from 1 only at finitley many places. The inverse of this map defines for each $\tilde{\phi} \in A^{\mathbf{Q} \cap \mathbf{T}}$ a cocycle $\phi$. The set of such cocycles together with the unit cocycle forms a subgroup of all cocycles. Derrien has shown [12] that all of these cocycles except the unit element are not coboundaries. Therefore, they are pairwise not cohomologous. $\square$

Remarks.
1) There are situations, where the first cohomology group for *continuous* functions is known: for Markov chains $(X, T, m)$ with infinite $X$ and $U = \mathbf{R}$, the cohomology group $C(X)/\{f \in C(X) \mid f = g(T) - g\}$ is the free abelian group with a countable infinite number of generators ([35] p.62).
2) It would be interesting to know if $K(U)$ is already equal to $\mathcal{H}^1(U)$. Other guesses are that $\mathcal{H}^1(U)$ is isomorphic to the completion of $U^{\mathbf{Z}}$ of $K(U)$ or that $\mathcal{H}^1(U)$ is a countable set containing $K(U)$.

We want to illustrate now in the simplest situation in a self-contained way, why the cohomology group $\mathcal{H}^1(U)$ is countably infinite if $U$ is $U(1)$. We consider $\mathbf{Z}_2 = \{-1, 1\}$ as a discrete subgroup of $U(1)$.

**Lemma 10.5** *Assume $T$ is ergodic on $(X, \mathcal{F}, m)$. Given $B \in \mathcal{L}(X, \mathbf{Z}_2)$. If $B = A(T)A^{-1}$ with $A \in \mathcal{L}(X, U(1))$, then there exists $C \in \mathcal{L}(X, \mathbf{Z}_2)$, such that $B = C(T)C^{-1}$.*

*Proof.* ¿From $B = A(T)A^{-1}$, we get $A(Tx) = \pm A(x)$ so that $\arg(A(x))$ (mod $\pi$) is $T$-invariant. The ergodicity of $T$ implies that $\beta = \arg(A(x))(\text{mod } \pi)$ is constant almost everywhere so that $A(x) = C(x)e^{i\beta}$ with $C(x) \in \mathbf{Z}_2 = \{1, -1\}$. Now $A(x) = C(Tx)C^{-1}(x)$. □

Remark. This lemma shows that $\mathcal{H}^1(\mathbf{Z}_2)$ is a subgroup of $\mathcal{H}^1(U)$. Similarly, one can show that if $U$ is a subgroup of $V$, then $\mathcal{H}^1(U)$ is a subgroup of $\mathcal{H}^1(V)$.

A result of [20] illustrates in the simplest situation, how cohomology constraints emerge:

**Corollary 10.6** *If both $T$ and $T^2$ are ergodic, then $B(x) = -1$ can not be written as $B(x) = A(Tx)A(x)^{-1}$ with $A \in \mathcal{U}$.*

*Proof.* ¿From the above lemma, we have only to show that $B$ can not be written as $B = A(T)A^{-1}$ with $A \in \mathcal{L}(X, \mathbf{Z}_2)$. Assume, there exists a solution $A$ of this equation. Then, since $A(Tx) = -A(x)$ also $A(T^2x) = A(x)$, so that by the ergodicity of $T^2$, $A$ must be constant. There are only two possibilities: $A = 1$ or $A = -1$ and both are not solving $-1 = A(T)A^{-1}$. □

This leads to a special case of Derriens result in the case $U = \mathbf{Z}_2$, where the group of cocycles is the $\sigma$-algebra of the probability space $\mathcal{F}$ and the coboundaries is the additive subgroup of $\mathcal{F}$ consisting of elements which are of the form $Z \Delta T(Z)$.

**Corollary 10.7** *If $T$ is the irrational rotation of the circle $X$, then every dyadic interval $[k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$ is not a coboundary and two dyadic intervals of length $2^{-n}$ are pairwise not cohomologous.*

*Proof.* Corollary 10.6 shows that $X$, the dyadic interval of length $2^{-0}$ is not a coboundary. If a dyadic interval of length $2^{-n}$ were a coboundary then all of them were coboundaries and so also a disjoint union of $2^n$ such intervals contradicting that $X$ is not a coboundary. Inductively, assume the claim is true for coboundaries of length $2^{-n}$ and assume the union $I = I_1 \Delta I_2$ of two disjoint dyadic intervals of length $2^{-(n+1)}$ is a coboundary. If $I_1$ and $I_2$ are adjacent, then $I$ is a dyadic interval of length $2^{-n}$ and can not be a coboundary. Also in the other case, $I$ is not a coboundary because else $K = I \Delta (I + 2^{-(n+1)} \text{ mod } 1)$ is a coboundary which is the union of two dyadic intervals of length $2^{-n}$ which is not a coboundary by induction. □

Remarks.
1) Corollary 10.7 shows that $\mathcal{H}^1(\mathbf{Z}_2)$ is at least countably infinite and and so also every $\mathcal{H}^1(U)$ with a group $U$ containing $\mathbf{Z}_2$. Taking dyadic intervals is not essential and with some more effort the above argument allows to recover Derrien's result in the case $U = \mathbf{Z}_2$.
2) In the topology $d(Y, Z) = m(Y \Delta Z)$ on the $\sigma$-algebra $\mathcal{F}$, both the set $\{Y \in \mathcal{F} \mid Y = Z(T) \Delta Z\}$ of coboundaries as well as its complement are dense in $\mathcal{F}$ ([21]).
3) If $X$ is a finite probability space, $\mathcal{H}^1(U) = U$ for any group $U$.

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