Modular Inclusion, the Hawking Temperature and Quantum Field Theory in Curved Space-Time

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Abstract

A recent result by Borchers connecting geometric modular action, modular inclusion and the spectrum condition, is applied in quantum field theory on spacetimes with a bifurcate Killing horizon (these are generalizations of black hole space-times, comprising the familiar black hole spacetime models). Within this framework we give sufficient, model-independent conditions ensuring that the temperature of thermal equilibrium quantum states is the Hawking temperature.

1. Introduction

Since Hawking [11] suggested that quantum fields on black hole space-times have a thermal distribution corresponding to a certain temperature now come to be known as the Hawking temperature, there has been a series of papers attempting to understand this in a mathematically rigorous way (see e.g. [15], [10], [12], [9], [14]). In this note we wish to contribute to this discussion by pointing out a rigorous, model-independent and, we believe, clarifying connection between the geometrical action of certain modular objects and the necessity of thermal states having the Hawking temperature. We shall address this subject in the context of algebraic quantum field theory on a class of space-times with a bifurcate Killing horizon. Spacetimes with a bifurcate Killing horizon – see the next section for a definition – may be viewed as generalizations of black-hole spacetimes. Some important examples of such space-times are the Minkowski, the Schwarzschild-Kruskal, the Schwarzschild-deSitter, and a family of Kerr-Newman space-times (see [14]). A natural question of physical interest is under which conditions thermal equilibrium states of quantum fields propagating in such a background are forced to assume the Hawking temperature.

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We wish to emphasize that the Hawking temperature itself provides information about the geometry of the background space-time, since it is in one-one correspondence with the surface gravity of the bifurcate Killing horizon of the underlying space-time, which itself is closely related to the mass of the black hole [6]. This is mentioned because we are also interested in the degree to which the modular objects associated with certain states on (subalgebras of) elements of nets of C^*-algebras on a space-time with a bifurcate Killing horizon contain information about the underlying space-time geometry. In studying physically significant states from the point of view of the geometric content of their modular objects, we are motivated by a program outlined in [5] (see also [3], [4]) which hopes to characterize such states by their geometric modular action. The tools we shall be employing here are due to Borchers [1] and Wiesbrock [16], who established the intimate interconnection of a weak form of geometric modular action, the spectrum condition and the notion of modular inclusion (to be discussed below).

In the next section we introduce the geometrical setting of globally hyperbolic spacetimes with a bifurcate Killing horizon according to the detailed exposition by Kay and Wald [14]. We define the central notion of a symmetry-improving restriction (to the bifurcate Killing horizon) of a net of local observable algebras over such a spacetime in Section 3. In that same section we use this notion to prove a theorem providing model-independent sufficient conditions for a KMS (thermal equilibrium) state on a net of local observable algebras over a spacetime with a bifurcate Killing horizon to have the Hawking temperature. We view these conditions as rather general – in the final section we show that the net of observable algebras associated with a linear Hermitean scalar field on such a space-time admits the required symmetry-improving restriction; moreover, there do exist states over these space-times which manifest the geometric modular action required by our theorem (see Section 4).

2. Space-times With a Bifurcate Killing Horizon

In this section we briefly summarize parts of the discussion in [14] on spacetimes with a bifurcate Killing horizon. We refer to this reference for further details and illustrations.

We recall the following notational conventions: Let (M, g) be a C^∞-spacetime manifold M with smooth Lorentzian metric g, admitting a time-orientation. Then for $\mathcal{O} \subset M$, $J^\pm(\mathcal{O})$ denotes the set of all points in M which can be reached by future/past directed causal curves emanating from $\mathcal{O}$. $I^\pm(\mathcal{O})$ is defined analogously for timelike curves. In contrast to [14], we do not use the abstract index notation for tensor fields in the present Letter.

A space-time with a *bifurcate Killing horizon* is characterized by a quintupel $(M, g, \tau_t, \Sigma, \mathbf{h})$, where (M, g) is a globally hyperbolic space-time, $\{\tau_t\}$ is a (nontrivial) one-parameter group of isometries of (M, g), Σ is a two-dimensional connected spacelike submanifold, contained in a spacelike Cauchy-surface of M, which is left pointwise invariant under the action of $\{\tau_t\}$, i.e. $\tau_t(p) = p$ for all $t \in \mathbf{R}$, $p \in \Sigma$, and $\mathbf{h}$ is the bifurcate Killing horizon, i.e. the union of two three-dimensional C^∞-manifolds in M formed by the lightlike geodesics emanating from Σ with Σ-orthogonal tangent vectors; it is assumed that a choice of two continuous, linearly independent, lightlike, future-directed, Σ-orthogonal vector fields χ_A, χ_B along Σ can be made.

Let γ_{Ap} and γ_{Bp} be the maximal geodesics defined by $\chi_A(p)$ and $\chi_B(p)$, respectively, for $p \in \Sigma$. These are lightlike geodesics which are left invariant under the action of $\{\tau_t\}$. This means, in particular, that the corresponding Killing field ξ is tangent to these geodesics, and geodesics starting at different points $p \in \Sigma$ cannot cross. One defines the following pieces of $\mathbf{h}$: $\mathbf{h}_X$ is the subset of $\mathbf{h}$ generated by γ_{Xp}, $p \in \Sigma$, and $\mathbf{h}_X^\pm \equiv \mathbf{h}_X \cap I^\pm(\Sigma)$ for $X \in \{A, B\}$. Then we set

$$\mathbf{h}_A^R \equiv \mathbf{h}_A^+ \,, \quad \mathbf{h}_A^L \equiv \mathbf{h}_A^-$$

and

$$\mathbf{h}_B^R \equiv \mathbf{h}_B^- \,, \quad \mathbf{h}_B^L \equiv \mathbf{h}_B^+ \,.$$

By convention, it will be assumed that ξ is future oriented on $\mathbf{h}_A^R$. A point $q \in \mathbf{h}_A$ ($\mathbf{h}_B$) can be coordinatized by a pair (U, p) (resp. (V, p)), where the point $p \in \Sigma$ determines on which geodesic q lies and the affine parameter U (resp. V) indicates where on the specified geodesic q lies, so that we have $\gamma_{Ap}(U) = q$ (resp. $\gamma_{Bp}(V) = q$). We assume that the affine parameters are chosen such that $\gamma_{Ap}(U = 0) = p$ (resp. $\gamma_{Bp}(V = 0) = p$). As we have already mentioned, ξ is tangent to the geodesics γ_{Ap} and γ_{Bp}, whose tangent vector fields will be denoted by χ_{Ap} and χ_{Bp}, and it can be shown that there exists a smooth function f_A^R, defined on $\mathbf{R}^+ \times \Sigma$,[1] positive and strictly increasing with U, such that

$$\xi(U, p) \;=\; f_A^R(U, p)\, \chi_{Ap}(U) \,, \tag{1}$$

for points $(U, p) \in \mathbf{h}_A^R \equiv \mathbf{R}^+ \times \Sigma$, and the quantity

$$\kappa \;\equiv\; \xi \cdot \ln(f_A^R) \;>\; 0 \tag{2}$$

is a constant, i.e. independent of U and p (see [14]). κ is called the *surface gravity* of $\mathbf{h}_A$. Similar arguments apply with functions f_A^L, f_B^R and f_B^L for the other parts of $\mathbf{h}$, yielding the same κ.[2] This implies that the action of $\{\tau_t\}$ on points of the bifurcate Killing horizon is of the following form, which relates the action of the Killing flow on the horizon to the action of the affine dilatations:

Lemma 1: Under the stated assumptions, one has

$$\tau_t(U, p) \;=\; (e^{\kappa t} U, p) \qquad \text{and} \qquad \tau_t(V, p) \;=\; (e^{-\kappa t} V, p).$$

[1] where $\mathbf{R}^+ = (0, \infty)$

[2] The $f_{\dots}^L$ are negative and so one has to take $-f_{\dots}^L$ in the argument of the logarithm in (2).

Proof: Choose arbitrary $U > 0$ and $p \in \Sigma$. The lightlike geodesic γ_{Ap} is left invariant under the action of $\{\tau_t\}$, thus by the properties of the chosen coordinatization we have that $\tau_t(U,p) = (\widehat{\tau}_t(U), p)$ with a group action $\{\widehat{\tau}_t\}$ on $\mathbf{R}^+$. Using this, one deduces from (1) that

$$\frac{d}{dt}\widehat{\tau}_t(U) = f_A^R(\widehat{\tau}_t(U), p)\,. \tag{3}$$

¿From (2) it follows that

$$\kappa = \frac{d}{dt}\ln(f_A^R(\widehat{\tau}_t(U), p)) = \frac{d}{dt}\widehat{\tau}_t(U) \cdot \frac{1}{f_A^R(\widehat{\tau}_t(U), p)} \cdot \frac{\partial}{\partial x} f_A^R(x, p)\bigg|_{x=\widehat{\tau}_t(U)}\,.$$

Hence it follows that $f_A^R(\widehat{\tau}_t(U), p) = e^{\kappa t + C(U)}$, and thus, with (3),

$$\widehat{\tau}_t(U) \;=\; \frac{1}{\kappa}\, e^{\kappa t + C(U)} + \tilde{C}(U)\,.$$

By the group property $\widehat{\tau}_{t'}\,\widehat{\tau}_t(U) = \widehat{\tau}_{t'+t}(U)$ for all $t', t \in \mathbf{R}$, one easily obtains $\tilde{C}(U) = 0$. Then $U = \widehat{\tau}_0(U) = \frac{1}{\kappa}e^{C(U)}$, implying $\widehat{\tau}_t(U) = e^{\kappa t}U$. The argument for $\mathbf{h}_A^L$ $(U < 0)$ and for $\mathbf{h}_B$ (i.e. $\tau_t(V,p) = (e^{-\kappa t}V, p)$) is similar. $\Box$

Another natural group action on $\mathbf{h}_A$ is that of the affine translations:

$$\ell_a(U,p) = (U - a, p), \qquad \ell_a(V,p) = (V - a, p),$$

which we shall see also play an important role in our considerations.

3. Symmetry-Improving Restrictions of Nets of von Neumann Algebras, Modular Inclusion and the Hawking Temperature

Let $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ be a net[3] of C^*-algebras over M indexed by a basis $\mathcal{R}$ for the topology of M and contained in a C^*-algebra $\mathcal{A}$. The net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ need not be covariant with respect to any transformation group associated with the manifold M. We shall denote by $\mathcal{K}$ a set of open subsets of $\mathbf{h}_A$ (resp. $\mathbf{h}_B$) which is invariant under the Killing flow and the affine translations. In other words, for every $\mathcal{G} \in \mathcal{K}$ and $t, a \in \mathbf{R}$, there exist $\mathcal{G}_t, \mathcal{G}_a \in \mathcal{K}$ such that $\tau_t\mathcal{G} = \mathcal{G}_t$ and $\ell_a\mathcal{G} = \mathcal{G}_a$. In addition, with $\mathcal{K}_X$ defined as the set $\{\mathbf{h}_A^X \cap \mathcal{G} \mid \mathcal{G} \in \mathcal{K}\}$ (resp. $\{\mathbf{h}_B^X \cap \mathcal{G} \mid \mathcal{G} \in \mathcal{K}\}$), we require that $\mathcal{K}_X \subset \mathcal{K}$, for $X = L, R$. If $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$ is a net of C^*-algebras indexed by $\mathcal{K}$ and contained in the C^*-algebra $\mathcal{N}$, we shall denote by $\mathcal{N}_X$ the sub-C^*-algebra of $\mathcal{N}$ generated by the algebras $\{\mathcal{N}(\mathcal{G} \cap \mathbf{h}_A^X) \mid \mathcal{G} \in \mathcal{K}\}$ (resp. $\{\mathcal{N}(\mathcal{G} \cap \mathbf{h}_B^X) \mid \mathcal{G} \in \mathcal{K}\}$), for $X = L, R$. The central notion to be introduced here is that of a symmetry-improving restriction.

Definition: Let $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$ be a net of C^*-algebras (contained in the C^*-algebra

[3] strictly speaking, an inclusion-preserving map will do

$\mathcal{N}$) over $\mathbf{h}_A$ (resp. $\mathbf{h}_B$) indexed by a collection $\mathcal{K}$ as described above. Such a net $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$ is said to be a *symmetry-improving restriction* to $\mathbf{h}_A$ (resp. $\mathbf{h}_B$) of the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ if:

(1) it is covariant with respect to the Killing flow and the affine translations, i.e. if there exist (suitably) continuous[4] group representations

$$\mathbf{R} \ni t \mapsto \alpha_t \in \mathrm{Aut}\,\mathcal{N} \quad , \quad \mathbf{R} \ni a \mapsto \lambda_a \in \mathrm{Aut}\,\mathcal{N} \quad ;$$

such that for all $a, t \in \mathbf{R}$ and $\mathcal{G} \in \mathcal{N}$ one has

$$\alpha_t(\mathcal{N}(\mathcal{G})) = \mathcal{N}(\tau_t\mathcal{G}) \quad , \quad \lambda_a(\mathcal{N}(\mathcal{G})) = \mathcal{N}(\ell_a\mathcal{G}) \quad ;$$

(2) one has the inclusion $\mathcal{K} \subset \{\mathcal{O}\cap\mathbf{h}_A \mid \mathcal{O}\in\mathcal{R}\}$ (resp. $\mathcal{K} \subset \{\mathcal{O}\cap\mathbf{h}_B \mid \mathcal{O}\in\mathcal{R}\}$) and for each $\mathcal{K} \ni \mathcal{G} = \mathcal{O}\cap\mathbf{h}_A$ (resp. $\mathcal{K} \ni \mathcal{G} = \mathcal{O}\cap\mathbf{h}_B$), $\mathcal{O}\in\mathcal{R}$, also $\mathcal{N}(\mathcal{G})$ is a subalgebra of $\mathcal{A}(\mathcal{O})$.

We shall show in the next section that there exist examples of such symmetry-improving restrictions on general space-times with bifurcate Killing horizons. The name symmetry-improving restriction is, of course, motivated by the fact that the original net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ need not be covariant under the Killing flow and certainly not under the affine translations, which only act on the horizons, whereas both conditions are required for $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$. One can well imagine that this concept could be useful in any space-time which possesses a submanifold having a group of isometries which are not the restriction of isometries of the larger manifold.

Remark: We note here that in the case where the spacetime (M, g) is Minkowski spacetime and the bifurcate Killing horizon $\mathbf{h}$ is formed by the horizons of adjacent and causally complementary wedge regions, each Poincaré covariant net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ (of von Neumann algebras) in the vacuum representation possesses symmetry-improving (or rather, symmetry-preserving, in this case) restrictions to $\mathbf{h}$. A detailed investigation of such a situation has been carried out by Driessler [8].

Beginning with [1] and continuing with [16], [17], [18], [5], [3], [2] and [4], interesting connections between the spectrum condition and the 'geometric' action of modular objects have been established. We recall the first result of this nature.

Theorem 2 [1]: Let $\mathcal{M}$ be a von Neumann algebra acting on some Hilbert space $\mathcal{H}$ and assume that $\Omega\in\mathcal{H}$ is cyclic and separating for $\mathcal{M}$. Then let Δ, J be the modular operator and modular conjugation corresponding to $(\mathcal{M},\Omega)$. Let $\mathsf{U}(a)$, $a\in\mathbf{R}$, be a strongly continuous one-parameter group with positive generator leaving Ω invariant. If, in addition, $\mathsf{U}(a)\mathcal{M}\mathsf{U}(a)^* \subset \mathcal{M}$ for $a \geq 0$, then it follows that

$$\Delta^{it}\mathsf{U}(a)\Delta^{-it} \;=\; \mathsf{U}(e^{-2\pi t}a) \;\text{ and }\; \mathrm{J}\mathsf{U}(a)\mathrm{J} \;=\; \mathsf{U}(-a)\,,$$

[4] We shall indicate later which continuity we require for our present purposes; however, since we feel that this notion will have other uses besides the application presented here, we shall not at this point further specify the continuity.

for all $t, a \in \mathbf{R}$; if, instead, $\mathsf{U}(a)\mathcal{M}\mathsf{U}(a)^* \subset \mathcal{M}$ for $a \leq 0$, then it follows that

$$\Delta^{it}\mathsf{U}(a)\Delta^{-it} = \mathsf{U}(e^{2\pi t}a) \text{ and } \mathrm{J}\mathsf{U}(a)\mathrm{J} = \mathsf{U}(-a)\,,$$

for all $t, a \in \mathbf{R}$.

(For the proof, see Theorem II.9 in [1].)

Remark: In [16] Wiesbrock has proven an interesting converse to Borchers' result. He showed that if $\mathsf{U}(a)$ is a continuous unitary group such that $\mathsf{U}(a)\mathcal{M}\mathsf{U}(a)^* \subset \mathcal{M}$ for $a \geq 0$, and if

$$\Delta^{it}\mathsf{U}(a)\Delta^{-it} = \mathsf{U}(e^{-2\pi t}a) \text{ and } \mathrm{J}\mathsf{U}(a)\mathrm{J} = \mathsf{U}(-a)\,,$$

for all $t, a \in \mathbf{R}$, then it follows that the generator of $\mathsf{U}(a)$ is positive. In [5] it was shown that in this converse it is not necessary to assume the condition $\mathrm{J}\mathsf{U}(a)\mathrm{J} = \mathsf{U}(-a)$.

We shall use Theorem 2 to show that any state on a symmetry-improving restriction $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$ which is a ground state with respect to $\{\lambda_a\}$ [5] and which is also a KMS-state with respect to $\{\alpha_t\}$ on $\mathcal{N}_R$ *must* have the (inverse) Hawking temperature $\beta = 2\pi/\kappa$.

Theorem 3: Let ω be a state on a net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ admitting a symmetry-improving restriction $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$ to $\mathbf{h}_A$ (or $\mathbf{h}_B$) with the additional properties:

(i) $\mathcal{G}_1 \neq \mathcal{G}_2$ entails $\mathcal{N}(\mathcal{G}_1) \neq \mathcal{N}(\mathcal{G}_2)$;

(ii) the restriction of ω to $\mathcal{N}$ is a ground state with respect to the affine translations λ_a;

(iii) the restriction of ω to $\mathcal{N}_R$ (resp. $\mathcal{N}_L$) is a KMS-state at inverse temperature $\beta \neq 0$ with respect to the Killing flow α_t.

Then $\beta = 2\pi/\kappa$ (resp. $\beta = -2\pi/\kappa$).

Proof: 1. Note that the collection $\mathcal{K}_R$ is invariant under the induced action of the Killing flow and under nonpositive affine translations. This is readily seen by noting that one has the decomposition $\mathbf{h}_A \cong \mathbf{R} \times \Sigma$ and $\mathbf{h}_A^R \cong \mathbf{R}^+ \times \Sigma$. Let pr_1 denote the corresponding projection onto the first component, $\mathrm{pr}_1(U,p) = U$, for $(U,p) \in \mathbf{h}_A$. In order to show that $\tau_t(\mathbf{h}_A^R \cap \mathcal{G}) \in \mathcal{K}_R$ for $\mathcal{G} \in \mathcal{K}$, it suffices to verify that $\mathrm{pr}_1(\tau_t(\mathbf{h}_A^R \cap \mathcal{G})) \subset \mathbf{R}^+$, which is evident from Lemma 1. Similarly, it is evident that $\mathrm{pr}_1(\ell_a(\mathbf{h}_A^R \cap \mathcal{G})) \subset \mathbf{R}^+$ whenever $a \leq 0$. It may be seen in a like manner that $\mathcal{K}_L$ is invariant under the induced action of the Killing flow and under nonnegative affine translations.

The C^*-algebras $\mathcal{N}(\mathcal{G})$ will be notationally identified with their images under π_ω, $\pi_\omega(\mathcal{N}(\mathcal{G}))$, where $(\mathcal{H}_\omega, \pi_\omega, \Omega)$ is the GNS-representation of $\mathcal{N}$ with respect to

[5]Here we use the term in the sense of algebraic quantum field theory; hence the states in question are invariant with respect to $\{\lambda_a\}$ and the generator of the corresponding strongly continuous unitary representation of $\{\lambda_a\}$ in the GNS-representation space must have nonnegative spectrum.

the state ω. $\mathsf{U}(\mathbf{R})$ will denote the continuous unitary group implementing the affine translations in the GNS-representation of ω, whose existence is assured by assumption (ii). The previous paragraph entails that the von Neumann algebra $\mathcal{N}_R''$ is invariant under the action of $\mathrm{Ad}\mathsf{U}(a)$, $a \leq 0$. Moreover, $\mathcal{N}_R''$ is also invariant under the action of the Killing flow, which by assumption (iii) is implemented on $\mathcal{N}_R''$ by $\mathrm{Ad}\Delta^{it/\beta}$, for all $t \in \mathbf{R}$, where Δ is the modular operator associated with $(\mathcal{N}_R'', \Omega)$.[6] Since assumption (ii) implies that the generator of the group $\mathsf{U}(\mathbf{R})$ satisfies the spectrum condition, one may appeal to Theorem 2 to conclude that

$$\Delta^{it}\mathsf{U}(a)\Delta^{-it} \ = \ \mathsf{U}(e^{2\pi t}a)\,,$$

for all $t, a \in \mathbf{R}$. Similarly, when considering $\mathcal{N}_L''$ these arguments lead to the equality $\Delta^{it}\mathsf{U}(a)\Delta^{-it} = \mathsf{U}(e^{-2\pi t}a)$.

2. For an arbitrary $\mathcal{G} \in \mathcal{K}_R$ one therefore has

$$\begin{aligned}
\mathcal{N}((\tau_t \circ \ell_a)\mathcal{G}) &= (\alpha_t \circ \lambda_a)\mathcal{N}(\mathcal{G}) \\
&= \Delta^{it/\beta}\mathsf{U}(a)\mathcal{N}(\mathcal{G})\mathsf{U}(a)^*\Delta^{-it/\beta} \\
&= \mathsf{U}(e^{2\pi t/\beta}a)\Delta^{it/\beta}\mathcal{N}(\mathcal{G})\Delta^{-it/\beta}\mathsf{U}(e^{2\pi t/\beta}a)^* \\
&= (\lambda_{e^{2\pi t/\beta}a} \circ \alpha_t)\mathcal{N}(\mathcal{G}) \\
&= \mathcal{N}((\ell_{e^{2\pi t/\beta}a} \circ \tau_t)\mathcal{G}) \quad .
\end{aligned}$$

In light of assumption (i), this entails

$$(\tau_t \circ \ell_a)\mathcal{G} = (\ell_{e^{2\pi t/\beta}a} \circ \tau_t)\mathcal{G},$$

for all $\mathcal{G} \in \mathcal{K}_R$. Applying the projection pr_1 to both sides of this equation and using the fact that both τ_t and ℓ_a act only on the first component, one finds

$$e^{\kappa t}(\mathrm{pr}_1\mathcal{G} - a) = e^{\kappa t}(\mathrm{pr}_1\mathcal{G}) - e^{2\pi t/\beta}a,$$

for all $t \in \mathbf{R}$, $a \leq 0$, and all $\mathcal{G} \in \mathcal{K}_R$. One concludes that $\beta = 2\pi/\kappa$. If one had considered $\mathcal{N}_L$, one would have concluded $\beta = -2\pi/\kappa$. □

Remark: Suppose that we assume in (ii) of Theorem 3 only that ω induces a λ_a-invariant state on $\mathcal{N}$. Then the automorphisms

$$\alpha_t \circ \lambda_a \circ \alpha_t^{-1} \circ \lambda_{e^{\kappa t}a}^{-1} \tag{4}$$

are, in the GNS-representation of $\omega|\mathcal{N}$, implemented by unitaries $V_{t,a}$ which leave the GNS-vector Ω invariant, and which are inner automorphisms of the net, $V_{t,a}\mathcal{N}(\mathcal{G})V_{t,a}^{-1} = \mathcal{N}(\mathcal{G})$ for all $\mathcal{G} \in \mathcal{K}$, $t, a \in \mathbf{R}$. If these automorphisms are trivial and if $\omega|\mathcal{N}_R$ is already at inverse temperature $\beta = 2\pi/\kappa$ with respect to the Killing flow, then using Wiesbrock's converse (extended in [5]) to Theorem 2, one concludes that $\omega|\mathcal{N}$ is actually a ground state for the affine translations. We point out that in the case that the net $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$ is generated by a quantum field Φ over $\mathbf{h}$, more precisely by an injective, operator valued distribution

[6] Note that (iii) implicitly assumes that the vector Ω is cyclic and separating for $\mathcal{N}_R''$.

$\mathcal{T} \ni f \mapsto \Phi(f)$, with a sufficiently large class $\mathcal{T}$ of testfunctions on $\mathbf{h}$, the automorphisms (4) are, in fact, trivial.

One therefore sees that the Hawking temperature emerges quite naturally in a model-independent manner [7] on space-times with a bifurcate Killing horizon – if the states on the horizon are "hot", then they are at the Hawking temperature. However, what is necessarily a model-dependent question is how large the region of dependence of the horizon is, i.e. how large the algebra $\mathcal{N}_R$ is in $\mathcal{A}$. This question is, in principle, of physical interest, since one wishes to know where one may observe the temperature of the horizon.

The first paper [15] to connect the Hawking temperature with modular objects can be seen in hindsight to have posited the existence of something which looks very much like a symmetry-improving restriction (though in the setting of Wightman fields) to a horizon. However, we feel that Theorem 3 more clearly isolates the structures required.

Note that the situation adressed in Theorem 3 is analogous to the Rindler-Fulling-scenario (see, e.g., [12]), where the Minkowski vacuum, which is a ground state with respect to lightlike affine translations, restricts to a thermal equilibrium state on the algebra of local observables localized in the right Rindler wedge with respect to the Lorentz boosts leaving this region invariant. In fact, this scenario is simply an *example* of a space-time with a bifurcate Killing horizon where the assumptions of our theorem hold.

In their fundamental work [14], Kay and Wald have studied thermal and uniqueness properties of Killing-flow invariant, Hadamard states of the linear Klein-Gordon field on manifolds with a bifurcate Killing horizon. In the next section we illustrate that their setting provides an example of a net of local algebras over a spacetime with a bifurcate Killing horizon admitting symmetry-improving restrictions to $\mathbf{h}_A$ (or $\mathbf{h}_B$) to which our Theorem 3 applies. It is, in particular, worth noting that assumptions (ii) and (iii) have been shown in [14] to follow from the Hadamard condition and Killing-flow invariance for quasifree states of the Klein-Gordon field on a spacetime with a bifurcate Killing horizon; see also Section 4 below.

A further result of relevance here is that of Haag, Narnhofer and Stein [10], who proved that if a (linear) quantum field theory on a Lorentzian manifold with timelike Killing vector field and horizons satisfying certain conditions admits a KMS-state with respect to the Killing flow and the state satisfies their conditions of local definiteness and stability on the horizon, then the temperature of the said state must equal the Hawking temperature. In [9] the collapse of a star into a black hole was treated dynamically, and it was shown that for a linear quantum field there is radiation at large times at the Hawking temperature.

[7] In particular, one sees that the statement and proof of the theorem do not rely upon using the special structure of free quantum fields, as do most of the rigorous discussions of the Hawking temperature in the literature.

4. Example: The Weyl Algebra of the Linear Hermitean Scalar Field in a Space-time with a Bifurcate Killing Horizon

We next provide an example which illustrates the notions and satisfies the assumptions of the previous section by considering a net of local algebras corresponding to the linear Hermitean scalar field, following Dimock [7] and Kay and Wald [14], and showing that it does admit a symmetry-improving restriction. The relevant field equation on (M, g) is the Klein-Gordon equation:

$$(\Box_g + m^2)\varphi \;=\; 0\,, \tag{5}$$

for $m \geq 0$. Here, $\Box_g$ is the D'Alembertian with respect to the Lorentzian metric g (which equals $\nabla^\mu\nabla_\mu$ in index notation with ∇ = covariant derivative of g). As explained in the "Note added in proof" in [14], it is necessary to consider special spaces of solutions of (5), since we wish to view certain Weyl algebras associated with symplectic spaces formed by characteristic data of (5) on the bifurcate Killing horizon as subalgebras of the Weyl algebra over the symplectic space of solutions of (5) whose restrictions to Cauchy surfaces have compact support.

Let $\mathcal{C}$ be a Cauchy surface for (M, g) and let, for C^1-functions ψ on M, $\rho_0\psi \equiv \psi|\mathcal{C}$ and $\rho_1\psi \equiv n \cdot \psi|\mathcal{C}$, with n denoting the future-directed unit-normal field of $\mathcal{C}$. We define S as the space of all *real-valued* C^2-solutions φ of (5) such that their Cauchy-data $\rho_0\varphi$ and $\rho_1\varphi$ are of compact support on any Cauchy surface $\mathcal{C}$. The set S will be endowed with the symplectic form

$$\sigma(\varphi, \psi) \;\equiv\; \int_{\mathcal{C}} (\varphi\,(n \cdot \psi) - \psi\,(n \cdot \varphi))\, d\eta_{\mathcal{C}}\,,$$

where $d\eta_{\mathcal{C}}$ denotes the induced measure on $\mathcal{C}$. That σ is indeed a symplectic form and independent of $\mathcal{C}$ follows from standard theorems on existence and uniqueness of initial-value solutions of (5) in globally hyperbolic space-times (cf. [7]) and from Green's formula.

We next introduce some symplectic subspaces of (S, σ) (following the "Note added in proof" of [14]). Let S_A consist of all solutions φ in S such that there is a function $f \in C_0^\infty(\mathbf{h}_A)$ so that the characteristic data $\varphi|\mathbf{h}_A$ of φ on $\mathbf{h}_A$ have the form

$$(\varphi|\mathbf{h}_A)(U, p) \;=\; U^5 \frac{\partial^5}{\partial U^5} f(U, p)\,, \tag{6}$$

for all $U \in \mathbf{R}$ and all $p \in \Sigma$ (by the results of the "Note added in proof" in [14], one obtains that equation (6) indeed implies $\varphi \in S$). Moreover, we shall say that φ is in the set S_A^R if the function f in (6) lies in $C_0^\infty(\mathbf{h}_A^R)$. The subspaces S_B and S_X^Y for $X = A, B$ and $Y = R, L$ are defined analogously. The symplectic form $\sigma(\varphi, \psi)$ for elements $\varphi, \psi \in S_A$ takes the form

$$\sigma(\varphi, \psi) \;=\; \int_{\mathbf{h}_A} \left(\tilde{\varphi}(U, p)\, \frac{\partial}{\partial U} \tilde{\psi}(U, p) - \tilde{\psi}(U, p)\, \frac{\partial}{\partial U} \tilde{\varphi}(U, p) \right) dU\, d\eta_\Sigma(p)\,,$$

where $\tilde{\varphi}, \tilde{\psi}$ denote the restrictions of φ, ψ to $\mathbf{h}_A$ in the coordinatization of $\mathbf{h}_A$ which we have chosen; note that we shall henceforth maintain this notation. The expression for $\sigma(\varphi,\psi)$ when $\varphi,\psi \in S_B$ is analogous. Therefore one can show that S_A, S_B and S_X^Y ($X = A, B$; $Y = R, L$) are symplectic subspaces of (S,σ) (see [14] for further details). If we write

$$T_t\varphi \;\equiv\; \varphi \circ \tau_{-t}\,,$$

then $\{T_t\}$ is a symplectomorphism group on (S,σ). It is also clear from Lemma 1 that the action of $\{T_t\}$ on S_X^Y ($X = A, B$; $Y = R, L$) and on S_A and S_B leaves these symplectic subspaces of (S,σ) invariant. Another group of symplectomorphisms on S_A, and on S_B, is given by the *affine translations*,

$$(\widetilde{\Lambda_a\varphi})(U,p) \;\equiv\; \tilde{\varphi}(U+a,p)\,,$$

for $a \in \mathbf{R}$. Observe that S_X^Y ($X = A, B$; $Y = R, L$) are not left invariant under the action of $\{\Lambda_a\}$, but are left *half-sided invariant*:

$$\Lambda_a\varphi \in S_A^R \quad \text{for} \quad \varphi \in S_A^R\,, a \in \mathbf{R}^- \tag{7}$$

$$\Lambda_a\varphi \in S_A^L \quad \text{for} \quad \varphi \in S_A^L\,,\; a \in \mathbf{R}^+\,,\; \text{etc.}$$

By $\mathcal{A}$, $\mathcal{A}_A$, $\mathcal{A}_B$, $\mathcal{A}_X^Y$ we shall denote the Weyl algebras corresponding to the symplectic spaces (S,σ), $(S_A,\sigma|S_A)$, $(S_B,\sigma|S_B)$, $(S_X^Y,\sigma|S_X^Y)$, and by α_t, λ_a the induced actions of τ_t, Λ_a on the appropriate Weyl algebras.

It is known (cf. [7] and references quoted there) that there are unique advanced (+)/retarded(−) distributional fundamental solutions of the Klein-Gordon equation (5), that is, continuous linear operators

$$(E')^{\pm} : \mathcal{E}'(M) \longrightarrow \mathcal{D}'(M)$$

with the property that $(\Box_g + m^2)(E')^{\pm}u = u = (E')^{\pm}(\Box_g + m^2)u$ for all $u \in \mathcal{E}'(M)$ (to be understood in the sense of distributions), and $\mathrm{supp}((E')^{\pm}u) \subset J^{\mp}(\mathrm{supp}\, u)$[8]. Their difference gives the distributional propagator $E' := (E')^+ - (E')^-$ of the Klein-Gordon equation. Generalizing the the proofs presented in [7] for smooth functions, one can show that, whenever φ is in S, with data $\rho_0\varphi, \rho_1\varphi$ on some Cauchy-surface $\mathcal{C}$, there exists for every open neighborhood $\mathcal{O}$ of $\mathrm{supp}\,\rho_0\varphi \cup \mathrm{supp}\,\rho_1\varphi$ a continuous function u on M with $\mathrm{supp}\, u \subset \mathcal{O}$ and $E'u = \varphi$. We define for each open, relatively compact subset $\mathcal{O}$ of M the set $S(\mathcal{O})$ as consisting of all $\varphi \in S$ such that there is $u \in C^0(M)$ with $E'u = \varphi$ and $\mathrm{supp}\, u \subset \mathcal{O}$. Then, defining $\mathcal{A}(\mathcal{O})$ as the C^*-subalgebra of $\mathcal{A}$ generated by Weyl-operators $W(\varphi)$ with $\varphi \in S(\mathcal{O})$, one proves by an appropriate generalization of the methods given in [7] that the family of C^*-algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, as $\mathcal{O}$ ranges through the set $\mathcal{R}$ of open, relatively compact subsets of M, is a net of local algebras. (The local net of the Klein-Gordon field.)

We shall need the following lemma.

[8] This is in contrast to $\mathrm{supp}(E^{\pm}f) \subset J^{\pm}(\mathrm{supp} f)$ for smooth functions f, where $E^{\pm}$ are the fundamental solutions defined on smooth test functions, which is due to the way the smooth test functions are embedded in the distributions.

Lemma 4: Let $\varphi \in S_A$, let E' denote the distributional propagator of the Klein-Gordon equation (5), and let $\overline{\mathcal{G}}$ be the support of the restriction of φ to $\mathbf{h}_A$. Given any neighborhood $\mathcal{O}$ of $\overline{\mathcal{G}}$ in M, there exists a continuous function u with support in $\mathcal{O}$ such that $\varphi = E'u$. The same result is true if the subscripts A are everywhere replaced by B.

Proof: For any $p, q \in M$ with $p \in I^+(q)$, let $D_{p,q}$ denote the interior of the set $J^-(p) \cap J^+(q)$. Since the space-time M is assumed to be globally hyperbolic, the set of all such "double cones" $D_{p,q}$ forms a basis for the topology of M. Let $\mathcal{O}$ be a neighborhood of $\overline{\mathcal{G}}$ in M. Since $\overline{\mathcal{G}}$ is compact, it possesses a finite cover $\{D^{(k)}\}_{k=1,\dots,n}$ of double cones $D^{(k)} = D_{p_k,q_k}$ satisfying $\underline{D}^{(k)} \subset \mathcal{O}$, $k = 1,\dots,n$. The sets $L^{(k)} \equiv D^{(k)} \cap \mathbf{h}_A$ form a finite open cover of $\overline{\mathcal{G}}$ in the submanifold $\mathbf{h}_A$. For this open cover, there exists a smooth decomposition of 1; denote this by the collection $\{\chi^{(k)}\}_{k=1,\dots,n}$. Then one has $\varphi = \sum_{k=1}^n \varphi_k$, with $\varphi_k \in S_A$,

$$(\varphi_k|\mathbf{h}_A)(U,p) \;=\; U^5 \frac{\partial^5}{\partial U^5}(\chi_k f)(U,p)\,,$$

whenever

$$(\varphi|\mathbf{h}_A)(U,p) \;=\; U^5 \frac{\partial^5}{\partial U^5} f(U,p)\,.$$

Of course, the restriction $\varphi_k|\mathbf{h}_A$ has compact support contained in $L^{(k)}$. It remains to be shown that for each φ_k there exists a continuous u_k with support in $D^{(k)}$ such that $\varphi_k = E'u_k$.

For a given $k = 1,\dots,n$, choose a Cauchy surface C_k such that $\overline{L^{(k)}} \subset I^+(C_k)$. The submanifold M_k, defined to be the interior of the set $J^-(p_k) \cup J^-(C_k)$, is also a globally hyperbolic space-time with respect to the restricted metric, so there exists a Cauchy surface C such that the (compact) closure of $L^{(k)}$ is contained in the interior (with respect to M_k) of $J^-(C)$ (which coincides with the interior with respect to M of $J^-(C)$). C is also a Cauchy surface for M. Since the intersection of the support of φ_k with C is compact and contained in $D^{(k)}$, there exists a continuous u_k with support in $D^{(k)}$ satisfying $\varphi_k = E'u_k$. $\Box$

This entails that if $\mathcal{K}$ denotes the set of open, relatively compact subsets of $\mathbf{h}_A$ (or $\mathbf{h}_B$), and if for each $\mathcal{G} \in \mathcal{K}$, $\mathcal{N}(\mathcal{G})$ is defined as the C^*-subalgebra of $\mathcal{A}_A$ (or $\mathcal{A}_B$) generated by all Weyl-operators $W(\varphi)$ with $\varphi \in S_A$ (or S_B) and $\mathrm{supp}(\varphi|\mathbf{h}_A) \subset \mathcal{G}$ (resp., $\mathrm{supp}(\varphi|\mathbf{h}_B) \subset \mathcal{G}$), then the net $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$ is a restriction of the local net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ to $\mathbf{h}_A$ (or $\mathbf{h}_B$). Hence states on the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ may be restricted to the net $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$. Note that in this case one has $\mathcal{N} = \mathcal{A}_A$ (or $\mathcal{N} = \mathcal{A}_B$), and $\mathcal{N}_X = \mathcal{A}_A^X$ (or $\mathcal{N}_X = \mathcal{A}_B^X$), for $X = R, L$. Note also that the collection $\mathcal{K}$ is τ_t- and ℓ_a-invariant. In addition, it is evident from the construction that $\mathcal{N}(\mathcal{G}_1) \neq \mathcal{N}(\mathcal{G}_2)$ whenever $\mathcal{G}_1 \neq \mathcal{G}_2$. We therefore have the immediate corollary:

Corollary 5: The above-constructed net of C^*-algebras of the Klein-Gordon field admits a symmetry-improving restriction to the horizon $\mathbf{h}_A$ (and to $\mathbf{h}_B$) which satisfies the condition that $\mathcal{G}_1 \neq \mathcal{G}_2$ implies $\mathcal{N}(\mathcal{G}_1) \neq \mathcal{N}(\mathcal{G}_2)$.

We now come to the question whether states on this symmetry-improving restriction which satisfy the assumptions of Theorem 3 exist at all. In fact, we may take

$$w_\omega(\varphi, \psi) \;=\; -\frac{1}{\pi} \lim_{\epsilon \to 0+} \int \frac{\widetilde{\varphi}(U_1, p)\, \widetilde{\psi}(U_2, p)}{(U_1 - U_2 - i\epsilon)^2}\, d\eta_\Sigma(p)\, dU_1\, dU_2 \tag{8}$$

as the two-point function of a quasifree state ω on $\mathcal{A}_A$ (or $\mathcal{A}_B$), and this state has all the desired properties (cf. [14]) on the net $\{\mathcal{N}(\mathcal{G})\}_{\mathcal{G}\in\mathcal{K}}$. Note that Kay and Wald [14] proved that every quasifree Hadamard state which is invariant under the action of $\{\alpha_t\}$ on $\mathcal{A}$ must restrict to a quasifree state on $\mathcal{A}_A$ with the two-point function given in (8). (In fact, with the new argument of Kay [13], the assumption that the state be quasifree may be dropped). Hence, any extension of this state to the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ (which always exists, since $\mathcal{N}$ is a subalgebra of $\mathcal{A}$) yields an example satisfying the assumptions of Theorem 3.

Corollary 6: On every globally hyperbolic space-time with bifurcate Killing horizon there exist a state ω and a net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ satisfying the hypotheses of Theorem 3.

A different question, however, is if states on the larger algebra $\mathcal{A}$ exist such that their restrictions to more than one horizon algebra satisfy the assumptions of Theorem 3. It may well be that this is not possible if the space-time contains more than one bifurcate Killing horizon which have different surface gravities. And, in fact, as in Section 6.3 in [14], if one considers as an example the Schwarzschild-deSitter space-time, where there is a pair of neighboring bifurcate Killing horizons with unequal surface gravities, then it is clear from Theorem 3 that there cannot exist a state on any net, whose causal support contains both horizons and has symmetry-improving restrictions to both bifurcate Killing horizons, and which is simultaneously a ground state for the affine translations and a KMS-state for the Killing flow on these restrictions. There are other such examples, and it is obvious how a theorem analogous to Theorem 6.5 in [14] can be formulated in our setting.

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References

[1] H.-J. Borchers, The CPT-theorem in two-dimensional theories of local observables, *Commun. Math. Phys.,* **143**, 315-332 (1992).

[2] H.-J. Borchers, On modular inclusion and spectrum condition, *Lett. Math. Phys.,* **27**, 311-324 (1993).

[3] R. Brunetti, D. Guido and R. Longo, Modular structure and duality in quantum field theory, *Commun. Math. Phys.,* **156**, 201-219 (1993).

[4] R. Brunetti, D. Guido and R. Longo, Group cohomology, modular theory and space-time symmetries, *Rev. Math. Phys.,* **7**, 57-71 (1995).

[5] D. Buchholz and S.J. Summers, An algebraic characterization of vacuum states in Minkowski space, *Commun. Math. Phys.,* **155**, 449-458 (1993).

[6] B. Carter, The general theory of the mechanical, electromagnetic and thermodynamic properties of black holes, in: *General Relativity*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge) 1979, pp. 294-369.

[7] J. Dimock, Algebras of local observables on a manifold, *Commun. Math. Phys.,* **77**, 219-228 (1980).

[8] W. Driessler, On the structure of fields and algebras on null planes I, *Acta Phys. Austriaca,* **46**, 63-96 (1977).

[9] K. Fredenhagen and R. Haag, On the derivation of Hawking radiation associated with the formation of a black hole, *Commun. Math. Phys.,* **127**, 273-284 (1990).

[10] R. Haag, H. Narnhofer and U. Stein, On quantum field theory in gravitational background, *Commun. Math. Phys.,* **94**, 219-238 (1984).

[11] S.W. Hawking, Particle creation by black holes, *Commun. Math. Phys.,* **43**, 199-220 (1975).

[12] B.S. Kay, The double-wedge algebra for quantum fields on Schwarzschild and Minkowski spacetimes, *Commun. Math. Phys.,* **100**, 57-81 (1985).

[13] B.S. Kay, Sufficient conditions for quasifree states and an improved uniqueness theorem for quantum fields on space-times with horizons, *J. Math. Phys.,* **34**, 4519-4539 (1993).

[14] B.S. Kay and R.M. Wald, Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on space-times with a bifurcate Killing horizon, *Phys. Rep.,* **207**, 49-136 (1991).

[15] G.L. Sewell, Quantum fields on manifolds: PCT and gravitationally induced thermal states, *Ann. Phys.,* **141**, 201-224 (1982).

[16] H.-W. Wiesbrock, A comment on a recent work of Borchers, *Lett. Math. Phys.,* **25**, 157-159 (1992).

[17] H.-W. Wiesbrock, Symmetries and half-sided modular inclusions of von Neumann algebras, *Lett. Math. Phys.,* **28**, 107-114 (1993).

[18] H.-W. Wiesbrock, Half-sided modular inclusions of von Neumann algebras, *Commun. Math. Phys.,* **157**, 83-92 (1993).