THE MIXED PROBLEMS FOR MULTIDIMENSIONAL TIME POLYPARABOLIC OPERATORS

Brenner Alexander

Abstract

We introduce a new class of polyparabolic equations with multidimensional time and develop the existence theory for the respective nonhomogeneous mixed problems. Also, we obtain the uniqueness theorem for the problems satisfying the newly defined inductive polyparabolicity condition. Various equations of this type occur in the theory of probability, cosmology and physics. In particular, they include the class of the so-called ultraparabolic equations.

In order to obtain the existence and uniqueness for polyparabolic problems we study elliptic boundary value problems with parameters in a bounded domain. For example, the notion of the ellipticity with parameters for differential and pseudodifferential operator pencils can be applied to the resolvent construction which leads to the definition of the corresponding complex powers and $\zeta-$function, i.e. to a new functional calculus of operators with promising applications to the multiparameter spectral theory.

Contents

Some Notation

Here is the list of the conventional notation to be used below:

$\mathbb{N}$ - the set of all positive integers.
$\mathbb{Z}$ - the set of all integers.
$\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
$\mathbb{R}$ - the set of all real numbers.
$\mathbb{C}$ - the set of all complex numbers.
$\mathbb{R}^n$ - the standard real n-dimensional euclidean space.
$\mathbb{C}^m$ - the standard complex m-dimensional euclidean space.
$\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n : \ x_n > \ (\text{resp.} <) \ 0\}$; $\overline{\mathbb{R}^n_\pm}$ - the closures of the half-spaces $\mathbb{R}^n_\pm$ in $\mathbb{R}^n$; $(\overline{\mathbb{R}^n_+})^m = \overline{\mathbb{R}^n_+} \times \overline{\mathbb{R}^n_+} \times \cdots \times \overline{\mathbb{R}^n_+}$ $-$ m times. $\mathbb{Z}^n_+ = (\mathbb{Z}_+)^n$.
$\mathbb{C}_\pm = \{z \in \mathbb{C} : \ \mathrm{Im} z > \ (\text{resp.} <) \ 0\}$.
$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \ . \ . \ . \ , \frac{\partial}{\partial x_n}\right)$, where $(x_1, ..., x_n) \in \mathbb{R}^n$.
$D = i^{-1}\frac{\partial}{\partial x}$, $D_j = i^{-1}\frac{\partial}{\partial x_j}$, where $i = \sqrt{-1} \in \mathbb{C}$.
$\partial^l_{x_j} = \frac{\partial^l}{\partial x_j^l}$, $l \in \mathbb{Z}_+$.
$D^\alpha = D_1^{\alpha_1} \ . \ . \ . \ D_n^{\alpha_n}$, where α is a multiindex, i.e. $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j \in \mathbb{Z}_+$.
$\xi^\alpha = \xi_1^{\alpha_1} \ . \ . \ . \ \xi_n^{\alpha_n}$, where $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$.
$x \cdot \xi = x_1\xi_1 \ + \ \cdot \ \cdot \ \cdot \ + \ x_n\xi_n$, where $x = (x_1, ..., x_n) = (x', x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, ..., \xi_n) = (\xi', \xi_n) \in \mathbb{R}^n$.
$|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, where α is a multiindex, $\alpha = (\alpha', \alpha_n)$.
$|||q|||^l = \sum_{i=1}^m |q_i|^l$, if $q = (q_1, ..., q_m) \in \mathbb{C}^m$; $M_q = \min_{i=1,...,m} |q_i|$.
$\prod_{j=1}^k E_j = E_1 \times E_2 \times \cdots \times E_k$, where E_j are Hilbert spaces.
$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, $< x > = (1 + x_1^2 + \cdots + x_n^2)^{1/2}$ for $x =$ $= (x_1, ..., x_n) \in \mathbb{R}^n$.
$\frac{l}{2b} = \left\{\frac{l}{2b_1}, ..., \frac{l}{2b_m}\right\}$, where $l \in \mathbb{Z}_+$, $b \in \mathbb{N}^m$.
$\Delta = \Delta_x = \frac{\partial^2}{\partial x_1^2} \ + \ \cdot \ \cdot \ \cdot \ + \ \frac{\partial^2}{\partial x_n^2}$ - the Laplacian in $\mathbb{R}^n$.
δ_{jk} - the Kroneker symbol ($\delta_{jk} = 0$ as $j \neq k$ and 1 as $j = k$).
$[s]$ - the integer part of s .
$\triangleright$ means the end of the proof.
:= means "equals by definition".
$\subset\subset$ means the compact inclusion.
$\mathrm{supp}\, u$ means the support of the function u .
$\partial\Omega$ means the boundary of an open set $\Omega \subset \mathbb{R}^n$, $\overline{\Omega}$ means its closure.
$C_0^\infty(\mathbb{R}^n)$ - the space of all infinitely smooth complex - valued functions with compact support.
$C^\infty(\overline{\Omega}, Y)$ - the space of smooth functions on Ω with the values in Y , which are continuous up to the boundary.

$S(\mathbb{R}^n)$ - the Schwartz space of C^∞ - functions on $\mathbb{R}^n$, with derivatives vanishing faster than any power of $|x|$ as $|x| \to \infty$.

The direct and the inverse Fourier transformations F , F^{-1} : $S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ and the Fourier transformation in the variables $x' \in \mathbb{R}^{n-1}$ are defined by the formulae

$$\widehat{u}(\xi) \equiv F(u)(\xi) \equiv F_{x\to\xi}(u) = \int_{\mathbb{R}^n_x} e^{-ix\cdot\xi} u(x)\, dx \ ,$$

$$\widetilde{f}(x) \equiv F^{-1}(f)(x) \equiv F^{-1}_{\xi\to x}(f) = (2\pi)^{-n} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} f(\xi)\, d\xi \ ,$$

$$\widehat{u}(\xi') \equiv F'(u)(\xi') \equiv F'_{x'\to\xi'}(u) = \int_{\mathbb{R}^{n-1}_{x'}} e^{-ix'\cdot\xi'} u(x')\, dx' \ .$$

INTRODUCTION

0.1 History and previous results

The first multidimensional time problem to be considered was probably the Goursat problem for Wave equation:

$$\frac{\partial^2 U}{\partial t_1 \partial t_2} \equiv U_{t_1,t_2} = 0, \qquad t_1, t_2 > 0 \; ; \tag{0.1.1}$$

$$U\,|_{t_1=0} = \phi_1(t_2) \; , \quad U\,|_{t_2=0} = \phi_2(t_1) \; , \quad (\phi_1(0) = \phi_2(0)) \; .$$

The subsequent development of the Goursat problem theory was connected with successive approximations method of Picard (see, for example, (Sobolev S.L. (1966))). This method allowed to prove existence and uniqueness of the solution to the Goursat problem for Hyperbolic equation with bounded coefficients

$$U_{t_1,t_2} + a(t_1,t_2)U_{t_1} + b(t_1,t_2)U_{t_2} + c(t_1,t_2)U = f(t_1,t_2) \; , \tag{0.1.2}$$

$$U\Big|_{t_1=t_1^0} = \phi_1(t_2) \; , \quad U\Big|_{t_2=t_2^0} = \phi_2(t_1) \; ,$$

where $t_1^0 \leq t_1 \leq a$ and $t_2^0 \leq t_2 \leq b$.

The analytic Goursat problem for systems with two independent variables and the corresponding nonanalytic Goursat problem for systems of such kind with constant coefficients, but with boundary conditions of polynomial growth, were presented in the works (Mikhailov V.P. (1957a) , (1957b)). The next step in the study of multidimensional time problems was made by Friedman and Littman (see Friedman A. (1962), Friedman A., Littman W. (1963)). The term *Several time variables* itself was probably for the first time introduced in the paper (Friedman A. (1962)), where the uniqueness and existence (under certain conditions) theorems were proved for the Cauchy problem

$$\frac{\partial U_i(x,t)}{\partial t_i} = \sum_{j=1}^{\nu} P_{i,j}\Big(\frac{\partial}{\partial x}, t\Big) U_j(x,t) \; , \quad 1 \leq i \leq \nu \; , \tag{0.1.3}$$

$$U_i(x,t)\,|_{t_i=0} = U_i^0(x, t_1, .., t_{i-1}, t_{i+1}, .., t_\nu) \; ,$$

where U_i are vectors of order N_i , and $P_{i,j}$ are matrices of order $N_i \times N_j$ with elements which are differential operators in $x \in \mathbb{R}^n$ smoothly depending on $t \in (\mathbb{R}_+)^\nu$. In the particular case where $P_{i,i+1} \neq 0$, $P_{i,j} = 0$, $j \neq i + 1$, $(P_{\nu,\nu+1} := P_{\nu,1})$, and all $P_{i,j}$ are independent of t , the problem (0.1.3) for sufficiently smooth functions is equivalent to multidimensional Goursat problem

$$\frac{\partial^\nu U(x,t)}{\partial t_1 \ldots \partial t_\nu} = P\Big(\frac{\partial}{\partial x}\Big) U(x,t) \; , \tag{0.1.4}$$

$$U(x,t)\mid_{t_i=0} \;=\; U_{0i}(x,t_1,..,t_{i-1},t_{i+1},..,t_\nu)\ ,$$

where $P = P_{12}P_{23}...P_{\nu 1}$.

A version of the problem (0.1.3) with complex coefficients was studied in the work (Kirichenko Yu. V. (1983)).

Starting with the problem (0.1.4), Friedman A. and Littman W. (1963) have solved the following mixed problem with two-dimensional time :

$$U_{t_1 t_2} \;-\; LU \;=\; F(x,t) \tag{0.1.5}$$

$$U\mid_{t_1=0} \;=\; U\mid_{t_2=0} \;=\; U\mid_{x\in\partial D} \;=\; 0\ ,$$

where ∂D is the boundary of domain D . It happens that, if L is a second order elliptic self-adjoint differential operator, then the problem (0.1.5) is correctly posed and the existence and uniqueness of the solution are obtained.

The further development of the theory was elaborated in the series of papers by Brish and Yurchuk (see Brish N.I., Yurchuk N.I. (1968), (1970), (1971), Yurchuk. N.I. (1968a), (1968b), (1969)). The Mixed and Goursat problems for hyperbolic equations of a special kind, and also Goursat problem for an abstract linear second order differential equation were studied in these works with the help of the energy inequality method. We shall describe these results briefly, beginning from (Brish N.I., Yurchuk N.I. (1971)).

Let $J = (0,T_1)\times(0,T_2)$ be a bounded rectangle in the plane $\mathbb{R}^2$ with coordinates $t=(t_1,t_2)$ and let $\mathcal{E}$ be a Hilbert space. We consider the equation

$$\frac{\partial^2 U(t)}{\partial t_1 \partial t_2} + A_1(t)\frac{\partial U(t)}{\partial t_1} + A_2(t)\frac{\partial U(t)}{\partial t_2} + A_0(t)U(t) \;=\; f(t)\ , \tag{0.1.6}$$

where U and f are $\mathcal{E}$ - valued functions on J , A_i , $i=0,1,2$ - are the given functions on J with values in the space of bounded linear operators in $\mathcal{E}$. It was proved in (Brish N.I., Yurchuk N.I. (1971)) that equation (0.1.6) with initial conditions of the Goursat type (as in (0.1.1)) admits a unique solution in the corresponding functional spaces. A special case of equation (0.1.6) was introduced in the work (Brish N.I., Yurchuk N.I. (1968)), where

$$A_1(t) = a_1(x,t)\ ,\quad A_2(t) = a_2(x,t)\ ,\quad A_0(t) = a_0(x,t,D) \tag{0.1.7}$$

and a_1 , a_2 were operators of multiplication by functions and a_0 was a uniformly elliptic positive differential operator of order $2k$ with respect to $x\in\Omega\subset\mathbb{R}^n$. Different types of mixed boundary value problems for equation (0.1.6) under conditions (0.1.7) were studied in the works (Brish N.I., Yurchuk N.I. (1968), Yurchuk. N.I. (1968a), (1968b), (1969)). In all cases boundary conditions in x formed a system of boundary differential equations with operators $\{B_j\}_{j=1}^k$:

$$B_j U\big|_\Gamma = \ 0\ ,\quad j=1,..,m\ ,\quad \Gamma = \partial\Omega\times J\ , \tag{0.1.8}$$

where

$$B_j = \sum_{|\alpha| \leq k_j}^{r} b_{j,\alpha}(x) D^\alpha \ , \quad j = 1,..,m \ ,$$

are operators of orders $k_j \leq 2k-1$ $(0 \leq k_1 < ... < k_m \leq 2k)$. It had been supposed that the system (0.1.8) was associated with operator A_0 through the well known Shapiro - Lopatinskij condition (see, for example, Egorov Yu.V., Shubin M.A. (1988)). The following types of boundary conditions in t were introduced in (Brish N.I., Yurchuk N.I. (1968)):

1) Cauchy conditions.

Let $\mathbf{T}' = \{t \in \mathbb{R}^2 : 0 < t_1 < T_1 \ , \ \nu(t_1) < t_2 < T_2\}$, where $\nu \in C^1[0, T_1]$, $\nu'(t_1) < 0$, $0 < t_1 < T_1$, $\nu(0) = T_2, \nu(T_1) = 0$. The following conditions are required to as the Cauchy ones

$$l_0 U \ \equiv \ U \,|_{t_2=\nu(t_1)} \ = \ \phi_0(x, t_1) \ , \quad l_1 U \ \equiv \ \frac{\partial U}{\partial t_2}\Big|_{t_1=\chi(t_2)} \ = \ \phi_1(x, t_2) \ ,$$

where $\chi = \nu^{-1}$.

2) Picard conditions.

Let $\mathbf{T}'' = \{t \in \mathbb{R}^2 : \mu(t_2) < t_1 < T_1, \ 0 < t_2 < T_2\}$, where $\mu \in C^1[0, T_2]$, $\mu'(t_2) > 0$, $\mu(0) = 0$, $\mu(T_2) = T_1$. The Picard conditions are:

$$\gamma_1 U \ \equiv \ U \,|_{t_2=0} \ = \ \phi(x, t_1) \ , \quad \gamma_2 U \ \equiv \ U \,|_{t_1=\mu(t_2)} \ = \ \psi(x, t_2) \ ,$$

where ϕ and ψ are the known functions satisfying $\phi(x, 0) = \psi(x, 0)$.

3) Mixed conditions.

Let $\mathbf{T}''' = t \in \mathbb{R}^2 : 0 < t_1 < T_1 \ , \ \nu(t_1) < t_2 < \omega(t_1)$, where
$\nu, \omega \in C^1[0, T_1]$, $\nu'(t_1) < 0$, $\omega'(t_1) > 0$, $\mu = \omega^{-1}$.
It is required that

$$l_0 U \equiv U \,|_{t_2=\omega(t_1)} = \phi_0(x, t_1) \ , \quad l_1 U \equiv \frac{\partial U}{\partial t_2}\Big|_{t_2=\nu(t_1)} = \phi_1(x, t_1) \ ,$$

$$\gamma_2 U \equiv U \,|_{t_1=\mu(t_2)} = \psi(x, t_2) \ ,$$

under the natural assumption $\phi_0(x, 0) = \psi(x, 0)$.

4) Goursat conditions.

$$l_1 U \equiv U \,|_{t_2=0} = \phi(x, t_2) \, , \quad l_2 U \equiv U \,|_{t_1=0} = \psi(x, t_2) \ ,$$

with $\phi(x, 0) = \psi(x, 0)$.

The existence and uniqueness theorems for the problems (0.1.6) - (0.1.8) with initial conditions 1)-4) were proved in (Brish N.I., Yurchuk N.I. (1968), Yurchuk N.I. (1968b), (1969)) by the energy inequality method and in specially designed functional classes.

A little bit apart is lying the work (Brish N.I., Yurchuk N.I. (1970)) where equation

$$\sum_{j=1}^{m} \frac{\partial U}{\partial t_j} + A_0(t)U = F(t) \tag{0.1.9}$$

was considered together with boundary conditions (0.1.8) and Goursat initial conditions (here $A_0(t)$ is the same as in (0.1.7)). The authors refer to (0.1.9) as to pluriparabolic differential equation and give the proof of the appropriate uniqueness and existence theorems under certain parabolicity conditions.

The next step in the development of the theory is contained in the work (Tersenov S.A. (1987)), where the author considered the *ultraparabolic* equation

$$\sum_{j=1}^{m} \lambda_j(t) \frac{\partial U}{\partial t_j} + A_0 U = F(t) \ , \tag{0.1.10}$$

where $\lambda_j(t) \geq \alpha > 0$ and A_0 was independent of t .

We have to point out that equation (0.1.10) with $\lambda_j = \mathrm{const}$ and operator A_0 of the second order (being called *ultraparabolic*) is of interest since the 30 - th, when they arose in the works of Kolmogorov A.N. (1933), (1934) connected with the theory of probability and Brownian motion. Also, equations of the type (0.1.10) appear in Cosmology, the theory of boundary layer, the theory of neutrons slowing - down etc. (see Chandrasekhar S. (1943), Genchev T.G. (1963), Vladimirov V.S., Drozhzhinov Yu.N. (1967), Drozhzhinov Yu.N. (1969), Marchuk G.I. (1958) and others).

For example, equation (0.1.10) with $\lambda_j = \mathrm{const}$ and $\mathrm{ord}\, A_0 = 2$ appears after linearization of the kinetic Boltzmann equation, where the role of "second time" is played by the lethargy, i.e. the logarithm of energy (see Marchuk G.I. (1958)).

Finally, in the paper (Iliev I.D. (1977)) we see the existence and uniqueness theorem for (0.1.10) with $\lambda_j \neq \mathrm{const}$, homogeneous Goursat initial conditions and A_0 and B_j (in (0.1.8)) being pseudodifferential operators.

On the other hand, mixed problems for the equation

$$A\big(x, D_x, \frac{\partial}{\partial t}\big)\, u(x,t) = f(x,t) \ , \tag{0.1.11}$$

with the one time variable, where $(x,t) \in \Omega \times [0,T]$ and $0 < T \leq \infty$, were considered in (Agranovich M.S., Vishik M.I. (1964)). The following boundary conditions

$$B_\mu\big(x, D_x, \frac{\partial}{\partial t}\big) u(x,t)\big|_{x=x'} = g_\mu(x',t) \ , \tag{0.1.12}$$

where $x' \in \partial\Omega$, $\mu = 1, ..., k$, and initial conditions

$$\frac{\partial^j u}{\partial t^j}\Big|_{t=0} = \phi_j(x) \ , \tag{0.1.13}$$

where $j = 0, ..., \kappa - 1$ and $\kappa = k/b \in \mathbb{Z}$, were attached to (0.1.11). Here A and B_μ are linear differential operators in (x, t) with coefficients being C^∞ - functions in $x \in \overline{\Omega}$, $\operatorname{ord} A = 2k$, $\operatorname{ord} B_\mu = k_\mu$ $(\mu = 1, ..., k)$.

Under the assumption that operator A is parabolic in generalized sense and together with operators B_μ satisfies the Generalized Shapiro - Lopatinskij Condition, the authors have developed the complete theory of such problems together with the theory of corresponding parameter - elliptic boundary value problems .

In this way, if we start with the situation, where t in (0.1.11) - (0.1.12) belongs to $(\mathbb{R}_+)^m$, we arrive to a mixed problem containing most of the problems for the equations (0.1.2), (0.1.3), (0.1.5), (0.1.7), (0.1.9), (0.1.10). This problem was stated in the work (Agranovich M.S., Vishik M.I. (1964)) and we shall refer to it as to Agranovich - Vishik Problem.

Finally, we refer to the increasing number of investigations containing different promising applications of equations with multidimensional time and the related ones to mechanics, physics and cosmology (see Baez J.C., Segal I.E., Zhon Z.-F. (1990), Hillion P. (1990a), (1990b), Newton R.G. (1991), Rendall A.D. (1992), Uglum J. (1992), Polubinskij A.S. (1989), Fushchich V.I., Galitsyn A.S., Polubinskij A.S. (1990)). In particular, the last two works are devoted to the so called Biparabolic equation

$$L_2 U = 0 \ , \quad L_2 = L_1 \circ L_1 \ , \quad L_1 = \frac{\partial}{\partial t} - k^2 \frac{\partial^2}{\partial x^2} \tag{0.1.14}$$

with interesting applications. Thus, it is natural to consider as a Model Example for (0.1.11) the operator

$$A\big(x, D_x, \frac{\partial}{\partial t}\big) \ = \ \frac{\partial^2}{\partial t_1 \partial t_2} - B(x, D_x) \frac{\partial}{\partial t_1} - C(x, D_x) \frac{\partial}{\partial t_2} + E(x, D_x) \, ,$$

where all operators are differential with respective orders $(1/2 \operatorname{ord} E \ = \ \operatorname{ord} B \ = \operatorname{ord} C \ = \ 2)$, and study the respective problems for it (see Brenner A.(1995)). We refer to equations with such operators as to **Polyparabolic equation** with two-dimensional time.

0.2 Description of results

We start with the study of operators with parameters, i.e. we solve an elliptic boundary value problem with parameters in a bounded domain Ω corresponding to the problem (0.1.11) - (0.1.13):

$$Au \ \equiv \ A(x, D, q)\, u(x, q) \ = \ f(x, q) \ , \quad x \in \Omega \subset \mathbb{R}^n \ , \quad q \in \mathbb{C}^m \ , \tag{0.2.1}$$

with boundary conditions on $\Gamma = \partial\Omega$:

$$B_\nu u|_\Gamma \ \equiv \ B_\nu(x, D, q) u(x, q)\big|_{x = x'} \ = \ g_\nu(x', q) \ , \quad x' \in \Gamma \ , \ \nu = 1, ..., k \ , \tag{0.2.2}$$

where

$$A(x,D,q) \;=\; \sum_{|\alpha|+|\beta|\le 2k} a_{\alpha\beta}(x) q^{\beta} D_x^{\alpha} \ ,$$

$$B_\nu(x,D,q) \;=\; \sum_{|\alpha|+|\beta|\le k_\nu} b_{\alpha\beta}(x) q^{\beta} D_x^{\alpha} \ ,$$

i.e. $A(x,D,q)$ and $B_\nu(x,D,q)$ are differential operators with coefficients being C^∞ - functions in $x \in \overline{G}$ and polynomials in q_i for $i = 1,...,m$, where

$$q_i \in Q_i \;=\; \{q_i \in \mathbb{C}^m \ : \ \arg q_i \in [\alpha_i, \beta_i]\} \ ,$$

thus $A(x,\xi,q)$ and $B_\nu(x,\xi,q)$ are polynomials in (ξ,q) of degrees $s = 2k$ and k_ν $(\nu = 1,...,k)$ respectively.

We denote by $A_0(x,\xi,q)$ and $B_{0,\nu}(x,\xi,q)$ the leading parts of the corresponding polynomials which are obtained by dropping the terms with degrees relatively to (ξ,q) being less then $2k$ and k_ν respectively.

We suppose the following conditions of the unique solvability of the problem (0.2.1) - (0.2.2) to be executed.

Condition 0.2.1 $\forall\, x \in \overline{\Omega}$, $\forall\, q \in Q \;=\; Q_1 \times ... \times Q_m$ *and* $\forall\, \xi \in \mathbb{R}^n$

$$A_0(x,\xi,q) \;\neq 0 \quad for \quad |\xi| + M_q \neq 0 \ , \tag{0.2.3}$$

where $M_q \;=\; \min_{i=1,...,m} |q_i|$.

Condition 0.2.2 *Let us fix a boundary point* $x' \in \Gamma$ *and suppose that all operators in (0.2.1) - (0.2.2) are written in a local coordinate system* connected with this point *(i.e. the boundary is straightened and the* x_n-*axis is looking inside* Ω *together with the inner normal to* Γ *).*

We consider the problem on the half-axis $x_n > 0$

$$A_0\big(0,\xi',-i\frac{d}{dx_n},q\big)\, v(x_n) \;=\; 0 , \quad x_n > 0 \ , \tag{0.2.4}$$

$$B_{0,\nu}\big(0,\xi',-i\frac{d}{dx_n},q\big)\, v(x_n)\big|_{x_n=0} = \; h_\nu \ , \quad \nu = 1,...,k \ , \tag{0.2.5}$$

smoothly depending on parameters ξ' *and* q *, and suppose that* $\forall\, q \in Q$ *for* $|\xi'| + M_q \neq 0$ *this problem has a unique solution in the space* $\mathcal{M}$ *of all stable solutions to the equation (0.2.4) for all* $h_\nu \in \mathbb{C}$ *.*

Let us note that for $q = 0$ Condition 0.2.1 turns into the ordinary ellipticity condition and Condition 0.2.2 - into the classical Shapiro - Lopatinskij condition. Here are the main examples.

Example 0.2.3 *Let* $A(x,D)$ *be an elliptic operator* $(\mathrm{ord}A = 2k)$ *with a positive real leading part. Then the operator*

$$A(x,D) + q_1^k q_2^k$$

with Dirichlet boundary conditions

$$B_\nu = \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1}, \quad \nu = 1, ..., k,$$

satisfies Conditions 0.2.1 and 0.2.2 in the cone $Q = Q_1 \times Q_2$, *where* $Q_j = \left\{|\arg q_j| \le \frac{\pi-\varepsilon}{2k}\right\}$, $\varepsilon \in (0,\pi)$, $j = 1,2$.

Example 0.2.4 *If* A_1 *and* A_2 $(\mathrm{ord}A_j = 2k_j\,,\ j = 1,2)$ *are the two such operators as in Example 0.2.3, then the operator*

$$\left(A_1(x,D) + q_1^{2k_1}\right)\left(A_2(x,D) + q_2^{2k_1}\right)$$

with Dirichlet boundary conditions

$$B_\nu = \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1}, \quad \nu = 1, ..., k_1 + k_2,$$

also satisfies Conditions 0.2.1 and 0.2.2, but in the cone $Q = Q_1 \times Q_2$, *where* $Q_j = \left\{|\arg q_j| \le \frac{\pi-\varepsilon}{2k_j}\right\}$, $\varepsilon \in (0,\pi)$, $j = 1,2$.

The main result of Chapter 1 is the unique solvability of the problem (0.2.1) - (0.2.2) in the scale of Sobolev spaces equipped with special norms depending on parameters q for sufficiently large $\sum_{j=1}^m |q_j|$ (and $M_q \ge q^{(0)}$ for some $q^{(0)} > 0$) under Conditions 0.2.1 and 0.2.2 (Proposition 1.3.6 and Theorem 1.4.2). The proof is obtained via a priori estimates which follow in a standard manner from the parametrix construction. The difference with the one - parameter case is in the fact, that though $\sum_{j=1}^m |q_j|$ is sufficiently large, some of q_j may be rather small (but not zero!).

Chapter 2 is devoted to the solution of the problem (0.1.11) - (0.1.13) in the multidimensional time case.

We start with the description of functional spaces. By definition the space $\mathbf{P}^\alpha\left(e^{-\gamma\cdot t}\right)$ (where $\alpha,\gamma,t \in (\overline{\mathbb{R}_+})^m$) consists of all functions $u(t) = u(t_1, ..., t_m)$ supported in the positive "octant" $(\bar{\mathbb{R}}_+)^m$ and such that $e^{-\gamma\cdot t}u(t) \in H^{\vec{\alpha}}(\mathbb{R}^m)$, where $H^{\vec{\alpha}}(\mathbb{R}^m)$ is the anisotropic Sobolev space (see, for example, Slobodetskij L.N. (1958), Kudryavtsev L.D., Nikolskij S.M. (1988)).

The space of functions $\mathbf{E}_\alpha(\gamma)$ is defined as the set of functions $U(p) = U(p_1, ..., p_m)$ $(p = \sigma + i\gamma,\ p_j = \sigma_j + i\gamma_j,\ j = 1, ..., m)$ which are holomorphic in the direct product of the right half-planes in $\Pi_\gamma = \{p \in \mathbb{C}^m : \mathrm{Re}\,p_j > \gamma_j\,,\ j = 1, ..., m\}$. Obviously, we have to prove that Laplace transformation defines an isomorphism between the spaces $\mathbf{P}^\alpha\left(e^{-\gamma\cdot t}\right)$ and $\mathbf{E}_\alpha(\gamma)$. The key to the desired isomorphism is lying in the slight modification we give to Cauchy integral formula in several complex variables (see in this connection Shabat B.V. (1985), Henkin G.M., Leiterer J. (1984)).

Consider now the problem (0.1.11) - (0.1.13) in a "polycylinder" domain $\Omega_I = \Omega \times \prod_{j=1}^m (0, T_j)$, $0 < T_j \le \infty$. Only initial conditions (0.1.13) have to be modified:

$$\frac{\partial^{\nu_j} u}{\partial t_j^{\nu_j}}\Big|_{t_j=0} = \phi_{j,\nu_j}(x, t[j]) , \quad \text{for} \quad \nu_j = 0, ..., \kappa_j - 1 , \tag{0.2.6}$$

where κ_j equals to the maximal order of the t_j-derivative in the expression for operator $A\big(x, D_x, \frac{\partial}{\partial t}\big)$, $\kappa_j \le k/b_j$, $j = 1, ..., m$. Suppose for the time being, that all $\phi_{j,\nu_j} = 0$ and $T_j = +\infty$, $j = 1, ..., m$. Then Laplace transformation in the variables T_j transfers the problem (0.1.11), (0.1.12), (0.2.6) to the following "stationary" problem

$$A(x, D_x, p)\, U(x, p) = F(x, p) , \quad x \in \Omega , \tag{0.2.7}$$

$$B_\mu(x, D_x, p) U(x, p)\big|_{x=x'} = G_\mu(x', p) , \quad x' \in \Gamma , \tag{0.2.8}$$

$\mu = 1, ..., k$. Putting $p_j = q_j^{2b_j}$, $j = 1, ..., m$, we see, that in the same time as p_j varies in the right half-plane $\operatorname{Re} p_j \ge 0$, q_j varies in its turn in the angle

$$Q_j = \big\{ q_j \in \mathbb{C} : |\arg q_j| \le \frac{\pi}{4b_j} \big\} . \tag{0.2.9}$$

The problem (0.1.11), (0.1.12), (0.2.6) is called *polyparabolic* if the problem (0.2.7) - (0.2.8) with p_j substituted by $q_j^{2b_j}$ for all $j = 1, ..., m$, satisfies in the cone $Q = Q_1 \times \cdots \times Q_m$ the Conditions 0.2.1 - 0.2.2.

In order to illustrate this definition we consider the inverse Laplace images of Examples 0.2.3, 0.2.4.

Example 0.2.5 *Let* $A_1(x, D)$ *and* $A_2(x, D)$ $(\operatorname{ord} A_j = 2k_j$, $j = 1, 2)$ *be two elliptic operators with positive real leading parts. Consider an operator*

$$\Big(\frac{\partial}{\partial t_1} + A_1(x, D)\Big)\Big(\frac{\partial}{\partial t_1} + A_2(x, D)\Big)$$

with Dirichlet boundary conditions

$$B_\nu = \Big(-i\frac{\partial}{\partial \vec{n}}\Big)^{\nu-1} , \quad \nu = 1, ..., k_1 + k_2,$$

and zero initial conditions (0.2.6).

This problem is polyparabolic in the cone $Q' = Q'_1 \times Q'_2$, $Q'_j = \big\{q_j \in \mathbb{C} : |\arg q_j| \le \frac{\pi-\varepsilon}{2k_j}\big\}$, *including the cone* $Q = Q_1 \times Q_2$, $Q_j = \big\{q_j \in \mathbb{C} : |\arg q_j| \le \frac{\pi}{4k_j}\big\}$, *for sufficiently small* ε .

Example 0.2.6 *Let* $A(x,D)$ *be an elliptic operator* $(\mathrm{ord}A = 4k)$ *with a positive real leading part. Consider the operator (cf. Friedman A., Littman W. (1963))*

$$\frac{\partial^2}{\partial t_1 \partial t_2} \; + \; A(x,D)$$

with Dirichlet boundary conditions

$$B_\nu \; = \; \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1} \quad , \quad \nu = 1, ..., 2k,$$

and zero initial conditions.

This problem is not polyparabolic *in the above defined sense. Indeed, after substituting of* p_j *by* q_j^{2k} $(j = 1,2)$ *in the corresponding system (0.2.7) - (0.2.8) we obtain that, if* p_j *varies in the right half-plane* $\mathrm{Re}\, p_j \geq 0$, q_j *varies in its turn in the angle* $Q_j \; = \; \left\{q_j \in \mathbb{C} \, : \, |\arg q_j| \, \leq \, \frac{\pi}{4k}\right\}$ *. And thus, the ellipticity condition (Condition 0.2.1) is violated for our problem is elliptic only in the cone* $Q' = Q_1' \times Q_2'$ *, where*

$$Q_j' \; = \; \left\{|\arg q_j| \, \leq \, \frac{\pi - \varepsilon}{4k}\right\} \; .$$

In order to overcome this difficulty , we have to introduce the notion of ellipticity with parameters in conical domains and show, how to improve the theory of Chapter 1 to include this class of operators. This work is in progress now. We also refer to (Brenner A.(to appear)) for applications to multivariable operator $\zeta-$function and related spectral asymptotics.

Here we also have to improve results of Chapter 1 and obtain (see Theorem 2.2.5) that there exists a vector γ with $\gamma_j > 0$ such that for $\mathrm{Re}\, p_j \geq \gamma_j$ the problem (0.2.7) - (0.2.8) possesses a unique solution $U(x,p) \in \mathbf{E}_\alpha(\gamma)$ (suitably depending on x). Using Laplace isomorphism, we obtain (see Theorem 2.2.7) the unique solvability of the initial polyparabolic problem (0.1.11), (0.1.12), (0.2.6) (with $\phi_{j,\nu_j} \equiv 0$) in the scale of spaces $\mathbf{P}^\alpha\left(e^{-\gamma \cdot t}\right)$ (suitably depending on x).

The rest part of Chapter 2 is devoted to the existence and uniqueness theorems for the problem (0.1.11), (0.1.12), (0.2.6) in the following cases:

A) nonhomogeneous initial conditions (Theorems 2.2.11 and 2.2.16);

B) finite (all $T_j < \infty$) polycylinder (Theorem 2.3.3);

C) the coefficients of operators A and B_μ are dependent of t (Theorem 2.4.1), (here we include equation (0.1.10) into our theory);

D) operators A and B_μ are square $N \times N-$matrices (Theorem 2.4.2).

The main difficulty in A) -D) is lying in the uniqueness theorem for the case of nonhomogeneous initial conditions.

It happens that usual Compatibility Condition 2.2.10 is sufficient only for the existence theorem. The uniqueness is guaranteed by the new *Inductive Polyparabolicity Condition* (see Condition 2.2.14) which is, roughly speaking, the condition of the unique solvability of certain additional boundary value problems determininig the functions $w_j = \frac{\partial^{\kappa_j} u}{\partial t_j^{\kappa_j}}\Big|_{t_j=0}$, where κ_j is the highest order of the t_j – derivative in $A\big(x, D_x, \frac{\partial}{\partial t}\big)$.

More precisely, we denote by $A_{\kappa_j}\big(x, D_x, \frac{\partial}{\partial t}[j]\big)$ the coefficient before the t_j–derivative of the maximal order $\frac{\partial^{\kappa_j}}{\partial t_j^{\kappa_j}}$ in the expression for $A\big(x, D_x, \frac{\partial}{\partial t}\big)$ (see (0.2.1)).

We now deduce from (0.2.1), (0.2.2), (0.2.6) the boundary value problem for the operator A_{κ_j} :

$$A_{\kappa_j}\big(x, D_x, \frac{\partial}{\partial t}[j]\big)\, w(x, t[j]) \;=\; \tilde f(x, t[j]) \ , \tag{0.2.10}$$

$$B_{\mu,j}\big(x, D_x, \frac{\partial}{\partial t}[j]\big) w(x, t[j])\big|_{x=x'} \;=\; \tilde g_{\mu,j}(x', t[j]) \ , \tag{0.2.11}$$

$$\frac{\partial^{\nu_i} w}{\partial t_i^{\nu_i}}\Big|_{t_i=0} \;=\; \tilde\phi_{i,\nu_i}(x, t[i,j]) \quad \text{for } \nu_i = 0, ..., \kappa_i - 1\,,\ i \neq j \ , \tag{0.2.12}$$

where $w = \frac{\partial^{\kappa_j} u}{\partial t_j^{\kappa_j}}\big|_{t_j=0}$, $\tilde f$ is the sum of $f(x,0)$ and the replaced terms of $A\big(x, D_x, \frac{\partial}{\partial t}\big)u$, which do not contain the highest t_j–derivative $\frac{\partial^{\kappa_j} u}{\partial t_j^{\kappa_j}}$ with the subsequent restriction of these terms to the plane $t_j = 0$ and the substitution of the appropriate derivatives of u by $(x, t[j])$– derivatives of ϕ_{j,ν_j} . Further, $\tilde g_{\mu,j}$ is a linear combination of $(x, t[j])$– derivatives of ϕ_{j,ν_j} and $\frac{\partial^{\nu_j} g_\mu}{\partial t_j^{\nu_j}}\Big|_{t_j=0}$ for $\nu_j \le \kappa_j$, at last, $\tilde\phi_{i,\nu_i} = \frac{\partial^{\kappa_j}\phi_{i,\nu_i}}{\partial t_j^{\kappa_j}}\Big|_{t_j=0}$. Certainly, there are more boundary conditions in (0.2.11) than is needed. We have to choose $k - \kappa_j b_j$ of them to satisfy the following

Condition 0.2.7 *(***Inductive polyparabolicity***).*

Suppose the number of times $m = 2$. In this case we say that the problem (0.2.1), (0.2.2), (0.2.6) is inductively polyparabolic *for $t_j = 0$ if, for all $j \in \{1, ..., m\}$ the order of the highest t_j-derivative in B_μ ($\mu = 1, ..., k$) is less or equal to κ_j , and the corresponding system (0.2.1), (0.2.2), (0.2.6) (after the sufficient choice of $k - \kappa_j b_j$ boundary conditions) is parabolic.*

For general $m = k$ we say that the problem (0.2.1), (0.2.2), (0.2.6) is inductively polyparabolic *if, for all $j \in \{1, ..., m\}$ the order of the highest t_j-derivative in B_μ ($\mu = 1, ..., k$) is less or equal to κ_j , and the corresponding system (0.2.10) - (0.2.12) (after the sufficient choice of $k - \kappa_j b_j$ boundary conditions) is polyparabolic and inductively polyparabolic for $m = k - 1$.*

We note that though *Inductive Polyparabolicity Condition* is sufficient, it is not necessary, as it may be seen from the examples below.

Example 0.2.8 *Suppose operator* $A\big(x, D_x, \frac{\partial}{\partial t}\big)$ *be as in Example 0.2.5, but not decomposed into multipliers i.e.*

$$A\big(x, D_x, \frac{\partial}{\partial t}\big) \;=\; \frac{\partial^2}{\partial t_1 \partial t_2} - B(x, D_x)\frac{\partial}{\partial t_1} - C(x, D_x)\frac{\partial}{\partial t_2} + E(x, D_x) \ ,$$

$(1/2 \operatorname{ord} E \;=\; \operatorname{ord} B \;=\; \operatorname{ord} C \;=\; 2)$. *Also, we suppose that together with Dirichlet boundary conditions it forms a polyparabolic problem*

$$A\big(x, D_x, \frac{\partial}{\partial t}\big)u \;=\; f \ ,$$

$$u\big|_\Gamma \;=\; h_0(t_1, t_2, x') \ , \quad \frac{\partial u}{\partial x_n}\Big|_\Gamma \;=\; h_1(t_1, t_2, x') \ ,$$

$$u\big|_{t_1=0} \;=\; \phi_1(t_2, x) \ , \quad u\big|_{t_2=0} \;=\; \phi_2(t_1, x) \ . \tag{0.2.13}$$

Then the respective systems for functions $w_j = \frac{\partial u}{\partial t_j}\big|_{t_j=0}$, *are*

$$\Big(\frac{\partial}{\partial t_2} - B\Big)w_1 \;=\; C\frac{\partial \phi_1}{\partial t_2} - E\phi_1 + f_1 \;:=\; \tilde{f}_1 \ ,$$

$$w_1\big|_\Gamma \;=\; h_{0,t_1}(0, t_2, x') \ , \qquad w_1\big|_{t_2=0} \;=\; \phi_{2,t_1}(0, x) \tag{0.2.14}$$

and

$$\Big(\frac{\partial}{\partial t_1} - C\Big)w_2 \;=\; B\frac{\partial \phi_2}{\partial t_1} - E\phi_2 + f_2 \;:=\; \tilde{f}_2 \ ,$$

$$w_2\big|_\Gamma \;=\; h_{0,t_2}(t_1, 0, x') \ , \qquad w_2\big|_{t_1=0} \;=\; \phi_{1,t_2}(0, x) \tag{0.2.15}$$

for w_2 , *where* $f_1 = f(0, t_2, x)$, $f_2 = f(t_1, 0, x)$ *and* z_{t_1} *means* $\frac{\partial z}{\partial t_1}$.

If B *and* C *are elliptic operators then these systems are parabolic, and hence, it means that the initial problem (0.2.13) is inductively polyparabolic and possesses the unique solution (see Theorem 2.2.16).*

Though Example 0.2.6 gives us a nonpolyparabolic problem in the above sense, it shows, that Inductive Parabolicity Condition is sufficient, but not necessary. Indeed, in this case the systems (0.2.14), (0.2.15) turn into Cauchy problems for ordinary differential equations with constant coefficients.

Example 0.2.9 *Consider the equation (cf. (0.1.5)):*

$$\frac{\partial^2 u}{\partial t_1 \partial t_2} \;+\; a(x, D)u \;=\; f \ ,$$

where $a(x, D)$ *is an elliptic operator of order 4 with boundary and initial conditions*

$$u\big|_\Gamma \;=\; h_0(t_1, t_2, x') \ , \quad \frac{\partial u}{\partial x_n}\Big|_\Gamma \;=\; h_1(t_1, t_2, x') \ ,$$

$$u\big|_{t_1=0} = \phi_1(t_2,x) \ , \quad u\big|_{t_2=0} = \phi_2(t_1,x) \ .$$

The respective systems (0.2.14), (0.2.15) are

$$\frac{\partial w_1}{\partial t_2} = f_1 - a(x,D)\phi_1 \ , \qquad w_1\big|_{t_2=0} = \phi_{2,t_1}(0,x) \ ,$$

and

$$\frac{\partial w_2}{\partial t_1} = f_2 - a(x,D)\phi_2 \ , \qquad w_2\big|_{t_1=0} = \phi_{1,t_2}(0,x) \ .$$

Thus, we obtain Cauchy problems for simplest ordinary differential equations with respect to w_1 *and* w_2 *, i.e.* w_1 *and* w_2 *can be easily estimated through the functions* $f\,, h_0\,, h_1\,, \phi_1$ *and* ϕ_2 *, possibly with increased* γ *in appropriate* $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$.

In general, we observe that in the cases, where (0.2.14) and (0.2.15) are Cauchy problems for ordinary differential equations with constant coefficients, we obtain the desired uniqueness in our functional classes automatically (this is true, i.e., for (0.1.9)). It is not true in the variable coefficients case, for example, if we obtain for $w_{1,2}$ first order linear equations of the form $\frac{\partial w}{\partial t_2} + a(x,t)w = f(x,t)$ (this is exactly the case of Brish - Yurchuk equation (0.1.6), (0.1.7)): we have to demand the function $a(x,t_1,t_2)$ to be bounded from below in t_2 uniformly in (x,t_1) , otherwise the solution does not fall into $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$ even with increased γ .

Obviously, equation (0.1.10) falls into our case for positive $\lambda_j(t)$. That is why, the uniqueness may not occur in nonhomogeneous Iliev's case (only homogeneous initial conditions were introduced in (Iliev I.D. (1977))).

We conclude Chapter 2 with the discussion of Compatibility Conditions for the nonhomogeneous problem (0.1.11), (0.1.12), (0.2.6) in the spirit of (Lions J.L., Magenes E. (1972)).

ACKNOWLEDGEMENTS

I would like to thank my Supervisor Prof. Boris Paneah, who have introduced me to the Theory of Multidimensional Time Problems, for fruitful discussions and kindest advices. I wish to express my gratitude to Technion and Ministry of Science and Technology for generous financial support.

Chapter 1

Differential operators with parameters

1.1 Main functional spaces and interpolation inequalities

1.1.1

We denote by $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ (see, for example, Besov O.V., Il'in V.P., Nikolskij S.M. (1975), Kudryavtsev L.D., Nikolskij S.M. (1988), Egorov Yu.V., Shubin M.A. (1988)) the completion of the space $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_s = \|u\|_{s,\mathbb{R}^n} = \left(\int_{\mathbb{R}^n_\xi} \left(1 + |\xi|^{2s}\right) |Fu|^2 \, d\xi \right)^{1/2}, \tag{1.1.1}$$

and also denote by $H^l(\mathbb{R}^n_+)$, $l \in \mathbb{Z}$ – the completion of the set of restrictions to $\mathbb{R}^n_+$ of the functions from $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{l,\mathbb{R}^n_+} = \left(\sum_{|\alpha| \le l} \int_{\mathbb{R}^n_+} |D^\alpha u|^2 \, dx \right)^{1/2}, \tag{1.1.2}$$

or according to the equivalent norm

$$\left(\sum_{j+k \le l} \int_{\mathbb{R}^{n-1}_{\xi'}} |\xi'|^{2j} \int_0^\infty |D_n^k F' u|^2 \, dx_n d\xi' \right)^{1/2}. \tag{1.1.3}$$

Note that, if $u \in H^l(\mathbb{R}^n)$ then $u\,|_{\mathbb{R}^n_+} \in H^l(\mathbb{R}^n_+)$ and

$$\|u\|_{l,\mathbb{R}^n_+} \;\le\; \|u\|_{l,\mathbb{R}^n} \;. \tag{1.1.4}$$

Inversely, there exists an extension operator (see Egorov Yu.V., Shubin M.A. (1988)) from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, such that

$$\|Lu(x)\|_{l,\mathbb{R}^n} \;\le\; C\,\|u(x)\|_{l,\mathbb{R}^n_+} \;, \tag{1.1.5}$$

Here and further we denote by C an arbitrary positive constant, which is independent of $u(x)$ and the parameters $q \in \mathbb{C}^m$ (if there are any) and may differ from inequality to inequality.

Also, we shall use the so-called trace inequality: let $u \in H^l(\mathbb{R}^n_+)$, $l > 1/2$, then there exists $u(x',0) \in H^{l-1/2}(\mathbb{R}^{n-1}_{x'})$ such that

$$\|u(x',0)\|'_{l-1/2} \;\le\; C\,\|u(x)\|_{l,\mathbb{R}^n_+} \;, \tag{1.1.6}$$

where we denote by $\|\cdot\|'_s$ the norm in $H^s(\mathbb{R}^{n-1}_{x'})$.

1.1.2

The down-stated interpolation lemmas generalize the corresponding results of (Agranovich M.S., Dynin A.S. (1962), Agranovich M.S., Vishik M.I. (1964)) to the multiparameter case.

Lemma 1.1.1 *Let $q \in \mathbb{C}^m$, then for $0 < k < l$ the following interpolation inequalities are valid:*

$$\sum_{i=1}^m |q_i|^{l-k}\|u\|_k \;\le\; C_{kl}\left[\|u\|_l + \sum_{i=1}^m |q_i|^l\|u\|_0\right] , \tag{1.1.7}$$

$$\sum_{i=1}^m |q_i|^{2(l-k)}\|u\|_k^2 \;\le\; C'_{kl}\left[\|u\|_l^2 + \sum_{i=1}^m |q_i|^{2l}\|u\|_0^2\right] , \tag{1.1.8}$$

where the norms are taken over $\mathbb{R}^n$ or $\mathbb{R}^n_+$, and the constants C_{kl} and C'_{kl} are independent of $u(x)$ and q .

Proof. It is sufficient to prove (1.1.8), whence follows (1.1.7). From the one-dimensional case we have

$$|q_i|^{2(l-k)}\left(1+|\xi|^{2k}\right) \;\le\; C\left[\left(1+|\xi|^{2l}\right) + |q_i|^{2l}\right]$$

for $i = 1,...,m$, and thus

$$\sum_{i=1}^m |q_i|^{2(l-k)}\left(1+|\xi|^{2k}\right) \;\le\; C_{kl}\left[\left(1+|\xi|^{2l}\right) + \sum_{i=1}^m |q_i|^{2l}\right] ,$$

from where (1.1.8) follows after multiplication by $|Fu|^2$ and integration over ξ . With the help of (1.1.4), (1.1.5) we also obtain (1.1.7) and (1.1.8) in $\mathbb{R}^n_+$. ▷

Lemma 1.1.2 *Let* $l = 1/2$ *and* $u \in H^1(\mathbb{R}^n_+)$ *, then*

$$\sum_{i=1}^{m} |q_i|^{1/2} \, \|u(x',0)\|_0' \;\le\; C\Big[\|u\|_1 + \sum_{i=1}^{m}|q_i| \, \|u\|_0\Big] \tag{1.1.9}$$

and

$$\sum_{i=1}^{m} |q_i| \, \|u(x',0)\|_0'^{\,2} \;\le\; C'\Big[\|u\|_1^2 + \sum_{i=1}^{m}|q_i|^2 \|u\|_0^2\Big] \, , \tag{1.1.10}$$

where the norms are taken over $\mathbb{R}^n$ *or* $\mathbb{R}^n_+$ *, and the constants* C *and* C' *are independent of* $u(x)$ *and* q *.*

Proof. As in the previous lemma it is enough to prove (1.1.10) for smooth functions and in the case of $\mathbb{R}^n$ - norms in the right - hand side. Rewriting the left - hand side in the form

$$\sum_{i=1}^{m} |q_i| \int |v(\xi')|^2 \, d\xi' \, ,$$

where

$$v(\xi') \;=\; F'\left[u(x',0)\right] \;=\; \frac{1}{2\pi} \int e^{ix_n\xi_n} Fu(\xi) \, d\xi_n \, ,$$

we see that

$$|v(\xi')|^2 \;\le\; \frac{1}{2\pi} \left(\int |Fu(\xi)|^2 \, d\xi_n\right)^2$$

$$\le\; \frac{1}{2\pi} \int \Big(1 + \Big(\sum_{i=1}^{m}|q_i|\Big)^2 + |\xi_n|^2\Big)|Fu|^2 \, d\xi_n \cdot \int \frac{d\xi_n}{1 + \left(\sum_{i=1}^{m}|q_i|\right)^2 + |\xi_n|^2} \tag{1.1.11}$$

$$\le\; C \int \Big(1 + |\xi|^2 + \sum_{i=1}^{m}|q_i|^2\Big)|Fu|^2 \, d\xi_n \;\cdot\; \frac{\pi}{\sqrt{1 + \left(\sum_{i=1}^{m}|q_i|\right)^2}} \, ,$$

where we have used the Schwartz inequality twice and calculated the last integral in (1.1.11). Now, as trivially $\sum_{i=1}^{m}|q_i| \;<\; \sqrt{1 + \left(\sum_{i=1}^{m}|q_i|\right)^2}$, we get after integrating over ξ' that the left-hand side in (1.1.10) is less than

$$C \int \Big(<\xi>^2 + \sum_{i=1}^{m}|q_i|\Big)|Fu|^2 \, d\xi \, .$$

$\triangleright$

Also we use the parameter-dependent norms

$$|||u|||_l^2 \ := \ \|u\|_l^2 + \sum_{i=1}^{m} |q_i|^{2l} \, \|u\|_0^2 \ = \ \|u\|_l^2 \ + \ |||q|||^{2l} \|u\|_0^2 \ , \tag{1.1.12}$$

where the norms in the right - hand side are taken taken either over $\mathbb{R}^n$ or over $\mathbb{R}^n_+$. As follows from (1.1.8), the norms $\|\cdot\|_l$ and $|||\cdot|||_l$ are equivalent for the fixed $q \in \mathbb{C}^m$ and

$$|||u|||_l^2 \ \leq \ \sum_{k=0}^{l} \sum_{i=1}^{m} |q_i|^{2(l-k)} \, \|u\|_k^2 \ \leq \ C|||u|||_l^2 \ . \tag{1.1.13}$$

Now we can rewrite (1.1.4) - (1.1.6) in the new terms

$$|||u(x)|||_{l,\mathbb{R}^n_+} \ \leq \ C|||u(x)|||_{l,\mathbb{R}^n} \ , \tag{1.1.14}$$

$$|||Lu(x)|||_{l,\mathbb{R}^n} \ \leq \ C|||u(x)|||_{l,\mathbb{R}^n_+} \ , \tag{1.1.15}$$

$$|||u(x',0)|||'_{l-1/2} \ \leq \ C|||u(x)|||_{l,\mathbb{R}^n_+} \ , \tag{1.1.16}$$

where $l \in \mathbb{N}$.

Remark 1.1.3 *The inequalities of this paragraph remain true if instead of* $\mathbb{R}^n_+$ *we consider a bounded domain* $G \subset \mathbb{R}^n$ *with sufficiently smooth boundary* Γ *, which is an* $(n-1)-$*dimensional surface admitting in every point local straightenings under local coordinate transformations.*

In this case the Sobolev norms are defined in a standard way as follows: for $l \in \mathbb{N}$ the space $H^l(G)$ is defined as a completion of the space $C^\infty(\overline{G})$ with respect to the norm

$$\|u\|_{l,G} \ = \ \left(\sum_{|\alpha| \leq l} \int_G |D^\alpha u|^2 \, dx \right)^{1/2} \ , \tag{1.1.17}$$

or, equivalently,

$$|||u|||_{l,G}^2 \ = \ \|u\|_{l,G}^2 \ + \ |||q|||^{2l} \, \|u\|_{0,G}^2 \ , \tag{1.1.18}$$

or with respect the equivalent norm which is defined by a partition of unity as follows: suppose $\{U_\alpha\}_{\alpha=1}^N$ is an open covering of G and $\sum_{\alpha=1}^N \varphi_\alpha(x) \equiv 1$ is the corresponding smooth partition of unity, then the norm (1.1.17) is equivalent to the norm $\left(\sum_{\alpha=1}^N \|\varphi_\alpha u\|_l^2\right)^{1/2}$, where the norms are taken in local coordinates either over $\mathbb{R}^n$ or over $\mathbb{R}^n_+$, with the additional assumption in the last case that, if $\operatorname{supp}\varphi_\alpha \cap \Gamma \neq \emptyset$, then $\operatorname{supp}\varphi_\alpha \subset U_j$ for some U_j from the covering of the boundary Γ .

Let $\{\varphi_\beta\}$, $(\beta = 1, ..., r)$ be all functions from $\{\varphi_\alpha\}$ such that $\operatorname{supp}\varphi_\beta \cap \Gamma \neq \emptyset$. For $l \in \mathbb{N}$ the Sobolev space $H^l(\Gamma)$ is defined as a completion of the space $C^\infty(\overline{\Gamma})$ with respect to the norm

$$\|u\|'_{l,G} \;=\; \left(\sum_{\beta=1}^{r} \|\varphi_\beta u\|'^{2}_{l,\mathbb{R}^{n-1}}\right)^{1/2} . \tag{1.1.19}$$

This space is independent of local coordinates and the partition of unity for sufficiently small coverings. Of course, we also need the equivalent norm

$$|||u|||'^{2}_{l,\Gamma} \;=\; \|u\|'^{2}_{l,\Gamma} \;+\; |||q|||^{2l}\, \|u\|'^{2}_{0,\Gamma} . \tag{1.1.20}$$

Further, we define the spaces $H^s(I)$, where $I \subset \mathbb{R}$ is an open interval in the real line. It can be the line $\mathbb{R}$ itself or the half-line $\mathbb{R}_+$. We introduce the space $H^s(I)$ as a completion of the space of restrictions to I of smooth functions on $\mathbb{R}$, relatively to the norm

$$\|u\|_s \;=\; \left(\sum_{k=0}^{[s]} \int_I \left|u^{(k)}(t)\right|^2 dt \;+\; \int_I \int_I \frac{\left|u^{[s]}(t) \,-\, u^{[s]}(\tau)\right|^2}{|t \,-\, \tau|^{2s-2[s]+1}}\, dt\, d\tau\right)^{1/2} , \tag{1.1.21}$$

The equivalent norm can be introduced in terms of the operator L (see (1.1.5) and (Slobodetskij L.N. (1958)) for details). Let $I = (a, b)$, $b < +\infty$ and $J = (a, +\infty)$, then

$$\langle\, u \,\rangle \;:=\; \inf \|v\|_{s,J} , \tag{1.1.22}$$

where $v = Lu$ and "inf" is taken over all $v \in H^s(J)$ such that $v = u$ on I almost everywhere.

At last, if I and J are the two such intervals, then the norm in the corresponding anisotropic Sobolev space (see Slobodetskij L.N. (1958)) $H^{l,k}(I \times J)$ is

$$\|u\|^2_{l,k,I\times J} \;=\; \int_I \left\|u(t_1, t_2)\right\|^2_{k,J} dt_1 \;+\; \int_J \left\|u(t_1, t_2)\right\|^2_{l,I} dt_2 , \tag{1.1.23}$$

and, in general, by induction, if $I = I_1 \times \cdots \times I_m$ (in particular, we shall consider $I = (\mathbb{R}_+)^m$), then the norm in the space $H^{\vec{\alpha}}(I)$ is defined by

$$\|u\|^2_{\vec{\alpha},I} \;=\; \sum_{i=1}^{m} \int_{I[i]} \left\|u(t)\right\|^2_{\alpha_i, I_i} dt[i] , \tag{1.1.24}$$

where $I[i] \,=\, I_1 \times \cdots \times I_{i-1} \times I_{i+1} \times \cdots \times I_m$, $dt[i] \,=\, dt_1 ... dt_{i-1} dt_{i+1} ... dt_m$.

1.2 Some model examples of operators with constant coefficients

1.2.1

Let us consider in $\mathbb{R}^n$ the equation

$$A(D,q)u(x,q) \ = \ f(x,q) \qquad x \in \mathbb{R}^n \ , \ q \in \mathbb{C}^m \ , \tag{1.2.1}$$

where

$$A(D,q) \ = \sum_{|\alpha|+|\beta|=s} a_{\alpha\beta} q^\beta D_x^\alpha$$

is a differential operator with constant coefficients and $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j \in \ \mathbb{Z}_+$, $\beta = (\beta_1, ..., \beta_m)$, $\beta_j \in \mathbb{Z}_+$ are multiindices.

We suppose that the symbol $A(\xi,q)$, which is obtained from $A(D,q)$ after substitution of D by ξ , is a homogeneous polynomial in (ξ,q) of degree s , and that each parameter q_i is varying in a closed angle

$$Q_i \ \subset \ \mathbb{C} \ : \ Q_i \ = \ \{q_i \in \mathbb{C} \ : \ \delta_i \leq \arg q_i \leq \gamma_i\} \tag{1.2.2}$$

with the vertex in the origin.

Proposition 1.2.1 *For* $l \geq s$ *operator* $A(D,q) \ : \ H^l(\mathbb{R}^n) \ \to \ H^{l-s}(\mathbb{R}^n)$ *is bounded and*

$$|||f|||_{l-s} \ \leq \ C|||u|||_l \ . \tag{1.2.3}$$

Proof. We easily obtain (1.2.3) applying (1.1.7) to the inequality

$$\left\|A(D,q)u\right\|_{l-s} \ \leq \ C\sum_{k=0}^{s} |||q|||^{l-k} \|u\|_k \ .$$

$\triangleright$

Now we formulate the *ellipticity condition* for the operator $A(D,q)$.

Condition 1.2.2 $\forall \, q \in Q \ = \ Q_1 \times ... \times Q_m$ *and* $\forall \, \xi \in \mathbb{R}^n$

$$A(\xi,q) \ \neq 0 \quad for \quad |\xi| + M_q \neq 0 \ , \tag{1.2.4}$$

where $M_q \ = \ \min_{i=1,...,m} |q_i|$.

We note that, if $q = (q_1, ..., q_m) = 0$, then (1.2.4) turns into the standard ellipticity condition for the operator $A(D,0)$.

Also we denote by $Q_M \ = \ \{q \in Q \ : \ M_q > 0\}$.

Proposition 1.2.3 *Let* $l \geq s$ *. Then under Condition 1.2.2 ,* $\forall\, q \in Q_M$ *and* $f \in H^{l-s}(\mathbb{R}^n)$ *there exists the unique solution* $u = \mathcal{R}_0 f \in H^l(\mathbb{R}^n)$ *to the equation (1.2.1). Moreover for arbitrary* $q^{(0)} > 0$ *and arbitrary*

$$q \in Q_M^0 \;=\; \left\{q \in Q_M :\, M_q \geq q^{(0)}\right\}$$

the following a priori estimate takes place

$$|||u|||_l \;=\; |||\mathcal{R}_0 f|||_l \;\leq\; C|||f|||_{l-s} \;. \qquad \textit{(1.2.5)}$$

Proof. Let $u \in H^l(\mathbb{R}^n)$, then $F\left[A(D,q)u\right] = A(\xi,q)Fu = Ff$ and it follows from (1.2.4) that $Fu = A^{-1}(\xi,q)Ff$. Then from the homogenity of A we obtain

$$|Fu| \;\leq\; C\left(|\xi| + |||q|||\right)^{-s}|Ff| \qquad (1.2.6)$$

for $q \in Q_M$, and thus

$$\begin{aligned} &\left(1 + |\xi|^{2l} + |||q|||^{2l}\right)|Fu|^2 \\ &\leq\; C\,\frac{1 + |\xi|^{2l} + |||q|||^{2l}}{\left(|\xi| + |||q|||\right)^{2s}}\,|Ff|^2 \\ &\leq\; C\,\left(1 + |\xi|^{2l-2s} + |||q|||^{2l-2s}\right)|Ff|^2 \;, \end{aligned}$$

where $M_q \geq q^{(0)}$ and the constants C are independent of u, ξ, q . Integrating over ξ we obtain (1.2.5) and the uniqueness follows. The existence is the consequence of the formula

$$u \;=\; \mathcal{R}f \;=\; F^{-1}A^{-1}(\xi,q)Ff \;, \qquad (1.2.7)$$

which is valid $\forall\, q \in Q_M$, and, as follows from (1.2.6), it defines $\forall\, f \in H^{l-s}(\mathbb{R}^n)$ the solution $u \in H^l(\mathbb{R}^n)$ to the equation (1.2.1). ▷

Remark 1.2.4 *Let* $n > 1$ *and suppose (1.2.4) is satisfied, then* $s = \mathrm{ord}A = 2k$ *and* $A(\xi + \lambda\xi_0, q)$ *as the polynomial in* λ *has for* $q \in Q_M$ *,* $|\xi| + M_q \neq 0$ *,* ξ_0 *orthogonal to* ξ *and* $\xi_0 \neq 0$ *, exactly* k *roots in the upper* $\lambda-$*half-plane and* k *roots in the lower half-plane, taking into account their multiplicities (for the proof see, for example, Agranovich M.S., Vishik M.I. (1964), Agranovich M.S. (1990)).*

Example 1.2.5 *The first main example of the above situation is the operator* $A(D) + q_1^k q_2^k$ *, where* $A(D)$ *is a homogeneous elliptic operator of order* $2k$ *with real coefficients and such that* $A(\xi) > 0$ *for* $\xi \neq 0$ *(for example,* $A = -\Delta$*). Then (1.2.4) is satisfied in the cone* $Q = Q_1 \times Q_2$ *, where*

$$Q_i \;=\; \left\{q_i \in \mathbb{C} :\; |\arg q_i| \leq \frac{\pi - \varepsilon}{2k}\right\} \,,$$

with arbitrary $\varepsilon > 0$ *.*

Example 1.2.6 *The second example concerns the pair of operators* $A_1(D)$ *and* $A_2(D)$ *of the above type:*

$$\left(A_1(D)+q_1^{2k_1}\right)\left(A_2(D)+q_2^{2k_2}\right) ,$$

where $\operatorname{ord} A_1 = 2k_1$ *and* $\operatorname{ord} A_2 = 2k_2$ *. Obviously, (1.2.4) is satisfied in the cone* $Q_1 \times Q_2$ *, where*

$$Q_i = \left\{q_i \in \mathbb{C}: \ |\arg q_i| \le \frac{\pi-\varepsilon}{2k_i}\right\} .$$

1.2.2

Let us consider equation (1.2.1) in $\mathbb{R}^n_+$

$$A(D,q)u(x,q) = f(x,q) , \qquad (1.2.8)$$

where $x=(x',x_n)\in\mathbb{R}^n_+$, $x_n\ge 0,\ \ q\in Q\subset\mathbb{C}^m$, with boundary conditions

$$B_\nu(D,q)u(x,q)\big|_{x_n=0} = g_\nu(x') , \ \nu=1,...,k. \qquad (1.2.9)$$

As in (1.2.1) $A(D,q)$ and $B_\nu(D,q)$ are differential operators with constant coefficients such that after substitution of D by ξ we obtain polynomials $A(\xi,q)$ and $B_\nu(\xi,q)$ which are homogeneous in $(\xi_1,...,\xi_n,q_1,...,q_m)\in\mathbb{R}^n\times Q$ of orders $2k$ and k_ν respectively. We denote by $\mathcal{A}$ a boundary value problem operator

$$\mathcal{A}u = \left\{A(D,q)u,\ B_1(D,q)u\big|_{x_n=0},...,\ B_k(D,q)u\big|_{x_n=0}\right\} \qquad (1.2.10)$$

and by l_0 – the $\max(2k,k_\nu+1)$.

Proposition 1.2.7 *Let* $l\in\mathbb{Z}$, $l\ge l_0=\max(2k,k_\nu+1)$. *Then the operator*

$$\mathcal{A}:\ H^l(\mathbb{R}^n_+)\ \longrightarrow\ H^{l-2k}(\mathbb{R}^n_+)\times\prod_{\nu=1}^{k}H^{l-k_\nu-1/2}(\mathbb{R}^{n-1}) \qquad \textit{(1.2.11)}$$

is bounded and

$$|||f|||_{l-2k} + \sum_{\nu=1}^{k}|||g_\nu|||'_{l-k_\nu-1/2} \le C|||u|||_l . \qquad \textit{(1.2.12)}$$

Proof. For the operator A the proof is the same as in Proposition 1.2.1. Using (1.1.6) we obtain for B_ν that

$$|||B_\nu(D,q)u|||'_{l-k_\nu-1/2} \le C|||B_\nu(D,q)u|||_{l-k_\nu} \le C|||u|||_l ,$$

where the last inequality is the analogue of (1.2.3) for B_ν and the proof follows. $\triangleright$

Let us formulate the analog of the Shapiro - Lopatinskij condition in our case, i.e. for the problem (1.2.8) - (1.2.9).

Suppose $f = 0$ there. After the Fourier transformation F' in x' we obtain from (1.2.8) - (1.2.9) the boundary value problem for an ordinary differential equation on the half-axis:

$$A\big(\xi', -i\frac{d}{dx_n}, q\big)\, v(x_n) \;=\; 0\,, \quad x_n > 0\,, \tag{1.2.13}$$

$$B_\nu\big(\xi', -i\frac{d}{dx_n}, q\big)\, v(x_n)\big|_{x_n=0} = \; h_\nu\,, \quad \nu = 1, ..., k\,, \tag{1.2.14}$$

smoothly depending on parameters ξ' and q .

Denote by $\mathcal{M}$ $-$ the space of all stable solutions to the equation (1.2.13) for fixed ξ' and q . It follows from Remark 1.2.4 that $\dim \mathcal{M} = k$ for $n > 1$. For $n = 1$ we additionally assume that the equation $A(\lambda, q) = 0$ has relatively to λ equally k solutions in the upper and the lower half-planes for $q \in Q_M$.

Condition 1.2.8 *$\forall\, q \in Q$ for $|\xi'| + M_q \neq 0$ the problem (1.2.13) - (1.2.14) has the unique solution in the space $\mathcal{M}$ for all $h_\nu \in \mathbb{C}$.*

Remark 1.2.9 *We do not lose the generality assuming $\big(F'_{x'\to\xi'} f\big)(\xi', x_n) \equiv 0$ in (1.2.13). Indeed, if for fixed ξ' it equals to $g(x_n)$, $g(x_n) \in \mathcal{M}$, we note that the equation*

$$A\big(\xi', -i\frac{d}{dx_n}, q\big)\, v(x_n) \;=\; g(x_n)$$

always has a particular solution $v \in \mathcal{M}$. Subtracting it we return to (1.2.13) with $g(x_n) \equiv 0$.

Example 1.2.10 *Let $A(D_1, ..., D_n, D_{n+1}, ..., D_{n+m})$ and $B_\nu(D_1, ..., D_n, D_{n+1}, ..., D_{n+m})$, $(n, m > 1)$ be homogeneous differential operators in $\mathbb{R}^{n+m}$ of orders $2k$ and k_ν respectively, let A be elliptic and A and B_ν satisfy together the classical Shapiro - Lopatinskij condition (see, for example, Agranovich M.S., Vishik M.I. (1964), Egorov Yu.V., Shubin M.A. (1988)) for $x_n = 0$. Then the operators $A(D, q)$ and $B_\nu(D, q)$ $(D = (D_1, ..., D_n)$, $q = (q_1, ..., q_m)$) satisfy Conditions 1.2.2 and 1.2.8 for $q \in \mathbb{R}^m \cap Q_M$ or, equivalently, for each q_i varying in a real ray $\{\arg q_i = 0$ or $\pi\}$. We are really interested not in rays, but in angles. The corresponding examples are given below (see Examples 1.3.4 and 1.3.5).*

In the sequel we need the algebraic form of Condition 1.2.8 . Let

$$\omega_1(x_n, \xi', q)\,, \;\ldots\,, \omega_k(x_n, \xi', q)$$

be an arbitrary basis in $\mathcal{M}$. Then an arbitrary solution of (1.2.13) - (1.2.14) can be written in the form

$$v(x_n) \;=\; \alpha_1\omega_1 \;+\; \cdots \;+\; \alpha_k\omega_k \;.$$

Substituting v into boundary conditions (1.2.14), we obtain the system of k linear algebraic equations with k unknowns $\alpha_1, ..., \alpha_k$. The condition of its unique solvability is

$$\det\Big\{ B_j\big(\xi', -i\frac{d}{dx_n}, q\big)\, \omega_\nu(x_n, \xi', q)\Big|_{x_n=0}\Big\} \;\neq\; 0$$

for $|\xi'| + M_q \neq 0$, $q \in Q$ and $j, \nu = 1, ..., k$.

The crucial role in the following will be played by the canonical basis

$$\Omega_1(x_n, \xi', q)\,,\; \ldots, \Omega_k(x_n, \xi', q) \tag{1.2.15}$$

in $\mathcal{M}$ which is defined by the boundary conditions

$$B_j\big(\xi', -i\frac{d}{dx_n}, q\big)\, \Omega_l(x_n, \xi', q)\Big|_{x_n=0} \;=\; \delta_{jl} \;.$$

Obviously, a solution to the problem (1.2.13) - (1.2.14) with arbitrary h_ν is expressed in terms of the canonical basis as follows:

$$v(x_n) \;=\; h_1\Omega_1 \;+\; \cdots \;+\; h_k\Omega_k \;.$$

We also use the following integral representation for $\Omega_\nu(x_n, \xi', q)$ (see Agranovich M.S., Vishik M.I. (1964), Agranovich M.S. (1990) for the proof):

$$\Omega_\nu(x_n, \xi', q) \;=\; \int_{\gamma_+} e^{i\lambda x_n}\, A_+^{-1}(\xi', \lambda, q)\, N_\nu(\xi', \lambda, q)\, d\lambda \;, \tag{1.2.16}$$

for $|\xi'| + M_q \neq 0$, $q \in Q$ and $\nu = 1, ..., k$, where γ_+ is a smooth curve in the upper half-plane $\operatorname{Im}\lambda > 0$, depending on (ξ', q) and surrounding all λ - zeroes of the polynomial

$$A_+(\xi', \lambda, q) \;=\; \prod_{j=1}^{k} \big(\lambda - \lambda_j^+(\xi', q)\big) \;, \tag{1.2.17}$$

where λ_j^+ - are all λ - zeroes of the polynomial $A(\xi', \lambda, q)$ with positive imaginary parts, $N_\nu(\xi', \lambda, q)$ are the C^∞-functions positively homogeneous of order $k - k_\nu - 1$ in all their variables and polynomial in λ .

Proposition 1.2.11 *Let $l \geq l_0$ and let Conditions 1.2.2 and 1.2.8 are fulfilled. Then, for arbitrary $q \in Q_M$, $f \in H^{l-2k}(\mathbb{R}^n_+)$ and $g_\nu \in H^{l-k_\nu-1/2}(\mathbb{R}^{n-1})$ the unique solution to the problem (1.2.8) - (1.2.9) exists.*

Moreover, for an arbitrary $q^{(0)} > 0$ *and* $\forall q \in Q_M^0$ *, we have the a priori estimate*

$$|||u|||_l \;\leq\; C\{|||f|||_{l-2k} \;+\; \sum_{\nu=1}^{k}|||g_\nu|||'_{l-k_\nu-1/2}\} \tag{1.2.18}$$

Proof. We follow here the scheme of the proof of Theorem 3.1 from (Agranovich M.S., Vishik M.I. (1964)). As in Proposition 1.2.3 we start with the proof of an a priori estimate.

1). First, we reduce the problem to the case $f = 0$. Let L be the extension operator (see (1.1.5)). It follows from Proposition 1.2.3 that we can solve the equation $A(D,q)u = Lf$ in $\mathbb{R}^n$, and in the view of (1.2.7)

$$u_0 \;=\; F^{-1}A^{-1}(\xi,q)FLf \;, \tag{1.2.19}$$

where $q \in Q_M$, is its particular solution and $u_0 \in H^l(\mathbb{R}^n)$.

Thus, $u_0|_{\mathbb{R}^n_+} \in H^l(\mathbb{R}^n_+)$, and from (1.2.5), (1.1.14) and (1.1.15) we obtain, that

$$|||u_0|||_{l,\mathbb{R}^n_+} \;\leq\; C\,|||f|||_{l-2k,\mathbb{R}^n_+} \tag{1.2.20}$$

for $q \,:\, M_q \geq q^{(0)}$. We now suppose that $u = u_0 + u_1$, where $u_1 \in H^l(\mathbb{R}^n_+)$ is a solution to the problem

$$\begin{cases} A(D,q)u_1 \;=\; 0 & \text{in } \mathbb{R}^n_+ \;, \\ B_\nu(D,q)u_1\big|_{x_n=0} \;=\; g_\nu - g_{0\nu} & \text{on } \mathbb{R}^{n-1}_{x'} \;, \end{cases}$$

where $g_{0\nu} \;=\; B_\nu(D,q)u_0\big|_{x_n=0}$. Suppose, we can estimate u_1 as follows:

$$|||u_1|||_l \;\leq\; C\sum_{\nu=1}^{k}|||g_\nu - g_{0\nu}|||'_{l-k_\nu-1/2} \;. \tag{1.2.21}$$

Then,

$$|||u|||_l \;\leq\; |||u_0|||_l \;+\; |||u_1|||_l \;\leq$$

$$\leq\; \{|||f|||_{l-2k} \;+\; \sum_{\nu=1}^{k}|||g_\nu|||'_{l-k_\nu-1/2} \;+\; \sum_{\nu=1}^{k}|||g_{0\nu}|||'_{l-k_\nu-1/2}\}$$

Using (1.2.12) with the function u_0 and (1.2.20) we obtain

$$\sum_{\nu=1}^{k}|||g_{0\nu}|||'_{l-k_\nu-1/2} \;\leq\; C\,|||u|||_l \;\leq\; C\,|||f|||_{l-2k} \;.$$

Thus, the estimate (1.2.18) follows from (1.2.21), and we are able to suppose $f = 0$.

2). Applying the Fourier transform F' with respect to x' in (1.2.8), (1.2.9), we arrive to the problem on the half-axis:

$$A(\xi', D_n, q)\, F'u \ = \ 0 \ , \tag{1.2.22}$$

$$B_\nu(\xi', D_n, q)\, F'u\big|_{x_n=0} \ = \ F'g_\nu \ , \tag{1.2.23}$$

where $\nu = 1, ..., k$. Since $u \in H^l(\mathbb{R}^n_+)$, then $F'u$ is a stable solution of (1.2.22) for almost all (ξ', q) . So we can write the decomposition of $F'u$ in the canonical basis Ω_ν (see (1.2.15), (1.2.16)):

$$F'u \ = \ \sum_{\nu=1}^{k} F'g_\nu \times \Omega_\nu \ ,$$

or, equivalently,

$$u \ = \ {F'}^{-1} \sum_{\nu=1}^{k} F'g_\nu \times \Omega_\nu \ . \tag{1.2.24}$$

In order to obtain the estimate of the H^l-norm of u , we have to estimate $D^j_{x_n}\Omega_\nu$. Differentiating (1.2.16) inside the integral, we get

$$|D^j_{x_n}\Omega_\nu|^2 \ \le \ \int_{\gamma_+} \left|\lambda^j e^{i\lambda x_n}\, A_+^{-1}(\xi', \lambda, q)\, N_\nu(\xi', \lambda, q)\right| |d\lambda| \ ,$$

or, after substitution of λ by $\lambda_1 \left(|\xi'|^2 + |||q|||^2\right)^{1/2}$ and integration over λ_1 ,

$$|D^j_{x_n}\Omega_\nu|^2 \ \le$$

$$\le \ C \left[\left(|\xi'|^2 + |||q|||^2\right)^{j/2 - k/2 + (k-k_\nu-1)/2 + 1/2}\right]^2 \exp\left\{-C_1 x_n \left(|\xi'|^2 + |||q|||^2\right)^{1/2}\right\}$$

$$= \ C \left(|\xi'|^2 + |||q|||^2\right)^{j-k_\nu} \ \exp\left\{-C_1 x_n \left(|\xi'|^2 + |||q|||^2\right)^{1/2}\right\}$$

with C and C_1 independent of ξ', q, x_n and $M_q \ge q^{(0)}$ (we need it to estimate A_+^{-1}). Finally, integrating over x_n we get

$$\int_0^\infty |D^j_{x_n}\Omega_\nu|^2\, dx_n \ \le C \left(|\xi'|^2 + |||q|||^2\right)^{j-k_\nu-1/2}$$

and thus

$$\sum_{j+j_1\le l} |\xi'|^{2j_1} \int_0^\infty |D^j_{x_n}\Omega_\nu|^2\, dx_n \ + |||q|||^{2l} \int_0^\infty |\Omega_\nu|^2\, dx_n$$

$$\le \ C \left(|\xi'|^2 + |||q|||^2\right)^{l-k_\nu-1/2} \ \le \ C \left(1 + |\xi'|^{2l-2k_\nu-1} \ + \ |||q|||^{2l-2k_\nu-1}\right) \ .$$

Multiplying the left- and the right-hand sides of the last inequality by $|F'g_\nu|^2$ and integrating over ξ' we obtain the desired estimate

$$\big\| \big|F'^{-1}\{F'g_\nu \cdot \Omega_\nu\}\big|\big\|^2_{l,\mathbb{R}^n_+} \;\le\; C\||g_\nu\||'^2_{l-k_\nu-1/2} \,. \tag{1.2.25}$$

The existence follows from the formulae (1.2.19) and (1.2.24) applied to u_0 and u_1 We put

$$u \;=\; u_0 \,+\, u_1 \;=\; F^{-1}A^{-1}(\xi,q)FLf \,+$$

$$+\sum_{\nu=1}^{k} F'^{-1}\Big\{F'\big[g_\nu - \big(B_\nu(D,q)F^{-1}A^{-1}(\xi,q)FLf\big)\big|_{x_n=0}\big]\cdot\Omega_\nu\Big\}$$

$$=\; \mathcal{R}_0 f \;+\; \sum_{\nu=1}^{k} \mathcal{R}_\nu\big[g_\nu - \big(B_\nu(D,q)\mathcal{R}_0 f\big)\big|_{x_n=0}\big] \;:=\; \mathcal{R}(f,g_1,...,g_k)\;, \tag{1.2.26}$$

where

$$\mathcal{R}_0 f \;:=\; F^{-1}A^{-1}(\xi,q)FLf \tag{1.2.27}$$

and

$$\mathcal{R}_\nu g_\nu \;:=\; F'^{-1}\{F'g_\nu \cdot \Omega_\nu\}\;. \tag{1.2.28}$$

For $f \in H^{l-2k}(\mathbb{R}^n_+)$, $g_\nu \in H^{l-k_\nu-1/2}(\mathbb{R}^{n-1})$ and $q \in Q_M$ these formulae define the solution $u \in H^l(\mathbb{R}^n_+)$ to the problem (1.2.8) - (1.2.9). $\triangleright$

1.3 Elliptic boundary value problems with parameters in a bounded domain

Let G be a bounded domain in $\mathbb{R}^n_x$ with boundary $\Gamma \in C^\infty$, which is an $(n-1)$ - dimensional surface admitting a local straightening with the help of a smooth nondegenerate coordinate transformations.

Let us consider in G the equation

$$Au \;\equiv\; A(x,D,q)\,u(x,q) \;=\; f(x,q) \tag{1.3.1}$$

with boundary conditions on Γ :

$$B_\nu u|_\Gamma \;\equiv\; B_\nu(x,D,q)u(x,q)\big|_{x=x'} \;=\; g_\nu(x',q)\;, \tag{1.3.2}$$

where $x' \in \Gamma$ and $\nu = 1,...,k$. Here $A(x,D,q)$ and $B_\nu(x,D,q)$ are differential operators with coefficients, which are C^∞- functions in $x \in \overline{G}$ and polynomials in q_i for $i = 1,...,m$, where

$$q_i \,\in\, Q_i \;=\; \{q_i \,:\, \arg q_i \,\in\, [\alpha_i,\beta_i]\}\;.$$

Hence, $A(x,\xi,q)$ and $B_\nu(x,\xi,q)$ are polynomials in (ξ,q) of degrees $s=2k$ and k_ν $(\nu=1,...,k)$ respectively.

We denote by $A_0(x,\xi,q)$ and $B_{0,\nu}(x,\xi,q)$ the leading parts of the corresponding polynomials which are obtained by dropping the terms with degrees relatively to the variables (ξ,q) being less than $2k$ and k_ν respectively.

The operators $A_0(x,D,q)$ and $B_{0,\nu}(x,D,q)$ are called the *leading terms* of the corresponding operators . By analogy with the previous section, we introduce the operator of the boundary value problem

$$\mathcal{A}u = \{Au,\ B_1u|_\Gamma,...,\ B_ku|_\Gamma\} : H^l(G) \longrightarrow H^{l-2k}(G) \times \prod_{\nu=1}^{k} H^{l-k_\nu-1/2}(\Gamma)$$

and the number $l_0 = \max(2k, k_\nu+1)$.

Proposition 1.3.1 *Let $l\in\mathbb{Z}$, $l\geq l_0$. Then the operator $\mathcal{A}$ is bounded and*

$$|||f|||_{l-2k} + \sum_{\nu=1}^{k}|||g_\nu|||'_{l-k_\nu-1/2} \leq C|||u|||_l , \qquad (1.3.3)$$

where the norms are taken over G and Γ (see (1.1.18) and (1.1.20)).

bf Proof does not differ from the proof of Proposition 1.2.7. $\triangleright$

Let us now formulate conditions of the unique solvability of the problem (1.3.1) - (1.3.2) which are analogous to Conditions 1.2.2 and 1.2.8 from the previous section.

Condition 1.3.2 $\forall x\in\overline{G}$, $\forall q\in Q = Q_1\times...\times Q_m$ *and* $\forall\xi\in\mathbb{R}^n$

$$A_0(x,\xi,q) \neq 0 \quad for \quad |\xi|+M_q\neq 0 , \qquad (1.3.4)$$

where $M_q = \min_{i=1,...,m}|q_i|$.

In the case $n=1$ we suppose in addition that the λ-roots of the polynomial $A_0(x,\lambda,q)$ are equally distributed between the upper and the lower half-planes.

Condition 1.3.3 *Let us fix a boundary point $x'\in\Gamma$ and suppose that the operators in (1.3.1) - (1.3.2) are written in a local coordinate system* connected with this point *(i.e. the boundary is straightened and the x_n-axis looks in one side with the inner normal to Γ).*

Consider the following problem on the half-axis $x_n>0$:

$$A_0\big(0,\xi',-i\frac{d}{dx_n},q\big)\,v(x_n) = 0, \quad x_n>0 , \qquad (1.3.5)$$

$$B_{0,\nu}\big(0,\xi',-i\frac{d}{dx_n},q\big)\,v(x_n)\big|_{x_n=0} = h_\nu , \quad \nu=1,...,k , \qquad (1.3.6)$$

smoothly depending on parameters ξ' *and* q *.*

We suppose that $\forall\, q \in Q$ *for* $|\xi'| + M_q \neq 0$ *this problem has the unique solution in the space* $\mathcal{M}$ *of all stable solutions of the equation (1.3.5) for all* $h_\nu \in \mathbb{C}$ *.*

Note that for $q = 0$ Condition 1.3.2 turns into the ordinary ellipticity condition as well as Condition 1.3.3 turns into the classical Shapiro - Lopatinskij condition.

Example 1.3.4 *Let* $A(x, D)$ *be an elliptic operator* $(\mathrm{ord} A = 2k)$ *with a positive real leading part. Then the operator*

$$A(x, D) + q_1^k q_2^k$$

with Dirichlet boundary conditions

$$B_\nu \;=\; \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1} \;, \quad \nu = 1, ..., k,$$

satisfies Conditions 1.3.2 and 1.3.3 in the cone $Q = Q_1 \times Q_2$ *, where* $Q_j = \left\{|\arg q_j| \leq \frac{\pi-\varepsilon}{2k}\right\}$ *,* $\varepsilon \in (0, \pi)$ *and* $j = 1, 2$ *.*

Example 1.3.5 *If* A_1 *and* A_2 $(\mathrm{ord} A_j = 2k_j\,,\ j = 1, 2)$ *are the two such operators as in Example 1.3.4, then the operator*

$$\left(A_1(x, D) + q_1^{2k_1}\right)\left(A_2(x, D) + q_2^{2k_1}\right)$$

with Dirichlet boundary conditions

$$B_\nu \;=\; \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1} \;, \quad \nu = 1, ..., k_1 + k_2,$$

also satisfies Conditions 1.3.2 and 1.3.3 in the cone $Q = Q_1 \times Q_2$ *, where* $Q_j = \left\{|\arg q_j| \leq \frac{\pi-\varepsilon}{2k_j}\right\}$ *,* $\varepsilon \in (0, \pi)$ *and* $j = 1, 2$ *.*

Proposition 1.3.6 *Let* $l \in \mathbb{Z}$ *,* $l \geq l_0$ *,* $q^{(0)} > 0$ *and suppose Conditions 1.3.2 and 1.3.3 being fulfilled. Then there exists a constant* $q^{(1)} > 0$ *such that for all* $q \in Q_M^0 = \{q \in Q_M :\ M_q \geq q^{(0)}\}$ *with the property* $|||q||| = \sum_{i=1}^m |q_i| \geq q^{(1)}$ *, the a priori estimate*

$$|||u|||_l \;\leq\; C\{|||f|||_{l-2k} \;+\; \sum_{\nu=1}^{k} |||g_\nu|||'_{l-k_\nu-1/2}\} \tag{1.3.7}$$

is valid for the solutions of (1.3.1) - (1.3.2), where the norms are taken over G *and* Γ *(see (1.1.18) and (1.1.20)).*

Proof. We begin with the local situation and then glue the required estimate from local ones using a partition of unity.

A). Suppose $x_0 \in U \subset \operatorname{int} G$, $u \in H^l(G)$ and $\operatorname{supp} u \subset\subset U$, where U is an open set. Then (1.3.7) is equivalent to the inequality $|||u|||_l \le C|||A(x)u|||_{l-2k}$, where for brevity we write $A(x) = A(x, D, q)$, and analogously $A(x_0) = A(x_0, D, q)$.

It follows from Proposition 1.2.3 that

$$|||u|||_l \le C_1 |||A_0(x_0)u|||_{l-2k} ,$$

where the norms are taken over G . And thus

$$|||u|||_l \le C_1 |||A(x)u|||_{l-2k} + \mathcal{R} ,$$

where

$$\begin{aligned} \mathcal{R} &= C_1 \big|\big|\big| \big(A(x) - A_0(x_0)\big) u \big|\big|\big|_{l-2k} \\ &\le C_1 \big|\big|\big| \big(A(x) - A_0(x)\big) u \big|\big|\big|_{l-2k} + C_1 \big|\big|\big| \big(A_0(x) - A_0(x_0)\big) u \big|\big|\big|_{l-2k} . \end{aligned}$$

We note that in $A(x) - A_0(x)$ the leading terms of the form $a(x)q^\alpha D^\beta$ $(|\alpha| + |\beta| = 2k)$ are absent. Hence, it follows from (1.1.13) that

$$\big|\big|\big| \big(A(x) - A_0(x)\big) u \big|\big|\big|_{l-2k} \le C_2 |||u|||_{l-1} .$$

Conversely, $A_0(x) - A_0(x_0)$ consists only of leading terms, but for sufficiently small U the coefficients $a(x)$ may be made sufficiently small. Thus

$$|||a(x)q^\alpha D^\beta u|||_{l-2k} \le \varepsilon |||u|||_l + C_3 |||u|||_{l-1} .$$

Now we have from (1.1.13)

$$|||q||| \cdot |||u|||_{l-1} = \sum_{i=1}^m |q_i| \, |||u|||_{l-1} \le C_4 |||u|||_l , \tag{1.3.8}$$

hence

$$\mathcal{R} \le C_1 \Big[\varepsilon + \frac{(C_2 + C_3)C_4}{\sum_{i=1}^m |q_i|} \Big] |||u|||_l ,$$

where C_i are independent of u and q . Now, for sufficiently small ε and sufficiently large $|||q||| = \sum_{i=1}^m |q_i|$, $q \in Q_M^0$, we see that $\mathcal{R} \le 1/2 |||u|||_l$ and thus, $|||u|||_l \le C|||A(x)u|||_{l-2k}$ for $C = 2C_1$.

B). We now suppose that $x_0 \in \Gamma$ and choose $U \ni x_0$ such that $U \subset U_\alpha$, where U_α is one of the boundary covering coordinate neighbourhoods. For the functions $u \in H^l(G)$ with $\operatorname{supp} u \subset U \cap \overline{G}$ we obtain from (1.2.18) (in local coordinates) that

$$|||u|||_l \le C \Big\{ |||A_0(x_0)u|||_{l-2k} + \sum_{\nu=1}^k |||B_{\nu,0}(x_0)u|||'_{l-k_\nu-1/2} \Big\} .$$

In the same way as in part A) of the proof we obtain

$$|||u|||_l \le C_1\{|||A(x)u|||_{l-2k} + \sum_{\nu=1}^{k}|||B_\nu(x)u|||'_{l-k_\nu-1/2}\} + \mathcal{R} ,$$

where

$$\mathcal{R} = C_1\Big\{\big|\big|\big|(A(x)-A_0(x_0))u\big|\big|\big|_{l-2k} + \sum_{\nu=1}^{k}\big|\big|\big|(B_\nu(x)-B_{\nu,0}(x_0))u\big|\big|\big|'_{l-k_\nu-1/2}\Big\} .$$

The first norm in $\mathcal{R}$ is estimated as in part A) of the proof. For each term in the sum over ν we obtain applying the trace inequality (1.1.6) :

$$\big|\big|\big|(B_\nu(x)-B_{\nu,0}(x_0))u\big|\big|\big|'_{l-k_\nu-1/2} \le \big|\big|\big|(B_\nu(x)-B_{\nu,0}(x_0))u\big|\big|\big|_{l-k_\nu} ,$$

and now the right hand-side can be estimated as $\big|\big|\big|(A(x)-A_0(x_0))u\big|\big|\big|_{l-2k}$ in part A) of the proof.

C). In order to end the proof we have to arrange a standard procedure with a partition of unity. Let $\{U_\alpha\}$ be a covering of $\bar{G}$ such that the results of items A) and B) are valid in each U_α , and let $\{U_\mu\}_{\mu=1}^{\mu_0}$ be its finite subcovering. Denote by C and $q^{(1)}$ the largest local ones. Now (1.3.7) is valid for all q : $|||q||| \ge q^{(1)}$, $q \in Q_M^0$ for all $u \in H^l(G)$ such that $\exists \mu \in \{1,...,\mu_0\}$: $\operatorname{supp} u \subset U_\mu$.

We now take the corresponding to $\{U_\mu\}$ partition of unity $\{\varphi_\mu\}_{\mu=1}^{\mu_0}$. As for the function $\varphi_\mu u$ (1.3.7) is already proved, we have

$$\begin{aligned} |||u|||_l &\le \sum_{\mu=1}^{\mu_0} |||\varphi_\mu u|||_l \\ &\le C\sum_{\mu=1}^{\mu_0}\Big\{|||A(x)[\varphi_\mu u]|||_{l-2k} + \sum_{\nu=1}^{k}|||B_\nu(x)[\varphi_\mu u]|||'_{l-k_\nu-1/2}\Big\} ,\end{aligned}$$

but $A(x)[\varphi_\mu u] = \varphi_\mu A(x)u + ...$, where in the terms denoted by "..." φ_μ is differentiated at least once and thus

$$\begin{aligned} &|||u|||_l \\ &\le C\sum_{\mu=1}^{\mu_0}\Big\{|||\varphi_\mu A(x)u|||_{l-2k} + \sum_{\nu=1}^{k}|||\varphi_\mu B_\nu(x)u|||'_{l-k_\nu-1/2}\Big\} + \\ &\qquad + C_1|||u|||_{l-1} .\end{aligned}$$

Once again we use (1.3.8) (together with the fact that the multiplication by $\varphi_\mu \in C^\infty$ is a bounded operator in H^s and $\|\varphi u\|_l^2 \le \|u\|_l^2 + C_\varphi\|u\|_{l-1}^2$) and obtain

$$|||u|||_l$$

$$\leq\; C_3 \sum_{\mu=1}^{\mu_0} \Big\{ |||A(x)u|||_{l-2k} \;+\; \sum_{\nu=1}^{k} |||B_\nu(x)u|||'_{l-k_\nu-1/2} \Big\} \;+$$

$$+\; \frac{C_2}{|||q|||} |||u|||_l \;.$$

Finally, we suppose $q^{(1)} \geq 2C_2$ and the proof follows. ▷

Naturally, the *uniqueness of the solution to the problem (1.3.1) - (1.3.2)* follows from (1.3.7) for sufficiently large $|||q|||$, $q \in Q^0_M$.

We now show that Conditions 1.3.2 and 1.3.3 are also necessary for the estimate (1.3.7) to take place for large $|||q|||$, $q \in Q^0_M$.

Proposition 1.3.7 *Let $l \in \mathbb{Z}$, $l \geq l_0$, $q^{(0)} > 0$ and suppose there exists a large constant $q^{(1)} > 0$ such that the estimate (1.3.7) is true for all solutions from $H^l(G)$ to the problem (1.3.1) - (1.3.2) when $|||q||| \geq q^{(1)}$ and $q \in Q^0_M$.*

Then Condition 1.3.2 is satisfied in every interior point $x_0 \in G$ and Condition 1.3.3 is satisfied in every boundary point $x_0 \in \Gamma$ where Condition 1.3.2 is already satisfied (if $n = 1$, we assume additionally the regular distribution of $\lambda-$roots of the equation $A_0(x, \lambda, q) = 0$).

Proof. A). First, we prove that $\forall x_0 \in G$ there exists $U_{x_0} \subset G$ and $q^{(1)}$: $\forall q \in Q^0_M$: $|||q||| \geq q^{(1)}$ and $\forall u \in H^l(G)$ with $\operatorname{supp} u \subset U_{x_0}$ the estimate

$$|||u|||_l \;\leq\; C_1 |||A_0(x_0)u|||_{l-2k} \;. \tag{1.3.9}$$

takes place. Indeed,

$$|||u|||_l \;\leq\; C|||A(x)u|||_{l-2k}$$

$$\leq\; C|||A_0(x_0)u|||_{l-2k} \;+\; \mathcal{R} \;,$$

where

$$\mathcal{R} \;=\; C|||(A(x) - A_0(x_0))u|||_{l-2k} \;,$$

but this was already estimated in the proof of part A) of Proposition 1.3.6, and thus, if U_{x_0} is sufficiently small and $|||q|||$ - sufficiently large , we obtain that $\mathcal{R} \leq 1/2|||u|||_l$, and the required estimate follows for $C_1 = 2C$.

B). By analogy, $\forall x_0 \in \Gamma$ there exists U_{x_0} and $q^{(2)}$ such that $\forall q \in Q^0_M$: $|||q||| \geq q^{(2)}$ and $\forall u \in H^l(G)$ with $\operatorname{supp} u \subset U_{x_0} \cap \overline{G}$ we have

$$|||u|||_l \;\leq\; C_2 \Big\{ |||A_0(x_0)u|||_{l-2k} \;+\; \sum_{\nu=1}^{k} |||B_{\nu,0}(x_0)u|||'_{l-k_\nu-1/2} \Big\} \;. \tag{1.3.10}$$

C). Suppose now Condition 1.3.2 is not satisfied in an interior point $x_0 \in G$, that is,

$$A_0(x_0, \xi_0, q_0) \;=\; 0 \tag{1.3.11}$$

for some (ξ_0, q_0) : $|\xi_0| + M_{q_0} \neq 0$, and let the estimate (1.3.9) be valid in the neighbourhood U_{x_0} and for $q \in Q_M^0$ we have $|||q||| \geq q^{(2)}$. Let $\varphi(x) \in C_0^\infty(U_{x_0})$ and $\varphi(x_0) \geq 0$. If $\xi_0 = 0$ then $M_{q_0} \neq 0$ and (1.3.11) means that in $A_0(x_0, D, q)$ the terms $C_\alpha q_1^{\alpha_1} ... q_m^{\alpha_m}$ with $\alpha_1 + \cdots + \alpha_m = 2k$ are absent. Putting $u = \varphi$ in (1.3.9) we get the estimate

$$\sum_{i=1}^{m} |q_i|^l \|\varphi\|_0 \ \leq \ C|||\varphi|||_l \ \leq \ C|||A_0(x_0)\varphi|||_{l-2k}$$

$$\leq \ C\Big[\|A_0(x_0)\varphi\|_{l-2k} \ + \ \sum_{i=1}^{m} |q_i|^{l-2k} \|A_0(x_0)\varphi\|_0\Big] \ \leq \ C\sum_{i=1}^{m} |q_i|^{l-1}$$

which is impossible for large $|||q|||$.

Let $\xi_0 \neq 0$, and introduce the function

$$u_\lambda(x) \ = \ e^{i\lambda x\cdot\xi_0}\varphi(x)$$

where $\lambda > 0$. Then we obtain from (1.3.9) with $q = \lambda q_0$ that

$$\|u_\lambda\|_l \ \leq \ |||u_\lambda|||_l \ \leq \ C\Big|\Big|\Big|A(x_0, D, \lambda q_0)u_\lambda\Big|\Big|\Big|_{l-2k}$$

$$\leq \ C\big\{\lambda^{l-2k}\|A(x_0, D, \lambda q_0)u_\lambda\|_0 \ + \ \|A(x_0, D, \lambda q_0)u_\lambda\|_{l-2k}\big\} \ , \tag{1.3.12}$$

but $\|u_\lambda\|_l = O(\lambda^l)$ for large λ . As $A(x_0, D, \lambda q_0)e^{i\lambda x\cdot\xi_0} = 0$, we see that the right-hand side in (1.3.12) is $O(\lambda^{l-1})$, and again we approach to the contradiction.

D). Now let $x_0 \in \Gamma$ and suppose the coordinates in U_{x_0} are chosen to straighten the boundary. Suppose Condition 1.3.2 is satisfied in the point x_0 and Condition 1.3.3 is violated. In this case there exists a nontrivial stable solution $v(x_n)$ to the problem (1.3.5) - (1.3.6) with zero right-hand sides for certain (ξ_0, q_0) such that $|\xi_0'| + M_{q_0} \neq 0$. Then the function

$$w_\lambda(x) \ = \ e^{i\lambda x'\cdot\xi_0'}v(\lambda x_n) \ , \quad \lambda > 0 \ ,$$

is the solution to the problem

$$A_0(x_0, D, \lambda q_0)u \ = \ 0 \ , \tag{1.3.13}$$

$$B_{\nu,0}(x_0, D, \lambda q_0)u\Big|_{x_n=0} \ = \ 0 \ , \quad \nu = 1, ..., k. \tag{1.3.14}$$

In order to apply part B) of the proof, we introduce cut-off functions $\varphi(x')$, $\psi(x_n)$ such that $\varphi\psi \neq 0$ in some small neighbourhood U'_{x_0} , $\mathrm{supp}\,(\varphi\psi) = \overline{U'_{x_0}}$ and for some $\delta, h > 0$: $\psi(x_n) \equiv 1$ as $0 \leq x_n \leq \delta$ and $\psi(x_n) \equiv 0$ for $x_n \geq h$.

Let $u_\lambda(x) = w_\lambda(x)\varphi(x')\psi(x_n)$. Using the fact that w_λ is the solution of (1.3.13) and substituting u_λ in (1.3.10) with $q = \lambda q_0$, we compute the order of the right-hand side with respect to λ :

$$\| | A(x_0, D, \lambda q_0) u_\lambda | \|^2_{l-2k}$$

$$\| | A(x_0, D, \lambda q_0)\varphi\psi w_\lambda \ - \ \varphi\psi A(x_0, D, \lambda q_0) w_\lambda | \|^2_{l-2k}$$

$$\| | D_1(\varphi\psi)\, \mathrm{Diff}^{2k-2} w_\lambda \ + \ \varphi\psi\, \mathrm{Diff}^{2k-1} w_\lambda | \|^2_{l-2k}$$

$$\leq \ C\lambda^{2l-2} \int_0^h |v(\lambda x_n)|^2 \, dx_n \ + \ C\lambda^{2l-4} \ \leq \ C\lambda^{2l-3} \ ,$$

where Diff^j is a homogeneous in D, λ differential operator of order j with constant coefficients.

By analogy,

$$\| | B_{\nu,0}(x_0, D, \lambda q_0) u_\lambda | \|'^2_{l-k_\nu-1/2} \ \leq \ C\lambda^{2l-3} \ .$$

As for the left-hand side, there for $\xi_0 = 0$, $M_{q_0} \neq 0$ and large λ we have

$$\| | u_\lambda | \|^2_l \ \geq \ \sum_{i=1}^m \| | q | \|^{2l} \, \| u_\lambda \|^2_0$$

$$\geq \ C\lambda^{2l} \int_0^\delta |v(\lambda x_n)|^2 \, dx_n \ \geq \ C\lambda^{2l-1} \ ,$$

and in the case $\xi_0 \neq 0$ and large λ we have

$$\| | u_\lambda | \|^2_l \ \geq \ \| u_\lambda \|^2_l$$

$$\geq \ C\lambda^{2l} \int_0^\delta |v(\lambda x_n)|^2 \, dx_n \ \ + \ C\lambda^{2l-2} \ \geq \ C\lambda^{2l-1} \ .$$

Consequently, the left-hand side in (1.3.10) has the order $l - 1/2$ in contradiction with the order $l - 3/2$ of the right-hand side. ▷

1.4 The existence theorems

We start with the corresponding theorem for operators with coefficients, sufficiently close to the constant ones, and denote by $\mathcal{A}$ an operator in $\mathbb{R}^n_+$ of the form

$$\mathcal{A}u \ = \Big\{ A(x, D, q)u, \ B_1(x, D, q)u\big|_{x_n=0}, \ldots, \ B_k(x, D, q)u\big|_{x_n=0} \Big\}.$$

Theorem 1.4.1 *A). Suppose* $l \in \mathbb{Z}$, $l \geq 2k$ *and let*

$$A(x,D,q) \;=\; \sum_{|\alpha|+|\beta|=2k} a_{\alpha,\beta}(x) q^\beta D^\alpha \;, \qquad a_{\alpha,\beta} \in C^\infty(\mathbb{R}^n) \;,$$

be a homogeneous in (D,q) *differential operator in* $\mathbb{R}^n$. *We assume that* $A(0,D,q)$ *satisfies Condition 1.3.2. There exists* $\varepsilon > 0$ *such that, if*

$$\bigl|a_{\alpha,\beta}(x) \;-\; a_{\alpha,\beta}(0)\bigr| \;<\; \varepsilon \;, \qquad \forall\, x \in \mathbb{R}^n \;, \tag{1.4.1}$$

then, for $q \in Q^0_M$ *with sufficiently large* $|||q|||$, *the right inverse* $\mathcal{R}$ *to the operator* $A(x,D,q)$ *is defined and*

$$|||\mathcal{R} f|||_l \;\leq\; C|||f|||_{l-2k} \;. \tag{1.4.2}$$

B). Suppose $l \in \mathbb{Z}$, $l \geq \max(2k, k_j + 1)$ *and let the operator* $A(x,D,q)$ *be as in part A) of the theorem with* $a_{\alpha,\beta} \in C^\infty(\mathbb{R}^n_+)$, *and*

$$B_j(x,D,q) \;=\; \sum_{|\alpha|+|\beta|=k_j} b_{j,\alpha,\beta}(x) q^\beta D^\alpha \;,$$

where $b_{j,\alpha,\beta} \in C^\infty(\mathbb{R}^n_+)$. *Suppose that* $A(0,D,q)$ *satisfies Condition 1.3.2, and together with* $B_j(0,D,q)$ $(j = 1,...,k)$ *– Condition 1.3.3. Then there exists* $\varepsilon > 0$ *such that, if*

$$\bigl|a_{\alpha,\beta}(x) \;-\; a_{\alpha,\beta}(0)\bigr| \;<\; \varepsilon \quad (|\alpha| + |\beta| = 2k) \;, \qquad \forall\, x \in \mathbb{R}^n \;, \tag{1.4.3}$$

$$\bigl|b_{j,\alpha,\beta}(x) \;-\; b_{j,\alpha,\beta}(0)\bigr| \;<\; \varepsilon \quad (|\alpha| + |\beta| = k_j) \;, \qquad \forall\, x \in \mathbb{R}^n \;, \tag{1.4.4}$$

then, for $q \in Q^0_M$ *with sufficiently large* $|||q|||$, *the right inverse* $\mathcal{R}$ *to the operator* $\mathcal{A}$ *is defined and*

$$|||\mathcal{R}(f, g_1, ..., g_k)|||_l \;\leq\; C\bigl\{|||f|||_{l-2k} \;+\; \sum_{\nu=1}^{k} |||g_\nu|||'_{l-k_\nu-1/2}\bigr\} \;. \tag{1.4.5}$$

Proof. We show how item A) follows from Proposition 1.2.3, item B) follows from Proposition 1.2.7 in the same way.

Let $\mathcal{R}_0$ be the inverse to $A(0,D,q)$ being built in Proposition 1.2.3 $\forall\, q \in Q^0_M$. As follows from (1.2.5)

$$|||\mathcal{R}_0 f|||_l \;\leq\; C|||f|||_{l-2k} \;. \tag{1.4.6}$$

Consider an operator

$$A(x,D,q)\mathcal{R}_0 \;=\; A(0,D,q)\mathcal{R}_0 \;+\; T \;=\; I \;+\; T \;,$$

where

$$T \;=\; \bigl[A(x,D,q) \;-\; A(0,D,q)\bigr]\mathcal{R}_0 \;.$$

By analogy with the proof of Proposition 1.3.6, part A), we see that

$$|||Tf|||_{l-2k} \le \eta(\varepsilon)|||\mathcal{R}_0 f|||_l + C|||\mathcal{R}_0 f|||_{l-1} ,$$

where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, using the inequalities (1.3.8) and (1.4.6), we obtain

$$|||Tf|||_{l-2k} \le C_1\Big(\eta(\varepsilon) + \frac{C_3}{|||q|||}\Big)|||\, f|||_{l-2k} .$$

Thus, for sufficiently large $|||q|||$ and sufficiently small ε , we obtain that

$$|||Tf|||_{l-2k} \le 1/2|||\, f|||_{l-2k} ,$$

and thus, operator T has a small norm. It implies , that the inverse $(I+T)^{-1}$ to the operator $(I+T)$ exists and is bounded in the norm $|||\cdot|||_{l-2k}$ and hence, in the norm $\|\cdot\|_{l-2k}$. Now we put

$$\mathcal{R} = \mathcal{R}_0(I+T)^{-1}$$

and observe that, the inequality (1.4.2) is valid for $\mathcal{R}$ and

$$A(x,D,q)\mathcal{R}f = f ; \quad f \in H^{l-2k}(\mathbb{R}^n) .$$

$\triangleright$

Now we are ready to formulate the main theorem about elliptic operators with parameters.

Theorem 1.4.2 *Let* $l \in \mathbb{Z}$, $l \ge l_0 = \max(2k, k_j+1)$ *and suppose the boundary value problem (1.3.1) - (1.3.2) satisfy Conditions 1.3.2 and 1.3.3.*

Then for all $q \in Q^0_M$ *with sufficiently large* $|||q|||$ *a bounded inverse*

$$\mathcal{A}^{-1} : H^{l-2k}(G) \times \prod_{j=1}^{k} H^{l-k_j-1/2}(\Gamma) \longrightarrow H^l(G) \qquad (1.4.7)$$

to the operator $\mathcal{A}$ *is defined.*

Proof. is rather standard. Using a sufficiently small partition of unity $\{\varphi_\nu\}_{\nu=1}^N$ such that $\varphi_\nu \in C_0^\infty(U_\nu)$ and $\sum_{\nu=1}^N \varphi_\nu(x) \equiv 1$ in a neighbourhood of G , we reduce the problem to the cases considered in Theorem 1.4.1.

Suppose that either $\operatorname{supp}\varphi_\nu \cap \Gamma = \emptyset$, or $\operatorname{supp}\varphi_\nu \subset U_j$, where $\bigcup_j U_j \supset \Gamma$. Moreover, for each U_ν we define an infinitely smooth function $\psi_\nu \in C_0^\infty(U_\nu)$ such that $\psi_\nu \equiv 1$ in a neighbourhood of $\operatorname{supp}\varphi_\nu(x)$,

$$D^\alpha\big[\varphi_\nu(x)\psi_\nu(x)\big] \equiv D^\alpha\varphi_\nu(x)$$

and ψ_ν has the same support properties as $\varphi_\nu(x)$. Obviously,

$$\mathcal{A}u = \sum_{\nu=1}^{N} \varphi_\nu \mathcal{A}(\psi_\nu u) . \qquad (1.4.8)$$

There are ingredients of two types in this sum: in those with $\operatorname{supp}\varphi_\nu, \psi_\nu \bigcap \Gamma \neq \emptyset$ we overcome to the local coordinates straightening the boundary in U_ν ; if $\operatorname{supp}\varphi_\nu, \psi_\nu \subset \operatorname{int} G$, we associate $\mathcal{A}$ with A for here $B_j(\psi_\nu u)\big|_\Gamma = 0$.

If $\operatorname{supp}\psi_\nu$ is sufficiently small the coefficients of the operator $\mathcal{A}$ on $\operatorname{supp}\psi_\nu$ are almost constant, and we construct an extension $\mathcal{A}_\nu$ of $\mathcal{A}$ from $\operatorname{supp}\psi_\nu$ to the whole $\mathbb{R}^n$ or $\mathbb{R}^n_+$ such that

$$\varphi_\nu \mathcal{A}(\psi_\nu u) \;=\; \varphi_\nu \mathcal{A}_\nu(\psi_\nu u) \;,$$

and its coefficients are almost constant too. We put

$$\mathcal{A}_\nu(x) \;=\; \alpha(x)\mathcal{A}(x) \;+\; [1-\alpha(x)]\mathcal{A}(x^{(\nu)}) \;,$$

where $\alpha(x) \in C_0^\infty$, $0 \leq \alpha(x) \leq 1$, $\alpha(x) \equiv 1$ on $\operatorname{supp}\psi_\nu$, $x^{(\nu)} \in \operatorname{supp}\psi_\nu$ and $x^{(\nu)} \in \Gamma$ if $\operatorname{supp}\psi_\nu \bigcap \Gamma \neq \emptyset$.

Operator

$$\mathcal{A}^0 u \;=\left\{ A_0 u,\; B_{1,0}u\big|_\Gamma, \ldots,\; B_{k,0}u\big|_\Gamma \right\}$$

is called a leading part of operator $\mathcal{A}$. Here and further $\mathcal{A}^0_\nu, A_0, B_{j,0}$ are the leading parts of operators $\mathcal{A}_\nu, A, B_j$.

If $\operatorname{supp}\psi_\nu$ is sufficiently small, then it follows from Theorem 1.4.1 that the right inverse $\mathcal{R}^0_\nu$ to $\mathcal{A}^0_\nu$ is defined for $q \in Q^0_M$ with sufficiently large $|||q|||$. For these q we define a bounded operator

$$\mathcal{R}(f,g) \;=\; \sum_{\mu=1}^{N} \psi_\mu \mathcal{R}^0_\mu(\varphi_\mu f, \varphi_\mu g) \;:\; H^l(G,\Gamma) \longrightarrow H^l(G) \;, \tag{1.4.9}$$

where $g = (g_1, \ldots, g_k)$ and $H^l(G,\Gamma) \;=\; H^{l-2k}(G) \times \prod_{j=1}^k H^{l-k_j-1/2}(\Gamma)$ is equipped with a norm

$$|||(f,g)|||_l \;=\; |||f|||_{l-2k} \;+\; \sum_{j=1}^{k} |||g_j|||'_{l-k_j-1/2} \;. \tag{1.4.10}$$

As usual, we have to prove that $\mathcal{A}\mathcal{R} = I + T$, where $T = T(f,g)$ is an operator with the sufficiently small norm in $H^l(G,\Gamma)$. Thus we have

$$\mathcal{A}\mathcal{R}(f,g) \;=\; \sum_\mu \sum_\nu \varphi_\nu \mathcal{A}_\nu \psi_\nu \psi_\mu \mathcal{R}_\mu(\varphi_\mu f, \varphi_\mu g) \;,$$

where we consider only the terms with $\psi_\nu\psi_\mu \neq 0$. We put

$$u_\mu \;=\; \mathcal{R}^0_\mu(\varphi_\mu f, \varphi_\mu g)$$

and replace $\mathcal{A}_\nu$ by $\mathcal{A}_\mu$ in the last sum:

$$\mathcal{A}\mathcal{R}(f,g) \;=\; \sum_\mu \sum_\nu \varphi_\nu \mathcal{A}_\mu \psi_\nu \psi_\mu u_\mu$$

$$= \sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu \mathcal{A}_{\mu,0} u_\mu \ + \ T(f,g) \ ,$$

where

$$T(f,g) \ = \ \sum_\mu \sum_\nu \varphi_\nu [\mathcal{A}_\mu \psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu \mathcal{A}_\mu] u_\mu \ +$$

$$+ \ \sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu [\mathcal{A}_\mu \ - \ \mathcal{A}_{\mu,0}] u_\mu \ . \tag{1.4.11}$$

As $\mathcal{A}_{\mu,0} \mathcal{R}^0_\mu = I$, it follows that,

$$\sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu \mathcal{A}_{\mu,0} u_\mu \ = \ \sum_\mu \sum_\nu \varphi_\nu \psi_\nu \psi_\mu \varphi_\mu (f,g)$$

$$= \ \sum_\mu \sum_\nu \varphi_\nu \varphi_\mu (f,g) \ = \ (\sum_\nu \varphi_\nu)^2 (f,g) \ = \ (f,g) \ .$$

In order to estimate $T(f,g)$, we fix a pair of indices μ and ν and suppose $\mathcal{A}_\mu$ includes boundary operators (otherwize the work is easier).

For each addend in the first sum in (1.4.11) we have in terms of the norm (1.4.10)

$$\| | \varphi_\nu [\mathcal{A}_\mu \psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu \mathcal{A}_\mu] u_\mu | \|_l$$

$$\leq \ \Big\{ \| | [A \psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu A] u_\mu | \|_{l-2k} \ +$$

$$+ \ \sum_{j=1}^{k} \| | [B_j \psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu B_j] u_\mu | \|_{l-k_j} \Big\}$$

$$\leq \ C \| | u_\mu | \|_{l-1} \ , \tag{1.4.12}$$

where as in the proof of Theorem 1.4.1 we use the fact that operators in square brackets do not contain the leading terms.

Now, it follows from (1.1.13) and (1.4.5) for $\mathcal{R}_\mu$ that

$$\| | \varphi_\nu [\mathcal{A}_\mu \psi_\nu \psi_\mu \ - \ \psi_\nu \psi_\mu \mathcal{A}_\mu] u_\mu | \|_l$$

$$\leq \ \frac{C}{\| | q | \|} \| | (\varphi_\mu f, \varphi_\mu g) | \|_l \ . \tag{1.4.13}$$

As $\|\varphi u\|_l^2 \ \leq \ \|u\|_l^2 \ + \ C_\varphi \|u\|_{l-1}^2$, we can get reed of φ_μ in the right - hand side of (1.4.13) and return to the initial coordinates .

The addends in the second sum in (1.4.11) we estimate in the same way, for the operator $\mathcal{A}_\mu - \mathcal{A}^0_\mu$ does not contain leading terms too. We arrive to inequality

$$\| | T(f,g) \ | \|_l \ \leq \ \frac{C}{\| | q | \|} \| | (f,g) | \|_l \ .$$

Consequently, for $q \in Q_M^0$ with sufficiently large $|||q|||$ we obtain that $\|T\| \leq 1/2$ and $\mathcal{AR} = I + T$ (I is the identity map in $H^l(G,\Gamma)$) possess the bounded inverse $(\mathcal{AR})^{-1}$. Finally, we define

$$\mathcal{A}^{-1} = \mathcal{R}(\mathcal{AR})^{-1}$$

being the bounded operator from $H^l(G,\Gamma)$ to $H^l(G)$ and the right inverse to $\mathcal{A}$. Thus, it follows, that $\mathcal{A}$ maps $H^l(G)$ onto $H^l(G,\Gamma)$, and is one-to-one for $q \in Q_M^0$ with sufficiently large $|||q|||$, and for these q , $\mathcal{A}^{-1}$ also is a left inverse to $\mathcal{A}$. $\triangleright$

We conclude this section with a few general remarks about possible generalizations and further results.

Remark 1.4.3 *A). If the problem (1.3.1) - (1.3.2) satisfies Conditions 1.3.2 and 1.3.2 on the rays* $\{\arg q_i = c_i\}$ *then, for all* $q \in Q_M^0$ *with sufficiently large* $|||q|||$ *, this is a Fredholm problem with zero index (this result follows directly from the work (Agranovich M.S., Dynin A.S. (1962))).*

B). All propositions and theorems form Sections 2-4 of this chapter can be reformulated for elliptic boundary value problems for systems with parameters (only the formulae for canonical basis need to be changed).

C). The above results can be reformulated for operators acting on manifolds and in sections of bundles over them.

D). It would be interesting to consider pseudodifferential boundary problems depending on a number of parameters. The one - parameter case was considered in the book (Grubb, G. (1986)), also see (Grubb, G. (1989)).

Remark 1.4.4 *We note that the parameter set* $\{q \in \mathbb{C}^n : q \in Q_M^0, |||q||| \geq q^{(1)}\}$ *is a relatively complete Reinhart domain. It can be made logarithmically convex if* $q^{(1)}$ *is slightly increased (see Shabat B.V. (1985)).*

Chapter 2

Polyparabolic mixed problems

2.1 Main functional spaces

2.1.1

Let $\alpha = (\alpha_1, ..., \alpha_m)$, $\gamma = (\gamma_1, ..., \gamma_m)$ be multi-indices with nonnegative coordinates.

By definition the space $\mathbf{P}^\alpha\big(e^{-\gamma\cdot t}\big)$ consists of all functions $u(t) = u(t_1, ..., t_m)$ supported in the positive "octant" $(\bar{\mathbb{R}}_+)^m$ (i.e. these functions are equal to zero if at least one $t_j < 0$) and such that

$$e^{-\gamma\cdot t}u(t) \ \in \ H^{\vec{\alpha}}(\mathbb{R}^m) \ , \tag{2.1.1}$$

where $H^{\vec{\alpha}}(\mathbb{R}^m)$ denotes the anisotropic Sobolev space, defined as a completion of the space $C_0^\infty(\mathbb{R}^m)$ with respect to the norm

$$\|u\|_{\vec{\alpha}}^2 \ = \ \sum_{j=1}^{m} \int_{(\mathbb{R}_+)^{m-1}} \int_{\mathbb{R}_{\xi_j}} \left|(F_{t_j\to\xi_j}u)(t_1, ..., t_{j-1}, \xi_j, t_{j+1}, ..., t_m)\right|^2 \times$$

$$\times \left(1 + |\xi_j|^2\right)^{\alpha_j} d\xi_j dt[j], \tag{2.1.2}$$

where $dt[j] \ = \ dt_1...dt_{j-1}dt_{j+1}...dt_m$. ¿From this definition and the embedding theorems (see, for example, Slobodeckij L.N. (1958), Besov O.V., Il'in V.P., Nikolskij S.M. (1975), Kudryavtsev L.D., Nikolskij S.M. (1988)) follows that, if $u(t) \in \mathbf{P}^\alpha\big(e^{-\gamma\cdot t}\big)$ then the derivatives $\partial_{t_i}^{k_i}u(t)$ of orders $k_i \le [\alpha_i] - 1$, for $\alpha_i \le [\alpha_i] + 1/2$, and of orders $k_i \le [\alpha_i]$, for $\alpha_i \ge [\alpha_i] + 1/2$, are continuous and thus, vanish when $t_i = 0$.

For the norm of a function $u(t)$ in the space $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$ we take the norm $\big\|e^{-\gamma\cdot t}u(t)\big\|_{\vec{\alpha}}$ of the function $e^{-\gamma\cdot t}u(t)$ in $H^{\vec{\alpha}}(\mathbb{R}^m)$.

We can give the "interior" definition of spaces $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$ for $\alpha_i \neq [\alpha_i]+1/2$, that is, the space $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$ consists of functions in positive "octant" $\hat{Q} = (\bar{\mathbb{R}}_+)^m$ such that

1) $e^{-\gamma\cdot t}u(t) \in H^{\vec{\alpha}}(\hat{Q})$

and

2) $e^{-\gamma\cdot t}u(t) \in H^{\vec{\alpha}}(\mathbb{R}^m)$ after zero extension into the domain, where at least one $t_i < 0$ (the continuation may be executed serially in $t_1, t_2, ..., t_m$ and the result is independent of the order of the operations).

In this case we take the value $\big\|e^{-\gamma\cdot t}u(t)\big\|_{\vec{\alpha},\hat{Q}}$ as the norm of the function $e^{-\gamma\cdot t}u(t)$ in the space $H^{\vec{\alpha}}(\hat{Q})$ (see (1.1.24)).

Condition 2) is equivalent to the following one:

2)$'$ for almost all $(t_1, ..., t_{i-1}, t_{i+1}, ..., t_m) = t[i]$ we have

$$\partial_{t_i}^{k_i} u(t)\big|_{t_i=0} \; = \; 0 \quad \text{for} \quad k_i \leq [\alpha_i]-1 \tag{2.1.3}$$

and

$$\int_0^1 t_i^{2(\alpha_i-[\alpha_i])}\big|\partial_{t_i}^{[\alpha_i]} u(t)\big|^2 \, dt_i \; < \; +\infty \quad \text{for} \quad \alpha_i > [\alpha_i] \, . \tag{2.1.4}$$

The functional space $\mathbf{E}_{\alpha}(\gamma)$ is defined as the set of functions $U(p) = U(p_1, ..., p_m)$, where $p = \sigma + i\gamma$ and $p_j = \sigma_j + i\gamma_j$, for $j = 1, ..., m$, which are holomorphic in the direct product of right half-planes $\Pi_{\gamma} = \{p \in \mathbb{C}^m : \operatorname{Re} p_j > \gamma_j \, , \; j = 1, ..., m\}$ and have finite norms

$$\big\|U(p_1, ..., p_m)\big\|^2_{\mathbf{E}_{\alpha}(\gamma)} \; = \; \sum_{j=1}^{m} \sup_{\sigma\in\Pi_{\gamma}} \int_{\mathbb{R}^m} \big|U(\sigma+i\tau)\big|^2 \, |\sigma_j + i\tau_j|^{2\alpha_j} \, d\tau \, . \tag{2.1.5}$$

Proposition 2.1.1 *Suppose* $p = (p_1, ..., p_m) \in \Pi_{\gamma}$ *with* $\gamma_j > 0$ *for* $j = 1, ..., m$. *Then the Laplace transformation*

$$Lu(t) \; := \; \int_0^{\infty} \cdots \int_0^{\infty} e^{-p\cdot t} u(t) \, dt \; = \; U(p) \tag{2.1.6}$$

defines a one-to-one mutually continuous correspondence of spaces $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$ *and* $\mathbf{E}_{\alpha}(\gamma)$.

Proof. 1). We start with the case $\alpha = 0$.

Let $u(t) \in \mathbf{P}^{0}\big(e^{-\gamma\cdot t}\big)$. Then

$$\begin{aligned} Lu(t) \; = \; U(p) \; &= \; \int_0^{\infty} \cdots \int_0^{\infty} e^{-i\tau\cdot t}\big[e^{-\sigma\cdot t}u(t)\big] \, dt \qquad (2.1.7) \\ &= \; F_{t\to\tau}\big[e^{-\sigma\cdot t}u(t)\chi(t)\big] \; , \end{aligned}$$

where $\chi(t) = \prod_{j=1}^{m} \chi(t_j)$ and $\chi(t_j)$ is Heaviside function (i.e. the characteristic function of the ray $0 \le t_j < \infty$).

For all $\sigma \in \overline{\Pi}_\gamma$ we have $e^{-\sigma \cdot t} u(t) \in L_2(\hat{Q})$, consequently $U(p) \in L_2$ in τ and in the view of Parseval equality

$$\frac{1}{(2\pi)^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| U(\sigma + i\tau) \right|^2 d\tau \ = \ \int_0^{\infty} \cdots \int_0^{\infty} e^{-2\sigma \cdot t} \left| u(t) \right|^2 dt \quad (2.1.8)$$

with $\sigma \in \overline{\Pi}_\gamma$, we obtain

$$\sup_{\sigma \in \Pi_\gamma} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| U(\sigma + i\tau) \right|^2 d\tau \ = \ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| U(\gamma + i\tau) \right|^2 d\tau$$

$$= \ (2\pi)^m \left\| e^{-\gamma \cdot t} u(t) \right\|_0^2 . \quad (2.1.9)$$

We note that for $\sigma = \gamma$ the integral in (2.1.7) is defined as the L_1-limit of integrals over $\prod_{j=1}^{m} [0, N_j]$ as $N_j = N \to \infty$, the same concerns the case where some of $\sigma_j = \gamma_j$. If $\sigma_j > \gamma_j$ for all j then the integral in (2.1.7) is absolutely convergent and $U(p)$ is holomorphic in each p_j . Thus, if $\mathrm{Re}\, p = \sigma \in \mathrm{int}\, \Pi_\gamma$ (i.e. $\sigma_j > \gamma_j$ for all $j \in \{1, ..., m\}$) then $U(p)$ is holomorphic in each p_j and, in the view of Hartogs theorem, is indeed holomorphic in Π_γ .

Thus, we have proved that L maps continuously the space $\mathbf{P}^0 \left(e^{-\gamma \cdot t} \right)$ into $\mathbf{E}_0(\gamma)$.

2). We show now that there exists L^{-1} , which is defined on $\mathbf{E}_0(\gamma)$ as a whole and maps continuously $\mathbf{E}_0(\gamma)$ into $\mathbf{P}^0 \left(e^{-\gamma \cdot t} \right)$.

In order to prove it, we need a slight modification of Cauchy integral formula in several complex variables (see, for example, Shabat B.V. (1985), Henkin G.M., Leiterer J. (1984)).

Lemma 2.1.2 *Suppose* $\sigma_j > \sigma_j^0 > \gamma_j$ *, for each* $j = 1, ..., m$ *and* $U(p) \in \mathbf{E}_0(\gamma)$ *, then*

$$U(p) \ = \ \frac{1}{(2\pi i)^m} \int_{\sigma_1^0 - i\infty}^{\sigma_1^0 + i\infty} \cdots \int_{\sigma_m^0 - i\infty}^{\sigma_m^0 + i\infty} \frac{U(\zeta)}{(p_1 - \zeta_1) \cdots (p_m - \zeta_m)} \, d\zeta_1 \cdots d\zeta_m \ . \quad \textit{(2.1.10)}$$

Proof. We start with Cauchy integral formula in a polycircle domain (see Shabat B.V. (1985), Henkin G.M., Leiterer J. (1984)) $\Gamma^N = \Gamma_1^N \times \cdots \times \Gamma_m^N$, where Γ_j^N is a rectangle with the sides $\sigma_j = \sigma_j^0$, $\sigma_j = \sigma_j^1$, $\tau_j = N_j$,

$\tau_j = -N_j$, $\sigma_j^1 > \sigma_j^0$. Integrating over Shilov boundary (i.e. the frame) of Γ^N , we obtain Cauchy formula

$$U(p) \ = \ \frac{1}{(2\pi i)^m} \int_{\Gamma^N} \frac{U(\zeta)}{(\zeta - p)}\, d\zeta \ , \tag{2.1.11}$$

where $d\zeta \ = \ d\zeta_1 \cdots d\zeta_m$, and $(\zeta - p) \ = \ \prod_{j=1}^m (\zeta_j - p_j)$. Integrating (2.1.11) over $N_j \in [N_j', N_j' + 1]$, for the fixed $j \in \{1, ..., m\}$, we obtain the integrands of the two kinds. In those with the integration over $[\sigma_j^0 \pm iN_j, \sigma_j^1 \pm iN_j]$ we obtain integrals of the form

$$\mp \int_{N_j'}^{N_j'+1} dN_j \int_{\Gamma'} \frac{d\zeta'}{(\zeta' - p')} \int_{\sigma_j^0}^{\sigma_j^1} \frac{U(..., \sigma_j \pm iN_j, ...)\, d\sigma_j}{p_j - (\sigma_j \pm iN_j)} \ , \tag{2.1.12}$$

where the integral over ζ' denotes the rest integrations. Changing the order of integration in (2.1.12) and applying Schwartz inequality, we obtain in the view of (2.1.5) that this term vanishes as $N_j' \to \infty$. Consequently, after integration over all N_j , only the terms with the integrations over $[\sigma_j^0 + iN_j, \sigma_j^0 - iN_j]$ or over $[\sigma_j^1 - iN_j, \sigma_j^1 + iN_j]$ (before passing to the limit as $N_j' \to \infty$) are left in the right-hand side of (2.1.11). Finally, after passing to the limits as all $N_j' \to \infty$ we obtain

$$\frac{1}{(2\pi i)^m} \int_{\sigma_1^{0\vee 1} \mp i\infty}^{\sigma_1^{0\vee 1} \pm i\infty} \cdots \int_{\sigma_m^{0\vee 1} \mp i\infty}^{\sigma_m^{0\vee 1} \pm i\infty} \frac{U(\zeta)\, d\zeta}{(\zeta - p)} \ ,$$

where for $\sigma_j = \sigma_j^0$ we integrate over $(\sigma_j^0 + i\infty, \sigma_j^0 - i\infty)$ and for $\sigma_j = \sigma_j^1$ respectively over $(\sigma_j^1 - i\infty, \sigma_j^1 + i\infty)$. If we have at least one integration over $(\sigma_j^1 - i\infty, \sigma_j^1 + i\infty)$, we again estimate the appropriate integral with the help of Schwartz inequality and, in the view of (2.1.5) we see that, this term vanishes as $\sigma_j^1 \to \infty$ in the same way as the integral

$$\int_{\sigma_j^1 - i\infty}^{\sigma_j^1 + i\infty} \frac{d\tau_j}{|\sigma_j^1 + i\tau_j - p_j|^2} \ .$$

Finally, the only term which is left in (2.1.11) after the procedures of passing to the limits as $\sigma_j^1 \to \infty$ for all j , is the one with only σ_j^0 left:

$$U(p) \ = \ \frac{1}{(2\pi i)^m} \int_{\sigma_1^0 + i\infty}^{\sigma_1^0 - i\infty} \cdots \int_{\sigma_m^0 + i\infty}^{\sigma_m^0 - i\infty} \frac{U(\zeta)\, d\zeta}{(\zeta - p)} \ .$$

Changing the order of the limits we obtain (2.1.10). ▷

Estimating (2.1.10) we easily obtain the following

Lemma 2.1.3 *Let* $U(p) \in \mathbf{E}_0(\gamma)$ *. Then*

$$U(p) \to 0 \quad \text{as} \quad |\tau| \to \infty$$

uniformly in σ *as* $\sigma_j \geq \sigma_j^1$ *for arbitrary* $\sigma_j^1 > \gamma_j$, $\forall j \in \{1, ..., m\}$.

The last lemma gives the opportunity to consider the operator M such that

$$MU(p) \;=\; \frac{1}{(2\pi i)^m} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \cdots \int_{\sigma_m - i\infty}^{\sigma_m + i\infty} U(p)\, e^{p\cdot t}\, dp \tag{2.1.13}$$

$$= \; \frac{1}{(2\pi i)^m} \operatorname*{l.i.m.}_{N\to\infty} \int_{\sigma_1 - iN}^{\sigma_1 + iN} \cdots \int_{\sigma_m - iN}^{\sigma_m + iN} U(p)\, e^{p\cdot t}\, dp \;,$$

where $\sigma_j > \gamma_j$, for all $j \in \{1, ..., m\}$. Operator M is defined on the whole $\mathbf{E}_0(\gamma)$, is independent of the choice of $\sigma_j > \gamma_j$ and maps continuously from $\mathbf{E}_0(\gamma)$ into $\mathbf{P}^0\big(e^{-\gamma\cdot t}\big)$ in the view of Parseval inequality (2.1.8). Further, $M = L^{-1}$ on smooth functions and thus, by continuity, we obtain from (2.1.8) that L and M are mutually inverse and provide the isomorphism between $\mathbf{P}^0\big(e^{-\gamma\cdot t}\big)$ and $\mathbf{E}_0(\gamma)$.

3). Whence the case with $\alpha = 0$ is proved, the case $\alpha > 0$ follows in a standard way (see Agranovich M.S., Vishik M.I. (1964)). $\triangleright$

2.1.2

We are ready now to define main functional spaces in "Polycylinder" domains . Let $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma \in C^\infty$, and let $I = \prod_{j=1}^m [T_j, T_j']$, where $-\infty \leq T_j < T_j' \leq +\infty$. Denote by Ω_I the "polycylinder" domain $G \times I$, and by Ω_I' its "side" surface $\Gamma \times I$.

We use the notation $H^{k,\vec{\alpha}}\big(\Omega_I\big)$ for a completion of the space $C_0^\infty\big(\Omega_I\big)$ with respect to the norm

$$\big\|u(x,t)\big\|_{k,\vec{\alpha}}^2 \;=\; \int_I \big\|u(x,t)\big\|_{k,G}^2\, dt \;+\; \int_G \big\|u(x,t)\big\|_{\vec{\alpha},I}^2\, dx \;, \tag{2.1.14}$$

where $dt = dt_1 ... dt_m$, and $\|\cdot\|_{\vec{\alpha},I}$ is the norm in $H^{\vec{\alpha}}(I)$ (see (1.1.24)).

Also, the norm in $H^{k,\vec{\alpha}}\big(\Omega_I'\big)$ is defined by

$$\big\|u(x,t)\big\|_{k,\vec{\alpha}}'^{\,2} \;=\; \int_I \big\|u(x,t)\big\|_{k,\Gamma}'^{\,2}\, dt \;+\; \sum_\beta \int_\Gamma \big\|\varphi_\beta(x')u(x',t)\big\|_{\vec{\alpha},I}^2\, dx' \;, \tag{2.1.15}$$

where $\{\varphi_\beta\}$ with the property $\operatorname{supp}\varphi_\beta \bigcap \Gamma \neq \emptyset$ are the functions from the partition of unity on G defining the norm $\|\cdot\|_{k,\Gamma}'$ (see (1.1.19)). Certainly, these spaces are the particular cases of anisotropic Sobolev spaces, being considered in (Slobodeckij L.N. (1958)). The down-stated Remark contains their properties in the special case of $\alpha_j = \frac{l}{2b_j}$.

Remark 2.1.4 *Hereafter we consider the spaces*

$H^{l,\frac{l}{2\vec{b}}}(\Omega_I) \ := \ H^{l,\{\frac{l}{2b_1},...,\frac{l}{2b_m}\}}(\Omega_I)$ and $H^{\lambda,\frac{\lambda}{2\vec{b}}}(\Omega_I') \ := \ H^{\lambda,\{\frac{\lambda}{2b_1},...,\frac{\lambda}{2b_m}\}}(\Omega_I')$, *(2.1.16)*

$(l+1, \lambda+1/2, b_j \ \in \ \mathbb{N}$), *with the following properties (see Slobodeckij L.N. (1958)):*

1). Any given function $u \in H^{l,\frac{l}{2\vec{b}}}(\Omega_I)$ *has generalized derivatives*

$$v(x,t) \ = \ D_x^{\alpha}\partial_t^{\beta}u(x,t) \ \in \ H^{r,\frac{r}{2\vec{b}}}(\Omega_I) \ , \qquad (2.1.17)$$

where $r = l - |\alpha| - 2b\cdot\beta \geq 0$, *and*

$$\|v\|_{r,\frac{r}{2\vec{b}}} \ \leq \ C\,\|u\|_{l,\frac{l}{2\vec{b}}} \ . \qquad (2.1.18)$$

2). For $r \geq 1$ *the function* $v(x,t)$ *has boundary values* $v(x',t)$ *belonging to* $H^{\lambda,\frac{\lambda}{2\vec{b}}}(\Omega_I')$ *with* $\lambda = r - 1/2$ *and*

$$\|v(x',t)\|'_{\lambda,\frac{\lambda}{2\vec{b}}} \ \leq \ C\,\|v(x,t)\|_{r,\frac{r}{2\vec{b}}} \ . \qquad (2.1.19)$$

3). For $r \geq b_j$ *there exists a trace* $v(x,t[j],t_j^0)$ *of the function* $v(x,t)$ *on the hyperplane* $\Omega_I \bigcap \{t_j = t_j^0\}$, *where* $t_j^0 \in [T_j, T_j']$. *In this case* $v(x,t[j],t_j^0)$ *belongs to* $H^{r-b_j,\frac{r-b_j}{2\vec{b}}[j]}(\Omega_I \bigcap \{t_j = t_j^0\})$ *and*

$$\|v(x,t[j],t_j^0)\|_{r-b_j,\frac{r-b_j}{2\vec{b}}[j]} \ \leq \ C\,\|v(x,t)\|_{r,\frac{r}{2\vec{b}}} \ , \qquad (2.1.20)$$

where we denote by $(t[j],t_j^0)$ *the vector* $(t_1,...,t_{j-1},t_j^0,t_{j+1},...,t_m)$ *and by* $\frac{l}{2\vec{b}}[j]$ *the vector* $\left(\frac{l}{2b_1},...,\frac{l}{2b_{j-1}},\frac{l}{2b_{j+1}},...,\frac{l}{2b_m}\right)$.

In particular, if $t_j = 0 \in [T_j, T_j']$, *the following initial conditions are defined for a function* $u(x,t) \in H^{l,\frac{l}{2\vec{b}}}(\Omega_I)$:

$$u\Big|_{t_j=0} \ = \ \phi_{j,0}(x,t[j]) \ , \quad \frac{\partial u}{\partial t_j}\Big|_{t_j=0} \ = \ \phi_{j,1}(x,t[j]) \ ,...,$$

$$\frac{\partial^{\kappa_j-1}u}{\partial t_j^{\kappa_j-1}}\Big|_{t_j=0} \ = \ \phi_{j,\kappa_j}(x,t[j]) \ , \qquad (2.1.21)$$

where

$$\kappa_j \ = \ \begin{cases} \left[\frac{l}{2b_j}\right] \ , & \text{if } \frac{l}{2b_j} - \left[\frac{l}{2b_j}\right] \ \leq \ 1/2 \ , \\ \left[\frac{l}{2b_j}\right] + 1 \ , & \text{if } \frac{l}{2b_j} - \left[\frac{l}{2b_j}\right] \ > \ 1/2 \ . \end{cases}$$

In this case

$$\phi_{j,\nu_j}(x,t[j]) \ \in \ H^{l_{\nu,j},\frac{l_{\nu,j}}{2\vec{b}}[j]}(\Omega_I \bigcap \{t_j = 0\}) \ ,$$

where $l_{\nu,j} = l - 2b_j\nu_j - b_j > 0$, $0 \le \nu_j \le \kappa_j - 1$ *and*

$$\|\phi_{j,\nu_j}\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\bar b}[j]} \;\le\; C\,\|u\|_{l,\frac{l}{2\bar b}} \;. \qquad (2.1.22)$$

4). Conversely, for a given set of functions

$$\phi_{j,\nu_j}(x,t[j]) \;\in\; H^{l_{\nu,j},\frac{l_{\nu,j}}{2\bar b}[j]}\big(\Omega_I \bigcap \{t_j = 0\}\big) \;,$$

where $0 \le \nu_j \le \kappa_j - 1$, *there exists* $u \in H^{l,\frac{l}{2\bar b}}(\Omega_I)$ *such that (2.1.21) are valid and*

$$\|u\|_{l,\frac{l}{2\bar b}} \;\le\; C\sum_{j=1}^{m}\sum_{\nu_j=0}^{\kappa_j-1} \|\phi_{j,\nu_j}\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\bar b}[j]} \;. \qquad (2.1.23)$$

5). Suppose $I = \Pi_{j=1}^m [a_j,b_j]$, *where* $b_j < +\infty$ *and* $J = \Pi_{j=1}^m [a_j,+\infty]$. *Then there exists an extension operator* L *(resp.* L' *) which is bounded from* $H^{l,\frac{l}{2\bar b}}(\Omega_I)$ *to* $H^{l,\frac{l}{2\bar b}}(\Omega_J)$ *(resp.* $H^{\lambda,\frac{\lambda}{2\bar b}}(\Omega'_I)$ *to* $H^{\lambda,\frac{\lambda}{2\bar b}}(\Omega'_J)$*) . We construct the operator* L *by induction over the number of times; for the one-dimensional time it is described in (Agranovich M.S., Vishik M.I. (1964)).*

In the same way as for spaces $\mathbf{P}^{\alpha}(e^{-\gamma\cdot t})$ and $\mathbf{E}_{\alpha}(\gamma)$ we introduce the spaces $\mathbf{P}^{l,\frac{l}{2\bar b}}(e^{-\gamma\cdot t})$ and $\mathbf{E}_{l,\frac{l}{2\bar b}}(\gamma)$ in "Polycylinder" domains (for simplicity we bound ourselves with the case $l \in \mathbb{N}$).

By definition, $u \in \mathbf{P}^{l,\frac{l}{2\bar b}}(e^{-\gamma\cdot t})$, if $u(x,t)$ is defined in the domain $\Omega = G \times \mathbb{R}^m$, $(I = \mathbb{R}^m)$, $u = 0$ if at least one of t_j is negative and $e^{-\gamma\cdot t}u \in H^{l,\frac{l}{2\bar b}}(\Omega_I)$. We take the value $\left\|e^{-\gamma\cdot t}u\right\|_{l,\frac{l}{2\bar b}}$ for the norm in $\mathbf{P}^{l,\frac{l}{2\bar b}}(e^{-\gamma\cdot t})$.

Equivalently, $u(x,t) \in \mathbf{P}^{l,\frac{l}{2\bar b}}(e^{-\gamma\cdot t})$ if, for almost all $x \in G$, the function $u(x,t) \in \mathbf{P}^{\frac{l}{2\bar b}}(e^{-\gamma\cdot t})$ relatively to t and, for almost all $t \in (\bar{\mathbb{R}}_+)^m$, it belongs to $H^l(G)$ relatively to x and, finally, the norm $\left\|e^{-\gamma\cdot t}u(x,t)\right\|_{l,\frac{l}{2\bar b}}$ is finite.

Suppose $\lambda + 1/2 \ge 1$ is an integer. We define the space $\mathbf{P'}^{\lambda,\frac{\lambda}{2\bar b}}(e^{-\gamma\cdot t})$ as a space of functions $u(x',t)$ defined in the "Polycylinder" $\Omega' = \Gamma \times \mathbb{R}^m$, equal to zero if at least one of t_j is negative and such that $e^{-\gamma\cdot t}u \in H^{\lambda,\frac{\lambda}{2\bar b}}(\Omega'_I)$.

Now let $l \in \mathbb{N}$ and $\gamma_j \ge 0$. Denote by $\mathbf{E}_{l,\frac{l}{2\bar b}}(\gamma)$ a set of functions $U(x,p)$ defined for almost all $x \in G$ and $p :\ \mathrm{Re}\,p_j \;\ge\; \gamma_j$, $j = 1,...,m$, and possessing the following properties:

1). For all $p = \sigma + i\tau$ with $\mathrm{Re}\,p_j \;>\; \gamma_j$, $j = 1,...,m$ (i.e. $p \in \Pi_\gamma$), and for almost all p on hyperplanes $\{\sigma_j = \gamma_j\}$, $j = 1,...,m$ (i.e. $p \in \partial\Pi_\gamma$), we have $U(\cdot,p) \in H^l(G)$.

2). For almost all $x \in G$ we have $U(x,\cdot) \in \mathbf{E}_{\frac{l}{2\bar b}}(\gamma)$, and its generalized derivatives $D_x^\alpha U$ belong to $\mathbf{E}_0(\gamma)$ for $|\alpha| \le l$ (in particular, it means that

$U(x,p)$ is holomorphic in domain $\Pi_\gamma = \{p \in \mathbb{C}^m : \operatorname{Re} p_j >, \gamma_j \,,\ j=1,...,m\}$ for these x and has limit boundary values almost everywhere on $\partial\Pi_\gamma$).

3). The norm in $\mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$ is defined by the formula

$$\| |U(x,p)| \|_{l,\frac{l}{2\vec{b}}} = \int_{\Pi_{j=1}^m \{\sigma_j=\gamma_j\}} \left[\|U\|_{l,G}^2 + \sum_{i=1}^m |p_i|^{\frac{l}{b_i}} \|U\|_{0,G}^2 \right] d\tau \ , \qquad (2.1.24)$$

where $p_j = \sigma_j + i\tau_j$.

For $\lambda = [\lambda] + 1/2$ the space $\mathbf{E}'_{\lambda,\frac{\lambda}{2\vec{b}}}(\gamma)$ is defined as the space of functions $U(x',p)$, $x' \in \Gamma$ with the properties analogous to 1) -3).

The following Proposition is the analog of Proposition 2.1.1.

Proposition 2.1.5 *Suppose $\gamma \in \mathbb{R}^m$ is a vector with positive coordinates. Then Laplace transformation L (see (2.1.6)) defines a one-to-one mutually continuous correspondence of the spaces $\mathbf{P}^{l,\frac{l}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right)$ and $\mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$ $(l \in \mathbb{N} \cup \{0\})$, and the spaces $\mathbf{P}'^{\,\lambda,\frac{\lambda}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right)$ and $\mathbf{E}'_{\lambda,\frac{\lambda}{2\vec{b}}}(\gamma)$ $(\lambda + 1/2 \in \mathbb{N})$.*

Proof. Suppose $u(x,t) \in \mathbf{P}^{l,\frac{l}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right)$ then, by definition, for almost all $x \in G$, we have $u(x,t) \in \mathbf{P}^{\frac{l}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right)$ in t . Hence, it follows from Proposition 2.1.1 that, for these x , the function

$$U(x,p) = Lu(x,t) \in \mathbf{E}_{\frac{l}{2\vec{b}}}(\gamma)$$

is defined (for the rest x it may be defined arbitrarily). It follows from (2.1.9) that, for almost all $x \in G$,

$$\sup_{\sigma\in\Pi_\gamma} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left|U(x,\sigma+i\tau)\right|^2 d\tau = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left|U(x,\gamma+i\tau)\right|^2 d\tau$$

$$= (2\pi)^m \left\|e^{-\gamma\cdot t}u(x,t)\right\|_{0,t}^2 \ ,$$

where the lower index $_{0,t}$ in the last norm denotes the L_2-norm in t . Integrating over x we obtain

$$\sup_{\sigma\in\Pi_\gamma} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \|U\|_{0,G}^2 \, d\tau = (2\pi)^m \int_G \left\|e^{-\gamma\cdot t}u(x,t)\right\|_{0,t}^2 dx \ , \qquad (2.1.25)$$

and thus $\|U(x,\sigma+i\tau)\|_{0,G}^2 < +\infty$ for almost all $\tau \in \mathbb{R}^m$ on the planes $\{\sigma = \sigma_0 \geq \gamma\}$.

If, for all j , $\sigma_j = \operatorname{Re} p_j > \gamma_j$, i.e. $p \in \operatorname{int}\Pi_\gamma$ then

$$\left|U(x,p)\right|^2 \;=\; \left|\int_{(\mathbb{R}_+)^m} e^{-p\cdot t}u(x,t)\,dt\right|^2 \;\le$$

$$\le\; \int_{(\mathbb{R}_+)^m} e^{-2\gamma\cdot t}|u|^2\,dt \;\int_{(\mathbb{R}_+)^m} e^{-2(\sigma-\gamma)\cdot t}\,dt$$

for all $x \in G$, or, after integrating over x ,

$$\|U(x,p)\|^2_{0,G} \;=\; \frac{1}{2^m\prod_{j=1}^m(\sigma_j-\gamma_j)}\int_{(\mathbb{R}_+)^m} e^{-\gamma\cdot t}\|u\|^2_{0,G}\,dt \;<\; +\infty$$

for $\sigma_j > \gamma_j$, $j = 1,...,m.$ And thus $\|U(x,\sigma+i\tau)\|^2_{0,G} < +\infty$ for all $p \in \operatorname{int}\Pi_\gamma$.

In the same way we obtain corresponding estimates for generalized derivatives of $U(x,p)$, and the conditions of items 1) and 2) of the definition of spaces $\mathbf{E}_{l,\frac{l}{2\bar b}}(\gamma)$ are satisfied. As for item 3), it follows from the following estimates for U and its generalized derivatives being derived from Proposition 2.1.1:

$$\int_{\sigma_1=\gamma_1}\cdots\int_{\sigma_m=\gamma_m}\left(1+|p_1|^{\frac{l}{2b_1}}+\cdots+|p_m|^{\frac{l}{2b_m}}\right)U(x,p)\,d\tau_1\cdots d\tau_m \;\le$$

$$\le\; C\left\|e^{-\gamma\cdot t}u(x,t)\right\|^2_{\frac{l}{2\bar b},t}$$

for almost all $x \in G$ and the constant C being independent of x and u . Integrating over x we get

$$\int_{\sigma_1=\gamma_1}\cdots\int_{\sigma_m=\gamma_m}\left(1+|p_1|^{\frac{l}{2b_1}}+\cdots+|p_m|^{\frac{l}{2b_m}}\right)\|U\|_{0,G}\,d\tau_1\cdots d\tau_m$$

$$\le\; C\int_G \left\|e^{-\gamma\cdot t}u(x,t)\right\|^2_{\frac{l}{2\bar b},t}dx\;.$$

Thus L maps continuously the space $\mathbf{P}^{l,\frac{l}{2\bar b}}\left(e^{-\gamma\cdot t}\right)$ into $\mathbf{E}_{l,\frac{l}{2\bar b}}(\gamma)$. The proof of the fact that it is indeed the map "onto" and the existence of the continuous inverse L^{-1} can be obtained integrating over x the corresponding estimates in the proof of Proposition 2.1.1.

Whence the proof is completed for the spaces $\mathbf{P}^{l,\frac{l}{2\bar b}}\left(e^{-\gamma\cdot t}\right)$ and $\mathbf{E}_{l,\frac{l}{2\bar b}}(\gamma)$, the proof for the spaces with primes does not differ from the one - dimensional time case (see Agranovich M.S., Vishik M.I. (1964)). $\triangleright$

We conclude this Section with some remarks about the properties of the above spaces which are used below.

Remark 2.1.6 *The spaces* $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$, $\mathbf{E}_{\alpha}(\gamma)$, $\mathbf{P}^{l,\frac{l}{2b}}\big(e^{-\gamma\cdot t}\big)$, $\mathbf{E}_{l,\frac{l}{2b}}(\gamma)$ *are complete.*

Proof. A). Indeed, the completeness of $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$ is a straight - forward consequence of the completeness of $H^{\vec{\alpha}}(\mathbb{R}^m)$. The functions from $C_0^{\infty}(\mathbb{R}^m)$ with supports in $(\mathbb{R}_+)^m$ form an everywhere dense set in $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$.

B). We obtain the completeness of $\mathbf{E}_{\alpha}(\gamma)$ as a consequence of Proposition 2.1.1.

C). The completeness of $\mathbf{P}^{l,\frac{l}{2b}}\big(e^{-\gamma\cdot t}\big)$ is stated analogously to the completeness of $\mathbf{P}^{\alpha}\big(e^{-\gamma\cdot t}\big)$ and the functions from $C_0^{\infty}(\Omega)$ (resp. $C_0^{\infty}(\Omega')$ for $\mathbf{P'}^{\lambda,\frac{\lambda}{2b}}\big(e^{-\gamma\cdot t}\big)$, where $\lambda+1/2\in\mathbb{Z}$) with supports in $(\mathbb{R}_+)^m$ with respect to t form an everywhere dense set in $\mathbf{P}^{l,\frac{l}{2b}}\big(e^{-\gamma\cdot t}\big)$.

D). Now it follows from Proposition 2.1.5 that $\mathbf{E}_{l,\frac{l}{2b}}(\gamma)$ is also complete and the functions from $C_0^{\infty}(G\times\mathbb{C}^m)$ which are entire analytic in $p\in\Pi_\gamma$ for every $x\in\overline{G}$ and belong to the intersection of all $\mathbf{E}_{k,\frac{k}{2b}}(\gamma)$, $k=0,1,...$ (for the spaces with primes we have to change $x\in\overline{G}$ into $x\in\Gamma$ and l into λ) form an everywhere dense set in $\mathbf{E}_{l,\frac{l}{2b}}(\gamma)$. ▷

Remark 2.1.7 *Let* $U\in\mathbf{E}_0(\gamma)$. *Then*

$$U(\sigma+i\tau)\longrightarrow U(\sigma_0+i\tau)$$

in $L_2(\mathbb{R}^m)$ *as* $\sigma\to\sigma_0$, *where some of (or even all) the coordinates of* σ_0 *may be equal to* γ_j .

Proof. It follows from (2.1.7) and (2.1.8) that

$$\int_{\mathbb{R}^m}\big|U(\sigma+i\tau)-U(\sigma_0+i\tau)\big|^2\,d\tau \;=\; (2\pi)^m\int_{(\mathbb{R}_+)^m}\big|e^{-\sigma\cdot t}\;-\;e^{-\sigma_0\cdot t}\big|^2|u(t)|^2\,dt$$

$$=\;(2\pi)^m\sum_{j=1}^{m}\int_{[0,N]^j\times[N,\infty]^{m-j}}\big|e^{-\sigma\cdot t}\;-\;e^{-\sigma_0\cdot t}\big|^2|u(t)|^2\,dt$$

$$\le\; C\sum_{j=1}^{m}\sum_{P(j,m)}\Bigg[\max_{t'\in[0,N]^j}\big|e^{-\sigma'\cdot t'}\;-\;e^{-\sigma_0'\cdot t'}\big|^2\int_{[0,N]^j\times[N,\infty]^{m-j}}e^{-\sigma''\cdot t''}|u(t)|^2\,dt\;+$$

$$+\;\int_{[0,N]^j\times[N,\infty]^{m-j}}e^{-\sigma'\cdot t'}e^{-2\sigma_0''\cdot t''}|u(t)|^2\,dt\Bigg]\,,$$

where $\sigma=(\sigma',\sigma'')$, $t=(t',t'')$, $\sigma',t'\in\mathbb{R}^j$, $\sigma'',t''\in\mathbb{R}^{m-j}$, and the sum over $P(m,j)$ is taken over all combinations of j elements from m . Thus, the second term in the right-hand side of the last inequality may be made infinitely small as $N\to\infty$ and the first vanishes as $\sigma\to\sigma_0$ for fixed N . ▷

Remark 2.1.8 *Now let* $U \in \mathbf{E}_\alpha(\gamma)$ *, then*

$$\|U\|^2_{\mathbf{E}_\alpha(\gamma)} \;=\; \sum_{j=1}^{m} \int \bigl|U(\gamma+i\tau)\bigr|^2 \bigl|\gamma_j + i\tau_j\bigr|^{2\alpha_j}\, d\tau\ . \qquad (2.1.26)$$

Proof follows from (2.1.9) applied to $p_j^{\alpha_j} U$ as $\|U\|_{\mathbf{E}_\alpha(\gamma)} =$
$= \sum_{j=1}^m \|p_j^{\alpha_j} U\|_0$. ▷

In the following Remark we reformulate the properties (2.1.17) - (2.1.23) of spaces $H^{l,\frac{l}{2b}}\bigl(\Omega_I\bigr)$, $H^{\lambda,\frac{\lambda}{2b}}\bigl(\Omega_I'\bigr)$ (see Remark 2.1.4).

Remark 2.1.9 *Let* $u \in \mathbf{P}^{l,\frac{l}{2b}}\bigl(e^{-\gamma\cdot t}\bigr)$ *and* $|\alpha| + 2b\cdot\beta \le l$ *, then*

1). Any given function $u \in \mathbf{P}^{l,\frac{l}{2b}}\bigl(e^{-\gamma\cdot t}\bigr)$ *has generalized derivatives*

$$v(x,t) \;=\; D_x^\alpha \partial_t^\beta u(x,t) \;\in\; \mathbf{P}^{r,\frac{r}{2b}}\bigl(e^{-\gamma\cdot t}\bigr)\ , \qquad (2.1.27)$$

where $r = l - |\alpha| - 2b\cdot\beta \ge 0$ *, and*

$$\|e^{-\gamma\cdot t} v(x,t)\|_{r,\frac{r}{2b}} \;\le\; C\, \|e^{-\gamma\cdot t} u(x,t)\|_{l,\frac{l}{2b}}\ , \qquad (2.1.28)$$

and moreover $\forall\,\alpha:\ |\alpha| \le l$ *, we have* $D_x^\alpha u \in \mathbf{P}^0\bigl(e^{-\gamma\cdot t}\bigr)$ *for almost all* $x \in G$ *.*

2). The same properties are valid for boundary classes $\mathbf{P'}^{\lambda,\frac{\lambda}{2b}}\bigl(e^{-\gamma\cdot t}\bigr)$ *, where* $\lambda \ge 1/2$ *and* $r = \lambda + 1/2$ *. A function* $v(x,t) \in \mathbf{P}^{r,\frac{r}{2b}}\bigl(e^{-\gamma\cdot t}\bigr)$ *has boundary values* $v(x',t)$ *belonging to* $\mathbf{P'}^{\lambda,\frac{\lambda}{2b}}\bigl(e^{-\gamma\cdot t}\bigr)$ *and*

$$\|e^{-\gamma\cdot t} v(x',t)\|'_{\lambda,\frac{\lambda}{2b}} \;\le\; C\, \|e^{-\gamma\cdot t} v(x,t)\|_{r,\frac{r}{2b}}\ . \qquad (2.1.29)$$

The following Remarks concern the properties of spaces $\mathbf{E}_{l,\frac{l}{2b}}(\gamma)$ and $\mathbf{E}'_{\lambda,\frac{\lambda}{2b}}(\gamma)$.

Remark 2.1.10 *Suppose* $U = U(x,p) \in \mathbf{E}_{l,\frac{l}{2b}}(\gamma)$ *, then*

$$\bigl\|\,|U(x,p)|\,\bigr\|_{l,\frac{l}{2b}} \;=\; \int_{\Pi_{j=1}^m\{\sigma_j=\gamma_j\}} \Bigl[\|U\|^2_{l,G} \,+\, \sum_{i=1}^m |p_i|^{\frac{l}{b_i}} \|U\|^2_{0,G}\Bigr]\, d\tau$$

$$=\; \sup_{p^o\in\Pi_\gamma} \int_{\Pi_{j=1}^m\{\sigma_j=\sigma_j^o\}} \Bigl[\|U\|^2_{l,G} \,+\, \sum_{i=1}^m |p_i|^{\frac{l}{b_i}} \|U\|^2_{0,G}\Bigr]\, d\tau\ , \qquad (2.1.30)$$

where $p_j = \sigma_j + i\tau_j$, $p_j^o = \sigma_j^o + i\tau_j^o$ *. Also, this statement is valid for the spaces with primes. The norms in (2.1.30) in this case have to be taken with primes too.*

Proof. As follows from the definition of spaces $\mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$, for almost all x we have $U(x,p) \in \mathbf{E}_{\frac{l}{2\vec{b}}}(\gamma)$ and $D_x^\alpha U(x,p) \in \mathbf{E}_{\vec{0}}(\gamma)$ for $|\alpha| \le l$. Thus

$$\int_{\Pi_{j=1}^m \{\sigma_j=\gamma_j\}} \sum_{i=1}^m |p_i|^{\frac{l}{b_i}} |U|^2 \, d\tau \;=\; \sup_{p^o \in \Pi_\gamma} \int_{\Pi_{j=1}^m \{\sigma_j=\sigma_j^o\}} \sum_{i=1}^m |p_i|^{\frac{l}{b_i}} |U|^2 \, d\tau$$

(see Remark 2.1.8) and

$$\int_{\Pi_{j=1}^m \{\sigma_j=\gamma_j\}} |D_x^\alpha U|^2 \, d\tau \;=\; \sup_{p^o \in \Pi_\gamma} \int_{\Pi_{j=1}^m \{\sigma_j=\sigma_j^o\}} |D_x^\alpha U|^2 \, d\tau \ .$$

Integration over $x \in G$ ends the proof. ▷

Remark 2.1.11 *As follows from the proof of Proposition 2.1.5 and Remark 2.1.9 the Laplace image of the generalized derivative in* t_j *is the multiplication by* p_j *and the operations of generalized derivatives in* x *and traces on* Γ *commute with Laplace transform (2.1.6). We conclude that for* $U \in \mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$ *with* $|\alpha| + 2b\cdot\beta \le l$ *and* $l \in \mathbb{Z}$,

$$V(x,p) \;=\; p^\beta D_x^\alpha U(x,p) \;\in\; \mathbf{E}_{r,\frac{r}{2\vec{b}}}(\gamma) \ ,$$

where $r = l - |\alpha| - 2b\cdot\beta \ge 0$, *and*

$$\|V(x,p)\|_{r,\frac{r}{2\vec{b}}} \;\le\; C\,\|U(x,p)\|_{l,\frac{l}{2\vec{b}}} \ ,$$

moreover,

$$\|V(x',p)\|'_{\lambda,\frac{\lambda}{2\vec{b}}} \;\le\; C\,\|V(x,p)\|_{r,\frac{r}{2\vec{b}}} \ ,$$

where $\lambda = r - 1/2$ *and* $r \ge 1$.

Remark 2.1.12 *Suppose* $U = U(x,p) \in \mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$, *then*

A). For $\operatorname{Re} p_j > \gamma_j$ *and* $\forall\, j \in 1,...,m$

$$\left\|U(x,p+\Delta p) - U(x,p)\right\|_{l,G} \longrightarrow 0 \quad \text{as} \quad \Delta p \to 0 \ . \tag{2.1.31}$$

B). Suppose there exists at least one j_0 *such that in* $p = \sigma^o + i\tau$ *we have* $\sigma_{j_0}^o = \operatorname{Re} p_{j_0}^o = \gamma_{j_0}$, *then there exists a sequence* $\sigma^{(n)} \to \sigma^o$ *such that*

$$\left\|U(x,\sigma^{(n)} + i\tau) - U(x,\sigma^o + i\tau)\right\|_{l,G} \longrightarrow 0 \quad \text{as} \quad n \to \infty \ . \tag{2.1.32}$$

Proof. A). In the view of Remark 2.1.9 it is sufficient to consider the case $l = 0$. Thus, suppose $U = Lu$, where $U \in \mathbf{E}_{0,0}(\gamma)$ and $u \in \mathbf{P}^{0,0}(e^{-\gamma\cdot t})$. Then

$$\big|U(x,p+\Delta p)-U(x,p)\big|^2 \;=\; \Big|\int_{(\mathbb{R}_+)^m}\Big(e^{-(p+\Delta p)\cdot t}\;-\;e^{-p\cdot t}\Big)u(x,t)\,dt\Big|^2$$

$$\leq\;\int_{(\mathbb{R}_+)^m} e^{-2\gamma\cdot t}|u(x,t)|^2\,dt\;\int_{(\mathbb{R}_+)^m}\Big|e^{-(p+\Delta p-\gamma)\cdot t}\;-\;e^{-(p-\gamma)\cdot t}\Big|^2\,dt$$

for almost all $x \in G$. Integrating over x we obtain

$$\big\|U(x,p+\Delta p)-U(x,p)\big\|^2_{l,G}$$

$$\leq\;\Big\|e^{-\gamma\cdot t}u(x,t)\Big\|^2_{0,0}\;\int_{(\mathbb{R}_+)^m} e^{-2(\sigma-\gamma)\cdot t}\big|e^{-\Delta p\cdot t}\;-\;1\big|\,dt\;,$$

where, as usual, $\sigma = \mathrm{Re}\,p$ and the last integral vanishes as $\Delta p \to 0$.

B). From the proof of Remark 2.1.7 we obtain that the following inequality holds for a vector σ with the coordinates $\sigma_j \geq \gamma_j$:

$$\int \big\|U(x,\sigma+i\tau)-U(x,\gamma+i\tau)\big\|^2_{0,G}\,d\tau$$

$$\leq\;(2\pi)^m\int e^{-2\gamma\cdot t}\|u(x,t)\|^2_{0,G}\,\big|1-e^{-(\sigma-\gamma)\cdot t}\big|^2\,dt\;,$$

where the last integral evidently vanishes as $\sigma \to \gamma$, and the existence of the desired sequence $\{\sigma^{(n)}\}$ follows. $\triangleright$

2.2 Polyparabolic mixed problems in the spaces $\mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$

We consider in a "polycylinder" domain $\Omega_I \;=\; G\times\prod_{j=1}^2(0,T_j)$, where $0 < T_j \leq \infty$, the equation

$$A\big(x,D_x,\frac{\partial}{\partial t}\big)\,u(x,t)\;=\;f(x,t) \qquad (2.2.1)$$

with boundary conditions

$$B_\mu\big(x,D_x,\frac{\partial}{\partial t}\big)u(x,t)\big|_{x=x'}\;=\;g_\mu(x',t) \qquad (2.2.2)$$

on its "side surface" $\Omega_I' \;=\; \Gamma\times\prod_{j=1}^2(0,T_j)$, where $(x',t)\in\Omega_I'$, $\mu = 1,...,k$, A and B_μ are differential operators with coefficients being C^∞ - functions in

$x \in \overline{G}$, and $\operatorname{ord} A = 2k$, $\operatorname{ord} B_\mu = k_\mu$ $(\mu = 1, ..., k)$. As usual, the order of an operator is defined as the highest of the orders of its terms and the order of the term

$$D_x^\alpha \partial_t^\beta u = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} \partial_{t_1}^{\beta_1} \cdots \partial_{t_m}^{\beta_m} u$$

is computed in the view of the rule $|\alpha| + 2b \cdot \beta = \alpha_1 + \cdots + \alpha_n + 2(b_1\beta_1 + \cdots + b_m\beta_m)$. Here and further $b_j \in \{1, ..., k\}$ are the fixed integer numbers being divisors of k . Vector $2\vec{b}$ is called a "polyparabolic" weight of the problem.

We also add initial conditions to the equations (2.2.1) - (2.2.2):

$$\frac{\partial^{\nu_j} u}{\partial t_j^{\nu_j}}\Big|_{t_j=0} = \phi_{j,\nu_j}(x, t[j]) , \quad \text{for} \quad \nu_j = 0, ..., \kappa_j - 1 , \tag{2.2.3}$$

where κ_j is the maximal order of the t_j-derivative in the expression for operator $A\big(x, D_x, \frac{\partial}{\partial t}\big)$, $\kappa_j \le k/b_j$ for $j = 1, ..., m$. Suppose for the time being that $\phi_{j,\nu_j} = 0$ and $T_j = +\infty$ for $j = 1, ..., m$. Then Laplace transformation in T_j transfers the problem (2.2.1) - (2.2.3) into the "stationary" problem

$$A(x, D_x, p)\, U(x,p) = F(x,p) , \quad x \in G , \tag{2.2.4}$$

$$B_\mu(x, D_x, p) U(x,p)\big|_{x=x'} = G_\mu(x', p) , \quad x' \in \Gamma , \tag{2.2.5}$$

where $\mu = 1, ..., k$. Putting $p_j = q_j^{2b_j}$, $j = 1, ..., m$, we see that while p_j varies in the right half-plane $\operatorname{Re} p_j \ge 0$, q_j varies in its turn in the angle

$$Q_j = \{q_j \in \mathbb{C} : |\arg q_j| \le \frac{\pi}{4b_j}\} . \tag{2.2.6}$$

Definition 2.2.1 *The problem (2.2.1) - (2.2.3) is called "polyparabolic" if the problem (2.2.4) - (2.2.5) with* p_j *substituted by* $q_j^{2b_j}$ *for all* $j = 1, ..., m$ *satisfies in the cone* $Q = Q_1 \times \cdots \times Q_m$ *Conditions 1.3.2 - 1.3.3. In this case we call the problem (2.2.4) - (2.2.5) "semibounded" (following Agranovich M.S., Vishik M.I. (1964)).*

In order to illustrate this definition we continue our Examples 1.3.4, 1.3.5.

Example 2.2.2 *Let* $A_1(x,D)$ *and* $A_2(x,D)$ $(\operatorname{ord} A_j = 2k_j$, $j = 1,2)$ *be two elliptic operators with positive real leading parts. Consider an operator*

$$\Big(\frac{\partial}{\partial t_1} + A_1(x,D)\Big)\Big(\frac{\partial}{\partial t_1} + A_2(x,D)\Big)$$

with Dirichlet boundary conditions

$$B_\nu = \Big(-i\frac{\partial}{\partial \vec{n}}\Big)^{\nu - 1} , \quad \nu = 1, ..., k_1 + k_2,$$

and zero initial conditions (2.2.3).

This problem is polyparabolic as the corresponding problem (2.2.4)-(2.2.5) (with p_j *replaced by* $q_j^{2k_j}$ *for* $j=1,2$ *) satisfies Conditions 1.3.2 - 1.3.3 in the cone* $Q' = Q'_1 \times Q'_2$ *, where* $Q'_j = \left\{ q_j \in \mathbb{C} \,:\, |\arg q_j| \le \frac{\pi-\varepsilon}{2k_j} \right\}$ *, which includes the cone* $Q = Q_1 \times Q_2$ *, where* $Q_j = \left\{ q_j \in \mathbb{C} \,:\, |\arg q_j| \le \frac{\pi}{4k_j} \right\}$ *, for sufficiently small* ε *.*

Example 2.2.3 *Let* $A(x,D)$ *be an elliptic operator* ($\mathrm{ord} A = 4k$) *with a positive real leading part. Consider an operator*

$$\frac{\partial^2}{\partial t_1 \partial t_2} \;+\; A(x,D)$$

with Dirichlet boundary conditions

$$B_\nu \;=\; \left(-i\frac{\partial}{\partial \vec{n}}\right)^{\nu-1} \quad , \quad \nu = 1, ..., 2k,$$

and zero initial conditions (2.2.3).

This problem is not polyparabolic *in the sense of Definition 2.2.1. Indeed, after substituting of* p_j *by* q_j^{2k} $(j=1,2)$ *in the corresponding system (2.2.4) - (2.2.5) we obtain that, if* p_j *varies in the right half-plane* $\mathrm{Re}\, p_j \ge 0$ *,* q_j *varies in its turn in the angle* $Q_j = \left\{ q_j \in \mathbb{C} \,:\, |\arg q_j| \le \frac{\pi}{4k} \right\}$ *. And thus the ellipticity condition (Condition 1.3.2) is violated because our problem is elliptic only in the cone* $Q' = Q'_1 \times Q'_2$ *, where* $Q'_j = \left\{ |\arg q_j| \le \frac{\pi-\varepsilon}{4k} \right\}$ *.*

Remark 2.2.4 *Note that in terms of parameters* p_j *the parameter set* $\left\{ q \in \mathbb{C}^n \,:\, q \in M_q^0 \,,\; |||q||| \ge q^{(1)} \right\}$ *(see Remark 1.4.4 and Proposition 1.3.7) can be written in the form*

$$\left\{ p \in \mathbb{C}^n \,:\, (\mathrm{Re}\, p_j)^{\frac{1}{2b_j}} \;\ge\; q^{(0)} \,,\; \sum_{j=1}^{m} (\mathrm{Re}\, p_j)^{\frac{1}{2b_j}} \;\ge\; q^{(1)} \right\} . \tag{2.2.7}$$

In this section we consider "polyparabolic" mixed problems with $T_j = +\infty$ for $j = 1, ..., m$ and with zero initial conditions (2.2.3), i.e. in the spaces $\mathbf{P}^{l,\frac{l}{2\vec{b}}}\left(e^{-\gamma \cdot t}\right)$. We start with elliptic "semibounded" problems (2.2.4) - (2.2.5) in the spaces $\mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$.

Theorem 2.2.5 *Let* $l \in \mathbb{Z}$ *and* $l \ge l_0 = \max(2k, k_\mu + 1)$ *. Then there exists a vector* $\gamma \,:\, \gamma_j > 0$ *, such that the "semibounded" boundary value problem (2.2.4) - (2.2.5) possesses the unique solution* $U(x,p) \in \mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$ *for all right-hand sides* $F(x,p) \in \mathbf{E}_{r,\frac{r}{2\vec{b}}}(\gamma)$ *and* $G_\mu(x',p) \in \mathbf{E}'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}}(\gamma)$ *, where* $r = l - 2k$ *and* $\lambda_\mu = l - k_\mu - 1/2$ *. The solution* $U(x,p)$ *satisfies the two-sided a priori estimate*

$$C_1 \, \|U\|_{l,\frac{l}{2\vec{b}}} \;\le\; \|F\|_{r,\frac{r}{2\vec{b}}} \;+\; \sum_{\mu=1}^{k} \|G_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}} \;\le\; C_2 \, \|U\|_{l,\frac{l}{2\vec{b}}} \;, \tag{2.2.8}$$

where the constants C_1 and C_2 are independent of functions.

Remark 2.2.6 *Under a solution $U(x,p) \in \mathbf{E}_{l,\frac{l}{2\bar{b}}}(\gamma)$ we understand a function from this space satisfying the problem for all $p \in \Pi_\gamma$ and for almost all $p \in \partial\Pi_\gamma$.*

Proof (of Theorem 2.2.5). A). We start with the proof of estimate (2.2.8). Let $U \in \mathbf{E}_{l,\frac{l}{2\bar{b}}}(\gamma)$, so for all $p \in \operatorname{int}\Pi_\gamma$ and for almost all $p \in \partial\Pi_\gamma$ we have $U(\cdot,p) \in H^l(G)$ and we are in position to apply Propositions 1.3.1 and 1.3.6. Thus, for an arbitrary γ from the set

$$\mathcal{N} \;=\; \left\{\gamma \in (\mathbb{R}_+)^m \;:\; \gamma_j^{\frac{1}{2b_j}} \;\geq\; q^{(0)}\,,\; \sum_{j=1}^{m}\gamma_j^{\frac{1}{2b_j}} \;\geq\; q^{(1)}\right\}$$

(here $q^{(0)} > 0$ can be chosen infinitely small, and $q^{(1)} > 0$ must be sufficiently large (cf. (2.2.7)), the following estimate is valid

$$\begin{aligned} &C_1\,\|U\|_l^2 \;+\; |||p|||^{\frac{l}{\bar{b}}}\|U\|_0^2 \\ &\leq\; \|F\|_r^2 \;+\; |||p|||^{\frac{r}{\bar{b}}}\|F\|_0^2 \;+\; \sum_{\mu=1}^{k}\left[\|G_\mu\|'^{\,2}_{\lambda_\mu} \;+\; |||p|||^{\frac{\lambda_\mu}{\bar{b}}}\|G_\mu\|'^{\,2}_0\right] \\ &\leq\; C_1\,\|U\|_l^2 \;+\; |||p|||^{\frac{l}{\bar{b}}}\|U\|_0^2\,, \end{aligned} \tag{2.2.9}$$

where $|||p|||^{\frac{y}{\bar{b}}} \;=\; \sum_{j=1}^{m}|p_j|^{\frac{y}{b_j}}$.

Further, according to Remark 2.1.11,

$$F(x,p) \;=\; A(x,D_x,p)\,U(x,p) \;\in\; \mathbf{E}_{r,\frac{r}{2\bar{b}}}(\gamma) \tag{2.2.10}$$

and

$$G_\mu(x',p) \;=\; B_\mu(x,D_x,p)U(x,p)\big|_{x=x'} \;\in\; \mathbf{E}'_{\lambda_\mu,\frac{\lambda_\mu}{2\bar{b}}}(\gamma)\;. \tag{2.2.11}$$

Finally, integrating (2.2.9) over τ in the plane $\{\sigma_1 = \gamma_1, ..., \sigma_m = \gamma_m\}$ we obtain the inequalities which are equivalent to (2.2.8) in the view of Remark 2.1.10. As usual, the uniqueness of the solution is the consequence of (2.2.8).

B). We start the proof of the existence for smooth right-hand sides F and G_μ being described in Remark 2.1.6 D).

We fix $l \geq \max\{2k, k_j+1\}$ and put $l_1 = l + [n/2] + 1$. For $\gamma \in \mathcal{N}$, we obtain in the view of Theorem 1.4.2 that, for all $p \in \overline{\Pi}_\gamma$, the problem (2.2.4) - (2.2.5) possesses an unique solution $U \in H^l(G)$ for all right-hand sides $F \in H^{l-2k}(G)$, $G_\mu \in H^{l-k_\mu-1/2}(\Gamma)$, and an unique solution $U \in H^{l_1}(G)$ for all $F \in H^{l_1-2k}(G)$, $G_\mu \in H^{l_1-k_\mu-1/2}(\Gamma)$, with the estimate (2.2.9) being valid for l and l_1 .

We now prove that, if $U(\cdot,p) \in H^{l_1}(G)$ is the solution to (2.2.4) - (2.2.5) with the right-hand sides as in Remark 2.1.6 D), than it also belongs to $\mathbf{E}_{l,\frac{l}{2b}}(\gamma)$.

It is sufficient to prove, that $U(x,\cdot) \in \mathbf{E}_{\frac{l}{2b}}(\gamma)$ and $D_x^\alpha U(x,\cdot) \in \mathbf{E}_{\bar{0}}(\gamma)$ as $|\alpha| \le l$. First, we prove that the norm (2.1.24) is finite and then, that U and $D_x^\alpha U$ are holomorphic in Π_γ .

Evidently, for all $p \in \overline{\Pi}_\gamma$, it follows from $U \in H^{l_1}(G)$ that $U \in H^l(G)$ and hence $\|U\|_{l,\frac{l}{2b}} \le \|U\|_{l_1,\frac{l_1}{2b}} \le +\infty$, where the last inequality is obtained by integration of (2.2.9) (with l substituted by l_1) over the plane $\{\sigma_1 = \gamma_1, ..., \sigma_m = \gamma_m\}$.

Now it follows from Sobolev embedding theorem (see, for example, Egorov Yu.V., Shubin M.A. (1988)), that for $l_1 > l + n/2$ the functions U and $D_x^\alpha U$ ($|\alpha| \le l$) are continuous and $|U| \le C\|U\|_\nu$, $|D_x^\alpha U| \le C\|U\|_{l_1}$, where $\nu = l_1 - l$. Thus, for $p \in \overline{\Pi}_\gamma$ and $x \in G$ we obtain using (1.1.13) that

$$\int |||p|||^{\frac{l}{b}} |U|^2 \, d\tau \ \le \ C \int |||p|||^{\frac{l}{b}} \|U\|_k^2 \, d\tau$$

$$\le \ C \int \left[\|U\|_{l_1}^2 \ + \ |||p|||^{\frac{l_1}{b}} \|U\|_0^2 \right] d\tau \ .$$

Now, using (2.2.9) being integrated over the plane $\{\mathrm{Re}\, p_1 = \sigma_1, ..., \mathrm{Re}\, p_m = \sigma_m\}$ (and with $l = l_1$), we obtain the desired inequality:

$$\int \left| D_x^\alpha U \right|^2 d\tau \ \le \ C_1 \int \|U\|_{l_1}^2 \, d\tau$$

$$\le \ C_2 \int \left\{ \|F\|_{r_1}^2 \ + \ |||p|||^{\frac{r_1}{b}} \|F\|_0^2 \ + \ \sum_{\mu=1}^{k} \left[\|G_\mu\|_{\lambda_{1,\mu}}'^2 \ + \ |||p|||^{\frac{\lambda_{1,\mu}}{b}} \|G_\mu\|_0'^2 \right] \right\} d\tau$$

$$\le \ C_3 \left[|||F|||_{r_1,\frac{r_1}{2b}}^2 \ + \ \sum_{\mu=1}^{k} |||G_\mu|||_{\lambda_{1,\mu},\frac{\lambda_{1,\mu}}{2b}}'^2 \right] ,$$

where the last inequality holds in the view of Remark 2.1.8. Here $r_1 = l_1 - 2k$ and $\lambda_{1,\mu} = l_1 - k_\mu - 1/2$. The derivatives $D_x^\alpha U$ are estimated in the same way.

C).In order to prove that U and $D_x^\alpha U$ ($|\alpha| \le l$) are holomorphic in Π_γ , it is enough to state their holomorphicity in each p_j in the view of Hartogs theorem (see Shabat B.V. (1985)). Differentiating (2.2.10) and (2.2.11) with respect to p_j , we obtain the appropriate problem for the unknown function $U_j' = \partial U / \partial p_j$:

$$A(x,D_x,p)\, U_j'(x,p) \ = \ F_j'(x,p) \ - \ A_j'(x,D_x,p)\, U(x,p) \qquad (2.2.12)$$

and

$$B_\mu(x, D_x, p)U_j'(x,p)\big|_\Gamma = G_\mu(x', p) - B_{\mu_j}'(x, D_x, p)U(x,p)\big|_\Gamma , \qquad (2.2.13)$$

where $\mu = 1, ..., k$, and by X_j' we denote the derivative $\partial X/\partial p_j$, and U is the above found solution to the problem (2.2.4) - (2.2.5).

As $U \in H^{l_1}(G)$, it follows from Proposition 1.3.1 that

$$A_j' U \in H^{r_1}(G) \quad B_{\mu_j}' U\big|_\Gamma \in H^{\lambda_{1,\mu}}(\Gamma) ,$$

and thus, the problem (2.2.12) - (2.2.13) has a unique solution $U_j'(x,p) \in H^{l_1}(G)$ for all $p \in \overline{\Pi}_\gamma$. We have to prove that the functions U_j' and $D_x^\alpha U_j'$ are really the derivatives of U and $D_x^\alpha U$ for all $p \in \Pi_\gamma$.

We denote by

$$\Delta_j X := X(p_1, ..., p_{j-1}, p_j + \Delta p_j, p_{j+1}, ..., p_m) - X(p)$$

$$:= X(p[j], p_j + \Delta p_j) - X(p) ,$$

where $p \in \Pi_\gamma$, and note that $\Delta_j D_x^\alpha U = D_x^\alpha \Delta_j U$ and

$$A(x, D, p[j], p_j + \Delta p_j)U(x, p[j], p_j + \Delta p_j) - A(x, D, p)U(x,p)$$

$$= \Delta_j A U(x, p[j], p_j + \Delta p_j) + A(x, D, p)\Delta_j U ,$$

and thus, we obtain the system

$$A\Delta_j U = \Delta_j F - (\Delta_j A)U(x, p[j], p_j + \Delta p_j) , \qquad (2.2.14)$$

and

$$B_\mu \Delta_j U\big|_\Gamma = \Delta_j G_\mu - (\Delta_j B_\mu)U(x, p[j], p_j + \Delta p_j)\big|_\Gamma , \qquad (2.2.15)$$

for $\mu = 1, ..., k$. Subtracting from (2.2.14) equation (2.2.12) divided by Δp_j and from (2.2.15) equation (2.2.13) divided by Δp_j , we overcome to the system

$$A\Big(\frac{\Delta_j U}{\Delta p_j} - U_j'\Big) = \Big(\frac{\Delta_j F}{\Delta p_j} - F_j'\Big) - \Big(\frac{\Delta_j A}{\Delta p_j} - A_j'\Big)U(x, p[j], p_j + \Delta p_j) - A'\Delta_j U \qquad (2.2.16)$$

in G and

$$B_\mu\Big(\frac{\Delta_j U}{\Delta p_j} - U_j'\Big) = \Big(\frac{\Delta_j G_\mu}{\Delta p_j} - G_{\mu_j}'\Big) - \Big(\frac{\Delta_j B_\mu}{\Delta p_j} - B_{\mu_j}'\Big)U(x', p[j], p_j + \Delta p_j) -$$

$$- B_{\mu_j}' \Delta_j U \qquad (2.2.17)$$

on Γ .

Using once again Sobolev embedding theorem and the estimate (2.2.9) with $l = l_1$, we approach to the estimate:

$$\sum_{|\alpha|\le l}\Big|\frac{\Delta_j D_x^\alpha U}{\Delta p_j} - D_x^\alpha U_j'\Big| \;=\; \sum_{|\alpha|\le l}\Big|D_x^\alpha\Big(\frac{\Delta_j U}{\Delta p_j} - U_j'\Big)\Big|$$

$$\le\; C_1\,\Big\|\frac{\Delta_j U}{\Delta p_j} - U_j'\Big\|_{l_1}$$

$$\le\; C_2\Big\{\Big\|\frac{\Delta_j F}{\Delta p_j} - F_j'\Big\|_{r_1} \;+\; \sum_{\mu=1}^{k}\Big\|\frac{\Delta_j G_\mu}{\Delta p_j} - G_{\mu_j}'\Big\|_{\lambda_{1,\mu}} \;+$$

$$+\;\Big\|\Big(\frac{\Delta_j A}{\Delta p_j} - A_j'\Big)\,U(x,p[j],p_j+\Delta p_j)\Big\|_{r_1} \;+$$

$$+\;\sum_{\mu=1}^{k}\Big\|\Big(\frac{\Delta_j B_\mu}{\Delta p_j} - B_{\mu_j}'\Big)\,U(x',p[j],p_j+\Delta p_j)\Big\|_{\lambda_{1,\mu}} \;+$$

$$+\;\big\|A'\Delta_j U\big\|_{r_1} \;+\; \big\|B_{\mu_j}'\Delta_j U\big\|_{\lambda_{1,\mu}}\Big\}\,. \tag{2.2.18}$$

As the coefficients of operators

$$D_x^\beta\Big(\frac{\Delta_j A}{\Delta p_j} - A_j'\Big) \quad\text{and}\quad D_x^\beta\Big(\frac{\Delta_j B_\mu}{\Delta p_j} - B_{\mu_j}'\Big)$$

converge to zero uniformly in x and p, we see that the third and the fourth terms in the right - hand side of (2.2.18) vanish as $\Delta p_j \to 0$. Further, functions F and G_μ are infinitely smooth in x and p, that is why

$$\Big|D_x^\beta\Big(\frac{\Delta_j F}{\Delta p_j} - F_j'\Big)\Big| \;=\; \Big|\frac{\Delta_j D_x^\beta F}{\Delta p_j} - D_x^\beta F_j'\Big| \;\to\; 0$$

as $\Delta p_j \to 0$ uniformly in x. Thus, for $|\beta| \le l_1 - 2k = r_1$, we obtain that the first two terms in the right - hand side of (2.2.18) vanish.

As for the last two terms in (2.2.18), we note that, $\Delta_j U$ satisfies the problem (2.2.14) - (2.2.15), and thus, the estimate of the type (2.2.9) is valid for it. We remind that $\Delta_j A$ and $\Delta_j B_{\mu_j}$ vanish uniformly in x as $\Delta p_j \to 0$. For $\Delta_j F$ and $\Delta_j G_\mu$ we use Remark 2.1.12 A) and obtain that $\|\Delta_j U\|_{l_1} \to 0$ as $\Delta p_j \to 0$ and the last two terms in (2.2.18) vanish too. Finally, for all $x \in G$ and $p \in \Pi_\gamma$

$$\frac{\Delta_j D_x^\alpha U}{\Delta p_j} \longrightarrow D_x^\alpha U_j' \quad\text{as}\quad \Delta p_j \to 0\,,$$

for $|\alpha| \le l$, and hence, the functions $D_x^\alpha U$ are holomorphic for these p .

D). We now overcome to arbitrary right - hand sides F and G_μ from $\mathbf{E}_{r,\frac{r}{2b}}(\gamma)$ and $\mathbf{E}'_{\lambda_\mu,\frac{\lambda_\mu}{2b}}(\gamma)$ respectively. In the view of Remark 2.1.6 D) there exist sequences of smooth functions F_n and G_{μ_n} converging to F and G_μ in respective spaces.

Let $U_n(x,p) \in \mathbf{E}_{l,\frac{l}{2b}}(\gamma)$ be a solution to the problem (2.2.4) - (2.2.5) with right - hand sides F_n and G_{μ_n} . It follows from (2.2.8) that U_n form a fundamental sequence in $\mathbf{E}_{l,\frac{l}{2b}}(\gamma)$. As this space is complete we have $U_n \to U \in \mathbf{E}_{l,\frac{l}{2b}}(\gamma)$.

On the other hand, for all $p \in \overline{\Pi}_\gamma$ we can find a solution $U_0(x,p)$ with right - hand sides F and G_μ and such that $U_0(\cdot,p) \in H^l(G)$. Then the function $U_n - U_0$ is the solution to (2.2.4) - (2.2.5) with the right - hand sides $F_n - F$ and $G_{\mu_n} - G_\mu$. Applying the estimate (2.2.9) integrated over τ , we obtain

$$\int_{\sigma_1=\sigma_1^o} \cdots \int_{\sigma_m=\sigma_m^o} \left[\|U_n - U_0\|_l^2 + \||p|\|^{\frac{l}{b}} \|U_n - U_0\|_0^2\right] d\tau \longrightarrow 0$$

as $n \to \infty$, where $\sigma^o = \mathrm{Re}\, p^o$ and $p^o \in \overline{\Pi}_\gamma$. Thus

$$\int_{\sigma_1=\sigma_1^o} \cdots \int_{\sigma_m=\sigma_m^o} \|U_n - U_0\|_0^2 \, d\tau \longrightarrow 0$$

as $n \to \infty$, and by definition

$$\int_{\sigma_1=\sigma_1^o} \cdots \int_{\sigma_m=\sigma_m^o} \|U_n - U\|_0^2 \, d\tau \longrightarrow 0$$

as $n \to \infty$, for this integral is not larger than the norm $\|U_n - U\|^2_{l,\frac{l}{2b}}$ vanishing itself as $n \to \infty$.

Thus, for almost all $p \in \overline{\Pi}_\gamma$, we have $U(x,p) = U_0(x,p)$ and the above - constructed function U is the solution to the problem (2.2.4) - (2.2.5) for almost all $p \in \overline{\Pi}_\gamma$.

Finally, Remark 2.1.12 gives us an opportunity to state that $U(x,p)$ is the solution to the problem (2.2.4) - (2.2.5) for all $p \in \Pi_\gamma$ and for almost all p from its boundary $\partial\Pi_\gamma$. ▷

Now we overcome to the problem of solvability of our initial polyparabolic problem (2.2.1) - (2.2.2) in the spaces $\mathbf{P}^{l,\frac{l}{2b}}\left(e^{-\gamma\cdot t}\right)$, i.e. with zero initial conditions (2.2.3).

Theorem 2.2.7 *Let $l \in \mathbb{Z}$, where $l \ge l_0 = \max(2k, k_\mu+1)$. Then there exists a vector γ : $\gamma_j > 0$ such that the "polyparabolic" problem (2.2.1) - (2.2.2)* *with zero initial conditions (2.2.3) has a unique solution* $u \in \mathbf{P}^{l,\frac{l}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right)$ *for all right-hand sides*

$$f \in \mathbf{P}^{r,\frac{r}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right) \ , \ \textit{and} \ \ g_\mu \in \mathbf{P'}^{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right) \ ,$$

where $r = l-2k$, $\lambda_\mu = l - k_\mu - 1/2$. *The solution* $u(x,t)$ *satisfies a two-sided a priori estimate*

$$C_1 \, \|e^{-\gamma\cdot t}u\|_{l,\frac{l}{2\vec{b}}} \ \le \ \|e^{-\gamma\cdot t}f\|_{r,\frac{r}{2\vec{b}}} \ + \ \sum_{\mu=1}^{k} \|e^{-\gamma\cdot t}g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}} \ \le \ C_2 \, \|e^{-\gamma\cdot t}u\|_{l,\frac{l}{2\vec{b}}} \ , \qquad (2.2.19)$$

where the norms are taken over $G\times(\mathbb{R}_+)^m$ *and* $\Gamma\times(\mathbb{R}_+)^m$ *and the constants are independent of functions.*

Remark 2.2.8 *This Theorem remains valid not only for integer* $l/(2b_j)$ *but also for* $l/(2b_j) \neq [l/(2b_j)] + 1/2$. *In the case of the half - integer* $l/(2b_j)$ *the Theorem also remains true, but the functions are supposed to be continued by zero into the domain, where* $t_j < 0$ *for at least one* j , *and the respective norms are taken over the infinite in all sides "polycylinder"* $G\times\mathbb{R}^m$ *and its "side surface"* $\Gamma\times\mathbb{R}^m$.

Proof (of Theorem 2.2.7). Suppose $u(x,t) \in \mathbf{P}^{l,\frac{l}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right)$ is a solution of (2.2.1) - (2.2.2). Then it follows from Remark 2.1.9 that in (2.2.1) - (2.2.2) $f \in \mathbf{P}^{r,\frac{r}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right)$ and $g_\mu \in \mathbf{P'}^{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}}\left(e^{-\gamma\cdot t}\right)$, where $r = l-2k$ and $\lambda_\mu = l - k_\mu - 1/2$. Applying Laplace transform (2.1.6) in t to (2.2.1) - (2.2.2) we obtain in the force of Proposition 2.1.5 and Remark 2.1.11 the boundary value problem (2.2.4) - (2.2.5) with

$$U = Lu \in \mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma) \ , \quad F = Lf \in \mathbf{E}_{r,\frac{r}{2\vec{b}}}(\gamma)$$

and

$$G_\mu = Lg_\mu \in \mathbf{E}'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}}(\gamma) \ . \qquad (2.2.20)$$

The polyparabolicity of (2.2.1) - (2.2.2) means the "semiboundedness" of (2.2.4) - (2.2.5). Thus from Theorem 2.2.5 follows the existence of $\gamma \in (\mathbb{R}_+)^m$ such that under condition (2.2.20) the problem (2.2.4) - (2.2.5) possesses the unique solution $U \in \mathbf{E}_{l,\frac{l}{2\vec{b}}}(\gamma)$ satisfying the estimate (2.2.8). Applying to it the inverse Laplace transform and using once again Proposition 2.1.5 we overcome to the estimate (2.2.19) and the uniqueness of the solution of (2.2.1) - (2.2.2) follows.

The existence is the straightforward consequence of the existence of the solution to the problem (2.2.4) - (2.2.5) and $u = L^{-1}U$ in the view of Proposition 2.1.5. $\triangleright$

Now we overcome to the case of *nonhomogeneous initial conditions* (2.2.3). We denote by Ω the "two-sided polycylinder" $G\times\mathbb{R}^m$, by Ω_+ the "one-sided polycylinder" $G\times(\mathbb{R}_+)^m$ and by Ω' and Ω'_+ their "side-surfaces". We look for solutions $u(x,t)$ to the problem (2.2.1) - (2.2.3) such that $e^{-\gamma\cdot t}u(x,t)\in H^{l,\frac{l}{2b}}(\Omega_+)$. With this end of view we suppose that

$$e^{-\gamma\cdot t}f(x,t)\ \in\ H^{\lambda,\frac{\lambda}{2b}}(\Omega_+)\ ,\quad \lambda\ =\ l-2k\ , \tag{2.2.21}$$

$$e^{-\gamma\cdot t}g_\mu(x',t)\ \in\ H^{\lambda_\mu,\frac{\lambda_\mu}{2b}}(\Omega'_+)\ ,\quad \lambda_\mu\ =\ l-k_\mu-1/2\ , \tag{2.2.22}$$

$$e^{-\gamma[j]\cdot t[j]}\phi_{j,\nu_j}(x,t[j])\ \in\ H^{l_{\nu,j},\frac{l_{\nu,j}}{2b}[j]}\big(G\times(\mathbb{R}_+)^{m-1}\big)\ , \tag{2.2.23}$$

where $l_{\nu,j}\ =\ l-2b_j\nu_j-b_j>0$, $\nu_j\ =\ 0,...,\kappa_j-1$ and $\kappa_j\ \le\ k/b_j$ is the maximal order of the t_j-derivative in the expression for the operator A .

Clearly that on the planes $t_j=0$ the right-hand sides of (2.2.1) - (2.2.3) have to satisfy the compatibility conditions. The simplest compatibility conditions are the ones with "zero" (see Theorem 2.2.7 and Remark 2.2.8):

Condition 2.2.9 *(Condition of Compatibility with zero).*

Suppose all $\phi_{j,\nu_j}(x,t[j])\equiv 0$. *We demand that after the zero extension of the functions* $f(x,t)$ *and* $g_\mu(x',t)$ *into the domain where at least one of* t_j *is negative (i.e.* $\mathbb{R}^m\setminus(\mathbb{R}_+)^m$*), the following inclusions are satisfied:*

$$e^{-\gamma\cdot t}f(x,t)\ \in\ H^{\lambda,\frac{\lambda}{2b}}(\Omega)\ , \tag{2.2.24}$$

$$e^{-\gamma\cdot t}g_\mu(x',t)\ \in\ H^{\lambda_\mu,\frac{\lambda_\mu}{2b}}(\Omega')\ . \tag{2.2.25}$$

The above - formulated Condition means that the zero extension of the functions f and g_μ does not disturb their smoothness. For nonhomogeneous initial conditions we pose the following *General Compatibility Condition:*

Condition 2.2.10 *There exists a function* $u_0(x,t)$ *such that the following relations take place:*

$$e^{-\gamma\cdot t}u_0(x,t)\ \in\ H^{l,\frac{l}{2b}}(\Omega_+)\ , \tag{2.2.26}$$

$$\left.\frac{\partial^{\nu_j}u_0}{\partial t_j^{\nu_j}}\right|_{t_j=0}\ =\ \phi_{j,\nu_j}(x,t[j])\quad\text{for}\quad \nu_j=0,...,\kappa_j-1\ , \tag{2.2.27}$$

where ϕ_{j,ν_j} *are the functions from (2.2.3) and satisfy (2.2.23) for* $j=1,...,m$. *Moreover, if we put*

$$f_0(x,t)\ =\ Au_0\ \text{ in }\ \Omega_+\ \text{ and }\ g_{\mu 0}(x',t)\ =\ B_\mu u_0\big|_{\Omega'_+}$$

for $\mu\ =\ 1,...,k$, *then after zero extension into* $\mathbb{R}^m\setminus(\mathbb{R}_+)^m$, *the functions* $f-f_0$ *and* $g_\mu-g_{\mu_0}$ *being the new right-hand sides of (2.2.1) - (2.2.3) are compatible with zero, i.e.*

$$e^{-\gamma\cdot t}(f - f_0) \ \in \ H^{\lambda,\frac{\lambda}{2b}}(\Omega) \tag{2.2.28}$$

and

$$e^{-\gamma\cdot t}(g_\mu - g_{\mu_0}) \ \in \ H^{\lambda_\mu,\frac{\lambda_\mu}{2b}}(\Omega') \ . \tag{2.2.29}$$

Theorem 2.2.11 *Let (2.2.1) - (2.2.3) be a polyparabolic problem,* $l \in \mathbb{Z}$ *such that* $l \geq l_0 = \max(2k, k_\mu + 1)$ *and* $l/(2b_j) \in \mathbb{Z}$ *for all* $j = 1, ..., m$ *. Then there exists a vector* γ *with* $\gamma_j > 0$ *such that for arbitrary right-hand sides* f, g_μ, ϕ_{j,ν_j} *satisfying the relations (2.2.21) - (2.2.23) and General Compatibility Condition, there exists such a solution* $u(x,t)$ *to the problem (2.2.1) - (2.2.3) that*

$$e^{-\gamma\cdot t}u(x,t) \ \in \ H^{l,\frac{l}{2b}}(\Omega_+) \ .$$

Proof follows trivially from Theorem 2.2.7 and Condition 2.2.10. Indeed, suppose $v(x,t) = u(x,t) - u_0(x,t)$, then for the function v we have to deal with the problem

$$Av \ = \ f - f_0 \quad \text{in} \ \ \Omega_+ \ , \tag{2.2.30}$$

$$B_\mu v \ = \ g_\mu - g_{\mu 0} \quad \text{on} \ \ \Omega'_+ \ , \tag{2.2.31}$$

$$\frac{\partial^{\nu_j} v}{\partial t_j^{\nu_j}}\bigg|_{t_j=0} \ = \ 0 \ . \tag{2.2.32}$$

As $f - f_0 \in \mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$ and $g_\mu - g_{\mu 0} \in \mathbf{P}'^{\,\lambda_\mu,\frac{\lambda_\mu}{2b}}(e^{-\gamma\cdot t})$, then in the view of Theorem 2.2.7 there exists a single function $v \in \mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$ satisfying (2.2.30) - (2.2.32). Finally, $u = v + u_0$ is the solution to our initial problem (2.2.1) - (2.2.3). ▷

In order to formulate the uniqueness theorem, we introduce some new additional conditions to the problem (2.2.1) - (2.2.3).

Condition 2.2.12 *(or* **Time-ellipticity condition** *).*

We say that the operator $A\big(x, D_x, \frac{\partial}{\partial t}\big)$ *of the problem (2.2.1) - (2.2.3) is* time-elliptic *if for every* $j \in \{1, ..., m\}$ *the coefficient* $A_{\kappa_j}\big(x, D_x, \frac{\partial}{\partial t}[j]\big)$ *before the* t_j-*derivative of the maximal order* $\frac{\partial^{\kappa_j}}{\partial t_j^{\kappa_j}}$ *in the expression for* $A\big(x, D_x, \frac{\partial}{\partial t}\big)$ *(see (2.2.1)) is a polyparabolic operator with respect to* $(m-1)-$*times, i.e. after the Laplace transform of* A_{κ_j} *and substitution of* p_i *by* $q_i^{2b_i}$ *,* $i \neq j$ *, we obtain the parameter elliptic operator in the sense of Condition 1.3.2 in the cone* $Q[j] \ = \ Q_1 \times ... \times Q_{j-1} \times Q_{j+1} \times ... \times Q_m$ *.*

Example 2.2.13 *Suppose*

$$A\big(x, D_x, \frac{\partial}{\partial t}\big) \ = \ \frac{\partial^2}{\partial t_1 \partial t_2} - B(x, D_x)\frac{\partial}{\partial t_1} - C(x, D_x)\frac{\partial}{\partial t_2} + E(x, D_x) \ .$$

Then the time-ellipticity of A *means the parabolicity of* $\left(\frac{\partial}{\partial t_2} - B\right)$ *and* $\left(\frac{\partial}{\partial t_1} - C\right)$ *or simply the ellipticity of* B *and* C *.*

We now deduce from (2.2.1) - (2.2.3) the boundary value problem for the operator A_{κ_j} :

$$A_{\kappa_j}\left(x, D_x, \frac{\partial}{\partial t}[j]\right) w(x, t[j]) \ = \ \tilde{f}(x, t[j]) \ , \tag{2.2.33}$$

$$B_{\mu,j}\big(x, D_x, \frac{\partial}{\partial t}[j]\big) w(x, t[j])\big|_{x=x'} \ = \ \tilde{g}_{\mu,j}(x', t[j]) \ , \tag{2.2.34}$$

$$\left.\frac{\partial^{\nu_i} w}{\partial t_i^{\nu_i}}\right|_{t_i=0} \ = \ \tilde{\phi}_{i,\nu_i}(x, t[i,j]) \quad \text{for} \ \ \nu_i = 0, ..., \kappa_i - 1 \, , \ i \neq j \ , \tag{2.2.35}$$

where $w \ = \ \frac{\partial^{\kappa_j} u}{\partial t_j^{\kappa_j}}\big|_{t_j=0}$, $\tilde{f}$ is the sum of $f(x, 0)$ and the replaced terms of $A\big(x, D_x, \frac{\partial}{\partial t}\big)u$, which do not contain the highest t_j-derivative $\frac{\partial^{\kappa_j} u}{\partial t_j^{\kappa_j}}$ with the subsequent restriction of these terms to the plane $t_j = 0$ and the substitution of the appropriate derivatives of u by $(x, t[j])-$ derivatives of ϕ_{j,ν_j} in the way that

$$e^{-\gamma[j]\cdot t[j]} \tilde{f} \ \in \ H^{\lambda - b_j, \frac{\lambda - b_j}{2\vec{b}}[j]}\big(\Omega \cap \{t_j = 0\}\big) \ .$$

Further, $\tilde{g}_{\mu,j}$ is a linear combination of $(x, t[j])-$ derivatives of ϕ_{j,ν_j} and $\left.\frac{\partial^{\nu_j} g_\mu}{\partial t_j^{\nu_j}}\right|_{t_j=0}$ for $\nu_j \leq \kappa_j$, and

$$e^{-\gamma[j]\cdot t[j]} \tilde{g}_{\mu,j} \ \in \ H^{\lambda_\mu - b_j - 2b_j\kappa_j, \frac{\lambda_\mu - b_j - 2b_j\kappa_j}{2\vec{b}}[j]}\big(\Omega' \cap \{t_j = 0\}\big) \ .$$

At last, $\tilde{\phi}_{i,\nu_i} \ = \ \left.\frac{\partial^{\kappa_j} \phi_{i,\nu_i}}{\partial t_j^{\kappa_j}}\right|_{t_j=0}$ and

$$e^{-\gamma[i,j]\cdot t[i,j]} \tilde{\phi}_{i,\nu_i} \ \in \ H^{l_1, \frac{l_1}{2\vec{b}}[i,j]}\big(G \times (\mathbb{R}_+)^{m-2}\big) \ ,$$

where $l_1 \ = \ l - 2b_i\nu_i - 2b_j\kappa_j - b_i - b_j$. Certainly, there are more boundary conditions (being compatible in the view of General Compatibility Condition 2.2.10) in (2.2.34) than is needed. We have to choose $k - \kappa_j b_j$ of them to satisfy the following

Condition 2.2.14 *(or* **Inductive polyparabolicity condition** *).*
Suppose the number of times $m = 2$ *. In this case we say that the problem (2.2.1) - (2.2.3) is* inductively polyparabolic *for* $t_j = \alpha_j = 0$ *if, for all* $j \in \{1, ..., m\}$ *the order of the highest* t_j*-derivative in* B_μ $(\mu = 1, ..., k)$ *is less or equal to* κ_j *, and the corresponding system (2.2.33) - (2.2.35) (after the sufficient choice of* $k - \kappa_j b_j$ *boundary conditions) is parabolic.*

For general $m = k$ *we say that the problem (2.2.1) - (2.2.3) is* inductively polyparabolic *if, for all* $j \in \{1, ..., m\}$ *the order of the highest* t_j*-derivative in* B_μ $(\mu = 1, ..., k\ )$ *is less or equal to* κ_j *, and the corresponding system (2.2.33) - (2.2.35) (after the sufficient choice of* $k - \kappa_j b_j$ *boundary conditions) is polyparabolic and inductively polyparabolic for* $m = k - 1$ *.*

Remark 2.2.15 *Of course, we really need Condition 2.2.14 which implies Condition 2.2.12, but, as we shall see, sometimes the latter is of use too.*

Theorem 2.2.16 *Suppose the problem (2.2.1) - (2.2.3) is* inductively polyparabolic *and* $l > \sum b_j(2\kappa_j + 1)$ *additionally. Then under the conditions of Theorem 2.2.11 the solution* $u(x,t)$ *to the problem (2.2.1) - (2.2.3) is unique and satisfies the following two-sided a priori estimate*

$$C_1 \|e^{-\gamma \cdot t} u\|_{l, \frac{l}{2b}} \ \le \ \|e^{-\gamma \cdot t} f\|_{\lambda, \frac{\lambda}{2b}} \ +$$

$$+ \ \sum_{\mu=1}^{k} \|e^{-\gamma \cdot t} g_\mu\|'_{\lambda_\mu, \frac{\lambda_\mu}{2b}} \ + \ \sum_{j=1}^{m} \sum_{\nu_j=0}^{\kappa_j - 1} \|e^{-\gamma[j] \cdot t[j]} \phi_{j,\nu_j}(x, t[j])\|_{l_{\nu,j}, \frac{l_{\nu,j}}{2b}[j]}$$

$$\le \ C_2 \|e^{-\gamma \cdot t} u\|_{l, \frac{l}{2b}} \ , \qquad (2.2.36)$$

where $l_{\nu,j} = l - 2b_j\nu_j - b_j$ *, the norms are taken over* Ω_+ *,* Ω'_+ *and* $G \times (\mathbb{R}_+)^{m-1}$ *respectively and the constants are independent of functions.*

Proof. As usual, we prove only the left one of the inequalities (2.2.36), which implies uniqueness. The right is a trivial consequence of inequalities (2.1.18) - (2.1.22) (see Remark 2.1.4).

The main problem here is to substitute the function $u_0(x,t)$ by a function $u_1(x,t)$ such that $e^{-\gamma \cdot t} u_1(x,t) \in H^{l, \frac{l}{2b}}(\Omega_+)$, in the way that we can estimate its norm through the norms of functions f, g_μ, ϕ_{j,ν_j} (unfortunately, we are not able to do it with the norm of $u_0(x,t)$ itself).

We prove the estimate (2.2.36) by induction over the number of times. If we have only one time, then our statement is contained in Theorem 10.1 from (Agranovich M.S., Vishik M.I. (1964)). Thus, we suppose the estimate is true for $m-1$ times and the solution to (2.2.1) - (2.2.3) is unique in this case.

We wish to estimate the initial values of the derivatives

$$\phi_{j,\rho_j} \ := \ \frac{\partial^{\rho_j} u}{\partial t_j^{\rho_j}}\Big|_{t_j=0} \quad \text{for} \ \ \rho_j = \kappa_j, ..., \frac{l}{2b_j} - 1$$

through the norms of f, g_μ, ϕ_{j,ν_j} with $\nu_j < \kappa_j$.

In this way for $w = \left.\frac{\partial^{\kappa_j} u}{\partial t_j^{\kappa_j}}\right|_{t_j=0}$ we obtain the system (2.2.33) - (2.2.35) which is polyparabolic and inductively polyparabolic with $(m-1)$ times, and thus, by inductive assumption possesses the unique solution $w(x,t[j])$, and

$$C_1 \, \|e^{-\gamma[j]\cdot t[j]} w\|_{l_{\kappa,j}, \frac{l_{\kappa,j}}{2\vec{b}}[j]} \;\leq\; \|e^{-\gamma[j]\cdot t[j]} \tilde{f}\|_{\lambda - b_j, \frac{\lambda - b_j}{2\vec{b}}[j]} \; +$$

$$+ \; \sum_{\mu=1}^{k} \|e^{-\gamma[j]\cdot t[j]} \tilde{g}_{\mu,j}\|'_{\lambda_\mu - 2b_j\kappa_j - b_j, \frac{\lambda_\mu - 2b_j\kappa_j - b_j}{2\vec{b}}[j]} \; +$$

$$+ \; \sum_{i\neq j} \sum_{\nu_i=0}^{\kappa_i - 1} \|e^{-\gamma[i,j]\cdot t[i,j]} \tilde{\phi}_{i,\nu_i}\|_{l_{\kappa,j} - 2b_i\nu_i - b_i, \frac{l_{\kappa,j} - 2b_i\nu_i - b_i}{2\vec{b}}[i,j]} \,. \tag{2.2.37}$$

In the view of the structure of functions $\tilde{f}, \tilde{g}_{\mu,j}, \tilde{\phi}_{i,\nu_i}$ and the inequalities (2.1.18) - (2.1.22) (see Remark 2.1.4) we may overwrite (2.2.37) without waves (and with different C_1) :

$$C_1 \, \|e^{-\gamma[j]\cdot t[j]} w\|_{l_{\kappa,j}, \frac{l_{\kappa,j}}{2\vec{b}}[j]} \;\leq\; \|e^{-\gamma\cdot t} f\|_{\lambda, \frac{\lambda}{2\vec{b}}} \; +$$

$$+ \; \sum_{\mu=1}^{k} \|e^{-\gamma\cdot t} g_\mu\|'_{\lambda_\mu, \frac{\lambda_\mu}{2\vec{b}}} \; + \; \sum_{j=1}^{m} \sum_{\nu_j=0}^{\kappa_j - 1} \|e^{-\gamma[j]\cdot t[j]} \phi_{j,\nu_j}(x,t[j])\|_{l_{\nu,j}, \frac{l_{\nu,j}}{2\vec{b}}[j]} \,. \tag{2.2.38}$$

For the functions ϕ_{j,ρ_j} with $\rho_j = \kappa_j + 1, ..., \frac{l}{2b_j} - 1$ we obtain the same systems (2.2.33) - (2.2.35) but with different right-hand sides. Thus, they can be uniquely defined too and we are in position to overwrite (2.2.38) for them:

$$C_1 \, \|e^{-\gamma[j]\cdot t[j]} \phi_{j,\rho_j}\|_{l_{\rho,j}, \frac{l_{\rho,j}}{2\vec{b}}[j]} \;\leq\; \|e^{-\gamma\cdot t} f\|_{\lambda, \frac{\lambda}{2\vec{b}}} \; +$$

$$+ \; \sum_{\mu=1}^{k} \|e^{-\gamma\cdot t} g_\mu\|'_{\lambda_\mu, \frac{\lambda_\mu}{2\vec{b}}} \; + \; \sum_{j=1}^{m} \sum_{\nu_j=0}^{\kappa_j - 1} \|e^{-\gamma[j]\cdot t[j]} \phi_{j,\nu_j}(x,t[j])\|_{l_{\nu,j}, \frac{l_{\nu,j}}{2\vec{b}}[j]} \,. \tag{2.2.39}$$

Now, remembering that $v = u - u_0$ satisfies (2.2.30) - (2.2.32) and belongs to $\mathbf{P}^{l,\frac{l}{2\bar{b}}}\left(e^{-\gamma\cdot t}\right)$, we see that

$$\frac{\partial^{\nu_j} v}{\partial t_j^{\nu_j}}\Big|_{t_j=0} \;=\; 0 \quad \text{for} \;\; \nu_j = 0, ..., \frac{l}{2b_j} - 1 \;,$$

and thus

$$\frac{\partial^{\nu_j} u_0}{\partial t_j^{\nu_j}}\Big|_{t_j=0} \;=\; \phi_{j,\nu_j} \quad \text{for} \;\; \nu_j = 0, ..., \frac{l}{2b_j} - 1 \;.$$

Further, in the view of Remark 2.1.4 4), there exists $u_1(x,t)$ such that

$$e^{-\gamma\cdot t} u_1(x,t) \;\in\; H^{l,\frac{l}{2\bar{b}}}\left(\Omega_+\right) \;, \tag{2.2.40}$$

$$\frac{\partial^{\nu_j} u_1}{\partial t_j^{\nu_j}}\Big|_{t_j=0} \;=\; \phi_{j,\nu_j} \quad \text{for} \;\; \nu_j = 0, ..., \frac{l}{2b_j} - 1 \;, \tag{2.2.41}$$

and

$$\|e^{-\gamma\cdot t} u_1(x,t)\|_{l,\frac{l}{2\bar{b}}} \;\le\; C \sum_{j=1}^{m} \sum_{\nu_j=0}^{\frac{l}{2b_j}-1} \|e^{-\gamma[j]\cdot t[j]} \phi_{j,\nu_j}\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\bar{b}}[j]} \;. \tag{2.2.42}$$

As u_0 also satisfies (2.2.40) - (2.2.41), we have $u_1 - u_0 \;\in \mathbf{P}^{l,\frac{l}{2\bar{b}}}\left(e^{-\gamma\cdot t}\right)$ and thus

$$A[u_1 - u_0] \;=\; Au_1 \;-\; Au_0 \;\in\; \mathbf{P}^{\lambda,\frac{\lambda}{2\bar{b}}}\left(e^{-\gamma\cdot t}\right) \;,$$

and

$$B_\mu[u_1 - u_0]\big|_{\Omega'} \in \; \mathbf{P'}^{\lambda_\mu,\frac{\lambda_\mu}{2\bar{b}}}\left(e^{-\gamma\cdot t}\right) \;, \quad \mu = 1, ..., k \,.$$

It means that we can substitute the function u_0 by u_1 without violating Compatibility Condition 2.2.10, but now we gain the estimate

$$C_1 \, \|e^{-\gamma\cdot t} u_1\|_{l,\frac{l}{2\bar{b}}} \;\le\; \|e^{-\gamma\cdot t} f\|_{\lambda,\frac{\lambda}{2\bar{b}}} \;+$$

$$+ \; \sum_{\mu=1}^{k} \|e^{-\gamma\cdot t} g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\bar{b}}} \;+\; \sum_{j=1}^{m} \sum_{\nu_j=0}^{\kappa_j-1} \|e^{-\gamma[j]\cdot t[j]} \phi_{j,\nu_j}\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\bar{b}}[j]} \;. \tag{2.2.43}$$

Now let $u(x,t)$ be an arbitrary solution to the problem (2.2.1) - (2.2.3) from the previous theorem. Computing its derivatives $\frac{\partial^{\nu_j} u}{\partial t_j^{\nu_j}}\Big|_{t_j=0}$ for $\nu_j = \kappa_j +$

$1,\dots,\frac{l}{2b_j}-1$, in the same way, as we computed ϕ_{j,ρ_j} for $\rho_j=\kappa_j+1,\dots,\frac{l}{2b_j}-1$, we see in the force of inductive polyparabolicity assumption that

$$\frac{\partial^{\nu_j}(u-u_1)}{\partial t_j^{\nu_j}}\bigg|_{t_j=0} \;=\; 0\;, \quad \nu_j=0,\dots,\frac{l}{2b_j}-1\;.$$

It implies $u-u_1\in\mathbf{P}^{l,\frac{l}{2\vec{b}}}\big(e^{-\gamma\cdot t}\big)$ and thus, the estimate (2.2.19) from Theorem 2.2.7 is true for it:

$$C\,\|e^{-\gamma\cdot t}(u-u_1)\|_{l,\frac{l}{2\vec{b}}} \;\le\; \|e^{-\gamma\cdot t}(f-f_1)\|_{\lambda,\frac{\lambda}{2\vec{b}}} \;+\; \sum_{\mu=1}^{k}\|e^{-\gamma\cdot t}(g_\mu-g_{\mu,1})\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}}\;, \tag{2.2.44}$$

where $f_1=Au_1$ in Ω_+ and $g_{\mu,1}=B_\mu u_1$ on Ω'_+ . As , evidently,

$$\|e^{-\gamma\cdot t}f_1\|_{\lambda,\frac{\lambda}{2\vec{b}}} \;+\; \sum_{\mu=1}^{k}\|e^{-\gamma\cdot t}g_{\mu,1}\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}} \;\le\; C\,\|e^{-\gamma\cdot t}u_1\|_{l,\frac{l}{2\vec{b}}}\;,$$

it follows that

$$C\,\|e^{-\gamma\cdot t}u\|_{l,\frac{l}{2\vec{b}}} \;\le\; \|e^{-\gamma\cdot t}f\|_{\lambda,\frac{\lambda}{2\vec{b}}}\;+$$

$$+\;\sum_{\mu=1}^{k}\|e^{-\gamma\cdot t}g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}} \;+\; \|e^{-\gamma\cdot t}u_1\|_{l,\frac{l}{2\vec{b}}}\;.$$

Applying (2.2.43) we obtain the desired estimate (2.2.36). ▷

We now consider an example of inductively polyparabolic problems.

Example 2.2.17 *Let* $A\big(x,D_x,\frac{\partial}{\partial t}\big)$ *be as in Example 2.2.13, that is,*

$$A\big(x,D_x,\frac{\partial}{\partial t}\big) \;=\; \frac{\partial^2}{\partial t_1\partial t_2}-B(x,D_x)\frac{\partial}{\partial t_1}-C(x,D_x)\frac{\partial}{\partial t_2}+E(x,D_x)\;,$$

$(1/2\operatorname{ord}E\;=\;\operatorname{ord}B\;=\;\operatorname{ord}C\;=\;2)$. *Also, we suppose that together with Dirichlet boundary conditions in* x *and Goursat initial conditions in* t *it forms the polyparabolic problem*

$$A\big(x,D_x,\frac{\partial}{\partial t}\big)u \;=\; f\;,$$

$$u\big|_\Gamma \;=\; h_0(t_1,t_2,x')\;,\quad \frac{\partial u}{\partial\vec{n}}\Big|_\Gamma \;=\; h_1(t_1,t_2,x')\;,$$

$$u\big|_{t_1=0} \;=\; \phi_1(t_2,x)\;,\quad u\big|_{t_2=0} \;=\; \phi_2(t_1,x)\;. \tag{2.2.45}$$

Then the respective systems (2.2.33) - (2.2.35) with the chosen Dirichlet boundary conditions are

$$\Big(\frac{\partial}{\partial t_2} - B\Big) w_1 \;=\; C\frac{\partial \phi_1}{\partial t_2} - E\phi_1 + f_1 \;:=\; \tilde{f}_1 \ ,$$

$$w_1\big|_{\Gamma} \;=\; h_{0,t_1}(0, t_2, x') \ , \qquad w_1\big|_{t_2=0} \;=\; \phi_{2,t_1}(0, x) \ ,$$

and

$$\Big(\frac{\partial}{\partial t_1} - A\Big) w_2 \;=\; B\frac{\partial \phi_2}{\partial t_1} - E\phi_2 + f_2 \;:=\; \tilde{f}_2 \ ,$$

$$w_2\big|_{\Gamma} \;=\; h_{0,t_2}(t_1, 0, x') \ , \qquad w_2\big|_{t_1=0} \;=\; \phi_{1,t_2}(0, x) \ ,$$

where $w_1 = \frac{\partial u}{\partial t_1}\big|_{t_1=0}$, $w_2 = \frac{\partial u}{\partial t_2}\big|_{t_2=0}$, $f_1 = f(0, t_2, x)$, $f_2 = f(t_1, 0, x)$ *and* z_{t_1} *means* $\frac{\partial z}{\partial t_1}$,

If A *and* B *are elliptic operators these systems are parabolic, and thus, the initial problem (2.2.45) is inductively polyparabolic and possesses the unique solution due to Theorem 2.2.16. Hence, we see that in this example the notions of the time - ellipticity and the inductive polyparabolicity coincide.*

Remark 2.2.18 *While proving the last theorem we found out that for inductively polyparabolic problem the initial values of derivatives* $\frac{\partial^{\kappa_j} u}{\partial t_j^{\kappa_j}}$ *for* $\kappa_j = 0, ..., \frac{l}{2b_j} - 1$ *can be defined from initial conditions and equation (2.2.1), i.e. we do not need the function* $u(x,t)$ *itself. Moreover, if* u_0 *is the function from General Compatibility Condition 2.2.10, then it satisfies the same initial conditions as* u *and, thus, we obtain the following Compatibility Criterion: for* ϕ_{j,ν_j} *with* $\nu_j = 0, ..., \frac{l}{2b_j} - 1$ *being built from initial conditions (2.2.3) and equation (2.2.1), we construct according to Remark 2.1.4, item 4) (see also the discussion in Section 2.5 below) their continuation* u_0 *such that* $e^{-\gamma \cdot t} u_0 \in H^{l, \frac{l}{2b}}(\Omega_+)$ *. Then our Criterion consists of the execution of (2.2.28), (2.2.29).*

Remark 2.2.19 *Condition 2.2.14 is sufficient, but not necessary. Indeed, suppose in Condition 2.2.12 operators* A_{κ_j} *are simply nonvanishing bounded from below functions, then we are able to estimate* ϕ_{j,ν_j} $(\nu_j \geq \kappa_j)$ *without boundary value problems in* $(m-1)$ *times. We note that this case covers the multidimensional time operators being introduced in (Brish N.I., Yurchuk N.I. (1970)) and (Tersenov S.A. (1987)), see equations (0.1.9), (0.1.10). A new more interesting and more general class may be obtained if all* $A_{\kappa_j} = A_{\kappa_j}(x, D_x)$ *are elliptic and satisfy Shapiro-Lopatinskij Condition together with some of boundary operators* B_μ *.*

Though Example 2.2.3 gives us a nonpolyparabolic problem in the sense of Definition 2.2.1, it also shows that Inductive Parabolicity Condition is sufficient, but not necessary as the systems (2.2.33) - (2.2.35) in this case turn into Cauchy problems for ordinary differential equations with constant coefficients.

Example 2.2.20 *Consider the equation*

$$\frac{\partial^2 u}{\partial t_1 \partial t_2} \;+\; a(x,D)u \;=\; f$$

(cf. Friedman A., Littman W. (1963)), where $a(x,D)$ *is an elliptic operator of order 4 with boundary and initial conditions*

$$u\big|_\Gamma \;=\; h_0(t_1,t_2,x') \;, \quad \frac{\partial u}{\partial x_n}\Big|_\Gamma \;=\; h_1(t_1,t_2,x') \;,$$

$$u\big|_{t_1=0} \;=\; \phi_1(t_2,x) \;, \quad u\big|_{t_2=0} \;=\; \phi_2(t_1,x) \;.$$

Here we obtain the respective systems (2.2.33) - (2.2.35) in the form

$$\frac{\partial w_1}{\partial t_2} \;=\; f_1 - a(x,D)\phi_1 \;, \qquad w_1\big|_{t_2=0} \;=\; \phi_{2,t_1}(0,x) \;,$$

and

$$\frac{\partial w_2}{\partial t_1} \;=\; f_2 - a(x,D)\phi_2 \;, \qquad w_2\big|_{t_1=0} \;=\; \phi_{1,t_2}(0,x) \;.$$

They are the Cauchy problems for simplest ordinary differential equations with respect to w_1 *and* w_2 *, i.e.* w_1 *and* w_2 *can be estimated through the functions* f *,* h_0 *,* h_1 *,* ϕ_1 *and* ϕ_2 *(possibly with increased* γ *in appropriate* $\mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$ *) via inequalities (2.1.18) - (2.1.22).*

In general, we observe that in all the cases where (2.2.33) - (2.2.35) are Cauchy problems for ordinary differential equations with constant coefficients, we obtain the desired uniqueness in our functional classes automatically. It is not so in the variable coefficients case. For example, if we obtain for w_1 or w_2 first order linear equations of the form $\frac{\partial w}{\partial t_2} + a(x,t)w \;=\; f(x,t)$ (this is exactly the case of Brish - Yurchuk equation (0.1.6), (0.1.7)), then we have to demand the function $a(x,t_1,t_2)$ to be bounded from below in t_2 uniformly in (x,t_1) , otherwise the solution does not fall into $\mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$ even with increased γ .

2.3 The case of a finite polycylinder

Once again we consider the mixed problem (2.2.1) - (2.2.3):

$$A\big(x,D_x,\frac{\partial}{\partial t}\big)\,u(x,t) \;=\; f(x,t) \;, \tag{2.3.1}$$

$$B_\mu\big(x,D_x,\frac{\partial}{\partial t}\big)u(x,t)\Big|_{x=x'} \;=\; g_\mu(x',t) \;, \quad \text{for } \mu = 1,...,k \,, \tag{2.3.2}$$

$$\left.\frac{\partial^{\nu_j} u}{\partial t_j^{\nu_j}}\right|_{t_j=0} = \phi_{j,\nu_j}(x, t[j]) \ , \quad \text{for} \ \ \nu_j = 0, ..., \kappa_j - 1 \ , \tag{2.3.3}$$

where κ_j equals to the maximal order of the t_j-derivative in the expression for operator $A\big(x, D_x, \frac{\partial}{\partial t}\big)$, where $\kappa_j \leq k/b_j$ and $j = 1, ..., m.$

But now we consider the equation (2.3.1) in the domain $\Omega_T = G \times \prod_{j=1}^{m}(0, T_j)$, where $T_j < +\infty$ and $T = (T_1, ..., T_m)$, with boundary conditions (2.3.2) on its side surface $\Omega'_T = \partial G \times \prod_{j=1}^{m}(0, T_j) = \Gamma \times \prod_{j=1}^{m}(0, T_j)$ and the initial conditions (2.3.3) on polycylinders $\Omega_T[j] = G \times \prod_{\rho\neq j}(0, T_\rho)$, where $j = 1, ..., m$. The functions in (2.3.1) - (2.3.3) will be considered in the following spaces

$$u(x, t) \ \in \ H^{l, \frac{l}{2\vec{b}}}\big(\Omega_T\big) \ , \tag{2.3.4}$$

$$f(x, t) \ \in \ H^{\lambda, \frac{\lambda}{2\vec{b}}}\big(\Omega_T\big) \ \ \text{for} \ \ \lambda \ = \ l - 2k \ , \tag{2.3.5}$$

$$g_\mu(x', t) \ \in \ H^{\lambda_\mu, \frac{\lambda_\mu}{2\vec{b}}}\big(\Omega'_T\big) \ \ \text{for} \ \ \lambda_\mu \ = \ l - k_\mu - 1/2, \tag{2.3.6}$$

where $\mu = 1, ..., k$, and

$$\phi_{j,\nu_j}(x, t[j]) \ \in \ H^{l_{\nu,j}, \frac{l_{\nu,j}}{2\vec{b}}[j]}\big(\Omega_T[j]\big) \ , \tag{2.3.7}$$

where $l_{\nu,j} = l - 2b_j\nu_j - b_j$, $\nu_j = 0, ..., \kappa_j - 1$ and $\kappa_j \leq k/b_j$. Here $2k, k_\mu$ are the orders of operators A , B_μ , $2\vec{b}$ is the polyparabolic weight of the problem,

$$l, \frac{l}{2b_j}, \kappa_j \ \in \ \mathbb{Z} \ \ \text{and} \ \ l \ \geq \ \max(2k, k_\mu) \ . \tag{2.3.8}$$

Finally, $\frac{l}{2\vec{b}}[j] = \big(\frac{l}{2b_1}, ..., \frac{l}{2b_{j-1}}, \frac{l}{2b_{j+1}}, ..., \frac{l}{2b_m}\big)$ is an integer $(m-1)$-dimensional vector.

We denote by $\tilde{\Omega}_T$, $\tilde{\Omega}'_T$, $\tilde{\Omega}_T[j]$ the corresponding domains, where the times t_i for $i = 1, ..., m$ are extended to the sets of negative numbers, i.e. $t_i \in (-\infty \, , T_i)$. Then we are in position to overwrite Compatibility Conditions 2.2.9, 2.2.10 in the following form:

Condition 2.3.1 *(Condition of Compatibility with zero).*

Suppose $\phi_{j,\nu_j}(x, t[j]) \equiv 0$ *for* $\nu_j = 0, ..., \kappa_j - 1$. *We demand that after the zero extension of functions* $f(x, t)$ *and* $g_\mu(x', t)$ *into the domain where at least one of* t_j *is negative (i.e.* $\mathbb{R}^m \setminus (\mathbb{R}_+)^m$*), the following inclusions are satisfied:*

$$f(x, t) \ \in \ H^{\lambda, \frac{\lambda}{2\vec{b}}}\big(\tilde{\Omega}_T\big) \ ,$$

$$g_\mu(x', t) \ \in \ H^{\lambda_\mu, \frac{\lambda_\mu}{2\vec{b}}}\big(\tilde{\Omega}'_T\big) \ ,$$

where $\mu = 1, ..., k$.

The above Condition means that the zero extension of the functions f and g_μ does not disturb their smoothness. If $\phi_{j,\nu_j} \neq 0$ we pose the following *General Compatibility Condition:*

Condition 2.3.2 *There exists a function* $u_0(x,t)$ *such that the following relations take place:*

$$u_0(x,t) \in H^{l,\frac{l}{2\bar{b}}}(\Omega_T) ,$$

$$\frac{\partial^{\nu_j} u_0}{\partial t_j^{\nu_j}}\Big|_{t_j=0} = \phi_{j,\nu_j}(x,t[j]) \quad \text{for } \nu_j = 0,...,\kappa_j - 1 ,$$

where ϕ_{j,ν_j} *are the functions from (2.3.3) and satisfy (2.3.7) for* $j = 1,...,m$ *. Moreover, if we put*

$$f_0(x,t) = Au_0 \text{ in } \Omega_+ \text{ and } g_{\mu_0}(x',t) = B_\mu u_0\big|_{\Omega'_+}$$

for $\mu = 1,...,k$ *, then after zero extension into* $\mathbb{R}^m \setminus (\mathbb{R}_+)^m$ *, the functions* $f - f_0$ *and* $g_\mu - g_{\mu_0}$ *for* $\mu = 1,...,k$ *, being the new right-hand sides of (2.3.1) - (2.3.3) are compatible with zero:*

$$f - f_0 \in H^{\lambda,\frac{\lambda}{2\bar{b}}}(\tilde{\Omega}_T) ,$$

$$g_\mu - g_{\mu_0} \in H^{\lambda_\mu,\frac{\lambda_\mu}{2\bar{b}}}(\tilde{\Omega}'_T) .$$

Theorem 2.3.3 *Let (2.3.1)-(2.3.3) be a polyparabolic problem,* $l, k, k_\mu, b_j \in \mathbb{Z}$ *are the integers satisfying (2.3.8). Then for arbitrary right-hand sides* f, g_μ, ϕ_{j,ν_j} *satisfying the relations (2.3.5) - (2.3.7) and General Compatibility Condition, there exists a solution* $u(x,t)$ *to the problem (2.3.1) - (2.3.3) such that*

$$u(x,t) \in H^{l,\frac{l}{2\bar{b}}}(\Omega_T) .$$

If, moreover, the problem (2.3.1)-(2.3.3) is inductively polyparabolic and $l > \sum_{j=1}^m b_j(2\kappa_j + 1)$ *, then the above solution* $u(x,t)$ *is unique and satisfies the following two-sided a priori estimate*

$$C_1 \|u\|_{l,\frac{l}{2\bar{b}}} \leq \|f\|_{\lambda,\frac{\lambda}{2\bar{b}}} + \sum_{\mu=1}^{k} \|g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\bar{b}}} +$$

$$+ \sum_{j=1}^{m} \sum_{\nu_j=1}^{\kappa_j-1} \|\phi_{j,\nu_j}(x,t[j])\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\bar{b}}[j]} \leq C_2 \|u\|_{l,\frac{l}{2\bar{b}}} , \qquad (2.3.9)$$

where the norms are taken over Ω_T *,* Ω'_T *and* $\Omega_T[j]$ *respectively, the constants are independent of functions.*

Proof. 1). We prove the existence first. Starting with a stationary problem to be obtained from (2.3.1)-(2.3.3) (with zero initial conditions) by means of Laplace transformation, we fix a vector $\gamma \in (\mathbb{R}_+)^m$ such that the boundary value problem

$$A(x, D_x, p)\, U(x,p) \;=\; F(x,p) \;, \quad x \in G \;, \tag{2.3.10}$$

$$B_\mu(x, D_x, p) U(x,p)\big|_{x=x'} \;=\; G_\mu(x', p) \;, \quad x' \in \Gamma \;, \tag{2.3.11}$$

where $\mu = 1, ..., k$, is uniquely solvable. Now, in the view of Remark 2.1.4, section 5), the functions $e^{-\gamma\cdot t} f$, $e^{-\gamma\cdot t} g_\mu^*$, $e^{-\gamma\cdot t} u_0$, together with f, g_μ, u_0 belonging to $H^{\lambda, \frac{\lambda}{2b}}(\Omega_T)$, $H^{\lambda_\mu, \frac{\lambda_\mu}{2b}}(\Omega'_T)$, $H^{l, \frac{l}{2b}}(\Omega_T)$ respectively, may be continued to the domains with $t_j > T_j$. Thus, we obtain the functions $e^{-\gamma\cdot t} f^*$, $e^{-\gamma\cdot t} g_\mu^*$, $e^{-\gamma\cdot t} u_0^*$ belonging to the spaces $H^{\lambda, \frac{\lambda}{2b}}(\Omega_+)$, $H^{\lambda_\mu, \frac{\lambda_\mu}{2b}}(\Omega'_+)$, $H^{l, \frac{l}{2b}}(\Omega_+)$ respectively, where $\Omega_+ = G \times (\mathbb{R}_+)^m$, $\Omega'_+ = \Gamma \times (\mathbb{R}_+)^m$.

Consequently, we obtain a polyparabolic problem over Ω_+ with the right-hand sides $f^*, g_\mu^*, \phi_{j,\nu_j}$ and u_0^* providing the execution of Compatibility Condition 2.2.10. Applying Theorem 2.2.11 we conclude that there exists a solution u^* to the problem and $e^{-\gamma\cdot t} u^* \in H^{l, \frac{l}{2b}}(\Omega_+)$. Then $u(x,t) = u^*(x,t)\big|_{\Omega_T}$ is the solution to the initial problem (2.3.1)-(2.3.3) and $u \in H^{l, \frac{l}{2b}}(\Omega_T)$.

2). To prove the uniqueness we need some additional facts about spaces $\mathbf{P}^{a, \frac{a}{2b}}(e^{-\gamma\cdot t})$ ($a \in \mathbb{Z}_+$ for Ω_+ and $a = [a] + 1/2 > 0$ for Ω'_+).

Proposition 2.3.4 *Let* $u \in \mathbf{P}^{a, \frac{a}{2b}}(e^{-\gamma\cdot t})$, *then* $u = 0$ *while* $t_j < T_j$ *for some* $j \in \{1, ..., m\}$ *if and only if* $U = Lu$ *can be written in the form* $U = e^{-p_j T_j} U_1$, *where* $U_1 \in \mathbf{E}_{a, \frac{a}{2b}}(\gamma)$.

Proof. Suppose $u \in \mathbf{P}^{l, \frac{l}{2b}}(e^{-\gamma\cdot t})$ and $u = 0$ as $t_j < T_j$ for some $j \in \{1, ..., m\}$, then

$$L[u(x,t)] \;=\; \int_T^\infty \int_{(\mathbb{R}_+)^{m-1}} e^{-p\cdot t} u(x,t)\, dt \;=$$

$$= \int_{(\mathbb{R}_+)^m} e^{-p_j T_j} e^{-p\cdot t} u(x, t[j], t_j + T_j)\, dt \;=\; e^{-p_j T_j} L\big[u(x, t[j], t_j + T_j)\big] \;.$$

As obviously, $u(x, t[j], t_j + T_j) \in \mathbf{P}^{l, \frac{l}{2b}}(e^{-\gamma\cdot t})$, then $U_1 = L\big[u(x, t[j], t_j + T_j)\big] \in \mathbf{E}_{l, \frac{l}{2b}}(\gamma)$.

Inversely, let $U = e^{-p_j T_j} U_1$, where $U_1 \in \mathbf{E}_{l, \frac{l}{2b}}(\gamma)$. Then, $u(x,t) = u_1(x, t[j], t_j - T_j)$, where $u_1 = L^{-1} U_1 \in \mathbf{P}^{l, \frac{l}{2b}}(e^{-\gamma\cdot t})$, and thus, $u = 0$ as $t_j < T_j$. $\triangleright$

Proposition 2.3.5 *Consider the problem (2.3.1)-(2.3.3) in* Ω_+ *and suppose* $\phi_{j,\nu_j} = 0$, $f \in \mathbf{P}^{\lambda,\frac{\lambda}{2b}}(e^{-\gamma\cdot t})$, $g_\mu \in \mathbf{P}^{\lambda_\mu,\frac{\lambda_\mu}{2b}}(e^{-\gamma\cdot t})$ *, where* $\gamma \in (\mathbb{R}_+)^m$ *is such a vector, that this problem has a unique solution* $u \in \mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$.

Then, if $f, g_\mu = 0$ *while* $t_j < T_j$ *for all* $j \in \{1, ..., m\}$ *, we obtain that* $u = 0$ *if* $t \in \prod_{j=1}^m [0, T_j)$.

Proof. Considering (2.3.10)-(2.3.11), we see, that if $f, g_\mu = 0$ as $t_j < T_j$ for all $j \in \{1, ..., m\}$, then in the view of Proposition 2.3.4 $F = e^{-p\cdot T} F_1$, $G_\mu = e^{-p\cdot T} G_{\mu 1}$, and $F_1 \in \mathbf{E}_{\lambda,\frac{\lambda}{2b}}(\gamma)$, $G_{\mu 1} \in \mathbf{E}_{\lambda_\mu,\frac{\lambda_\mu}{2b}}(\gamma)$.

Now we substitute F and G in the right-hand sides of (2.3.10)-(2.3.11) by F_1 and $G_{\mu 1}$ and obtain the problem

$$A(x, D_x, p)\, U_1(x,p) = F_1(x,p) , \quad x \in G , \tag{2.3.12}$$

$$B_\mu(x, D_x, p) U_1(x,p)\big|_{x=x'} = G_{\mu 1}(x', p) , \quad x' \in \Gamma \tag{2.3.13}$$

with the unique solution $U_1 \in \mathbf{E}_{l,\frac{l}{2b}}(\gamma)$. Multiplying these equations by $e^{-p\cdot T}$, we see, that the function $e^{-p\cdot T} U_1 \in \mathbf{E}_{l,\frac{l}{2b}}(\gamma)$ is the solution to (2.3.10)-(2.3.11). As in the view of Theorem 2.2.5 this problem possesses the unique solution, we obtain $U = e^{-p\cdot T} U_1$. then, applying once again Proposition 2.3.4, we obtain, that $u = 0$ as $t \in \prod_{j=1}^m [0, T_j)$. ▷

We now return to the proof of Theorem 2.3.3.

In order to prove the uniqueness of the solution to the problem (2.3.1)-(2.3.3), we suppose that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ to the problem with the same right-hand sides f, g_μ, ϕ_{j,ν_j} . Reviewing the proof of Theorem 2.2.16 we see that for $t_j = 0$

$$\frac{\partial^{\nu_j} u_1}{\partial t_j^{\nu_j}} = \frac{\partial^{\nu_j} u_2}{\partial t_j^{\nu_j}} \text{ while } \nu_j = 0, ..., \frac{l}{2b_j} - 1 . \tag{2.3.14}$$

Thus, an extension u^* of the function $u = u_1 - u_2$ into Ω_+ (see Remark 2.1.4, sect.5)), such that $e^{-\gamma\cdot t} u^* \in H^{l,\frac{l}{2b}}(\Omega_+)$, indeed belongs to $\mathbf{P}^{l,\frac{l}{2b}}(e^{-\gamma\cdot t})$ (see (2.3.14)). We put

$$f^* = A\big(x, D_x, \frac{\partial}{\partial t}\big)\, u^* \text{ in } \Omega_+ \tag{2.3.15}$$

and

$$g_\mu^* = B_\mu\big(x, D_x, \frac{\partial}{\partial t}\big) u^*\big|_{\Omega'_+} . \tag{2.3.16}$$

The functions f^* and g_μ^* belong to the spaces $\mathbf{P}^{\lambda,\frac{\lambda}{2b}}(e^{-\gamma\cdot t})$ and $\mathbf{P}^{\lambda_\mu,\frac{\lambda_\mu}{2b}}(e^{-\gamma\cdot t})$ respectively and are equal to zero if $t_j < T_j$ for all $j \in \{1, ..., m\}$.

Hence, it follows from Proposition 2.3.5 that $u^* = 0$ for $t \in \prod_{j=1}^m [0, T_j)$, that is, $u_1 = u_2$ in Ω_T and the uniqueness follows.

3). The last fact we need to prove is the first inequality in the estimate (2.3.9). Let $u \in H^{l, \frac{l}{2b}}(\Omega_T)$ be the solution to the problem (2.3.1)-(2.3.3) and let u^* be its extension to Ω_+ such that $e^{-\gamma \cdot t} u^* \in H^{l, \frac{l}{2b}}(\Omega_+)$.

Let $Au^* = f^*$ in Ω_+ and $B_\mu u^*\big|_{\Omega'_+} = g^*_\mu$, $\mu = 1, ..., k$. The functions f^*, g^*_μ coincide with f, g_μ for $t \in \prod_{j=1}^m [0, T_j)$ and

$$e^{-\gamma \cdot t} f^* \ \in \ H^{\lambda, \frac{\lambda}{2b}}(\Omega_+) \ , \quad e^{-\gamma \cdot t} g^*_\mu \ \in \ H^{\lambda_\mu, \frac{\lambda_\mu}{2b}}(\Omega'_+) \ .$$

We shall now construct extensions f^{**} and g^{**}_μ of f and g_μ in the view of Remark 2.1.4, sect.5) and in the way that

$$\begin{aligned} \left\| e^{-\gamma \cdot t} f^{**} \right\|_{\lambda, \frac{\lambda}{2b}, \Omega_+} \ &\leq \ C_1 \left\| e^{-\gamma \cdot t} f \right\|_{\lambda, \frac{\lambda}{2b}, \Omega_T} \\ &\leq \ C_2 \| f \|_{\lambda, \frac{\lambda}{2b}, \Omega_T} \ , \end{aligned} \tag{2.3.17}$$

and

$$\begin{aligned} \left\| e^{-\gamma \cdot t} g^{**}_\mu \right\|_{\lambda_\mu, \frac{\lambda_\mu}{2b}, \Omega'_+} \ &\leq \ C_1 \left\| e^{-\gamma \cdot t} g_\mu \right\|_{\lambda_\mu, \frac{\lambda_\mu}{2b}, \Omega'_T} \\ &\leq \ C_2 \| g_\mu \|_{\lambda_\mu, \frac{\lambda_\mu}{2b}, \Omega'_T} \ . \end{aligned} \tag{2.3.18}$$

We denote by u^{**} the solution to the problem (2.3.1)-(2.3.3) with the right-hand sides f^{**}, g^{**}_μ ϕ_{j,ν_j}. As $f^* = f^{**} = f$ and $g^*_\mu = g^{**}_\mu = g_\mu$ on $\prod_{j=1}^m [0, T_j)$ we are able to apply Proposition 2.3.5 to the function $u^* - u^{**}$ and thus, obtain that $u = u^* = u^{**}$ for $t \in \prod_{j=1}^m [0, T_j)$.

Applying Theorem 2.2.16 to u^{**} we have

$$C \| e^{-\gamma \cdot t} u^{**} \|_{l, \frac{l}{2b}, \Omega_+} \ \leq \ \| e^{-\gamma \cdot t} f^{**} \|_{\lambda, \frac{\lambda}{2b}, \Omega_+} \ +$$

$$+ \sum_{\mu=1}^{k} \| e^{-\gamma \cdot t} g^{**}_\mu \|'_{\lambda_\mu, \frac{\lambda_\mu}{2b}, \Omega'_+} \ + \ \sum_{j=1}^{m} \sum_{\nu_j=1}^{\kappa_j - 1} \| e^{-\gamma[j] \cdot t[j]} \phi_{j,\nu_j}(x, t[j]) \|_{l_{\nu,j}, \frac{l_{\nu,j}}{2b}[j], G} \ .$$

Finally, applying (2.3.17), (2.3.18) and obvious inequalities

$$\| u \|_{l, \frac{l}{2b}, \Omega_T} \ \leq \ C_1 \| e^{-\gamma \cdot t} u \|_{l, \frac{l}{2b}, \Omega_T} \ \leq \ C_2 \| e^{-\gamma \cdot t} u \|_{l, \frac{l}{2b}, \Omega_+} \ ,$$

we obtain the desired estimate

$$C\,\|e^{-\gamma\cdot t}u\|_{l,\frac{l}{2b}} \;\le\; \|e^{-\gamma\cdot t}f\|_{\lambda,\frac{\lambda}{2b}} +$$

$$+\sum_{\mu=1}^{k}\|e^{-\gamma\cdot t}g_\mu\|'_{\lambda_\mu,\frac{\lambda_\mu}{2b}} + \sum_{j=1}^{m}\sum_{\nu_j=1}^{\kappa_j-1}\|e^{-\gamma[j]\cdot t[j]}\phi_{j,\nu_j}(x,t[j])\|_{l_{\nu,j},\frac{l_{\nu,j}}{2b}[j]}\;.$$

$\triangleright$

Corollary 2.3.6 *The following statement is true in the view of Proposition 2.3.5 and Theorem 2.3.3:*

Suppose $\phi_{j,\nu_j}\equiv 0$ *for all* $j\in\{1,...,m\}$. *Then for arbitrary* f,g_μ , *being compatible with zero, there exists a unique compatible with zero solution* $u\in H^{l,\frac{l}{2b}}(\Omega_T)$ *to the problem (2.3.1)-(2.3.3) satisfying (2.3.9) with* $\phi_{j,\nu_j}\equiv 0$.

In the rest of this section we show how to substitute the norms in (2.3.9) by equivalent ones in the way to obtain the constants C_1 and C_2 independent of $T=(T_1,...,T_m)$ with fixed $T_j\;:\;0\le T_j\;\le \mathrm{const}$.

Suppose $u(x,t)\in H^{l,\frac{l}{2b}}(\Omega_T)$. The norm

$$\langle u\rangle_{l,\frac{l}{2b}} \;=\; \langle u\rangle_{l,\frac{l}{2b},\Omega_T} \;=\; \inf\|v\|_{l,\frac{l}{2b},\Omega_+}\;, \tag{2.3.19}$$

is equivalent to the initial one. Here the infimum is taken over all functions $v(x,t)\in H^{l,\frac{l}{2b}}(\Omega_+)$ coinsiding with u almost everywhere in Ω_T . The norms $\langle\cdot\rangle'_{l,\frac{l}{2b},\Omega_T}$ over Ω'_T are defined in the same way. The equivalence of the norms $\langle\cdot\rangle_{l,\frac{l}{2b},\Omega_T}$ and $\|\cdot\|_{l,\frac{l}{2b},\Omega_T}$ is rather trivial. Indeed, if $v\in H^{l,\frac{l}{2b}}(\Omega_+)$ and $u=v$ on Ω_T , it is clear, that

$$\|u\|_{l,\frac{l}{2b},\Omega_T} \;=\; \|v\|_{l,\frac{l}{2b},\Omega_T} \;\le\; \|v\|_{l,\frac{l}{2b},\Omega_+}\;,$$

and thus,

$$\|u\|_{l,\frac{l}{2b}} \;\le\; \langle u\rangle_{l,\frac{l}{2b}}.$$

On the other hand, using section 5) of Remark 2.1.4, we obtain

$$\langle u\rangle_{l,\frac{l}{2b}} \;\le\; \|Lu\|_{l,\frac{l}{2b},\Omega_+} \;\le\; C\|u\|_{l,\frac{l}{2b}}\;.$$

Proposition 2.3.7 *Under the conditions of Theorem 2.3.3 it is possible to overwrite inequalities (2.3.9) in terms of norms (2.3.19)*

$$C_1\,\langle u\rangle_{l,\frac{l}{2b}} \;\le\; \langle f\rangle_{\lambda,\frac{\lambda}{2b}} \;+\; \sum_{\mu=1}^{k}\langle g_\mu\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2b}} \;+$$

$$+\sum_{j=1}^{m}\sum_{\nu_j=1}^{\kappa_j-1}\|\phi_{j,\nu_j}(x,t[j])\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\bar b}[j]} \;\le\; C_2\,\langle u\rangle_{l,\frac{l}{2\bar b}}\;, \tag{2.3.20}$$

where the constants C_1 *and* C_2 *are independent of* T .

Proof. 1). We start with the proof of the second inequality in (2.3.20). Let $u\in H^{l,\frac{l}{2\bar b}}(\Omega_T)$ be a solution to (2.3.1)-(2.3.3). Also, let u^* be an extension of u to Ω_+ such that

$$\left\|e^{-\gamma\cdot t}u^*\right\|_{l,\frac{l}{2\bar b},\Omega_+} \;\le\; 2\left\langle e^{-\gamma\cdot t}u\right\rangle_{l,\frac{l}{2\bar b},\Omega_T}$$

(the existence of such an extension follows from (2.3.19). Then the functions $f^*=Au^*$ and $g^*_\mu=B_\mu u^*\big|_{\Omega'_+}$ are extensions of f and g_μ and

$$\left\langle e^{-\gamma\cdot t}f\right\rangle_{\lambda,\frac{\lambda}{2\bar b},\Omega_T} \;\le\; \left\|e^{-\gamma\cdot t}f^*\right\|_{\lambda,\frac{\lambda}{2\bar b},\Omega_+}\;, \tag{2.3.21}$$

$$\left\langle e^{-\gamma\cdot t}g_\mu\right\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2\bar b},\Omega'_T} \;\le\; \left\|e^{-\gamma\cdot t}g^*_\mu\right\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\bar b},\Omega'_+}\;. \tag{2.3.22}$$

Applying to u^* , f^* , g^*_μ , ϕ_{j,ν_j} the second inequality in (2.2.36), we obtain in the view of (2.3.21), (2.3.22), that

$$\left\langle e^{-\gamma\cdot t}f\right\rangle_{\lambda,\frac{\lambda}{2\bar b}} \;+\; \sum_{\mu=1}^{k}\left\langle e^{-\gamma\cdot t}g_\mu\right\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2\bar b}} \;+$$

$$+\sum_{j=1}^{m}\sum_{\nu_j=1}^{\kappa_j-1}\left\|e^{-\gamma[j]\cdot t[j]}\phi_{j,\nu_j}(x,t[j])\right\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\bar b}[j]} \;\le\; C\left\langle e^{-\gamma\cdot t}u\right\rangle_{l,\frac{l}{2\bar b}}\;, \tag{2.3.23}$$

where C is independent of T for such is the constant in (2.2.36).

We now have to get reed of multipliers $e^{-\gamma\cdot t}$. Let $\varphi_j(t_j)\in C^\infty(\bar{\mathbb{R}}_+)$ and $\varphi_j(t_j)\equiv 1$ for $t_j\ge T^o_j+1$ and $T^o_j\ge T_j$. Suppose $u\in H^{l,\frac{l}{2\bar b}}(\Omega_T)$ and $u^*\in H^{l,\frac{l}{2\bar b}}(\Omega_+)$ be its arbitrary extension to Ω_+ then, for $\Phi(t)=\prod_{j=1}^{m}\varphi_j(t_j)$, we obtain that

$$\langle\Phi u\rangle_{l,\frac{l}{2\bar b},\Omega_T} \;\le\; \|\Phi u^*\|_{l,\frac{l}{2\bar b},\Omega_+} \;\le\; C\|u^*\|_{l,\frac{l}{2\bar b},\Omega_+}$$

and thus,

$$\langle\Phi u\rangle_{l,\frac{l}{2\bar b},\Omega_T} \;\le\; C\|u\|_{l,\frac{l}{2\bar b},\Omega_T}$$

with the constant C independent of T. The same inequality with primes is true too.

We put $\varphi_j(t_j) = e^{\pm\gamma_j t_j}$ for $t_j \le T_j^o$ and $\varphi_j(t_j) \equiv 1$ for $t_j \ge T_j^o + 1$. Then we easily may drop the exponents in (2.3.23) and obtain the second inequality in (2.3.20) with the constant C independent of T.

2). In order to prove the first inequality in (2.3.20) we construct such extensions f^* and g_μ^* of the functions f and g_μ that

$$\|e^{-\gamma\cdot t} f^*\|_{\lambda,\frac{\lambda}{2b},\Omega_+} \;\le\; 2\left\langle e^{-\gamma\cdot t} f\right\rangle_{\lambda,\frac{\lambda}{2b},\Omega_T} \qquad (2.3.24)$$

and

$$\|e^{-\gamma\cdot t} g_\mu^*\|_{\lambda_\mu,\frac{\lambda_\mu}{2b},\Omega_+} \;\le\; 2\left\langle e^{-\gamma\cdot t} g_\mu\right\rangle_{\lambda_\mu,\frac{\lambda_\mu}{2b},\Omega_T}\,, \qquad (2.3.25)$$

Now let u^* be a solution to the problem (2.3.1)-(2.3.3) with the right-hand sides $f^*, g_\mu^*, \phi_{j,\nu_j}$ such that $e^{-\gamma\cdot t}u \in H^{l,\frac{l}{2b}}(\Omega_+)$. It then follows from the proof of the existence in Theorem 2.3.3 that $u = u^*$ in Ω_T and

$$\left\langle e^{-\gamma\cdot t} u\right\rangle_{l,\frac{l}{2b},\Omega_T} \;\le\; \|e^{-\gamma\cdot t} u^*\|_{l,\frac{l}{2b},\Omega_+}\,.$$

Applying the first inequality in (2.2.36) to the functions u^*, f^*, g_μ^*, ϕ_{j,ν_j} and using (2.3.24) and (2.3.25), we overcome to the inequality

$$C\,\langle e^{-\gamma\cdot t} u\rangle_{l,\frac{l}{2b}} \;\le\; \langle e^{-\gamma\cdot t} f\rangle_{\lambda,\frac{\lambda}{2b}} \;+$$

$$+\;\sum_{\mu=1}^{k}\langle e^{-\gamma\cdot t} g_\mu\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2b}} \;+\; \sum_{j=1}^{m}\sum_{\nu_j=1}^{\kappa_j-1}\|e^{-\gamma[j]\cdot t[j]}\phi_{j,\nu_j}(x,t[j])\|_{l_{\nu,j},\frac{l_{\nu,j}}{2b}[j]}$$

with the constant C independent of T.

Dropping $e^{-\gamma\cdot t}$ and $e^{-\gamma[j]\cdot t[j]}$ as in the first part of this proof, we obtain the first inequality in (2.3.20) with the constant independent of T. $\triangleright$

2.4 Generalizations to the cases of time - dependent operators and polyparabolic systems

Suppose the operators A and B_μ in (2.3.1) - (2.3.2) have coefficients being infinitely smooth functions in (x,t) in the closed *finite* polycylinder $\overline{\Omega_T}$, i.e.

$$A(t) \;\equiv\; A\big(x,t,D_x,\frac{\partial}{\partial t}\big)\,, \quad B_\mu(t) \;\equiv\; B_\mu\big(x,t,D_x,\frac{\partial}{\partial t}\big) \qquad (2.4.1)$$

for $\mu = 1, ..., k$. The problem (2.3.1) - (2.3.3) with such operators is called polyparabolic in Ω_T if the problem to be obtained from it by means of freezing the values $t_j = \alpha_j$ in the coefficients of operators A and B_μ is polyparabolic in the sense of Definition 2.2.1 for all fixed α_j , $0 \le \alpha_j \le T_j$, in other words the operators

$$A\big(x, \alpha, D_x, q^{2\vec{b}}\big) , \quad B_\mu\big(x, \alpha, D_x, q^{2\vec{b}}\big) ,$$

where $q^{2\vec{b}} = (q_1^{2b_1}, ..., q_m^{2b_m})$, define an elliptic boundary value problem with parameters, i.e. the problem satisfying in the cone $\big\{|\arg q_j| \le \frac{\pi}{4b_j} , \ j = 1, ..., m\big\}$ the Conditions 1.3.2, 1.3.3.

The problem (2.3.1) - (2.3.3) with operators (2.4.1) is called inductively polyparabolic in Ω_T , if for all fixed $\alpha \in \prod_{j=1}^m [0, T_j)$ operators (2.4.1) form the inductively polyparabolic system in the sense of Condition 2.2.14 for $t = \alpha$ ($t_j = \alpha_j$ for $j = 1, ..., m$).

The compatibility conditions do not differ from Conditions 2.3.1, 2.3.2.

Theorem 2.4.1 *Theorem 2.3.3 remains valid for operators* $A\big(x, t, D_x, \frac{\partial}{\partial t}\big)$ *and* $B_\mu\big(x, t, D_x, \frac{\partial}{\partial t}\big)$ *forming the polyparabolic problem in* Ω_T *(resp. inductively polyparabolic problem in* Ω_T *for the validity of (2.3.9) with the consequent uniqueness).*

Proof. 1). As we need many polycylindrical domains, we introduce new special notations for them:

$$\Omega[\alpha, \beta] \ = \ G \times \prod_{j=1}^{m} (\alpha_j, \beta_j) \ , \quad \Omega'[\alpha, \beta] \ = \ \Gamma \times \prod_{j=1}^{m} (\alpha_j, \beta_j) \ ,$$

where $\alpha_j \ge -\infty$, $\beta_j \le +\infty$, and, for example, $\Omega_T = \Omega[0, T]$, $\Omega'_T = \Omega'[0, T]$.

We say that $u(x, t) \in H^{l, \frac{l}{2\vec{b}}}\big(\Omega[\alpha, \beta]\big)$ is compatible with zero for $t = \alpha$, if after zero extension into the domain where at least one of $t_j < \alpha_j$ we obtain the function $\tilde{u} \in H^{l, \frac{l}{2\vec{b}}}\big(\Omega[-\infty, \beta]\big)$.

We start the proof of the theorem with the uniqueness, that is, with the estimate (2.3.9). As usual, the second inequality in (2.3.9) is trivial and we have to prove only the first one in our case.

Suppose $0 \le \alpha_j \le T_j$ and $\alpha_j < \beta_j < T_j^1$, where $T_j < T_j^1 < T_j + \varepsilon$ for sufficiently small ε , and the coefficients of operators $A(t)$ and $B_\mu(t)$ are smoothly extended to $\Omega[0, T^1]$.

As follows from Proposition 2.3.7, for $u \in H^{l, \frac{l}{2\vec{b}}}\big(\Omega[\alpha, \beta]\big)$ with sufficiently large l , we have

$$C_1 \, \langle u \rangle_{l, \frac{l}{2\vec{b}}} \ \le \ \langle A(\alpha) u \rangle_{\lambda, \frac{\lambda}{2\vec{b}}} \ + \ \sum_{\mu=1}^{k} \langle B_\mu(\alpha) u \rangle'_{\lambda_\mu, \frac{\lambda_\mu}{2\vec{b}}} \ +$$

$$+\sum_{j=1}^{m}\sum_{\nu_j=1}^{\kappa_j-1}\|\partial_{t_j}^{\nu_j}u(x,t[j],\alpha_j)\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\vec{b}}[j]} \;\le\; C_2\,\langle u\rangle_{l,\frac{l}{2\vec{b}}}\ , \tag{2.4.2}$$

where $A(\alpha) = A\big(x,\alpha,D_x,\frac{\partial}{\partial t}\big)$, $B_\mu(\alpha) = B_\mu\big(x,\alpha,D_x,\frac{\partial}{\partial t}\big)$, $\partial_{t_j}^{\nu_j}u(x,t[j],\alpha_j) = \partial_{t_j}^{\nu_j}u(x,t)\big|_{t_j=\alpha_j}$, the norms are taken over $\Omega[\alpha,\beta]$, $\Omega'[\alpha,\beta]$, and $\Omega\big[\alpha[j],\beta[j]\big]$, and the constants C_1 , C_2 are independent of α,β and u .

Indeed, only independence of β was proved in Proposition 2.3.7. Independence of α follows trivially if we examine the proofs of Theorems 2.2.7, 2.2.11, 2.2.16, 2.3.3, and Proposition 2.3.7 and observe that the estimates obtained there are independent of α (in the view of Propositions 2.3.4, 2.3.5).

2). We now wish to substitute $A(\alpha)$ and $B_\mu(\alpha)$ in (2.4.2) by $A(t)$ and $B_\mu(t)$. First, we prove it for a small "polycylinder shell" $\Omega[\alpha,\beta]$ with $\beta_j-\alpha_j\le\delta<\varepsilon$ and $u(x,t)$ being compatible with zero for $t_j=\alpha_j$, i.e. for all $j=1,...,m$ the relation $\partial_{t_j}^{\nu_j}u(x,t[j],\alpha_j)=0$ holds for $\nu_j=1,...,\kappa_j-1$.

For such functions we obtain in the view of (2.4.2) that

$$C\,\langle u\rangle_{l,\frac{l}{2\vec{b}}} \;\le\; \langle A(\alpha)u\rangle_{\lambda,\frac{\lambda}{2\vec{b}}} \;+\; \sum_{\mu=1}^{k}\langle B_\mu(\alpha)u\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}}\ ,$$

and thus, in $\Omega[\alpha,\beta]$ we have

$$C\,\langle u\rangle_{l,\frac{l}{2\vec{b}}} \;\le\; \langle A(t)u\rangle_{\lambda,\frac{\lambda}{2\vec{b}}} \;+\; \sum_{\mu=1}^{k}\langle B_\mu(t)u\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}} \;+\; \mathcal{R}_{\alpha\beta}(u)\ , \tag{2.4.3}$$

where

$$\mathcal{R}_{\alpha\beta}(u) \;=\; \big\langle[A(\alpha)-A(t)]u\big\rangle_{\lambda,\frac{\lambda}{2\vec{b}}} \;+\; \sum_{\mu=1}^{k}\big\langle[B_\mu(\alpha)-B_\mu(t)]u\big\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}}\ . \tag{2.4.4}$$

Writing the expressions for operators $A(\alpha)-A(t)$ and $B_\mu(\alpha)-B_\mu(t)$

$$A(\alpha)-A(t) \;=\; \sum_{|r|+2b\cdot s\le 2k} a_{rs}(x,t)D_x^r\partial_t^s\ , \tag{2.4.5}$$

$$B_\mu(\alpha)-B_\mu(t) \;=\; \sum_{|r|+2b\cdot s\le k_\mu} b_{\mu rs}(x,t)D_x^r\partial_t^s\ , \tag{2.4.6}$$

we see that $a_{rs}(x,t)$, $b_{\mu rs}(x,t)\in C^\infty\big(\Omega[\alpha,T_1]\big)$ and are equal to zero for $t=\alpha$. In order to control their growth for large $|t|$, we shall change them for $t_j>\alpha_j+\delta$, where $j=1,...,m$, considering instead of a_{rs} and $b_{\mu rs}$ the functions $\tilde{a}_{rs} = a_{rs}\tilde{\chi}(t)$ and $\tilde{b}_{\mu rs} = b_{\mu rs}\tilde{\chi}(t)$, where $\tilde{\chi}(t) = \prod_{j=1}^m \chi(t_j)$ and $\chi(t_j) \in C^\infty(\bar{\mathbb{R}}_+)$ is the cut-off function with the properties: $\chi(t_j) = 1$ for $\alpha_j \le t_j \le \alpha_j + \delta$, $\chi(t_j) = 0$ for $t_j \ge \alpha_j + 2\delta$ and $\left|\partial_{t_j}^{\nu_j} \chi(t_j)\right| \le C\delta^{1-\nu_j}$ for $\nu_j \ge 1$ (for the construction of such a function see Sect.28 in the book (Shubin M.A. (1978))). We put

$$\varepsilon(\delta) \;=\; \max_{\Omega[\alpha,\alpha+\delta]} \left\{|D_x^N a_{rs}|\,,\; |D_x^N b_{\mu rs}|\right\}\;, \text{ where } |N| \le N_1\;,$$

and

$$M \;=\; \max_{\Omega[0,T^1]} \left\{|D_x^N \partial_t^\nu a_{rs}|\,,\; |D_x^N \partial_t^\nu b_{\mu rs}|\right\}\;, \text{ where } |N| + |\nu| \le N_1$$

for fixed $N_1 \in \mathbb{N}$, and observe that

$$|D_x^N \tilde{a}_{rs}|\,,\; |D_x^N \tilde{b}_{\mu rs}| \;\le\; \varepsilon(\delta) \tag{2.4.7}$$

for $|N| \le N_1$, and

$$|D_x^N \partial_t^\nu \tilde{a}_{rs}|\,,\; |D_x^N \partial_t^\nu \tilde{b}_{\mu rs}| \;\le\; CM\delta^{1-|\nu|} \tag{2.4.8}$$

for $|N| + |\nu| \le N_1$, $|\nu| \ge 1$, and $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Using the notation $u_{rs} = D_x^r \partial_t^s u$ we are in position to estimate the terms in (2.4.4):

$$\big\langle [A(\alpha) - A(t)]u \big\rangle_{\lambda,\frac{\lambda}{2b},\Omega[\alpha,\beta]} \;\le\; \Big\| \sum_{|r|+2b\cdot s \le 2k} \tilde{a}_{rs} u_{rs} \Big\|_{\lambda,\frac{\lambda}{2b},\Omega[\alpha,\infty]} \tag{2.4.9}$$

and

$$\big\langle [B_\mu(\alpha) - B_\mu(t)]u \big\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2b},\Omega[\alpha,\beta]} \;\le\; \Big\| \sum_{|r|+2b\cdot s \le k_\mu} \tilde{b}_{\mu rs} u_{rs} \Big\|'_{\lambda_\mu,\frac{\lambda_\mu}{2b},\Omega'[\alpha,\infty]} \;\le$$

$$\le\; C \sum_{|r|+2b\cdot s \le k_\mu} \|\tilde{b}_{\mu rs} u_{rs}\|_{l-k_\mu,\frac{l-k_\mu}{2b},\Omega'[\alpha,\infty]}\;, \tag{2.4.10}$$

where the constant C is resulting from the embedding theorem in $\Omega[\alpha,\infty]$ and thus, is independent of α, β and functions.

We now wish to estimate the right-hand sides of (2.4.9) and (2.4.10). Starting with (2.4.9) we see that

$$\|\tilde{a}_{rs} u_{rs}\|^2_{\lambda,\frac{\lambda}{2b},\Omega[\alpha,\infty]} \;=$$

$$=\; \sum_{|\sigma| \le \lambda} \left\|D_x^\sigma[\tilde{a}_{rs} u_{rs}]\right\|^2_{0,0} \;+\; \sum_{\tau:\, 2b\cdot\tau \le \lambda} \left\|\partial_t^\tau[\tilde{a}_{rs} u_{rs}]\right\|^2_{0,0}\;, \tag{2.4.11}$$

where the norms in the right-hand side are taken over $\Omega[\alpha,\infty]$. Using (2.4.7) we obtain

$$\sum_{|\sigma|\le\lambda} \left\|D_x^\sigma[\tilde a_{rs}u_{rs}]\right\|_{0,0}^2 \;\le\; C\,[\varepsilon(\delta)]^2\|u_{rs}\|_{|\sigma|,0}^2 \;\le$$

$$\le\; C\,|\varepsilon(\delta)|^2\|u\|_{l,\frac{l}{2b}}^2 \;. \tag{2.4.12}$$

Further, for $|\tau|\ge 1$

$$\left\|\partial_t^\tau[\tilde a_{rs}u_{rs}]\right\|_{0,0}^2 \;\le\; C \sum_{|\tau_1|+|\tau_2|=|\tau|} \left\|\partial_t^{\tau_1}\tilde a_{rs}\partial_t^{\tau_2}u_{rs}\right\|_{0,0}^2 \;.$$

The terms with $\tau_2=0$ are estimated as in (2.4.12). For $\tau_2\ne 0$ we have $\partial_t^{\tau_1}\tilde a_{rs}=0$ if $t_j>\alpha_j+2\delta$ and according to (2.4.8)

$$\left\|\partial_t^{\tau_1}\tilde a_{rs}\partial_t^{\tau_2}u_{rs}\right\|_{0,0}^2 \;\le\; C^2M^2\delta^{2-2|\tau_1|}\int\limits_G\int\limits_{\alpha_1}^{\alpha_1+2\delta}\cdots\int\limits_{\alpha_m}^{\alpha_m+2\delta}\left|\partial_t^{\tau_2}u_{rs}\right|^2 dxdt \;. \tag{2.4.13}$$

We now remind that $u(x,t)$ is compatible with zero for $t_j=\alpha_j$ and use the following property of such functions: for a smooth function v on $\mathbb{R}$ which is compatible with zero for $t=\alpha\in\mathbb{R}$ we have

$$v(t) \;=\; \int\limits_\alpha^t v'(\theta)\,d\theta \;,$$

and thus for $t\in[\alpha,\alpha+2\delta]$:

$$|v(t)| \;\le\; \left[\int_\alpha^{\alpha+2\delta}|v'(\theta)|\,d\theta\right]^2 \;\le$$

$$\le\; 2\delta\int_\alpha^{\alpha+2\delta}|v'(\theta)|^2\,d\theta \;.$$

Integrating over t we obtain

$$\int\limits_\alpha^{\alpha+2\delta}|v(t)|^2\,dt \;\le\; 4\delta^2\int\limits_\alpha^{\alpha+2\delta}|v'(\theta)|^2\,d\theta \;.$$

Applying this inequality to $\partial_t^{\tau_2}u_{rs}$ in (2.4.13), we get after integrating over x that

$$\int_G \int_{\alpha_1}^{\alpha_1+2\delta} \cdots \int_{\alpha_m}^{\alpha_m+2\delta} \left|\partial_t^{\tau_2} u_{rs}\right|^2 dxdt \;\le\; 2^{2|\tau_1|}\,\delta^{2|\tau_1|} \int_G \int_{\alpha_1}^{\alpha_1+2\delta} \cdots \int_{\alpha_m}^{\alpha_m+2\delta} \left|\partial_t^{\tau} u_{rs}\right|^2 dxdt \;\le$$

$$\le\; 2^{2|\tau_1|}\,\delta^{2|\tau_1|} \left\|\partial_t^{\tau} u_{rs}\right\|^2_{0,0,\Omega[\alpha,\infty]} \;,$$

and thus

$$\left\|\partial_t^{\tau_1}\tilde{a}_{rs}\partial_t^{\tau_2}u_{rs}\right\|^2_{0,0} \;\le\; CM^2\delta^2 \left\|\partial_t^{\tau} u_{rs}\right\|^2_{0,0} \;\le\; CM^2\delta^2 \left\|u\right\|^2_{l,\frac{l}{2\vec{b}}} \;. \tag{2.4.14}$$

Applying (2.4.12) and (2.4.14) to (2.4.11) we obtain

$$\left\|\tilde{a}_{rs}u_{rs}\right\|^2_{\lambda,\frac{\lambda}{2\vec{b}}} \;\le\; C\left[(\varepsilon(\delta))^2+\delta^2\right]\left\|u\right\|^2_{l,\frac{l}{2\vec{b}}} \tag{2.4.15}$$

and, by analogy,

$$\left\|\tilde{b}_{\mu rs}u_{rs}\right\|^2_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}} \;\le\; C\left[(\varepsilon(\delta))^2+\delta^2\right]\left\|u\right\|^2_{l,\frac{l}{2\vec{b}}} \tag{2.4.16}$$

with the constants being independent of α,β and u . We now see from (2.4.4), (2.4.9), (2.4.10), (2.4.15) and (2.4.16), that

$$\mathcal{R}_{\alpha\beta}(u) \;\le\; \varepsilon_1(\delta)\left\|u\right\|^2_{l,\frac{l}{2\vec{b}},\Omega[\alpha,\infty]} \;\le\; 2\varepsilon_1(\delta)\left\langle u\right\rangle^2_{l,\frac{l}{2\vec{b}},\Omega[\alpha,\beta]} \tag{2.4.17}$$

where $\varepsilon_1(\delta)$ is independent of α,β and u and vanishes as $\delta\to 0$ $(\alpha_j\le \beta_j\le\alpha_j+\delta)$.

In this way for $\varepsilon_1(\delta)<\frac{1}{4}$ we obtain that

$$\mathcal{R}_{\alpha\beta}(u) \;\le\; \frac{1}{2}\langle u\rangle^2_{l,\frac{l}{2\vec{b}},\Omega[\alpha,\beta]} \;,$$

and in the view of (2.4.3) we get the desired inequality

$$C\left\langle u\right\rangle_{l,\frac{l}{2\vec{b}}} \;\le\; \left\langle A(t)u\right\rangle_{\lambda,\frac{\lambda}{2\vec{b}}} \;+\; \sum_{\mu=1}^{k}\left\langle B_\mu(t)u\right\rangle'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}}} \tag{2.4.18}$$

in $\Omega[\alpha,\alpha+\delta]$.

We now fix $\delta>0$ such that in the "shell" $\Omega[\alpha,\alpha+\delta]$ with $0\le\alpha_j\le\alpha_j+\delta\le T_j$ the inequality (2.4.18) takes place and suppose $r_j\delta<T_j\le(r_j+1)\delta$, where $r_j\in\mathbb{Z}$. We cut the domain $\Omega[0,T]$ into "shells" $\Omega[\vec{i}]:=\Omega[\delta\vec{i},\delta(\vec{i}+\vec{1})]$, where $\vec{i}=(i_1,...,i_m)$, $\vec{i}+\vec{1}=(i_1+1,...,i_m+1)$, where $0\le i_j\le r_j$ and for $i_j=r_j$ we put $i_j+1=T_j$. Thus, the last shell in each row may be smaller

than the others. Also, we denote by $\Omega'[\vec{i}]$ the "side" surface of the shell $\Omega[\vec{i}]$ and by

$$G[\vec{i};0] \;=\; \cup_{j=1}^{m} G[\vec{i};j;0] \ , \ \text{and} \ \ G[\vec{i};1] \;=\; \cup_{j=1}^{m} G[\vec{i};j;1]$$

its "lower" and "upper" bases, where

$$G[\vec{i};j;0] \;=\; \bigl\{(x,t) \in \Omega[\vec{i}] \ : \ t_j = i_j\delta\bigr\}$$

and

$$G[\vec{i};j;1] \;=\; \bigl\{(x,t) \in \Omega[\vec{i}] \ : \ t_j = (i_j+1)\delta\bigr\} \ .$$

As for the fixed shell the norms $\|\cdot\|$ and $\langle\cdot\rangle$ are equivalent then, in the view of (2.4.18), we obtain

$$C\,\|u\|_{l,\frac{l}{2\vec{b}},\Omega[\vec{i}]} \;\le\; \|A(t)u\|_{\lambda,\frac{\lambda}{2\vec{b}},\Omega[\vec{i}]} \;+\; \sum_{\mu=1}^{k} \|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}},\Omega'[\vec{i}]}$$

for functions $u \in H^{l,\frac{l}{2\vec{b}}}\bigl(\Omega[\vec{i}]\bigr)$ being compatible with zero on $G[\vec{i};0]$. Finally, for all $u \in H^{l,\frac{l}{2\vec{b}}}\bigl(\Omega[\vec{i}]\bigr)$ we obtain in the view of inductive polyparabolicity the two-sided a priori estimate

$$C\,\|u\|_{l,\frac{l}{2\vec{b}},\Omega[\vec{i}]} \;\le\; \|A(t)u\|_{\lambda,\frac{\lambda}{2\vec{b}},\Omega[\vec{i}]} \;+\; \sum_{\mu=1}^{k} \|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}},\Omega'[\vec{i}]} \;+$$

$$+\sum_{j=1}^{m}\sum_{\nu_j=1}^{\kappa_j-1} \|\partial_{t_j}^{\nu_j} u(x,t[j],i_j\delta)\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\vec{b}}[j],G[\vec{i};j;0]} \ , \tag{2.4.19}$$

where the constant C is independent of u and $i_j = 0,1,...,r_j$.

3). We now overcome to the proof of the estimate (2.3.9) in our case.

Let $u \in H^{l,\frac{l}{2\vec{b}}}\bigl(\Omega[0,T]\bigr)$, then $u \in H^{l,\frac{l}{2\vec{b}}}\bigl(\Omega[\vec{0}]\bigr)$ (here all $i_j = 0$ for all $j = 1,...,m$) and $u \in H^{l,\frac{l}{2\vec{b}}}\bigl(\Omega[(1,0,...,0)]\bigr)$ with $i_1 = 1$, $i_j = 0$ for $j > 1$. In these shells we have (2.4.19) with $\vec{i} = (0,...,0)$ and $\vec{i} = (1,0,...,0)$ respectively. In the view of Remark 2.1.4, item 3)

$$\sum_{\nu_j=1}^{\kappa_j-1} \|\partial_{t_j}^{\nu_j} u(x,t[j],(i_j+1)\delta)\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\vec{b}}[j],G[\vec{i};j;1]} \;\le\; C\,\|u\|_{l,\frac{l}{2\vec{b}},\Omega[\vec{i}]} \ . \tag{2.4.20}$$

¿From (2.4.20) and (2.4.19) with $\vec{i} = (0,...,0)$ and in the view of the obvious property of the sets G : $G[(1,0,...,0);1;0] \;=\; G[(0,0,...,0);1;1]$ we obtain that

$$C \sum_{\nu_1=1}^{\kappa_1-1} \|\partial_{t_1}^{\nu_1} u(x,\vec{0}[1],\delta)\|_{l_{\nu,1},\frac{l_{\nu,1}}{2\vec{b}}[1],G[\vec{0};1;1]}$$

$$\leq \|A(t)u\|_{\lambda,\frac{\lambda}{2\vec{b}},\Omega[\vec{0}]} + \sum_{\mu=1}^{k} \|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}},\Omega'[\vec{0}]} +$$

$$+ \sum_{j=1}^{m} \sum_{\nu_j=1}^{\kappa_j-1} \|\partial_{t_j}^{\nu_j} u(x,t[j],0)\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\vec{b}}[j],G[\vec{0};j;0]} .$$

Now, as obviously,

$$\|u\|_{l,\frac{l}{2\vec{b}},\Omega[\vec{0},\delta(2,0,...,0)]} \leq \|u\|_{l,\frac{l}{2\vec{b}},\Omega[\vec{0}]} + \|u\|_{l,\frac{l}{2\vec{b}},\Omega[(1,0,...,0)]}$$

and on the other hand, for $\vec{i}=\vec{0}$ and $(1,0,...,0)$ in polycylinders

$$\Omega[\vec{0},\delta(2,0,...,0)] = G\times[0,2\delta]\times[0,\delta]\times\cdots\times[0,\delta]$$

and

$$G[\vec{0}\cup(1,0,...,0);j;0] = \Omega[\vec{0}[j],\delta(2,0,...,0)[j]]$$

we have

$$\|A(t)u\|_{\lambda,\frac{\lambda}{2\vec{b}},\Omega[\vec{i}]} \leq \|A(t)u\|_{\lambda,\frac{\lambda}{2\vec{b}},\Omega[\vec{0},\delta(2,0,...,0)]} ,$$

$$\|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}},\Omega'[\vec{i}]} \leq \|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}},\Omega'[\vec{0},\delta(2,0,...,0)]}$$

and

$$\|\partial_{t_j}^{\nu_j} u(x,t[j],0)\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\vec{b}}[j],G[\vec{i};j;0]} \leq \|\partial_{t_j}^{\nu_j} u(x,t[j],0)\|_{l_{\nu,j},\frac{l_{\nu,j}}{2\vec{b}}[j],G[\vec{0}\cup(1,0,...,0);j;0]} .$$

And thus, we overcome to the desired inequality in the joined shell $[\vec{0},\delta(2,0,...,0)]$ unifying the inequalities (2.4.19) for $\vec{i}=\vec{0}$ and $(1,0,...,0)$:

$$C\,\|u\|_{l,\frac{l}{2\vec{b}},\Omega[\vec{0},\delta(2,0,...,0)]} \leq$$

$$\|A(t)u\|_{\lambda,\frac{\lambda}{2\vec{b}},\Omega[\vec{0},\delta(2,0,...,0)]} + \sum_{\mu=1}^{k} \|B_\mu(t)u\|'_{\lambda_\mu,\frac{\lambda_\mu}{2\vec{b}},\Omega'[\vec{0},\delta(2,0,...,0)]} +$$

$$+\sum_{j=1}^{m}\sum_{\nu_j=1}^{\kappa_j-1}\|\partial_{t_j}^{\nu_j}u(x,t[j],0)\|_{l_{\nu,j},\frac{l_{\nu,j}}{2b}[j],\Omega[\vec{0}[j],\delta(2,0,\ldots,0)[j]]}\;. \tag{2.4.21}$$

Continuing to move over $i_1 = 2,...,r_1$ we obtain (2.4.21) in the strip $\Omega[\vec{0},(T_1,0,...,0)]$. Afterwards we continue the process by adding a new strip $\Omega[\vec{0},(T_1,\delta,0,...,0)]$ (in the view of (2.4.20)) and combining the appropriate inequalities (2.4.21) for them into one. Then we combine strips into layers..., until we obtain (2.4.21) in the whole polycylinder $\Omega[0,T]$. It means, that (2.3.9) is proved together with the uniqueness of the problem.

4). We start the proof of the existence again in a small shell $\Omega[\alpha,\beta]$. Denote by $\mathcal{A}(t)$ an operator which is mapping the functions $u \in H^{l,\frac{l}{2b}}(\Omega[\alpha,\beta])$ being compatible with zero for $t_j = \alpha_j$ into vectors

$$(f,g) \;=\; (f,g_1,...,g_k) \;=\; \big(A(t),B_1(t)\big|_{\Omega'},...,B_k(t)\big|_{\Omega'}\big)$$

with elements being compatible with zero for $t_j = \alpha_j$ and lying in the space

$$\mathcal{H}^l\big(\Omega[\alpha,\beta]\big) \;:=\; H^{\lambda,\frac{\lambda}{2b}}\big(\Omega[\alpha,\beta]\big) \times \prod_{\mu=1}^{k} H^{\lambda_\mu,\frac{\lambda_\mu}{2b}}\big(\Omega'[\alpha,\beta]\big)\,.$$

Again we use the norms $\langle\cdot\rangle$. For the norm $\langle f,g\rangle$ of a vector $(f,g) \in \mathcal{H}^l\big(\Omega[\alpha,\beta]\big)$ we take the sum of the norms $\langle\cdot\rangle$ of respective components.

Also, we note that the vectors which are compatible with zero for $t_j = \alpha_j$ form a closed subspace in $\mathcal{H}^l\big(\Omega[\alpha,\beta]\big)$.

We now prove (cf. Sect.2 of this proof) that there exists δ such that in the shell $[\alpha,\beta]$ with $\beta_j - \alpha_j \le \delta$ the operator $\mathcal{A}(t)$ possesses the right inverse $\mathcal{A}^{-1}(t)$ being defined on the subspace of vectors $(f,g) \in \mathcal{H}^l\big(\Omega[\alpha,\beta]\big)$ which are compatible with zero for $t_j = \alpha_j$. The operator $\mathcal{A}^{-1}(t)$ maps this subspace boundedly into the subspace of functions $u \in H^{l,\frac{l}{2b}}\big(\Omega[\alpha,\beta]\big)$ being also compatible with zero for $t_j = \alpha_j$.

Indeed, in the view of Corollary 2.3.6, in our shell exists a bounded inverse $\mathcal{A}^{-1}(\alpha)$ to the operator $\mathcal{A}(\alpha)$. This operator is defined on vectors $(f,g) \in \mathcal{H}^l\big(\Omega[\alpha,\beta]\big)$ which are compatible with zero for $t_j = \alpha_j$, and maps them boundedly to functions $u \in H^{l,\frac{l}{2b}}\big(\Omega[\alpha,\beta]\big)$ being compatible with zero for $t_j = \alpha_j$ too. Thus,

$$\mathcal{A}(t)\mathcal{A}^{-1}(\alpha)(f,g) \;=\; (f,g) \;+\; (f^1,g^1)\;,$$

where

$$(f^1,g^1) \;=\; \big(\mathcal{A}(t)-\mathcal{A}(\alpha)\big)\mathcal{A}^{-1}(\alpha)(f,g) \;=\; \big(\mathcal{A}(t)-\mathcal{A}(\alpha)\big)u\;,$$

for $u = \mathcal{A}(\alpha)(f, g)$, but it follows from (2.4.2) that

$$\langle u \rangle_{l, \frac{l}{2\vec{b}}} \;\le\; C \,\langle f, g \rangle_l,$$

where C is independent neither of α and β nor of the functions. Now, as follows from (2.4.4) and (2.4.17),

$$\langle f^1, g^1 \rangle_l \;\le\; \varepsilon(\delta) \langle u \rangle_{l, \frac{l}{2\vec{b}}},$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$ equicontinuously in α . Thus,

$$\langle f^1, g^1 \rangle_l \;\le\; \frac{1}{2} \langle f, g \rangle_l$$

for sufficiently small shell $[\alpha, \beta]$. An inverse operator $\left[\mathcal{A}(t)\mathcal{A}^{-1}(\alpha)\right]^{-1}$ is defined in such shells and we simply put

$$\mathcal{A}^{-1}(t) \;=\; \mathcal{A}^{-1}(\alpha)\left[\mathcal{A}(t)\mathcal{A}^{-1}(\alpha)\right]^{-1} .$$

As in Section 3) of this proof we divide $\Omega[0, t]$ into shells $\Omega[\vec{i}]$. The measures of the shells are fixed and we may return to the initial norms $\|\cdot\|$ in every shell. The operator $\mathcal{A}^{-1}(t)$ provides in each $\Omega[\vec{i}]$ the solvability of the problem under the condition of compatibility with zero on its lower base, and we have to review the proof of Theorem 2.2.11 in this case in order to obtain the solvability of our problem in each $\Omega[\vec{i}]$ under nonhomogeneous initial conditions on the lower base, satisfying General Compatibility Condition.

Suppose now f, g_μ, ϕ_{j,ν_j} belong to the corresponding spaces and satisfy General Compatibility Condition in the shell $\Omega[\vec{0}]$, and, hence, there exists a solution $u(x, t)$ to our problem, belonging to $H^{l, \frac{l}{2\vec{b}}}(\Omega[\vec{0}])$.

Consider the next shell $\Omega[(1, 0, ..., 0)]$. We take the corresponding values of the above-built in $\Omega[\vec{0}]$ solution and its derivatives while $t_1 = \delta$ and $t_j = 0$, $j > 1$, for the initial conditions. We have to check General Compatibility Condition in the shell $\Omega[(1, 0, ..., 0)]$. Indeed, let u_0 be an extension of the above solution from $\Omega[\vec{0}]$ into $\Omega[0, (2\delta, \delta, ..., \delta)]$ belonging to $H^{l, \frac{l}{2\vec{b}}}(\Omega[0, (2\delta, \delta, ..., \delta)])$. Then the functions f and $A(t)u_0$ belong to $H^{\lambda, \frac{\lambda}{2\vec{b}}}(\Omega[0, (2\delta, \delta, ..., \delta)])$ and coincide for $t \in \Omega[\vec{0}]$ $(t_j < \delta, \forall j \in \{1, ..., m\}$).

Thus, the difference $f - A(t)u_0$ is compatible with zero as $t_1 = \delta$; $t_j = 0$, $j > 1$. In the same way $g_\mu - B_\mu(t)u_0\big|_{\Omega'}$ are compatible with zero. Hence, it is possible to construct a solution $u_1(x, t)$ in $\Omega[(1, 0, ..., 0)]$. The difference $u_1 - u_0$ is compatible with zero on the common part of the boundaries of $\Omega[\vec{0}]$ and $\Omega[(1, 0, ..., 0)]$, thus u_1 together with the above-built in $\Omega[\vec{0}]$ solution u defines the joint solution belonging to $H^{l, \frac{l}{2\vec{b}}}(\Omega[0, (2\delta, \delta, ..., \delta)])$.

Moving this construction over the shells and layers as in Section 3) of this proof, we obtain the solution to the problem in the whole $\Omega[0, T]$. ▷

We now state an analog of Theorem 2.4.1 for systems of equations. Let $A\big(x,t,D_x,\frac{\partial}{\partial t}\big)$ be a square $N\times N-$matrix with elements, being differential operators of orders $\le 2k$. Also, let u,f,g_μ,ϕ_{j,ν_j} be the column vectors of functions of the height N and $B_\mu\big(x,t,D_x,\frac{\partial}{\partial t}\big)$ be the line of differential operators of orders $\le k_j$ with infinitely smooth coefficients in $\overline{\Omega}_T$. The total number of lines is kN . The polyparabolicity conditions are the same with the single exception that Condition 1.3.2 is replaced by the following one:

$$\det A_0(x,\alpha,\xi,q^{2\vec{b}})\ \neq 0$$

as $q\in Q$, $x\in\overline{G}$, $|\xi|+M_q\neq 0$.

Theorem 2.4.2 *Theorem 2.3.3 remains valid for matrix operators* $A\big(x,t,D_x,\frac{\partial}{\partial t}\big)$ *and* $B_\mu\big(x,t,D_x,\frac{\partial}{\partial t}\big)$ *forming the polyparabolic problem in* Ω_T *(resp. inductively polyparabolic problem in* Ω_T *for the validity of (2.3.9) with the consequent uniqueness).*

We conclude this section with a few general remarks about possible generalizations and further results.

Remark 2.4.3 *A). The above results can be reformulated for operators acting on manifolds and in sections of bundles over them.*

B). It would be interesting to consider pseudodifferential boundary problems depending on a number of times. The one-time case was considered in the book (Grubb G. (1986)), also see (Grubb G. (1989)).

2.5 Compatibility Conditions. The existence of function u_0.

This section is devoted to the discussion of the existence problem for the function u_0 satisfying both Compatibility Condition 2.2.10 and boundary conditions (2.2.2) in the special case of boundary operators $B_\mu=\frac{\partial^\mu}{\partial\vec{n}^\mu}$.

As this problem was not directly stated in the literature, but follows from the results of the works by Lions J.L., Magenes E. (1972), Nikolskij S.M. (1953), and Yakovlev G.N. (1967). We formulate the appropriate Theorem and sketch the proof.

Consider a function $u\in H^{l,\frac{l}{2\vec{b}}}\big(\Omega_T\big)$, where Ω_T is a finite polycylinder. A map

$$\Xi\ :\ u(x,t)\ \longrightarrow\ \Big\{\frac{\partial^i u}{\partial\vec{n}^i}(x',t),\ \frac{\partial^{\nu_j}u}{\partial t_j^{\nu_j}}(x,t[j])\Big\} \tag{2.5.1}$$

may be defined in the view of Remark 2.1.4. Here $i = 0,...,\kappa_0 - 1 := [l - 1/2]$, $\nu_j = 0,...,\kappa_j - 1 = [l/(2b_j) - 1/2]$, and $\vec{n}$ is the vector of the inner normal to boundary Γ. The map Ξ is continuous from $H^{l,\frac{l}{2b}}(\Omega_T)$ into

$$\mathbb{F}_T \;:=\; \prod_{i=0}^{[l-1/2]} H^{l-i-1/2,\frac{l-i-1/2}{2b}}(\Omega'_T) \;\times\; \prod_{j=1}^{m}\prod_{\nu_j=0}^{\kappa_j-1} H^{l_{\nu,j},\frac{l_{\nu,j}}{2b}[j]}(\Omega_T[j]) \;, \qquad (2.5.2)$$

where $l_{\nu,j} = l - 2b_j\nu_j - b_j > 0$, but is not surjective.

Necessarily, functions $\frac{\partial^i u}{\partial \vec{n}^i}(x',t)$ and $\frac{\partial^{\nu_j} u}{\partial t_j^{\nu_j}}(x,t[j])$ satisfy *additional compatibility relations.* In order to introduce them we need some notation:

$$\delta_{t_j}^{\nu_j} u \;:=\; \left.\frac{\partial^{\nu_j} u}{\partial t_j^{\nu_j}}\right|_{t_j=0} , \quad \delta_{t_0}^{\nu_0} u \;:=\; \left.\frac{\partial^{\nu_0} u}{\partial \vec{n}^{\nu_0}}\right|_{\Gamma} \;:=\; \left.\frac{\partial^{\nu_0} u}{\partial t_0^{\nu_0}}\right|_{t_0=0} ,$$

$$\delta_{t_{\vec{\nu}_r}}^{\vec{\nu}_r} u \;:=\; \delta_{t_{j_r}}^{\nu_{j_r}} \cdot\cdot\cdot \delta_{t_{j_1}}^{\nu_{j_1}} \delta_{t_{j_0}}^{\nu_{j_0}}\, u \;,$$

where $\vec{\nu}_r = \{\nu_{j_0},...,\nu_{j_r}\}$ is a multiindex of the length $1 \le r+1 \le m+1$, and $0 \le j_\alpha \le m$ are different integers, where $0 \le \alpha \le r$. Throughout this Section we suppose that $\nu_j = 0,...,\kappa_j - 1$ for $j = 1,...,m$. Also, we denote by $P(\vec{\nu}_r)$ a multiindex obtained from $\vec{\nu}_r$ by an arbitrary permutation of indices $j_0,...,j_r$, that is,

$$\delta_{t_{P(\vec{\nu}_r)}}^{P(\vec{\nu}_r)} u \;:=\; \delta_{t_{P(j_r)}}^{\nu_{P(j_r)}} \cdot\cdot\cdot \delta_{t_{P(j_1)}}^{\nu_{P(j_1)}} \delta_{t_{P(j_0)}}^{\nu_{P(j_0)}}\, u \;.$$

Lemma 2.5.1 *(Local compatibility relations, c.f. Theorem 4 in Slobodeckij L.N. (1958)).*

We suppose that

$$1 - \frac{1}{2l}\Big(1 + 2\sum_{\alpha=1}^{r} b_{j_\alpha}\Big) \;>\; 0 \;. \qquad (2.5.3)$$

Then

$$\delta_{t_{\vec{\nu}_r}}^{\vec{\nu}_r} u \;=\; \delta_{t_{P(\vec{\nu}_r)}}^{P(\vec{\nu}_r)} u \qquad (2.5.4)$$

for all permutations of multiindex $\vec{\nu}_r$ satisfying the condition

$$\frac{\nu_0}{l} + \sum_{\alpha=1}^{r} \frac{2b_{j_\alpha}\nu_{j_\alpha}}{l} \;<\; 1 - \frac{1}{2l}\Big(1 + 2\sum_{\alpha=1}^{r} b_{j_\alpha}\Big) \;. \qquad (2.5.5)$$

Proof. It is sufficient to prove that

$$\delta_{t_j}^{\nu_j}\delta_{t_0}^{\nu_0} u \;=\; \delta_{t_0}^{\nu_0}\delta_{t_j}^{\nu_j} u \qquad (2.5.6)$$

and

$$\delta_{t_j}^{\nu_j}\delta_{t_i}^{\nu_i} u \;=\; \delta_{t_i}^{\nu_i}\delta_{t_j}^{\nu_j} u \qquad (2.5.7)$$

with $i, j > 0$. We prove (2.5.6) (relations (2.5.7) follow in the same way). As the problem is local, suppose $\frac{\partial}{\partial \vec{n}} = \frac{\partial}{\partial x_n} = \frac{\partial}{\partial t_0}$. Applying Remark 2.1.4 we obtain

$$\delta_{t_0}^{\nu_0} u(x', t) \in H^{l-\nu_0-1/2, \frac{l-\nu_0-1/2}{2\vec{b}}}(\Omega'_T)$$

and

$$\delta_{t_j}^{\nu_j} \delta_{t_0}^{\nu_0} u(x', t[j]) \in H^{l_{\nu,j}-\nu_0-1/2, \frac{l_{\nu,j}-\nu_0-1/2}{2\vec{b}}[j]}(\Omega'_T[j]) .$$

Similarly,

$$\delta_{t_j}^{\nu_j} u(x, t[j]) \in H^{l_{\nu,j}, \frac{l_{\nu,j}}{2\vec{b}}[j]}(\Omega_T[j])$$

and

$$\delta_{t_0}^{\nu_0} \delta_{t_j}^{\nu_j} u(x', t[j]) \in H^{l_{\nu,j}-\nu_0-1/2, \frac{l_{\nu,j}-\nu_0-1/2}{2\vec{b}}[j]}(\Omega'_T[j]) .$$

Obviously, these traces are correctly defined in the view of relations (2.5.3) and (2.5.5) and coincide on the dense subspace of C^∞−functions. Hence, (2.5.6) follows. $\triangleright$

We now start the discussion of General Compatibility Relations. First, we remind a particular case due to Lions J.L., Magenes E. (1972).

Proposition 2.5.2 *(Theorem 4.2.2 from Lions J.L., Magenes E. (1972)).*
Let $\frac{1}{r} + \frac{1}{s} = 2$, $r, s > \frac{1}{2}$ *and* $u(x,t) \in H^{r,s}((\mathbb{R}_+)^2, X)$, *where* X *is a Hilbert space.*

Then, applying Lemma 2.5.1 (being valid without changes for Hilbert space - valued functions), we can define the traces

$$u(x, 0) = f(x) \in H^{1/2}(\mathbb{R}_+, X) , \quad \textit{and} \quad u(0, t) = g(t) \in H^{1/2}(\mathbb{R}_+, X)$$

and obtain for them a global compatibility relation

$$\int_0^\infty \left\| f(\sigma^s) - g(\sigma^r) \right\|_X^2 \frac{d\sigma}{\sigma} < \infty . \qquad \textit{(2.5.8)}$$

We consider General Compatibility Relations in the case $\Omega_T = \Omega_+ = (\mathbb{R}_+)^m$ with Dirichlet boundary conditions in x for $x_n = 0$.

Now let $u \in H^{l, \frac{l}{2\vec{b}}}(\Omega_+)$ and suppose $1 - \frac{1}{2l}\left(1 + 2\sum_{j=1}^m b_j\right) > 0$. We put

$$\left. \frac{\partial^{\nu_j} u}{\partial t_j^{\nu_j}} \right|_{t_j = 0} = \phi_{j,\nu_j}(x, t[j])$$

and

$$\left. \frac{\partial^{\nu_0} u}{\partial x_n^{\nu_0}} \right|_{x_n = 0} = g_{\nu_0}(x', t) := \phi_{0,\nu_0}(x', t[0]) ,$$

where $\nu_j = 0, ..., \kappa_j - 1$. Then, according to (2.5.1) and (2.5.2), we have

$$\left\{ \phi_{j,\nu_j} , \ \phi_{0,\nu_0} \right\} \in \mathbb{F}_+ ,$$

where $\mathbb{F}_+$ is defined in the same manner as $\mathbb{F}_T$, but with Ω_T being substituted by Ω_+. It follows from Lemma 2.5.1 that

$$\delta^{\vec{\nu}_r[j_\alpha]}_{t_{\vec{\nu}_r}[j_\alpha]}\,\phi_{j_\alpha,\nu_{j_\alpha}} \;=\; \delta^{\vec{\nu}_r[j_\beta]}_{t_{\vec{\nu}_r}[j_\beta]}\,\phi_{j_\beta,\nu_{j_\beta}}\;, \tag{2.5.9}$$

for all multiindices $\vec{\nu}_r$ satisfying (2.5.5), where $\vec{\nu}_r[j_\alpha]$ is the multiindex of length r which is obtained by dropping ν_{j_α} in $\vec{\nu}_r$.

Also, according to Proposition 2.5.2, for each $\vec{\nu}_r$ such that (2.5.5) turns into the equality, we have

$$\Delta(\phi,\alpha,\beta,\nu_{j_\alpha},\nu_{j_\beta}) \;:= \tag{2.5.10}$$

$$\int_0^\infty \int_{(\mathbb{R}_+)^{m-1}} \int_{\mathbb{R}^{n-1}} \left| \frac{\partial^{\nu_{j_\alpha}}\phi_{j_\beta,\nu_{j_\beta}}}{\partial t_{j_\alpha}^{\nu_{j_\alpha}}}\big(x', t[j_\beta, j_\alpha : \sigma^{\frac{l}{2b_{j_\alpha}}}]\big) \;-\; \frac{\partial^{\nu_{j_\beta}}\phi_{j_\alpha,\nu_{j_\alpha}}}{\partial t_{j_\beta}^{\nu_{j_\beta}}}\big(x', t[j_\alpha, j_\beta : \sigma^{\frac{l}{2b_{j_\beta}}}]\big)\right|^2 dx'dt[j_\alpha, j_\beta]\frac{d\sigma}{\sigma} \;<\; \infty\;,$$

where by $t[j_\beta, j_\alpha : \sigma]$ we denote the vector $(t_0 = x_n, t_1, ..., t_m)$ with t_{j_β} being dropped and t_{j_α} being replaced by σ, $dt[j_\alpha, j_\beta]$ denotes the product of all differentials $dt_0 \cdots dt_m$ with dt_{j_α} and dt_{j_β} being dropped and $b_0 = 1/2$.

We denote by $\mathbb{F}_0$ the subspace of elements $\{\phi_{j,\nu_j}\}_{j=0}^m$ from $\mathbb{F}_+$ satisfying (2.5.9) and (2.5.10).

We introduce the following norm in $\mathbb{F}_0$:

$$\big\|\{\phi_{j,\nu_j}\}\big\|^2_{\mathbb{F}_0} \;=\; \big\|\phi_{j,\nu_j}\big\|^2_{\mathbb{F}_+} \;+\; \sum_{\alpha,\beta}\;\sum_{(\nu_{j_\alpha},\nu_{j_\beta})} \Delta(\phi,\alpha,\beta,\nu_{j_\alpha},\nu_{j_\beta})\;, \tag{2.5.11}$$

where $0 \le \alpha, \beta \le m$, $\alpha \ne \beta$ and $(\nu_{j_\alpha}, \nu_{j_\beta})$ means that the interior sum is taken over ν_{j_α} and ν_{j_β} from all multiindices $\vec{\nu}_r$ such that (2.5.5) turns into the equality.

Theorem 2.5.3 *Mapping*

$$u \;\longrightarrow\; \{\phi_{j,\nu_j}\}_{j=0}^m \tag{2.5.12}$$

defines a continuous linear surjection

$$H^{l,\frac{l}{2\vec{b}}}(\Omega_+) \;\longrightarrow\; \mathbb{F}_0 \tag{2.5.13}$$

for all $l, b_j > 0$ *satisfying (2.5.3).*

Proof. Only surjectivity needs to be proved. Suppose we have the strict inequality in (2.5.5) for $\vec{\nu}_m = (\nu_0, ..., \nu_m)$, i.e.

$$\frac{\nu_0}{l} + \sum_{\alpha=1}^m \frac{2b_{j_\alpha}\nu_{j_\alpha}}{l} \;<\; 1 - \frac{1}{2l}\Big(1 + 2\sum_{\alpha=1}^m b_{j_\alpha}\Big)\,. \tag{2.5.14}$$

In this case the surjectivity follows from Remark 2.1.4, item 4), see Theorem 8 from the work (Slobodeckij L.N. (1958)) for the proof. In the article (Yakovlev G.N. (1967)) the theorem analogous to ours was proved in the isotropic case where $2b_j = 1$, $j = 1, ..., m$.

The method of the proof of Yakovlev's theorem is based on the local extensions of traces to neighbourhoods of "ribs" and the following pasting of them as it was done in the proof of Theorem 6.2. in the paper of Nikolskij S.M. (1953). On the other hand Theorem 4.2.3 in Lions J.L., Magenes E. (1972) (the one - dimensional time case of the present one) gives us an opportunity to construct local extensions of traces in anisotropic case to neghbourhoods of "ribs" in both cases $\{x_n = 0\,,\ t_j = 0\}$ and $\{t_i = 0\,,\ t_j = 0\}$. Thus, we end the proof referring to the pasting procedure due to Nikolskij S.M. (1953). $\triangleright$

Bibliography

[1] Agmon S., Nirenberg L. (1963): Properties of solutions of ordinary differential equations in Banach space. Comm. Pure Appl. Math. **16**, 121 - 139.

[2] Agranovich M.S. (1990): Elliptic operators on closed manifolds. Itogi Nauki i Techn., Akad.Nauk SSSR, VINITI, Moscow. Ser. Sovremennye Problemy Matematiki. Fundamentalnyie Napravleniya [Current Problems in Mathematics. Fundamental Directions]. **63**, 5–129 (Russian). [English transl.: Encyclopaedia of Mathematical Sciences. **63**, Springer-Verlag, Berlin, –]

[3] Agranovich M.S., Dynin A.S. (1962): General boundary value problems for elliptic systems in an n-dimensional domain. Dokl. Akad. Nauk SSSR **146**, 511-514 (Russian) [English transl.: Soviet Math. Doklady **3**, 1323 - 1327].

[4] Agranovich M.S., Vishik M.I. (1964): Elliptic problems with a parameter and parabolic problems of general type. Usp. Mat. Nauk **19**, 53–161 (Russian). [English transl.: Russ. Math. Surv. **19** (1964), 53-159]

[5] Aizenberg L.A., Yuzhakov A.P. (1979): Integral representations and residues in multidimensional complex analysis. Nauka, Novosibirsk (Russian).

[6] Baez J.C., Segal I.E., Zhon Z.-F. (1990): The global Goursat problem and scattering for nonlinear wave equations. J.Funct.Anal.,**93**, 239-269.

[7] Besov O.V.,Il'in V.P.,Nikolskij S.M. (1975): Integral Representations of Functions and Imbedding Theorems. Nauka, Moscow (Russian) [English transl.: Scripta Series in Math. Winston & Sons, Washington D.C., V.I (1978), V.II (1979)]

[8] Brenner A. (1995): Polyparabolic operators with two-dimensional time. "mp_arc@fireant.ma.utexas.edu" preprint No. 95 - 105.

[9] Brenner A. (to appear): Multiparameter pseudodifferential operators and related spectral asymptotics.

[10] Brish N.I., Yurchuk N.I. (1968): Some new boundary value problems for a class of partial differential equations. Part I. Differentsial'nye Uravneniya,**4**, 1081-1101 (Russian). [English. transl.: Differential Equations 560-570]. Part II. Diff. Uravn.,**4** (Russian). [English. transl.: Diff. Equat. 770-775]

[11] Brish N.I., Yurchuk N.I. (1970): A mixed problem for certain pluriparabolic differential equations. Diff. Uravn., **6**, 1624-1630 (Russian). [English. transl.: Diff. Equat. 1234-1239]

[12] Brish N.I., Yurchuk N.I. (1971): Goursat's problem for abstract second order linear differential equations. Diff. Uravn., **7**, 1017-1030 (Russian). [English. transl.: Diff. Equat. 770-779]

[13] Chandrasekhar S. (1943): Stochastic problems in physics and astronomy. Rev. Modern Phys. **15**, 1 - .

[14] Drozhzhinov Yu.N. (1969): The stabilization of solutions of the generalized Cauchy problem for ultraparabolic equations. Izv. Akad. Nauk SSSR Ser. Mat. **33**, 368 - 378 (Russian) [English transl.: Math. USSR - Izvestija **3**, 345 - 355].

[15] Egorov Yu.V., Shubin M.A. (1988): Linear partial differential equations. Elements of classical theory. Itogi Nauki i Techn., Akad. Nauk SSSR, VINITI, Moscow. Ser. Sovremennye Problemy Matematiki. Fundamentalnyie Napravleniya [Current Problems in Mathematics. Fundamental Directions]. **30**, 5-265 (Russian). [English transl.: Encyclopaedia of Mathematical Sciences. **30**, Springer-Verlag, Berlin, 1992, 1-259]

[16] Eidelman S.D. (1990): Parabolic equations. Itogi Nauki i Techn., Akad. Nauk SSSR, VINITI, Moscow. Ser. Sovremennye Problemy Matematiki. Fundamentalnyie Napravleniya [Current Problems in Mathematics. Fundamental Directions]. **63**, 201–313 (Russian). [English transl.: Encyclopaedia of Mathematical Sciences. **63**, Springer-Verlag, Berlin, –]

[17] Eskin G.I. (1973): Boundary value problems for elliptic pseudodifferential equations. Nauka, Moscow (Russian) [English transl.: Transl. Math. Monogr. **52** (1981), Amer. Math. Soc. Providence]

[18] Friedman A. (1962): The Cauchy Problem in Several Time Variables. Journ. of Math. and Mech.,**11**, 859-889.

[19] Friedman A., Littman W. (1963): Partially Characteristic Boundary Problems for Hyperbolic Equations. Journ. of Math. and Mech., **12**, 213-224.

[20] Fushchich V.I.,Galitsyn A.S., Polubinskii A.S. (1990): A new mathematical model of heat conducting processes. Ukr.Math.Zh., **42**, 210-216 (Russian). [English transl.: Ukr.Math.Journ., **42**]

[21] Genchev T.G. (1963): Ultraparabolic equations. Dokl. Akad. Nauk SSSR **151**, 265 - 268 (Russian) [English transl.: Soviet Math. Doklady **4**, 979 - 982].

[22] Grubb G. (1986): Functional calculus of pseudo-differential boundary problems. Progress in Math. **65**, Birkhäuser, Boston.

[23] Grubb, G. (1989): Parabolic pseudo-differential boundary problems and applications. Copenh. Univ. Math. Dept. Report. Series No.4, 79 pp.

[24] Henkin G.M., Leiterer J. (1984): Theory of functions on Complex Manifolds. Academie - Verlag, Berlin.

[25] Hillion P. (1990a): The Goursat problem for the homogeneous wave equation. J.Math.Phys.,**31**, 1939-1941.

[26] Hillion P. (1990b): The Goursat problem for Maxwell's equations. J.Math. Phys.,**31**, 3085-3088.

[27] Hörmander L. (1983, 1985): The analysis of linear partial differential operators. I-IV, Springer-Verlag, Berlin.

[28] Iliev I.D. (1977): Mixed problem for a class of parabolic equations with multidimensional time variable. C.R. Acad. Bulgare Sci. **30**, 1525-1527.

[29] Kirichenko Yu. V. (1983): Parabolic systems of linear partial differential equations with multidimensional time. Diff. Uravn., **19**, 1790-1793 (Russian). [English. transl.: absent]

[30] Kolmogorov A.N. (1933): Zufällige Bewegungen (Zur Theorie der Brownschen Bewegung). Ann. of Math. **35**, 116 - 117.

[31] Kolmogorov A.N. (1934): Zur Theorie der Stetigen Zufälligen Prozesse. Math. Ann. **108**, 149 - 160.

[32] Kolmogorov A.N., Fomin S.V. (1976): Elements of Theory of Functions and Functional Analysis, Nauka, Moscow (Russian).

[33] Kudryavtsev L.D., Nikolskij S.M. (1988): Spaces of differentiable functions of several variables and imbedding theorems. Itogi Nauki i Techn., Akad. Nauk SSSR, VINITI, Moscow. Ser. Sovremennye Problemy Matematiki. Fundamentalnyie Napravleniya [Current Problems in Mathematics. Fundamental Directions]. **26**, 5-157 (Russian). [English transl.: Encyclopaedia of Mathematical Sciences. **26** (1990), Springer-Verlag, Berlin, 1 - 140]

[34] Lions J.L., Magenes E. (1972): Non homogeneous Boundary Value Problems and Applications. **1**, **2**, Springer-Verlag, Berlin.

[35] Marchuk G.I. (1958): Numerical Methods for nuclear reactor calculations. Suppl. Nos 3-4 of the Soviet Journal of Atomic Energy, Atomnaya Energiya, Moscow (Russian) [English transl.: Consultants Bureau, N.-Y., (1959)].

[36] Mikhailov V.P. (1957a): The analytic solution of Goursat problem for the system of differential equations. Dokl. Akad. Nauk SSSR (Russian) [English transl.: Soviet Math. Doklady **115**, 450-453].

[37] Mikhailov V.P. (1957b): Non-analytical solutions of Goursat's problem for a system of differential equations in two independent variables. Dokl. Akad. Nauk SSSR (Russian) [English transl.: Soviet Math. Doklady **117**, 759-762].

[38] Nikolskij S.M. (1953): Properties of certain classes of functions of several variables on differentiable manifolds. Mat. Sbornik **33(75)**, 261-326 (Russian) [English transl.: Transl. Amer. Math. Soc. Transl. ser. 2, **80** (1969), 38 - 118].

[39] Nikolskij S.M. (1977): Approximation of functions of several variables and imbedding theorems. 2-d ed., Nauka, Moscow (Russian). [English transl. of the 1-st ed.: Springer-Verlag, Berlin (1975)]

[40] Newton R.G. (1991): Uniqueness in some quasi-Goursat problems in $3+1$ dimensions and the inverse scattering problem. J.Math. Phys. **32**, 3130 - 3134.

[41] Polubinskij A.S. (1989): Thermal waves with finite velocity for a biparabolic equation. In "Stability of motion of solids and deformable systems.VI".,105-110. Akad. Nauk Ukr.SSR. Inst.Mat., Kiev (Russian).

[42] Rendall A.D. (1992): The characteristic initial value problem for the Einstein equations. Pitman Res. Notes Math. Ser. **253**.

[43] Rosenblyum G.V., Solomyak M.Z., Shubin M.A. (1989): Spectral theory of differential operators . Itogi Nauki i Techn., Akad.Nauk SSSR, VINITI, Moscow. Ser. Sovremennye Problemy Matematiki. Fundamentalnyie Napravleniya [Current Problems in Mathematics. Fundamental Directions]. **64**, 1–248 (Russian). [English transl.: Encyclopaedia of Mathematical Sciences. **64**, Springer-Verlag, Berlin, –]

[44] Shabat B.V. (1985): Introduction to Complex Analysis II. 3-d edition, Nauka, Moscow (Russian).

[45] Shubin M.A. (1978): Pseudodifferential operators and spectral theory. Nauka, Mos - cow (Russian). [English transl.: Springer Series in Soviet Math., Springer-Verlag, Berlin (1987)]

[46] Slobodeckij L.N. (1958): Generalized S.L.Sobolev spaces and their applications to boundary problems for partial differential equations. Uch. Zap. Leningr. Gos. Ped. Inst. **197**, 54-112 (Russian). [English transl.: Amer. Math. Soc. Transl. (2) **57** (1966), 207 - 276].

[47] Sobolev S.L. (1966): Equations of Mathematical Physics Nauka, Moscow (Russian).

[48] Taylor M.E. (1981): Pseudodifferential operators. Princeton Univ. Press, Princeton, New Jersey.

[49] Tersenov S.A. (1987): Boundary value problems for a class of ultraparabolic equations and their applications. Mat. Sbornik **133(175)**, 539-555 (Russian) [English transl.: Math. USSR - Sbornik. (1988), **61**, 529 -544].

[50] Treves F. (1980): Introduction to pseudodifferential and Fourier integral operators.I,II. Plenum Press, N.-Y.

[51] Uglum J. (1992): Quantum cosmology of $\mathbb{R} \times S^2 \times S^1$. Physical Review. D (3). **46**, 4365 - 4372.

[52] Vladimirov V.S., Drozhzhinov Yu.N. (1967): Generalized Cauchy Problem for an ultraparabolic equation. Izv. Akad. Nauk SSSR Ser. Mat. **31**, 1341 - 1360 (Russian) [English transl.: Math. USSR - Izvestija **1**, 1285 - 1304].

[53] Yakovlev G.N. (1967) (= Jakovlev G.N.): Traces of functions from the space W_p^l onto piecewise smooth surfaces. Mat. Sbornik **74(116)**, 526-543 (Russian).

[54] Yurchuk. N.I. (1968a): The Goursat problem for second order hyperbolic equations of special kind. Diff. Uravn.,**4**, 1333-1345. (Russian). [English transl.: Diff. Equat. 694-700].

[55] Yurchuk N.I. (1968b): A partially characteristic boundary value problem for a particular type of partial differential equation .I. Diff. Uravn.,**4**, 2258-2267 (Russian). [English transl.: Diff.Equat. 1167-1172]. Part II. Diff. Uravn.,**5**, (1969), 531-542 (Russian). [English transl.: Diff. Equat. 421-429].

[56] Yurchuk N.I. (1969): A partially characteristic mixed boundary value problem with Goursat initial conditions for linear equations with two-dimensional time. Diff. Uravn.,**5**, 898-910 (Russian). [English transl.: Diff. Equat. 652-661].