Some New Results On The 2d Stochastic Ising Model In The Phase Coexistence Region

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Abstract. We consider a Glauber dynamics reversible with respect to the two dimensional Ising model in a finite square of side L with open boundary conditions, in the absence of an external field and at large inverse temperature β. We prove that the gap in the spectrum of the generator restricted to the invariant subspace of functions which are even under global spin flip is much larger than the true gap. As a consequence we are able to show that there exists a new time scale t_{even}, much smaller than the global relaxation time t_{rel}, such that, with large probability, any initial configuration first relaxes to one of the two "phases" in a time scale of order t_{even} and only after a time scale of the order of t_{rel} it reaches the final equilibrium by jumping, via a large deviation, to the opposite phase. It also follows that, with large probability, the time spent by the system during the first jump from one phase to the opposite one is much shorter than the relaxation time.

Key Words: Ising model, Glauber dynamics, Relaxation time

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0. Introduction

We consider a Glauber–type dynamics for the two dimensional Ising model in a finite square Λ_L of side L with open boundary conditions, zero external field and at large inverse temperature β .

The equilibrium Gibbs measure μ_{Λ_L} at inverse temperature β is given by

$$\mu_{\Lambda_L}(\sigma) = \frac{e^{-\beta H_{\Lambda_L}(\sigma)}}{Z_{\Lambda_L}} \quad ; \quad Z_{\Lambda_L} = \sum_{\sigma \in \{-1,1\}^{\Lambda_L}} e^{-\beta H_{\Lambda_L}(\sigma)}$$

where $H_{\Lambda_L}(\sigma) = -\sum_{<x,y> \in \Lambda_L} \sigma(x)\sigma(y)$ and, as usual, $\sum_{<x,y> \in \Lambda_L}$ denotes the sum of nearest neighbors pairs in Λ_L.

The associated reversible Glauber–type dynamics is characterized by its generator $\mathcal{L}_{\Lambda_L}$ of the form

$$\mathcal{L}_{\Lambda_L} f(\sigma) = \sum_{x \in \Lambda_L} \sum_{a = \pm 1} c_x(\sigma, a)[f(\sigma^{x,a}) - f(\sigma)]$$

where $\sigma^{x,a}$ is the configuration σ with the spin $\sigma(x)$ replaced by a and the jump rates $c_x(\sigma, a)$ satisfy the detailed balance condition w.r.t. the Gibbs measure μ_{Λ_L}

$$\mu_{\Lambda_L}(\sigma) c_x(\sigma, a) = \mu_{\Lambda_L}(\sigma^{x,a}) c_x(\sigma^{x,a}, \sigma(x))$$

and a natural symmetry property under global spin flip

$$c_x(\sigma, a) = c_x(-\sigma, -a)$$

If β is larger than the critical value β_c the system undergoes a phase transition and the infinite volume dynamics is not ergodic. It is therefore interesting to see how this absence of ergodicity in the thermodynamic limit affects the ergodic behaviour in finite volume, particularly when the boundary conditions, e.g. open or periodic, do not break the natural symmetry under global spin flip.

In the relaxation process of the dynamics generated by $\mathcal{L}_{\Lambda_L}$ to its equilibrium measure given by μ_{Λ_L}, there exist at least two physically relevant time scales that, in the sequel, we will denote by t_{rel} and t_{even}.

The first one, t_{rel}, can be identified with the inverse of the gap in the spectrum of the generator $\mathcal{L}_{\Lambda_L}$ and it is the time scale characterizing the global relaxation process to the equilibrium Gibbs measure μ_{Λ_L}. The second one, t_{even}, can be identified with the inverse of the gap in the spectrum of the generator $\mathcal{L}_{\Lambda_L}$ restricted to the invariant subspace $\mathcal{M}$ of observables that are even with respect to a global spin flip, and it characterizes the relaxation process as $t \to \infty$ of the probability distribution of the Peierls contours generated by the dynamics at time t.

In [M], our basic reference, the asymptotics of t_{rel} as $L \to \infty$ was analyzed in details and it was shown that, for any $\varepsilon \in (0, \frac{1}{4}]$, any β large enough and any L:

$$e^{\beta\tau(\beta)L - C\beta L^{\frac{1}{2}+\varepsilon}} \le t_{rel} \le e^{\beta\tau(\beta)L + C\beta L^{\frac{1}{2}+\varepsilon}} \tag{0.1}$$

for some numerical constant C, where $\tau(\beta)$ is the surface tension in the direction of e.g. the horizontal axis. Recently such a result has been extended in [CGMS] to any $\beta > \beta_c$.

The reason for such slow global approach to equilibrium is the following. If the dynamics starts from one stable phase, e.g. that in which the majority of the spins is $+1$, then, in order to relax to equilibrium, it has to make a "jump" to the opposite stable phase; in particular the system, during the time evolution, has to go through the "bottleneck" represented by the set of configurations of zero (or ± 1 if the cardinality of the square is odd) magnetization. Such a set has an equilibrium probability whose inverse is of the order of the leading term in (0.1) (see [Sh]). The difficult part of the proof of (0.1) was to show that the inverse of such equilibrium probability actually gives the right asymptotic for the relaxation time t_{rel} (the lower bound is easily obtained while the upper bound required new ideas and new techniques). Recently it has been shown in [HY] that a relaxation time exponentially large in L occurs also if the boundary conditions are present but, roughly speaking, they do not favor any one of the two phases. This is the case, in particular, if the boundary conditions are randomly distributed according to a $\{1/2, 1/2\}$ Bernoulli measure.

It is important to notice that if the symmetry of the Gibbs measure under global spin flip is broken by homogeneous boundary conditions, e.g. + b.c, and thus one of the two phases become unstable, then the relaxation time becomes much shorter than it was before and in particular (see theorem 3.1 in [M]) it can be bounded from above by $e^{C_\varepsilon \beta L^{\frac{1}{2}+\varepsilon}}$. Equilibrium is, in this case, induced by the boundary by means of some sort of spins wave with the same sign of the boundary conditions, initially attached to the boundary and shrinking to zero as time goes on. Recently in [YaV] it has been shown that the relaxation time with plus boundary conditions has to diverge in the thermodynamic limit at least as some small power of L.

The above discussion suggests that if we look at observable that are *even* under global spin flip, i.e. do not distinguish between the two phases, then their average over the dynamics at time t will relax to the equilibrium value in a time much shorter than the global relaxation time. This is indeed the case and its proof represents the main scope of our paper. More precisely we will show that

Theorem. *There exists a positive constant β_o such that for any $\beta \geq \beta_o$*

$$\lim_{L\to\infty} \frac{1}{L}\log(\frac{t_{rel}}{t_{even}}) > 0$$

Remark Unfortunately we are not able to prove that t_{even} is e.g. bounded above by a power of L, as it is naturale to conjecture if one neglects the interaction between Peierls contours and assumes a "mean curvature"–type of motion for each one of them. It would also be interesting to know whether the gap of $\mathcal{L}_{\Lambda_L}$ restricted to the subspace $\mathcal{M}$ coincides with the second non zero eigenvalue in the spectrum of $\mathcal{L}_{\Lambda_L}$. In this case $\mathcal{L}_{\Lambda_L}$ would fit in the general framework of "metastable Markov semigroup" discussed in [D].

Nevertheless the above result, besides being of independent interest, has some nice consequences that make the picture found in [M] more precise. The first one (see theorem 3.1) says that, under the dynamics, *any* initial configuration relaxes to one of the two phases in a time scale $L^2 t_{even}$ much shorter than t_{rel}. The second one (see theorem 3.2) says that, once the system decides to jump from one phase to the opposite one, then, with large probability, it does it on a time scale not larger than $L^3 t_{even}$, again much shorter than the average time one has to wait in order to see the jump. One could say that in our case the Glauber dynamics has a behavior similar, in some sense, to that of a finite dimensional reversible Markov processes with invariant measure having a symmetric double well structure in the low noise regime (see e.g.the fundamental work by Freidlin and Ventzel [FV]). These applications are discussed for simplicity only for the Heat Bath dynamics (see section 1), but they could actually be extended to any attractive Glauber dynamics.

The paper is organized as follows. In section 1 we define the model and recall some basic notions from the theory of the Ising model that will be useful later on. In section 2 we prove the main theorem, in section 3 we make precise the conclusions mentioned above while section 4 is devoted to the proof of several technical lemma there are needed in section 2.

1. The model

In this section we define the model and the random dynamics that will be the object of study in the next sections.

1.1. The Ising model in a finite square with open boundary conditions.

Let $\mathbb{Z}^2$ be the usual two dimensional square lattice with sites $x = (x_1, x_2)$, equipped with the norm $|x| = |x_1| + |x_2|$. We will sometime consider $\mathbb{Z}^2$ as a graph with vertices the sites $x \in \mathbb{Z}^2$ and edges all pairs of sites x and y such that $|x - y| = 1$. Given $V \subset \mathbb{Z}^2$, we define the interior and exterior boundaries of V as :

$$\partial_{int} V \equiv \{ x \in V ; \exists y \notin V ; \quad |x - y| = 1 \}$$

$$\partial_{ext} V \equiv \{ x \notin V ; \exists y \in V ; \quad |x - y| = 1 \}$$

and the boundary ∂V as:

$$\partial V = \{ (x, y); \; x \in \partial_{int} V, \; y \in \partial_{ext} V \quad |x - y| = 1 \}$$

We also denote by $|V|$ the cardinality of V. Next, for any finite subset V of the square $\Lambda_L = \{ x \in \mathbf{Z^2} \; : \; 0 < x_i \le L \; , \; i = 1, 2 \}$, we define the energy $H_V^\tau(\sigma)$ in V of a configuration $\sigma \in \Omega_V \equiv \{-1, 1\}^V$ with boundary conditions τ on $\partial V \setminus \partial \Lambda_L$ as

$$H_V^\tau(\sigma) = -\frac{1}{2} \sum_{\substack{x, y \in V \\ |x - y| = 1}} (\sigma(x)\sigma(y) - 1) - \sum_{(x,y) \in \partial V \setminus \partial \Lambda_L} (\sigma(x)\tau(y) - 1) \tag{1.1}$$

and the associated Gibbs probability measure at inverse temperature β:

$$\mu_V^\tau(\sigma) = \frac{e^{-\beta H_V^\tau(\sigma)}}{Z(V, \tau)} \tag{1.2}$$

where the partition function $Z(V, \tau)$ is given by

$$Z(V, \tau) = \sum_\sigma e^{-\beta H_V^\tau(\sigma)} \tag{1.3}$$

If the boundary condition τ is the special configuration $\tau(x) = 1 \; \forall \, x \in \mathbb{Z}^2$, then in all our notation the superscript τ will be replaced by a simple $+$. We also set, for any function $f : \Omega_V \to \mathrm{I\!R}$,

$$\mu_V^\tau(f) = \sum_\sigma \mu_V^\tau(\sigma) f(\sigma)$$

Notice that if the set V coincides with Λ_L then (1.2) describes the usual Ising model in Λ_L with open (free) boundary conditions. If the set V is a rectangle R (with sides parallel to the coordinate axes), we will sometimes denote,whenever confusion may arise, by $\mu_R^{\tau_1,\tau_2,\tau_3,\tau_4}$ the Gibbs measure on R with the boundary conditions $\tau_1,\tau_2,\tau_3,\tau_4$ on the external boundary of its four sides ordered clockwise starting from the bottom side. We use the convention that, if one of the configurations τ_i is identically equal to $+1$ or -1, then we replace it by a $+$ or a $-$ sign while it is replaced by the symbol $\emptyset$ if the i^{th} side lies on the i^{th} side of Λ_L. Thus for example $\tau_1,+,\emptyset,+$ means τ_1 boundary conditions on the bottom side, plus boundary conditions on the vertical ones and open boundary condition on the top one.

As a next step we recall some monotonicity properties enjoyed by the Gibbs measure μ_V^τ, which easily follow from the well known FKG inequalities (see [FKG]), which will play a crucial role in the next sections. Given two configurations τ_1, τ_2 in $\Omega_{\mathbb{Z}^2}$, we say that $\tau_1 \le \tau_2$ iff

$$\tau_1(x) \le \tau_2(x) \;\; \forall\, x \in \mathbb{Z}^2$$

Then, for any pair of finite subsets $V_1 \subset V_2 \subset \Lambda_L$, any pair of boundary conditions τ_1,τ_2 and any function $f : \Omega_{V_1} \to \mathbb{R}$ which is increasing with respect to the above partial order, we have:

$$\mu_{V_1}^{\tau_1}(f) \le \mu_{V_1}^{\tau_2}(f) \quad ; \quad \mu_{V_2}^{\tau_2}(f) \le \mu_{V_1}^{+}(f) \tag{1.4}$$

1.2. Chains, $*$-chains and Peierls contours.

Given a sequence of sites $\mathcal{C} = x^1 \dots x^n$ we say that $\mathcal{C}$ is a *chain* if $|x^i - x^{i+1}| = 1$ for any $i = 1 \dots n-1$. A $*$-chain is defined in a similar way but with $|x-y|$ substituted by

$$|x-y|_\infty \equiv \max\{|x_1-y_1|, |x_2-y_2|\}$$

A chain $\mathcal{C}$ is called a plus chain for the configuration σ if $\sigma(x) = +1 \; \forall\, x \in \mathcal{C}$ and similarly for a $*$-chain. Two disjoint sets A and B are said to be connected by a plus chain (plus $*$-chain) in the configuration σ if there exists a plus chain (plus $*$-chain) $\mathcal{C}$ with $x^1 \in A$ and $x^n \in B$.

Next, if we denote by $\mathbb{Z}^{2^*}$ the dual lattice of $\mathbb{Z}^2$, we call *bond* any closed segment in $\mathbb{R}^2$ connecting two neighboring sites of $\mathbb{Z}^{2^*}$ and we say that two neighboring sites x and y in $\mathbb{Z}^2$ are separated by the bond b if their distance (as sites in $\mathbb{R}^2$) from b is equal to $\frac{1}{2}$. We also say that a pair of orthogonal bonds intersecting in a given site x^* of the dual lattice $\mathbb{Z}^{2^*}$ are a *linked pair of bonds* iff they are both on the same side of the forty-five degrees line across x^*. Given $V \subset \Lambda_L$, $\tau \in \Omega_{\Lambda_L \setminus V}$ and $\sigma \in \Omega_V$, we denote by $\mathcal{G}_V^\tau(\sigma)$ the collection of all bonds separating sites $x, y \in V \cup \partial_{ext} V$ where either $\sigma(x) \ne \sigma(y)$ or

$\sigma(x) \neq \tau(y)$. It is easy to see that $\mathcal{G}_V^\tau(\sigma)$ splits up in a unique way in a collection of contours $\Gamma_1(\sigma), \Gamma_2(\sigma), \dots \Gamma_n(\sigma)$, where a contour Γ is a sequence e_o, e_1, $e_2 \dots e_n$ of bonds such that:

i) $e_i \neq e_j$ for all i and j
ii) for all $i = 1 \dots n-1$ the bonds e_i and e_{i+1} have a common vertex in $\mathbb{Z}^{2^*}$.
iii) if e_i, e_{i+1}, e_j, e_{j+1} intersect at a given site x^*, then both pairs (e_i, e_{i+1}) and (e_j, e_{j+1}) are linked pairs of bonds.

We will denote by $\delta\Gamma$ the set of sites of $\mathbb{Z}^{2^*}$ where an *odd* number of bonds in Γ meet and we will say that Γ is closed if $\delta\Gamma = \emptyset$ and open otherwise. Then it is easy to check that any $\Gamma \in \mathcal{G}_V^\tau(\sigma)$ is either closed or $\delta\Gamma = \{x^*, y^*\}$; moreover x^* is the endpoint of a bond $b \in \Gamma$ separating either two sites $x, y \in \partial_{int}V$ or $x \in \partial_{int}V$, $y \in \partial_{ext}V$ and the same for y^*. The length $|\Gamma|$ of a contour will simply be the number of bonds in Γ. Given a contour Γ, we denote by $\Delta(\Gamma)$ the set of sites in $\mathbb{Z}^2$ such that either their distance (in $\mathbb{R}^2$) from Γ is $\frac{1}{2}$ or their distance from the set of vertices of $\mathbb{Z}^{2^*}$ where two non-linked pair of bonds of Γ meet is equal to $\frac{1}{\sqrt{2}}$.

1.3. A Class of block-Glauber dynamics for the Ising model.

In this final paragraph we define a class of Markov processes on Ω_{Λ_L} which are all reversible with respect to the Gibbs measure μ_{Λ_L} with open boundary conditions.
Following [M] each one of these auxiliary Markov processes will be indexed by a certain covering of the set Λ_L by blocks (i.e. subsets of Λ_L) and at a given updating only the spins inside a particular block will be changed. More precisely, let $\{R_i\}_{i=1\dots n}$ be a covering of Λ_L and let us define the generator $\mathrm{L}^{\{R_i\}}$ of the Markov process $\sigma_t^{\{R_i\}}$ indexed by the covering $\{R_i\}_{i=1\dots n}$ by:

$$(\mathrm{L}^{\{R_i\}} f)(\sigma) = \sum_i \sum_{\eta \in \Omega_{R_i}} \mu^\sigma_{R_i}(\eta) \, [f(\sigma^\eta) - f(\sigma)] \tag{1.5}$$

where σ^η is the configuration in Ω_{Λ_L} equal to η in R_i and to σ in $\Lambda_L \setminus R_i$. As it is easy to check, the operator $L^{\{R_i\}}$ is symmetric in the Hilbert space $L^2(\Omega_{\Lambda_L}, d\mu_{\Lambda_L})$ with real non positive eigenvalues

$$0 = \lambda_o(\{R_i\}) > -\lambda_1(\{R_i\}) \geq \dots \geq -\lambda_k(\{R_i\}); \quad k = 2^{|V|} - 1$$

In the sequel we will call $\text{gap}(L^{\{R_i\}})$ the value $\lambda_1(\{R_i\})$ and we will refer to the Markov process generated by $\mathrm{L}^{\{R_i\}}$ as the $\{R_i\}$-dynamics. The particular generator $L^{\{R_i\}}$ in which the elements R_i of the covering are the sites x of Λ_L, in the sequel denoted simply by L_{Λ_L}, is known in the literature as the Heat Bath process (HB-dynamics in the sequel) and it is an example of a Glauber dynamics for the Ising model, that is a Markov process on Ω_{Λ_L} with generator of the form:

$$(\mathcal{L}_{\Lambda_L} f)(\sigma) \; = \; \sum_{x\in\Lambda_L}\sum_{a=\pm1} c_x(\sigma,a)\,[f(\sigma^{x,a}) \; - \; f(\sigma)\,] \tag{1.6}$$

where $\sigma^{x,a}$ is obtained from σ by substituting the value $\sigma(x)$ with a and the jump rates $c_x(\sigma,a)$ satisfy the detailed balance condition:

$$\mu_{\Lambda_L}(\sigma)\,c_x(\sigma,a) \; = \; \mu_{\Lambda_L}(\sigma^{x,a})\,c_x(\sigma^{x,a},\sigma(x)) \tag{1.7}$$

the short range condition:

$$c_x(\sigma,a) = c_x(\eta,a) \quad \text{if} \quad \sigma(y)=\eta(y) \quad \forall\,|x-y|\le R \tag{1.8}$$

for some finite R and the positivity and boundedness condition:

$$0 < k \le \min_{x,a,\sigma} c_x(\sigma,a) \le \max_{x,a,\sigma} c_x(\sigma,a) \le \frac{1}{k} \tag{1.9}$$

for a suitable constant k independent of L, the side of our square.

2. Proof of the main result

Let Λ_L be the square $\Lambda_L = \{x \in \mathbb{Z}^2 \, : \, 0 < x_i \leq L \, , \, i = 1,2\}$ and let $\mathcal{L}_{\Lambda_L}$ be of the form (1.6). We assume that the jump rates $c_x(\sigma, a)$ satisfy (1.7), (1.8), (1.9) for μ_{Λ_L} and the following additional symmetry condition:

$$c_x(\sigma, a) = c_x(-\sigma, -a) \tag{2.1}$$

If $\mathcal{M}$ is defined as

$$\mathcal{M} \equiv \{f : \Omega_{\Lambda_L} \to \mathbb{R}; \quad f(\sigma) = f(-\sigma) \, \forall \, \sigma \in \Omega_{\Lambda_L}$$

then, because of (2.1), $\mathcal{M}$ is left invariant by $\mathcal{L}_{\Lambda_L}$. Thus we can consider the eigenvalues of $-\mathcal{L}_{\Lambda_L}|_{\mathcal{M}}$ and in particular the first positive eigenvalue that we denote by $\mathrm{gap}_{even}(\mathcal{L}_{\Lambda_L})$. By the variational characterization of the eigenvalues, $\mathrm{gap}_{even}(\mathcal{L}_{\Lambda_L})$ is given by:

$$\mathrm{gap}_{even}(\mathcal{L}_{\Lambda_L}) \; = \; \inf_{f \in \mathcal{M}} \frac{\mathcal{E}(f,f)}{Var(f)} \tag{2.2}$$

where $\mathcal{E}(f,f)$ denotes the Dirichlet form associated to $\mathcal{L}_{\Lambda_L}$:

$$\mathcal{E}(f,f) = \frac{1}{2} \sum_{\sigma} \sum_{x,a} \mu_{\Lambda_L}(\sigma) c_x(\sigma, a) [f(\sigma^{x,a}) - f(\sigma)]^2 \tag{2.3}$$

and $Var(f)$ denotes the variance of f with respect to μ_{Λ_L}.

Theorem 2.1. *There exists a positive constant β_o such that for any $\beta \geq \beta_o$*

$$\lim_{L \to \infty} \frac{1}{L} \log\left(\frac{\mathrm{gap}_{even}(\mathcal{L}_{\Lambda_L})}{\mathrm{gap}(\mathcal{L}_{\Lambda_L})}\right) > 0$$

Remark We observe that observable $f \in \mathcal{M}$ are such that they depend *only* on the Peierls contours $\mathcal{G}_{\Lambda_L}(\sigma)$ of the configuration σ and not on the sign of the spins. Thus we can conclude from the theorem that the probability distribution of $\mathcal{G}_{\Lambda_L}(\sigma_t)$ converges to the equilibrium measure over the contours in a time of the order of $\mathrm{gap}_{even}(\mathcal{L}_{\Lambda_L})^{-1}$, which is much shorter than the relaxation time $\mathrm{gap}(\mathcal{L}_{\Lambda_L})^{-1}$ of the probability distribution of σ_t

Proof of theorem 2.1. First of all we observe that, because of (1.9), the Dirichlet form of $\mathcal{L}_{\Lambda_L}$ can be estimated, apart from a constant factor, from above and from below by the Dirichlet form of L_{Λ_L}, the "Heat Bath" generator. Therefore it is enough to prove the result only for L_{Λ_L}. Next we observe that

$$\lim_{\beta \to \infty} \lim_{L \to \infty} -\frac{1}{\beta L} \log(\mathrm{gap}(\mathrm{L}_{\Lambda_L})) \; = \; \lim_{\beta \to \infty} \tau(\beta) = 2 \tag{2.4}$$

where $\tau(\beta)$ is the surface tension in the horizontal direction. In (2.4) we used theorem 4.1 in [M] to derive the first main equality and standard results on the surface tension $\tau(\beta)$ (see e.g. [DKS]) to compute the limit $\beta \to \infty$. It is therefore enough to show that there exists a positive constant $\delta < 1$ such that, for all sufficiently large β, we have:

$$\lim_{L\to\infty} -\frac{1}{\beta L} \log(\text{gap}_{even}(\mathrm{L}_{\Lambda_L})) \leq 2(1-\delta) \tag{2.5}$$

In order to prove the above basic result we follow the strategy employed in [M] to prove the first limit in (2.4).

Given $0 < \delta < \frac{1}{20}$ let us consider the covering of Λ_L whose elements are the following six rectangles:

$$R_i = \{x \in \Lambda_L : (i-1)(L_1+\delta L)/2 < x_2 \leq (i+1)(L_1+\delta L)/2 - \delta L\} \quad i = 1,2,3$$

$$R_j = \{x \in \Lambda_L : (j-4)(L_2+\delta L)/2 < x_1 \leq (j-2)(L_2+\delta L)/2 - \delta L\} \quad j = 4,5,6$$

where $L_1 = L(1-\delta)/2$ and $L_2 = L(1-8\delta)/2$.
In the sequel we will denote by $\mathcal{E}^{\{R_i\}}(f,f)$ the Dirichlet form associated to the associated generator $L^{\{R_i\}}$:

$$\mathcal{E}^{\{R_i\}}(f,f) = \frac{1}{2}\sum_i \sum_{\sigma,\eta} \mu_{\Lambda_L}(\sigma)\mu^{\sigma}_{R_i}(\eta)[f(\sigma^\eta) - f(\sigma)]^2$$

where, according to section 1, $\mu^{\sigma}_{R_i}$ denotes the Gibbs measure in R_i with boundary condition σ along $\partial_{ext}R_i \setminus \partial_{ext}\Lambda_L$. It is quite easy to check (see e.g. proposition A1.1 in [CM]) that for any f we have:

$$\mathcal{E}(f,f) \geq \frac{1}{4}\inf_{j,\tau} \text{gap}(L^{\tau}_{R_j})\, \mathcal{E}^{\{R_i\}}(f,f) \tag{2.6}$$

Thus, since the subspace $\mathcal{M}$ is obviously invariant also under $L^{\{R_i\}}$, we have

$$\text{gap}_{even}(\mathrm{L}_{\Lambda_L}) \geq \frac{1}{4}\inf_{j,\tau} \text{gap}(L^{\tau}_{R_j})\, \text{gap}_{even}(L^{\{R_i\}}) \tag{2.7}$$

Finally, thanks to corollary 2.1 of [M], we have:

$$\inf_{j,\tau} \text{gap}(L^{\tau}_{R_j}) \geq \inf_j \frac{1}{2|R_j|}\frac{e^{-4\beta}}{e^{-4\beta}+e^{+4\beta}}\, e^{-2\beta(L(1-\delta)+2)} \tag{2.8}$$

If we now combine (2.7), (2.8) we conclude that (2.5) will follows once we prove the following result:

Proposition 2.1. *There exists $\delta_o \leq \frac{1}{20}$ such that for any $\delta \leq \delta_o$ there exists $\beta_o(\delta)$ such that for any $\beta \geq \beta_o$ there exists another positive constant $k(\beta,\delta)$ such that:*

$$\text{gap}_{even}(L^{\{R_i\}}) \geq k(\beta,\delta) \quad \forall\, L$$

Proof of proposition 2.1. The proposition follows immediately if we can show that, in the above range of parameters, there exists a number $\alpha(\beta,\delta) \in (0,1)$ such that for any large enough L the restriction to the subspace $\mathcal{M}$ of the semigroup generated by $L^{\{R_i\}}$ at time $t=1$ is a contraction in the sup norm, with norm less than $1-\alpha$. In more probabilistic terms if:

$$\sup_\sigma |E_\sigma f(\sigma_{t=1}^{\{R_i\}})| \leq (1-\alpha)|f|_\infty \quad \forall\, f \in \mathcal{M} \tag{2.9}$$

where $E_\sigma f(\sigma_t^{\{R_i\}})$ denotes the average over the process at time t starting from σ. Let now $\{t_i\}_{i=1\dots}$ be the random times at which the initial configuration σ is updated. Then (2.9) follows if we show that there exists a number $\varepsilon(\beta,\delta) \in (0,1)$ such that:

$$\sup_\sigma |E_\sigma f(\sigma_{t_5}^{\{R_i\}})| \leq (1-\varepsilon)|f|_\infty \quad \forall\, f \in \mathcal{M} \tag{2.10}$$

We will now concentrate on the proof of (2.10). Notice that, because of the definition of the block-dynamics, the following "multiple integral" formula holds for $E_\sigma f(\sigma_{t_5}^{\{R_i\}})$:

$$E_\sigma f(\sigma_{t_5}^{\{R_i\}}) =$$

$$\sum_{i_1\dots i_5 \in\{1\dots 6\}} \frac{1}{6^5} \int d\mu_{R_{i_1}}^{\sigma}(\sigma_1) \int d\mu_{R_{i_2}}^{\sigma_1}(\sigma_2) \int d\mu_{R_{i_3}}^{\sigma_2}(\sigma_3) \int d\mu_{R_{i_4}}^{\sigma_3}(\sigma_4) \int d\mu_{R_{i_5}}^{\sigma_4}(\sigma_5) f(\sigma_5) \tag{2.11}$$

where the factor $\frac{1}{6^5}$ stands for the probability that during the first five updatings the rectangles $R_{i_1}\dots R_{i_5}$ are chosen in the given order. Therefore, in order to prove (2.10), it is sufficient to show that for any initial configuration σ there exists a special sequence (in the sequel called good sequence) $i_1(\sigma)\dots i_5(\sigma)$ and a number $\bar{\varepsilon}(\beta,\delta) \in (0,1)$ such that

$$\sup_\sigma \Big| \int d\mu_{R_{i_1}}^{\sigma}(\sigma_1) \int d\mu_{R_{i_2}}^{\sigma_1}(\sigma_2) \int d\mu_{R_{i_3}}^{\sigma_2}(\sigma_3) \int d\mu_{R_{i_4}}^{\sigma_3}(\sigma_4) \int d\mu_{R_{i_5}}^{\sigma_4}(\sigma_5) f(\sigma_5)\Big| \leq$$

$$(1-\bar{\varepsilon})|f|_\infty \tag{2.12}$$

In order to define the set of good sequences we first need the following key result. Given a rectangle R with horizontal side L and vertical ones $\frac{L}{2}(1-\delta)$, $0<\delta<\frac{1}{20}$:

$$R = \{x;\ 0 < x_1 \leq L,\ 0 < x_2 \leq \frac{L}{2}(1-\delta)\}$$

let us denote by M_δ the vertical strip $\{x \in R : \frac{L}{2}(1-4\delta) \le x_1 \le \frac{L}{2}(1+4\delta)\}$ and let ∂_i, $i = 1 \dots 4$ be that part of $\partial_{int} R$ adjacent to the i^{th} side ordered clockwise starting from the bottom one. Given a vertical open contour Γ in R, namely an open contour whose first and last bond separate two sites in the top and bottom part of ∂R respectively, we will say that Γ is of type $(+,-)$ if the spins on the left part of $\Delta(\Gamma)$ are plus and the spins on the right part of $\Delta(\Gamma)$ are minus and similarly for $(-,+)$ type. Let us then consider the following four events:

$$\begin{aligned}
S^+ &\equiv \big\{\sigma; \exists \text{ a plus } *\text{-chain } \mathcal{C} \subset \{x \in R : dist(x, \partial_3) \le 3\delta L\} \text{ connecting } \partial_2 \text{ with } \partial_4\big\} \\
S^- &\equiv \big\{\sigma; \exists \text{ a minus } *\text{-chain } \mathcal{C} \subset \{x \in R : dist(x, \partial_3) \le 3\delta L\} \text{ connecting } \partial_2 \text{ with } \partial_4\big\} \\
C^{(+,-)} &\equiv \big\{\sigma; \exists \text{ an open } (+,-) \text{ vertical contour } \Gamma \text{ with } \Delta(\Gamma) \subset M_\delta\,;\ \exists \text{ a plus } *\text{-chain} \\
&\qquad \mathcal{C}_1 \subset \{x \in R : dist(x, \partial_3) \le 3\delta L\} \text{ connecting } \partial_2 \text{ with } \Delta(\Gamma);\ \exists \text{ a minus} \\
&\qquad *\text{-chain } \mathcal{C}_2 \subset \{x \in R : dist(x, \partial_3) \le 3\delta L\} \text{ connecting } \partial_4 \text{ with } \Delta(\Gamma)\big\} \\
C^{(-,+)} &\equiv \big\{\sigma; \exists \text{ an open } (-,+) \text{ vertical contour } \Gamma \text{ with } \Delta(\Gamma) \subset M_\delta\,;\ \exists \text{ a minus } *\text{-chain} \\
&\qquad \mathcal{C}_1 \subset \{x \in R : dist(x, \partial_3) \le 3\delta L\} \text{ connecting } \partial_2 \text{ with } \Delta(\Gamma);\ \exists \text{ a plus} \\
&\qquad *\text{-chain } \mathcal{C}_2 \subset \{x \in R : dist(x, \partial_3) \le 3\delta L\} \text{ connecting } \partial_4 \text{ with } \Delta(\Gamma)\big\}
\end{aligned} \tag{2.13}$$

Warning In the sequel, for notation convenience, we will denote with the same symbol $\varepsilon(L)$ any error term in our estimates which is exponentially small in the side L of our square. In particular, when adding two (or a finite number independent of L) error terms coming from two different estimates we will write $2\varepsilon(L)$ and so forth. Then we have:

Lemma 2.1. *There exists $\delta_o \le \frac{1}{20}$ such that for any $\delta \le \delta_o$ there exists $\beta_o(\delta)$, $k(\delta) > 0$ and L_o such that for any $\beta \ge \beta_o$ and any $L \ge L_o$:*

$$\sup_\sigma \mu_R^{\emptyset,\emptyset,\sigma,\emptyset}((S^+ \cup S^- \cup C^{(+,-)} \cup C^{(-,+)})^c) \le \varepsilon(L)$$

The proof, based on the Peierls argument, is postponed to section 4.

Remark Clearly an analogous result holds if the boundary condition σ is on the bottom side ∂_1 and, in the definition of the events $S^+, S^-, C^{(+,-)}, C^{(-,+)}$, the third side ∂_3 is substituted with ∂_1. For simplicity, however, we will keep the same notation $S^+, S^-, C^{(+,-)}, C^{(-,+)}$ for the modified events whenever confusion does not arise.

Using the above result we can conclude that for *any* σ :

$$\max\big\{\mu_{R_1}^\sigma(S^+)\,,\ \mu_{R_1}^\sigma(S^-)\,,\ \mu_{R_1}^\sigma(C^{(+,-)})\,,\ \mu_{R_1}^\sigma(C^{(-,+)})\big\} \ge \frac{1}{5}$$

and similarly for R_3:

$$\max\left\{\mu^\sigma_{R_3}(S^+)\ ,\ \mu^\sigma_{R_3}(S^-)\ ,\ \mu^\sigma_{R_3}(C^{(+,-)})\ ,\ \mu^\sigma_{R_3}(C^{(-,+)})\right\}\geq\frac{1}{5}$$

We are now in a position to define the set of good sequences $i_1\dots i_5$ for a given starting configuration σ.

Definition We say that the sequence $i_1\dots i_5$, $i_j\in\{1\dots 6\}$ is good if:

a) $i_1=1, i_2=2, i_3=3$ and i_4, i_5 arbitrary if

$$\mu^\sigma_{R_1}(S^+)\ >\ \frac{1}{5}\ \text{or}\ \mu^\sigma_{R_1}(S^-)\ >\ \frac{1}{5}$$

b) $i_1=3, i_2=2, i_3=1$ and i_4, i_5 arbitrary if condition (a) above is violated and

$$\mu^\sigma_{R_3}(S^+)\ >\ \frac{1}{5}\ \text{or}\ \mu^\sigma_{R_3}(S^-)\ >\ \frac{1}{5}$$

c) $i_1=1, i_2=3, i_3=4, i_4=5, i_5=6$ if conditions (a) and (b) above are violated and

$$\mu^\sigma_{R_1}(C^{(+,-)})\ >\ \frac{1}{5}\ \text{and}\ \mu^\sigma_{R_3}(C^{(+,-)})\ >\ \frac{1}{5}$$

or

$$\mu^\sigma_{R_1}(C^{(-,+)})\ >\ \frac{1}{5}\ \text{and}\ \mu^\sigma_{R_3}(C^{(-,+)})\ >\ \frac{1}{5}$$

d) $i_1=1, i_2=3, i_3=2, i_4=1, i_5=3$ if conditions (a), (b) and (c) above are violated and

$$\mu^\sigma_{R_1}(C^{(+,-)})\ >\ \frac{1}{5}\ \text{and}\ \mu^\sigma_{R_3}(C^{(-,+)})\ >\ \frac{1}{5}$$

or

$$\mu^\sigma_{R_1}(C^{(-,+)})\ >\ \frac{1}{5}\ \text{and}\ \mu^\sigma_{R_3}(C^{(+,-)})\ >\ \frac{1}{5}$$

Given now σ and a good sequence $i_1\dots i_5$, let us estimate the left hand side of (2.12). We have to distinguish among the different possibilities a)...d).

2.1. Case a).

Without loss of generality we can assume that $\mu^\sigma_{R_1}(S^+)\ >\ \frac{1}{5}$. Then we write:

$$|\int d\mu^\sigma_{R_1}(\sigma_1)\int d\mu^{\sigma_1}_{R_2}(\sigma_2)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)\int d\mu^{\sigma_3}_{R_{i_4}}(\sigma_4)\int d\mu^{\sigma_4}_{R_{i_5}}(\sigma_5)f(\sigma_5)|=$$

$$= \left| \int d\mu^{\sigma}_{R_1}(\sigma_1) \int d\mu^{\sigma_1}_{R_2}(\sigma_2) \int d\mu^{\sigma_2}_{R_3}(\sigma_3) g(\sigma_3) \right| \tag{2.14}$$

where we have set

$$g(\sigma_3) \equiv \int d\mu^{\sigma_3}_{R_{i_4}}(\sigma_4) \int d\mu^{\sigma_4}_{R_{i_5}}(\sigma_5) f(\sigma_5)$$

Notice that, by construction, $g \in \mathcal{M}$ and $|g|_\infty \le |f|_\infty$. If in (2.14) we write

$$1 = \chi_{S^+}(\sigma_1) + \chi_{(S^+)^c}(\sigma_1)$$

we get that the r.h.s of (2.14) is smaller or equal than:

$$\left| \int d\mu^{\sigma}_{R_1}(\sigma_1 | S^+) \mu^{\sigma}_{R_1}(S^+) \int d\mu^{\sigma_1}_{R_2}(\sigma_2) \int d\mu^{\sigma_2}_{R_3}(\sigma_3) g(\sigma_3) \right| +$$

$$+ \left| \int d\mu^{\sigma}_{R_1}(\sigma_1 | (S^+)^c) \mu^{\sigma}_{R_1}((S^+)^c) \int d\mu^{\sigma_1}_{R_2}(\sigma_2) \int d\mu^{\sigma_2}_{R_3}(\sigma_3) g(\sigma_3) \right| \tag{2.15}$$

The second term in (2.15) is trivially estimated by

$$\mu^{\sigma}_{R_1}((S^+)^c) |g|_\infty \le \mu^{\sigma}_{R_1}((S^+)^c) |f|_\infty \tag{2.16}$$

In order to estimate the first term we need the following technical easy lemma.

Lemma 2.2. *In the same range of parameters as proposition 2.1 and for any function F depending only on the spins in $R_1 \setminus R_2$ we have:*

$$\left| \int d\mu^{\sigma}_{R_1}(\sigma_1 | S^+) F(\sigma_1) - \int d\mu^{+}_{R_1}(\sigma_1) F(\sigma_1) \right| \le \varepsilon(L) |F|_\infty$$

The proof is postponed to section 4.

Thus we have that the first term in the r.h.s. of (2.15) is estimated by:

$$\left| \int d\mu^{\sigma}_{R_1}(\sigma_1 | S^+) \mu^{\sigma}_{R_1}(S^+) \int d\mu^{\sigma_1}_{R_2}(\sigma_2) \int d\mu^{\sigma_2}_{R_3}(\sigma_3) g(\sigma_3) \right| \le$$

$$\le \mu^{\sigma}_{R_1}(S^+) \left| \int d\mu^{+}_{R_1}(\sigma_1) \int d\mu^{\sigma_1,\sigma}_{R_2}(\sigma_2) \int d\mu^{\sigma_2}_{R_3}(\sigma_3) g(\sigma_3) \right| + \varepsilon(L) |f|_\infty \tag{2.17}$$

Notice that in the l.h.s of (2.17) the configuration σ_1 in the first sum must coincide with the initial configuration σ in $\Lambda_L \setminus R_1$. Thus the boundary conditions for the second rectangle

R_2 are σ_1 below and σ above. This fact justifies our notation $\mu_{R_2}^{\sigma_1,\sigma}$ in the r.h.s of (2.17). As before, we write in the sum over σ_2: $1 \;=\; \chi_{\bar{S}^+}(\sigma_2) \;+\; \chi_{(\bar{S}^+)^c}(\sigma_2)$, where:

$$\bar{S}^+ \;=\; \{\sigma; \exists\, \text{a plus } *\text{-chain } \mathcal{C} \subset \{x \in R_2 : dist(x, \partial_3) \;\le\; \frac{L}{8}(1-\delta)\} \text{ connecting } \partial_2 \text{ with } \partial_4\}$$

We thus get:

$$\mu_{R_1}^{\sigma}(S^+)| \int d\mu_{R_1}^{+}(\sigma_1) \int d\mu_{R_2}^{\sigma_1,\sigma}(\sigma_2) \int d\mu_{R_3}^{\sigma_2}(\sigma_3) g(\sigma_3)| \;\le$$

$$\le\; \mu_{R_1}^{\sigma}(S^+)| \int d\mu_{R_1}^{+}(\sigma_1) \int d\mu_{R_2}^{\sigma_1,\sigma}(\sigma_2 | \bar{S}^+) \mu_{R_2}^{\sigma_1,\sigma}(\bar{S}^+) \int d\mu_{R_3}^{\sigma_2}(\sigma_3) g(\sigma_3)| \,+$$

$$+\, \mu_{R_1}^{\sigma}(S^+)| \int d\mu_{R_1}^{+}(\sigma_1) \int d\mu_{R_2}^{\sigma_1,\sigma}(\sigma_2 | (\bar{S}^+)^c) \mu_{R_2}^{\sigma_1,\sigma}((\bar{S}^+)^c) \int d\mu_{R_3}^{\sigma_2}(\sigma_3) g(\sigma_3)| \qquad (2.18)$$

The second term in the r.h.s of (2.18) is easily seen to be bounded from above (see e.g. proposition 4.1 of [M] for a similar statement) by

$$|f|_\infty \mu_{R_1}^{\sigma}(S^+) \int d\mu_{R_1}^{+}(\sigma_1) \mu_{R_2}^{\sigma_1,\sigma}((\bar{S}^+)^c) \;\le$$

$$\le\; |f|_\infty \mu_{R_1}^{\sigma}(S^+)[\int d\mu_{\bar{R}_1}^{+,+}(\sigma_1) \mu_{R_2}^{\sigma_1,\sigma}((\bar{S}^+)^c) \;+\; \varepsilon(L)] \qquad (2.19)$$

where

$$\bar{R}_1 \;=\; \{x \in R_1 : dist(x, \partial_3) \le \frac{L}{4}(1+\delta)\}$$

and $\varepsilon(L)$ goes to zero exponentially fast in L.
Using the monotonicity properties discussed in section 1 and the DLR equations, the r.h.s. of (2.19) is in turn bounded from above by:

$$|f|_\infty \mu_{R_1}^{\sigma}(S^+)[\int d\mu_{\bar{R}_1 \cup R_2}^{+,+}(\sigma_1) \mu_{R_2}^{\sigma_1,-}((\bar{S}^+)^c) \;+\; \varepsilon(L)] \le$$

$$\le |f|_\infty \mu_{R_1}^{\sigma}(S^+)[\int d\mu_{\bar{R}_1 \cup R_2}^{+,-}(\sigma_1) \mu_{R_2}^{\sigma_1,-}((\bar{S}^+)^c) \;+\; \varepsilon(L)] =$$

$$= |f|_\infty \mu_{R_1}^{\sigma}(S^+)[\mu_{\bar{R}_1 \cup R_2}^{+,-}((\bar{S}^+)^c) \;+\; \varepsilon(L)] \qquad (2.20)$$

Let us now examine the first term in the r.h.s of (2.18). We write:

$$\mu_{R_1}^{\sigma}(S^+)| \int d\mu_{R_1}^{+}(\sigma_1) \int d\mu_{R_2}^{\sigma_1,\sigma}(\sigma_2 | \bar{S}^+) \mu_{R_2}^{\sigma_1,\sigma}(\bar{S}^+) \int d\mu_{R_3}^{\sigma_2}(\sigma_3) g(\sigma_3)| \le$$

$$\leq \mu^{\sigma}_{R_1}(S^+)\mu^{+,\sigma}_{R_2}(\bar S^+)|\int d\mu^+_{R_1}(\sigma_1)\int d\mu^{\sigma_1,\sigma}_{R_2}(\sigma_2|\bar S^+)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)|+$$

$$+\mu^{\sigma}_{R_1}(S^+)|\int d\mu^+_{R_1}(\sigma_1)\int d\mu^{\sigma_1,\sigma}_{R_2}(\sigma_2|\bar S^+)(\mu^{\sigma_1,\sigma}_{R_2}(\bar S^+)-\mu^{+,\sigma}_{R_2}(\bar S^+))\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)| \quad (2.21)$$

The second term in the r.h.s of (2.21) can be estimated from above by:

$$|f|_\infty \mu^{\sigma}_{R_1}(S^+)[(\mu^{+,\sigma}_{R_2}(\bar S^+)-\int d\mu^+_{R_1}(\sigma_1)\mu^{\sigma_1,\sigma}_{R_2}(\bar S^+))]\leq$$

$$|f|_\infty \mu^{\sigma}_{R_1}(S^+)[(\mu^{+,\sigma}_{R_2}(\bar S^+)-\mu^{+,\sigma}_{\bar R_1\cup R_2}(\bar S^+)+\varepsilon(L)] \quad (2.22)$$

by the same argument that was to derive (2.20). Notice that the difference in height between the two rectangles R_2 and $\bar R_1\cup R_2$ is δL. This observation leads to the following lemma:

Lemma 2.3. *There exists $\delta_o\leq\frac{1}{20}$ such that for any $\delta\leq\delta_o$ there exist $\beta_o(\delta)$, $k(\delta_o)>0$ and L_o such that for any $\beta\geq\beta_o$ and any $L\geq L_o$:*

$$0\leq \mu^{+,\sigma}_{R_2}(\bar S^+)-\mu^{+,\sigma}_{\bar R_1\cup R_2}(\bar S^+)\leq k(\delta_o)\delta \qquad \forall\,\sigma \quad (2.23)$$

$$\mu^{+,\sigma}_{\bar R_1\cup R_2}(\bar S^+)\geq\frac{1}{5} \qquad \forall\,\sigma \quad (2.24)$$

The proof of the lemma is postponed to section 4.

In conclusion, the r.h.s of (2.18) is bounded from above by:

$$\mu^{\sigma}_{R_1}(S^+)\mu^{+,\sigma}_{R_2}(\bar S^+)|\int d\mu^+_{R_1}(\sigma_1)\int d\mu^{\sigma_1,\sigma}_{R_2}(\sigma_2|\bar S^+)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)|+$$

$$+|f|_\infty \mu^{\sigma}_{R_1}(S^+)[\mu^{+,-}_{\bar R_1\cup R_2}((\bar S^+)^c)\,+\,k(\delta_o)\delta\,+\,2\varepsilon(L)] \quad (2.25)$$

for any $\delta\leq\delta_o$, any $\beta\geq\beta_o$ and any $L\geq L_o$. Our goal at this point is to show that the first of the two dominant terms in (2.25) is exponentially small in L thanks to the fact that $g(\sigma)\in\mathcal{M}$ and $\mu_{\Lambda_L}(g)=0$. The first result that we need is the following.

Lemma 2.4. *There exists $\delta_o\leq\frac{1}{20}$ such that for any $\delta\leq\delta_o$ there exist $\beta_o(\delta)$, $k(\delta_o)>0$ and L_o such that for any $\beta\geq\beta_o$ and any $L\geq L_o$:*

$$|\int d\mu^+_{R_1}(\sigma_1)[\int d\mu^{\sigma_1,\sigma}_{R_2}(\sigma_2|\bar S^+)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)-\int d\mu^{\sigma_1,+}_{R_2}(\sigma_2)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)]|\leq$$

$$\leq|f|_\infty\varepsilon(L)$$

with $\varepsilon(L)$ exponentially small in L.

The proof of the lemma is postponed to section 4.

Using the lemma we get that

$$\mu^{\sigma}_{R_1}(S^+)\mu^{+,\sigma}_{R_2}(\bar S^+)|\int d\mu^+_{R_1}(\sigma_1)\int d\mu^{\sigma_1,\sigma}_{R_2}(\sigma_2|\bar S^+)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)|\le$$

$$\le \mu^{\sigma}_{R_1}(S^+)\mu^{+,\sigma}_{R_2}(\bar S^+)|\int d\mu^+_{R_1}(\sigma_1)\int d\mu^{\sigma_1,+}_{R_2}(\sigma_2)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)|+$$

$$+\mu^{\sigma}_{R_1}(S^+)\mu^{+,\sigma}_{R_2}(\bar S^+)|f|_\infty\varepsilon(L)\le$$

$$\le \mu^{\sigma}_{R_1}(S^+)\mu^{+,\sigma}_{R_2}(\bar S^+)|\int d\mu^+_{R_1\cup R_2}(\sigma_2)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)|+2\mu^{\sigma}_{R_1}(S^+)\mu^{+,\sigma}_{R_2}(\bar S^+)|f|_\infty\varepsilon(L) \tag{2.26}$$

where we used lemma 4.1 of section 4 to replace the measure $\mu^+_{R_1}$ with the measure $\mu^+_{R_1\cup R_2}$, and the DLR equations.

We now observe that $F(\sigma_2)\equiv\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)$ is a zero average even function of σ_2 depending only on the spins $\sigma_2(x)$ for $x\in\Lambda_L\setminus R_3$; moreover $|F|_\infty\le|f|_\infty$. For such kind of functions we can safely replace in (2.26) the measure $\mu^+_{R_1\cup R_2}$ with the measure $\mu_{\Lambda_L}(\sigma|m>0)$, where, for an arbitrary configuration σ, $m(\sigma)=\frac{\sum_{x\in\Lambda_L}\sigma(x)}{|\Lambda_L|}$ denotes the normalized magnetization. More precisely the following holds:

Lemma 2.5. *There exists $\delta_o\le\frac{1}{20}$ such that for any $\delta\le\delta_o$ there exists $\beta_o(\delta)$, $k(\delta_o)>0$ and L_o such that for any $\beta\ge\beta_o$ and any $L\ge L_o$*

$$|\int d\mu^+_{R_1\cup R_2}(\sigma_2)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)-\int d\mu_{\Lambda_L}(\sigma_2|m>0)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)|\le|f|_\infty\varepsilon(L) \tag{2.27}$$

with $\varepsilon(L)$ exponentially small in L.

The proof of the lemma is postponed to section 4.

Finally we notice that for any even function $F(\sigma)$

$$\mu_{\Lambda_L}(F)=\int d\mu_{\Lambda_L}(\sigma|m>0)F(\sigma) \tag{2.28}$$

so that

$$\int d\mu_{\Lambda_L}(\sigma_2|m>0)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)=\int d\mu_{\Lambda_L}(\sigma_2)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)=$$

$$\int d\mu_{\Lambda_L}(\sigma) f(\sigma) = 0 \tag{2.29}$$

where we used the DLR equations, the definition of $g(\sigma)$ and the fact that f has zero mean. In conclusion, by putting together (2.26)...(2.29), we get that the first main term in (2.25) satisfies:

$$\mu^{\sigma}_{R_1}(S^+)\mu^{+,\sigma}_{R_2}(\bar{S}^+)|\int d\mu^{+}_{R_1}(\sigma_1)\int d\mu^{\sigma_1,\sigma}_{R_2}(\sigma_2|\bar{S}^+)\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)| \leq$$

$$3\varepsilon(L)|f|_\infty \tag{2.30}$$

This allows us to conclude that the r.h.s of (2.14), our starting point, is bounded from above by:

$$|f|_\infty\{\mu^{\sigma}_{R_1}((S^+)^c) + \mu^{\sigma}_{R_1}(S^+)[\mu^{+,-}_{\bar{R}_1\cup R_2}((\bar{S}^+)^c) \;+\; k(\delta_o)\delta] \;+\; 6\varepsilon(L)\} \leq$$

$$\leq |f|_\infty\{1 - \frac{1}{25} + k(\delta_o)\delta \;+\; 6\varepsilon(L)\} \tag{2.31}$$

where we used the starting hypothesis, $\mu^{\sigma}_{R_1}(S^+) \geq \frac{1}{5}$, and the bound

$$\mu^{+,-}_{\bar{R}_1\cup R_2}((\bar{S}^+)^c) \leq \frac{4}{5}$$

which follows immediately from lemma 2.3. Thus (2.12) follows for δ and L small and large enough respectively and the proof is complete. ■

2.2. Case b).

This case is related to case a) by a 180^o rotation. Thus the same proof applies but starting from the top rectangle R_3 and ending in the bottom one, R_1.

2.3. Case c).

We have to bound the quantity:

$$|\int d\mu^{\sigma}_{R_1}(\sigma_1)\int d\mu^{\sigma}_{R_3}(\sigma_2)\int d\mu^{\sigma}_{R_4}(\sigma_3)\int d\mu^{\sigma}_{R_5}(\sigma_4)\int d\mu^{\sigma_4}_{R_6}(\sigma_5)f(\sigma_5)| \tag{2.32}$$

Without loss of generality we can suppose that

$$\mu^{\sigma}_{R_1}(C^{(+,-)}) \ > \frac{1}{5} \text{ and } \mu^{\sigma}_{R_3}(C^{(+,-)}) \ > \frac{1}{5}$$

As in case a) we write

$$\begin{aligned} 1 &= \chi_{C^{(+,-)}}(\sigma_1) + \chi_{(C^{(+,-)})^c}(\sigma_1) \\ 1 &= \chi_{C^{(+,-)}}(\sigma_2) + \chi_{(C^{(+,-)})^c}(\sigma_2) \end{aligned}$$

in the first and second integral respectively, and we bound from above (2.32) by:

$$\begin{aligned} \mu^{\sigma}_{R_1}(C^{(+,-)})\mu^{\sigma}_{R_3}(C^{(+,-)})| \int d\mu^{\sigma}_{R_1}(\sigma_1|C^{(+,-)}) \int d\mu^{\sigma}_{R_3}(\sigma_2|C^{(+,-)}) \dots \int d\mu^{\sigma_4}_{R_6}(\sigma_5) f(\sigma_5)| + \\ + |f|_\infty \mu^{\sigma}_{R_1}(C^{(+,-)})(1-\mu^{\sigma}_{R_3}(C^{(+,-)})) + \\ + |f|_\infty (1-\mu^{\sigma}_{R_1}(C^{(+,-)})) \end{aligned} \tag{2.33}$$

We will now analyze the term

$$| \int d\mu^{\sigma}_{R_1}(\sigma_1|C^{(+,-)}) \int d\mu^{\sigma}_{R_3}(\sigma_2|C^{(+,-)}) ... \int d\mu^{\sigma_4}_{R_6}(\sigma_5) f(\sigma_5)| \tag{2.34}$$

Let S^+_v be the analogous of the event S^+ for the "vertical" rectangle R_4:

$$S^+_v \ = \ \big\{\sigma; \exists \text{ a plus } *\text{-chain } \mathcal{C} \subset \{x \in R_4 : dist(x, \partial_4) \ \le \ 3\delta L\} \text{ connecting } \partial_1 \text{ with } \partial_3\big\}$$

Then we write in (2.34) $1 = \chi_{S^+_v}(\sigma_3) + \chi_{(S^+_v)^c}(\sigma_3)$ and get that the r.h.s of (2.34) is bounded from above by:

$$\begin{aligned} \sup_{\sigma} \mu^{\sigma}_{R_4}(S^+_v)| \int d\mu^{\sigma}_{R_4}(\sigma_3|S^+_v) \int d\mu^{\sigma_3}_{R_5}(\sigma_4) \int d\mu^{\sigma_4}_{R_6}(\sigma_5) f(\sigma_5)| + \\ + \int d\mu^{\sigma}_{R_1}(\sigma_1|C^{(+,-)}) \int d\mu^{\sigma}_{R_3}(\sigma_2|C^{(+,-)}) \mu^{\sigma_2}_{R_4}((S^+_v)^c)|f|_\infty \end{aligned} \tag{2.35}$$

Notice that the first term in (2.35) is identical, after counterclockwise rotation of 90^o, to the first term in (2.15). Thus we can repeat the reasoning that led us from (2.15) to (2.31) and get

$$\begin{aligned} \sup_{\sigma} \mu^{\sigma}_{R_4}(S^+_v)| \int d\mu^{\sigma}_{R_4}(\sigma_3|S^+_v) \int d\mu^{\sigma_3}_{R_5}(\sigma_4) \int d\mu^{\sigma_4}_{R_6}(\sigma_5) f(\sigma_5)| \le \\ \le \sup_{\sigma} \mu^{\sigma}_{R_4}(S^+_v)(\frac{4}{5} + k(\delta_o)\delta + 6\varepsilon(L)) \end{aligned} \tag{2.36}$$

The second term in (2.35) is estimated by the next lemma

Lemma 2.6. *There exists $\delta_o \leq \frac{1}{20}$ such that for any $\delta \leq \delta_o$ there exists $\beta_o(\delta)$, $k(\delta_o) > 0$ and L_o such that for any $\beta \geq \beta_o$ and any $L \geq L_o$*

$$|\int d\mu^{\sigma}_{R_1}(\sigma_1|C^{(+,-)})\int d\mu^{\sigma}_{R_3}(\sigma_2|C^{(+,-)})\mu^{\sigma_2}_{R_4}((S^{+}_{v})^c) \leq \varepsilon(L)$$

The proof of the lemma is postponed to section 4.

In conclusion, the r.h.s of (2.33) is bounded from above by

$$|f|_\infty[\mu^{\sigma}_{R_1}(C^{(+,-)})\mu^{\sigma}_{R_3}(C^{(+,-)})(\frac{4}{5}+k(\delta_o)\delta+7\varepsilon(L))+1-\mu^{\sigma}_{R_1}(C^{(+,-)})\mu^{\sigma}_{R_3}(C^{(+,-)})] \leq$$

$$\leq |f|_\infty[1-\frac{1}{125}+k(\delta_o)\delta+7\varepsilon(L)] \tag{2.37}$$

and (2.12) follows also in this case. ■

2.4. Case d).

We have to bound the quantity:

$$|\int d\mu^{\sigma}_{R_1}(\sigma_1)\int d\mu^{\sigma}_{R_3}(\sigma_2)\int d\mu^{\sigma}_{R_2}(\sigma_3)\int d\mu^{\sigma}_{R_1}(\sigma_4)\int d\mu^{\sigma_4}_{R_3}(\sigma_5)f(\sigma_5)| \tag{2.38}$$

Without loss of generality we can suppose that

$$\mu^{\sigma}_{R_1}(C^{(+,-)}) \ > \ \frac{1}{5} \text{ and } \mu^{\sigma}_{R_3}(C^{(-,+)}) \ > \ \frac{1}{5}$$

and we first bound from above (2.38) by:

$$\mu^{\sigma}_{R_1}(C^{(+,-)})\mu^{\sigma}_{R_3}(C^{(-,+)})|\int d\mu^{\sigma}_{R_1}(\sigma_1|C^{(+,-)})\int d\mu^{\sigma}_{R_3}(\sigma_2|C^{(-,+)})\int d\mu^{\sigma_2}_{R_2}(\sigma_3)g(\sigma_3)|+$$

$$+(1-\mu^{\sigma}_{R_1}(C^{(+,-)})\mu^{\sigma}_{R_3}(C^{(-,+)}))|f|_\infty \tag{2.39}$$

where

$$g(\sigma_3) \equiv \int d\mu^{\sigma_3}_{R_1}(\sigma_4)\int d\mu^{\sigma_4}_{R_3}(\sigma_5)f(\sigma_5) \tag{2.40}$$

Let us now define the events in R_2:

$$
\begin{aligned}
S^{++} &= \big\{\sigma; \exists \text{ two plus } *\text{-chains, } \mathcal{C}_1,\, \mathcal{C}_2 \text{ connecting } \partial_2 \text{ with } \partial_4 \text{ and such that} \\
&\qquad \mathcal{C}_1 \subset \{x \in R_2 : \frac{L}{12}(1-\delta) \le dist(x, \partial_3) \le \frac{L}{6}(1-\delta)\}\,; \\
&\qquad \mathcal{C}_2 \subset \{x \in R_2 : \frac{L}{12}(1-\delta) \le dist(x, \partial_1) \le \frac{L}{6}(1-\delta)\big\} \\
S^{--} &= \big\{\sigma; \exists \text{ two minus } *\text{-chains, } \mathcal{C}_1,\, \mathcal{C}_2 \text{ connecting } \partial_2 \text{ with } \partial_4 \text{ and such that} \\
&\qquad \mathcal{C}_1 \subset \{x \in R_2 : \frac{L}{12}(1-\delta) \le dist(x, \partial_3) \le \frac{L}{6}(1-\delta)\}\,; \\
&\qquad \mathcal{C}_2 \subset \{x \in R_2 : \frac{L}{12}(1-\delta) \le dist(x, \partial_1) \le \frac{L}{6}(1-\delta)\big\}
\end{aligned}
\tag{2.41}
$$

Then we have the following basic lemma

Lemma 2.7. *There exists $\delta_o \le \frac{1}{20}$ such that for any $\delta \le \delta_o$ there exists $\beta_o(\delta)$, $k(\delta_o) > 0$ and L_o such that for any $\beta \ge \beta_o$ and any $L \ge L_o$*

$$
|\int d\mu^{\sigma}_{R_1}(\sigma_1 | C^{(+,-)}) \int d\mu^{\sigma}_{R_3}(\sigma_2 | C^{(-,+)}) \mu^{\sigma_2}_{R_2}((S^{++} \cup S^{--})^c) \le \varepsilon(L)
$$

The proof of the lemma is postponed to section 4.

Using the lemma we can assume, without loss of generality, that

$$
\int d\mu^{\sigma}_{R_1}(\sigma_1 | C^{(+,-)}) \int d\mu^{\sigma}_{R_3}(\sigma_2 | C^{(-,+)}) \mu^{\sigma_2}_{R_2}(S^{++}) \ge \frac{1}{3} \tag{2.42}
$$

and we can bound from above the integral in the first term in the r.h.s. of (2.39) by

$$
\begin{aligned}
&\int d\mu^{\sigma}_{R_1}(\sigma_1 | C^{(+,-)}) \int d\mu^{\sigma}_{R_3}(\sigma_2 | C^{(-,+)}) \mu^{\sigma_2}_{R_2}(S^{++}) \sup_{\sigma} |\int d\mu^{\sigma}_{R_2}(\eta | S^{++}) g(\eta)| + \\
&\qquad + \int d\mu^{\sigma}_{R_1}(\sigma_1 | C^{(+,-)}) \int d\mu^{\sigma}_{R_3}(\sigma_2 | C^{(-,+)}) \mu^{\sigma_2}_{R_2}((S^{++})^c) |f|_{\infty}
\end{aligned}
\tag{2.43}
$$

Notice that in the first term of (2.43) the function $g(\eta)$ depends only on the spins $\eta(x)$; $x \in R_2 \setminus (R_1 \cup R_3)$. Therefore, as in the discussion of case a) (see lemma 2.2 and 2.6), we can safely replace the measure $\mu^{\sigma}_{R_2}(\eta | S^{++})$ with the measure $\mu_{\Lambda_L}(\eta | m > 0)$. More precisely:

$$
\sup_{\sigma} |\int d\mu^{\sigma}_{R_2}(\eta | S^{++}) g(\eta)| \le |\int d\mu_{\Lambda_L}(\eta | m > 0) g(\eta)| + \varepsilon(L) |f|_{\infty} \tag{2.44}
$$

Since the function $g(\eta)$ is even with zero mean,the first term in the r.h.s. of (2.44) is zero. In conclusion, if we combine together (2.41)...(2.44), we have bounded from above (2.39) by:

$$\mu^{\sigma}_{R_1}(C^{(+,-)})\mu^{\sigma}_{R_3}(C^{(-,+)})[\int d\mu^{\sigma}_{R_1}(\sigma_1|C^{(+,-)})\int d\mu^{\sigma}_{R_3}(\sigma_2|C^{(-,+)})\mu^{\sigma}_{R_2}((S^{++})^c)]|f|_\infty+$$

$$+(1-\mu^{\sigma_2}_{R_1}(C^{(+,-)})\mu^{\sigma}_{R_3}(C^{(-,+)})+\varepsilon(L))|f|_\infty \tag{2.45}$$

By assumption we have:

$$\mu^{\sigma}_{R_1}(C^{(+,-)}) > \frac{1}{5} \quad ; \quad \mu^{\sigma}_{R_3}(C^{(-,+)}) > \frac{1}{5}$$

$$\int d\mu^{\sigma}_{R_1}(\sigma_1|C^{(+,-)})\int d\mu^{\sigma}_{R_3}(\sigma_2|C^{(-,+)})\mu^{\sigma_2}_{R_2}(S^{++}) \geq \frac{1}{3}$$

which implies that the r.h.s. of (2.45) is smaller or equal than

$$(1-\frac{1}{75}+\varepsilon(L))|f|_\infty \tag{2.46}$$

■

3. Applications

3.1. Pathwise relaxation to one of the two phases.

In this section we derive some consequences of theorem 2.1 for the Heat Bath dynamics that, we hope, make more precise the picture found in [M].

To begin with, we describe a global coupling for the dynamics starting from arbitrary initial configurations, that will be important for the formulation of our result. Our construction works as follows:

i) With rate $|\Lambda_L|$ we choose a site $x \in \Lambda_L$ and a random number $\xi_x \in [0,1]$ with a uniform distribution.

ii) Given an arbitrary configuration η, the value $\eta(x)$ of the spin at x is replaced by $+1$ if

$$\xi_x \le \mu_{\Lambda_L}(\sigma(x) = +1 | \eta(y),\, y \neq x) \tag{3.1}$$

and by -1 if the opposite inequality holds.

The above algorithm is of course nothing more than an explicit way to realize on a common probability space the HB-dynamics in Λ_L starting from different initial conditions. In the sequel we will denote by σ^{η,x,ξ_x} the output of i) and ii) and by σ_t^η ($\sigma_{s,t}^\eta$) the configuration obtained from η by iteratively repeating the above steps up to time t (between time s and time t). We will also denote by N_t the number of updatings that occurred up to time t. Clearly N_t is a Poisson random variable with mean tL^2. If η is one of the two special configurations identically equal to $+1$ or -1, then it will be replaced by a $+$ or a $-$ sign.

Two properties of the above coupling will be relevant for us. The first one is known as monotonicity in the initial configuration:

$$\sigma_t^\eta \le \sigma_t^\tau \qquad \text{if} \qquad \eta \le \tau \tag{3.2}$$

while the second expresses the symmetry of the problem under global spin flip:

$$\sigma^{\eta,x,\xi_x}(x) = -\sigma^{-\eta,x,1-\xi_x}(x) \qquad \text{if} \quad \xi_x \neq \mu_{\Lambda_L}(\sigma(x) = +1 | \eta(y),\, y \neq x) \tag{3.3}$$

We are now in a position to formulate our first result. Let

$$t_o \equiv \frac{10\beta L^2}{\mathrm{gap}_{even}(\mathrm{L}_{\Lambda_L})} \tag{3.4}$$

Then we have:

Theorem 3.1. *There exist positive constants β_o, L_o such that for any $\beta \geq \beta_o$ and any $L \geq L_o$*

$$\sup_\eta \mathbb{P}(\sigma_t^+ \neq \sigma_t^\eta \neq \sigma_t^-) \;\leq\; e^{-[t/t_o]} \quad \forall t \geq t_o$$

Corollary 3.1. *In the same hypotheses of the theorem*

$$\sup_\eta \int d\mu_{\Lambda_L}(\tau)\mathbb{P}(\sigma_t^\tau \neq \sigma_t^\eta \neq \sigma_t^{-\tau}) \;\leq\; e^{-[t/t_o]} \quad \forall t \geq t_o$$

Proof of corollary 3.1 By monotonicity, the probability appearing in the statement is obviously smaller than the corresponding probability appearing in the statement of the theorem. ■

Remark Notice that, thanks to theorem 2.1,

$$\lim_{L\to\infty} \frac{1}{L} \log(\frac{t_{rel}}{t_o}) \;>\; 0$$

if $t_{rel} \equiv \mathrm{gap}(\mathrm{L}_{\Lambda_L})^{-1}$. Thus the corollary says that any initial configuration relaxes to the dynamics started from one of the two phases in a time much shorter than the global relaxation time t_{rel}.

Corollary 3.2. *In the same hypotheses of the theorem let $t_1 = Lt_o$. Then*

$$\sup_\eta |\mathbb{P}(\sigma_{t_1}^+ = \sigma_{t_1}^\eta) + \mathbb{P}(\sigma_{t_1}^- = \sigma_{t_1}^\eta) - 1| \leq \varepsilon(L)$$

where $\varepsilon(L)$ is exponentially small in L.

Proof of corollary 3.2 Using theorem 3.1 we have that

$$\sup_\eta |\mathbb{P}(\sigma_{t_1}^+ = \sigma_{t_1}^\eta) + \mathbb{P}(\sigma_{t_1}^- = \sigma_{t_1}^\eta) - 1| \leq e^{-L} + \mathbb{P}(\sigma_{t_1}^+ = \sigma_{t_1}^-) \tag{3.5}$$

Clearly the event $\sigma_{t_1}^+ = \sigma_{t_1}^-$ implies that

$$m(\sigma_{t_1}^+) \leq 0 \quad \text{or} \quad m(\sigma_{t_1}^-) \geq 0$$

Thus the second term in the r.h.s. of (3.5) is bounded from above by

$$2\mathbb{P}(\sum_{x\in\Lambda_L} \sigma_{t_1}^+(x) \leq 0) \leq$$

$$\leq 4 \int_{m(\eta)>0} d\mu_{\Lambda_L}(\eta) \mathbb{P}\big(\sum_{x\in\Lambda_L} \sigma^\eta_{t_1}(x) \leq 0 \big) \leq \varepsilon(L)$$

because of monotonicity in the initial configuration, of the definition of t_1, of theorem 2.1 and of estimate (4.5) of [M]. ■

Proof of theorem 3.1 Let us set

$$\rho(t) = \sup_\eta \mathbb{P}(\sigma^+_t \neq \sigma^\eta_t \neq \sigma^-_t) \tag{3.6}$$

Then, by monotonicity in the initial configuration and the Markov property, $\rho(t)$ satisfies the inequality:

$$\rho(t+s) \;\leq\; \rho(t)\rho(s) \tag{3.7}$$

Thus in particular

$$\rho(t) \;\leq\; \rho(t_o)^{[t/t_o]} \tag{3.8}$$

It remains to prove that $\rho(t_o) \leq e^{-1}$. For this purpose we observe that, because of (3.2),

$$\sigma^+_{t_o} \leq \sigma^+_{\frac{t_o}{2},t_o} \qquad \text{and} \qquad \sigma^-_{t_o} \geq \sigma^-_{\frac{t_o}{2},t_o} \tag{3.9}$$

which implies that $\rho(t_o)$ can be bounded from above by :

$$\rho(t_o) \leq \sup_\eta \mathbb{P}(\sigma^+_{\frac{t_o}{2},t_o} \neq \sigma^\eta_{t_o} \neq \sigma^-_{\frac{t_o}{2},t_o}) =$$

$$= \sup_\eta E_\eta(f(\sigma^\eta_{\frac{t_o}{2}})) \tag{3.10}$$

where

$$f(\eta) = \mathbb{P}(\sigma^+_{\frac{t_o}{2}} \neq \sigma^\eta_{\frac{t_o}{2}} \neq \sigma^-_{\frac{t_o}{2}}) \tag{3.11}$$

Notice that, because of (3.3), $f \in \mathcal{M}$. Therefore we can bound from above the r.h.s of (3.10) by:

$$\Big(\frac{\int d\mu_{\Lambda_L}(\eta)|E_\eta(f(\sigma^\eta_{\frac{t_o}{2}})) - \mu_{\Lambda_L}(f)|^2}{\min_\eta \mu_{\Lambda_L}(\eta)}\Big)^{\frac{1}{2}} + \mu_{\Lambda_L}(f) \leq$$

$$\big(\frac{Var(f)}{\min_\eta \mu_{\Lambda_L}(\eta)}\big)^{\frac{1}{2}} e^{-\frac{t_o}{2}\text{gap}_{even}(\mathrm{L}_{\Lambda_L})} + \mu_{\Lambda_L}(f) \tag{3.12}$$

Both terms in the r.h.s of (3.12) tend to zero as $L \to \infty$. The first one because of our choice of t_o and the second one because of proposition 5.2 of [M].

■

3.2. Tunneling between the two phases: last excursion.

We will analyze in some detail the last excursion from one phase to the opposite one. In particular we will show that, once the system decides to make the transition, then it does it in a time much shorter than the average time one has to wait in order to see the transition. As discussed in the introduction, such a phenomenon is very common in stochastic dynamics problems with several stable equilibrium points in the small noise limit.

In order to formulate the problem, let us define recursively, for a fixed small δ, the following sequence of stopping times:

$$\begin{aligned} &s_0 \equiv 0 \\ &t_i \equiv \inf\{t > s_{i-1};\ ||m(\sigma_t)| - m^*| \geq 2\delta\} \\ &s_i \equiv \inf\{t > t_i;\ ||m(\sigma_t)| - m^*| < \delta\} \end{aligned} \tag{3.13}$$

where m^* is the spontaneous magnetization. We also define the random variable $\nu(\eta)$ as

$$\nu(\eta) \equiv \min\{i;\ |m(\eta_{s_i}) + m^*| < \delta\} \tag{3.14}$$

Then we have:

Theorem 3.2. *There exist positive constants β_o, L_o such that for any $\beta \geq \beta_o$ and any $L \geq L_o$*

$$\sup_\eta \mathbb{P}(s_{\nu(\eta)} - t_{\nu(\eta)} \geq t_1) \ \leq\ \varepsilon(L)$$

where $t_1 = Lt_o$ is as in corollary 3.2 and $\varepsilon(L)$ goes to zero exponentially fast in L.

Remark If η is such that $m(\eta) > m^* - 2\delta$, then, using (3.13) and (3.14), we may call $s_{\nu(\eta)} - t_{\nu(\eta)}$ and $s_{\nu(\eta)}$ the time scale of the last excursion before leaving the set $\{s; m(\sigma) \geq -m^* + \delta\}$ and the tunneling time for η respectively. It follows from theorem 5.1 of [M] that, if η is identically equal to $+1$, the average of the tunneling time is of the order of $\mathrm{gap}(\mathrm{L}_{\Lambda_L})^{-1}$ and the same if η is distributed according to the Gibbs measure restricted to the phase of positive magnetization. Thus in this case, using the definition of t_o together with theorem 2.1, we may conclude that the last excursion occurs on a time scale much shorter than the average tunneling time.

Proof of theorem 3.2 For any integer n we may estimate from above $\sup_\eta \mathbb{P}(s_{\nu(\eta)} - t_{\nu(\eta)} \geq t_1)$ by

$$n \sup_\eta \mathbb{P}(||m(\eta_t)| - m^*| \geq \delta \quad \forall t \leq t_1) \, + \, \sup_\eta \mathbb{P}(\nu(\eta) > n) \tag{3.15}$$

Using the definition of t_o, theorem 2.1 and the fact that the absolute value of the magnetization is an even function, it is immediate to check that

$$\lim_{L\to\infty} \sup_\eta \mathbb{P}(||m(\eta_{t_o})| - m^*| \geq \delta) = \lim_{L\to\infty} \mu_{\Lambda_L}(||m(\eta)| - m^*| \geq \delta) = 0 \tag{3.16}$$

Thus, using the Markov property, we get that

$$\sup_\eta \mathbb{P}(||m(\eta_t)| - m^*| \geq \delta \quad \forall t \leq t_1) \leq e^{-[\frac{t_1}{t_o}]} \leq e^{-cL} \tag{3.17}$$

with c arbitrarily large for L large enough.

Next we observe that, using the monotonicity in the initial configuration and theorem 5.1 of [M],

$$\sup_\eta E(s_{\nu(\eta)}) \leq E(s_{\nu(+)}) \leq e^{(\beta\tau(\beta)+\gamma)L} \tag{3.18}$$

where $\tau(\beta)$ is the surface tension in the horizontal direction and $\gamma > 0$ can be taken arbitrarily small for L large enough. Therefore we can estimate from above the second term in the r.h.s of (3.15) by

$$\sup_\eta \mathbb{P}(s_{\nu(\eta)} \geq \frac{n}{2L^2}) + \mathbb{P}(N_{\frac{n}{2L^2}} > n) \leq \frac{2L^2 e^{(\beta\tau(\beta)+\gamma)L}}{n} + 2^{-n} e^{\frac{n}{2}} \tag{3.19}$$

where we used the Chebyshev inequality, (3.18) and the fact that the variable $N_{\frac{n}{2L^2}}$ is Poisson with mean $\frac{n}{2}$.

We now choose the integer n as $n = [e^{(\beta\tau(\beta)+1)L}]$. Then, if we combine together (3.17) and (3.19), we get that (3.15) is bounded from above by:

$$[e^{(\beta\tau(\beta)+1)L}]e^{-cL} + 2L^2 e^{(-1+\gamma)L} + \varepsilon(L) \leq 3\varepsilon(L)$$

provided that L is large enough. ∎

4. Proof of the Lemma of section 2

Before starting with the proofs of the various lemma, let us remind a rather standard result, whose proof based on cluster expansion or on the Peierls argument is omitted, that will be used several times in what follows.

Lemma 4.1. *There exist $\beta_o > 0$ and $m > 0$ such that for all $\beta \geq \beta_o$, for all subsets $V_1 \subset V_2 \subset \Lambda_L$ with $|\partial V_2| \leq 4L$ and for all events A in the $\sigma-$algebra generated by the spins in V_1:*

$$\mu^+_{V_1}(\sigma(x)=+1) - \mu^+_{V_2}(\sigma(x)=+1) \leq C \sum_{x\in V_1} e^{-m dist(x,\partial V_1)} \tag{4.1}$$

$$|\mu^+_{V_2}(A) - \mu^+(A)| \leq Ce^{-m\, dist(V_1,\partial V_2)}$$

where μ^+ denotes the infinite volume plus phase.

4.1. Proof of Lemma 2.1

In the sequel we will denote by ∂_i^* the set of bonds b parallel to the side ∂_i and such that they separate one site $x \in \partial_i$ from a site $y \notin R$. Then, given an open contour $\Gamma \in \mathcal{G}_R^\tau(\sigma)$, we will say that Γ starts in ∂_i and ends in ∂_j, and we will write $\Gamma : \partial_i \to \partial_j$, if the first (last) bond e_1 (e_n) of Γ either separates two sites in ∂_i (∂_j) or $e_1 \in \partial_i^*$ ($e_n \in \partial_j^*$). Let us now define the following four events:

$$\begin{aligned} A_1 \equiv &\{\sigma; \exists\, \Gamma \in \mathcal{G}_R^\tau(\sigma) \text{ with } \delta\Gamma = \emptyset \text{ and } |\Gamma| \geq 3\delta L\} \\ A_2 \equiv &\{\sigma; \exists\, \Gamma : \partial_3 \to \partial_j,\ j \neq 1 \text{ with } \Delta(\Gamma) \cap R \setminus R_\delta \neq \emptyset\} \\ A_3 \equiv &\{\sigma; \exists\, \Gamma : \partial_3 \to \partial_1 \text{ with } \Delta(\Gamma) \cap R \setminus M_\delta \neq \emptyset\} \\ A_4 \equiv &\{\sigma; \exists\, \Gamma : \partial_3 \to \partial_1 \text{ and } \Gamma' : \partial_3 \to \partial_1\} \end{aligned} \tag{4.2}$$

where $R_\delta = \{x \in R;\ dist(x, \partial_3) \leq 3\delta L\}$. Then we have

Lemma 4.2. *In the hypotheses of lemma 2.1*

$$\sup_\tau \mu_R^\tau(A_1 \cup A_2 \cup A_3 \cup A_4) \leq \varepsilon(L)$$

Proof. By standard Peierls argument

$$\sup_\tau \mu_R^\tau(A_1) \leq \varepsilon(L) \tag{4.3}$$

Let us estimate $\sup_\tau \mu_R^\tau(A_2)$. We first observe that, given an open contour $\Gamma : \partial_3 \to \partial_j$ $j \neq 1$, we have

$$\mu_R^\tau(\Gamma) = e^{-2\beta|\Gamma|} \frac{Z(R_\Gamma^+, (+,\tau))Z(R_\Gamma^-, (-,\tau))}{Z(R,\tau)} \tag{4.4}$$

where R_Γ^+ and R_Γ^- denotes the regions (not necessarily connected) in $R \setminus \Delta(\Gamma)$ above and below Γ respectively, and, without loss of generality, we have assumed that the boundary conditions on $\partial_{ext} R_\Gamma^+ \cap \Delta(\Gamma)$ are $+1$ and -1 on $\partial_{ext} R_\Gamma^- \cap \Delta(\Gamma)$. If we now set $l_\Gamma = dist(x^*(\Gamma), \partial_j^*)$, where $x^*(\Gamma) \in \delta\Gamma$ belongs to $e_1 \in \Gamma$, then, using the estimate

$$Z(R,\tau) \geq Z(R_\Gamma^-, (-,\tau)) Z(R_\Gamma^+, (-,\tau)) \tag{4.5}$$

and the fact that we have open boundary conditions on $\partial_{ext} R \cap \partial_{ext} \Lambda_L$, we get

$$\frac{Z(R_\Gamma^+, (+,\tau)) Z(R_\Gamma^-, (-,\tau))}{Z(R,\tau)} \leq e^{2\beta l_\Gamma} \tag{4.6}$$

Thus we have the bound

$$\sum_{\substack{\Gamma : \partial_3 \to \partial_j \\ \Delta(\Gamma) \cap R \setminus R_\delta \neq \emptyset}} \mu_R^\tau(\Gamma) \leq \sum_{\substack{\Gamma : \partial_3 \to \partial_j \\ \Delta(\Gamma) \cap R \setminus R_\delta \neq \emptyset}} e^{-2\beta|\Gamma| + 2\beta l_\Gamma} \leq$$

$$\leq 2 \sum_{0 \leq l \leq L} e^{-(2\beta - \log(3))(l + 3\delta L) + 2\beta l} \leq \varepsilon(L) \tag{4.7}$$

for β large enough. Clearly (4.7) shows that

$$\sup_\tau \mu_R^\tau(A_2) \leq \varepsilon(L) \tag{4.8}$$

We now estimate $\sup_\tau \mu_R^\tau(A_3)$. As before, given an open contour $\Gamma : \partial_3 \to \partial_1$, let $x^*(\Gamma) \in \delta\Gamma$ be one of the endpoint of $e_1 \in \Gamma$. We distinguish between two cases:

a) $|x_1^*(\Gamma) - \frac{L}{2}| \leq \delta L$
b) $|x_1^*(\Gamma) - \frac{L}{2}| > \delta L$

In the first case we assume, without loss of generality, that $x_1^*(\Gamma) \leq \frac{L}{2}$. Then, using the same ideas as in (4.4)...(4.6), we get

$$\mu_R^\tau(\Gamma) \leq e^{-2\beta|\Gamma| + 2\beta x_1^*(\Gamma)} \leq e^{-2\beta|\Gamma| + \beta L} \tag{4.9}$$

We observe at this point that, since $\Delta(\Gamma) \cap R \setminus M_\delta \neq \emptyset$, the length $|\Gamma|$ is larger than $\frac{L}{2}(1-\delta) + \delta L$. Thus, using (4.9), we get

$$\sum_{\substack{\Gamma : \partial_3 \to \partial_1 \\ |x_1^*(\Gamma) - \frac{L}{2}| \leq \delta L \\ |\Gamma| \geq \frac{L}{2}(1-\delta) + \delta L}} \mu_R^\tau(\Gamma) \leq \varepsilon(L) \tag{4.10}$$

In the second case we proceed exactly in the same way but we exploit the fact that $|\Gamma| \geq \frac{L}{2}(1-\delta)$ while $\min(x_1^*(\Gamma), L - x_1^*(\Gamma)) < \frac{L}{2} - \delta L$, which implies that the contour Γ prefers to end on $\partial_2 \cup \partial_4$ instead of ending on ∂_1.

It remains to estimate $\sup_\tau \mu_R^\tau(A_4)$ or better, using the above bound on $\sup_\tau \mu_R^\tau(A_3)$, to prove that

$$\sup_\tau \mu_R^\tau(A_4 \cap (A_3)^c) \leq \varepsilon(L) \tag{4.11}$$

This easy estimate follows immediately from the Peierls argument. Lemma 4.2 is proved.

■

Using the above lemma we can conclude that

$$\sup_\tau \mu_R^\tau((S^+ \cup S^- \cup C^{(+,-)} \cup C^{(-,+)})^c \cap (A_1 \cup A_2 \cup A_3 \cup A_4)) \leq \varepsilon(L)$$

so that we need only to estimate

$$\sup_\tau \mu_R^\tau((S^+ \cup S^- \cup C^{(+,-)} \cup C^{(-,+)})^c \cap (A_1 \cup A_2 \cup A_3 \cup A_4)^c)$$

We observe that, if the event $(S^+ \cup S^-)^c$ occurs, then there exist a plus and a minus chains, $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, that connect ∂_3 with the set $R \setminus R_\delta$. In turn this implies the existence of a contour $\Gamma \in \mathcal{G}_R^\tau(\sigma)$ with

$$\Delta(\Gamma) \cap \partial_3 \neq \emptyset \quad \text{and} \quad \Delta(\Gamma) \cap R \setminus R_\delta \neq \emptyset$$

that is

$$(S^+ \cup S^-)^c \cap (A_1 \cup A_2 \cup A_3 \cup A_4)^c \subset F \cap (A_1 \cup A_2 \cup A_3 \cup A_4)^c$$

where $F = \{\sigma;\ \exists!\ \Gamma \in \mathcal{G}_R^\tau(\sigma),\ \Gamma : \partial_3 \to \partial_1 \text{ with } \Delta(\Gamma) \subset M_\delta\}$. We are left with the estimate of

$$\sup_\tau \mu_R^\tau((C^{(+,-)} \cup C^{(-,+)})^c \cap F \cap (A_1 \cup A_2 \cup A_3 \cup A_4)^c) \tag{4.12}$$

Because of the event F, we know that there exists a unique vertical contour $\Gamma : \partial_3 \to \partial_1$ in the strip M_δ that, without loss of generality, we can assume to have plus spins on his left and minus spins on his right. Now, in order not to have the event $C^{(+,-)}$, there must exists either a minus chain $\mathcal{C}_1$ or a plus chain $\mathcal{C}_2$ to the left or to the right of Γ respectively, connecting ∂_3 with the set $R \setminus R_\delta$. One easily checks that the presence of $\mathcal{C}_1$ to the left of Γ implies the existence of another contour Γ' with length $|\Gamma'| \geq 3\delta L$ and analogously for $\mathcal{C}_2$. However the presence of such a new contour is forbidden by the event $(A_1 \cup A_2 \cup A_3 \cup A_4)^c$; thus

$$(C^{(+,-)} \cup C^{(-,+)})^c \cap F \cap (A_1 \cup A_2 \cup A_3 \cup A_4)^c = \emptyset$$

and Lemma 2.1 follows.

■

4.2 Proof of Lemma 2.2

It is immediate to check, using DLR and (1.4), that the projection on $\Omega_{R_1\setminus R_2}$ of the measure $\mu^{\sigma}_{R_1}(\sigma_1|S^+)$ is larger than the same projection but of the measure $\mu^{+}_{R_1}(\sigma_1)$. Thus we can estimate the quantity appearing in the statement by

$$2|F|_\infty \sum_{x\in R_1\setminus R_2} (\mu^{\sigma}_{R_1}(\sigma_1(x)=+1|S^+) - \mu^{+}_{R_1}(\sigma_1(x)=+1)) \tag{4.13}$$

Because of the definition of the event S^+ and because of (1.4), each term in the sum appearing in the r.h.s. of (4.13) can be estimated from above by

$$\mu^{+}_{\hat{R}_1}(\sigma_1(x)=+1) - \mu^{+}_{R_1}(\sigma_1(x)=+1) \tag{4.14}$$

where $\hat{R}_1 \equiv R_1 \setminus \{x\in R_1;\ dist(x,\partial_3)\le 3\delta L\}$. Lemma 4.1 now shows that (4.14) goes to zero exponentially fast in L uniformly in x. ■

4.3. Proof of Lemma 2.3

Without loss of generality we assume that $\frac{1}{2}(1+\delta) = 2^N\delta$, where $N >> 1$ is an integer, and we define for $i = 1\dots 2^N$

$$S_i = \{x\in \bar{R}_1\cup R_2:\ 0<x_1\le L;\ \frac{L(1-3\delta)}{4} + (i-1)\delta L \le x_2 < \frac{L(1-3\delta)}{4} + i\delta L\}$$

$$I_i = [(i-1)\delta L + \frac{1}{2}, i\delta L + \frac{1}{2})\quad i=1\dots 2^N-1;\quad I_{2^N} = [(2^N-1)\delta L + \frac{1}{2}, 2^N\delta L + \frac{1}{2}]$$

$$I = \bigcup_{i=1}^{2^N} I_i\ = [\frac{1}{2}, \frac{L}{2}(1+\delta) + \frac{1}{2}]$$

Notice that, by construction, $\{S_i\}_1^{2^N}$ is a partition of $\bar{R}_1\cup R_2$ into 2^N disjoint horizontal strips of width δL. Let now A be the event

$$A = \{\sigma; \exists \text{ a plus } *\text{-chain } \mathcal{C}\subset \{x\in R_2; dist(x,\partial_1)\le \delta L\} \text{ connecting } \partial_2 \text{ with } \partial_4\}$$

Then

$$0 \le \mu^{+,\sigma}_{R_2}(\bar{S}^+) - \mu^{+,\sigma}_{\bar{R}_1\cup R_2}(\bar{S}^+) \le \mu^{+,\sigma}_{R_2}(\bar{S}^+) - \mu^{+,\sigma}_{\bar{R}_1\cup R_2}(\bar{S}^+|A)\mu^{+,\sigma}_{\bar{R}_1\cup R_2}(A) \le$$

$$\leq \mu^{+,\sigma}_{\bar R_1\cup R_2}(A^c) \leq \mu^{+,-}_{\bar R_1\cup R_2}(A^c) \tag{4.15}$$

since, by monotonicity, $\mu^{+,\sigma}_{\bar R_1\cup R_2}(\bar S^+|A) \geq \mu^{+,\sigma}_{R_2}(\bar S^+)$. We will now estimate from above $\mu^{+,-}_{\bar R_1\cup R_2}(A^c)$. Notice that, because of the $(+,-)$ boundary conditions on the bottom and top side of $\bar R_1 \cup R_2$, there exists an open contour Γ connecting the two lateral sides of $\bar R_1 \cup R_2$. In the sequel, given any such contour Γ, we will order its bonds $e_1, e_2 \dots e_n$ starting from the left side ∂_2 and we will denote by x_Γ the distance of e_1 from ∂_1 (as sets in $\mathbb{R}^2$) and by d_Γ the largest vertical excursion of Γ above or below the horizontal line in $\mathbb{R}^2$ containing the first bond e_1. Clearly x_Γ is a discrete random variable taking values $\frac{1}{2}, \frac{3}{2} \dots$ in the interval I. Let now B be the event

$$B \equiv \{\sigma;\ \exists \text{ a unique open contour } \Gamma : \partial_2 \to \partial_4\} \cap \{d_\Gamma \leq \frac{\delta L}{2}\}$$

The standard Peierls argument together with large deviation estimates on Γ (see [DKS] and lemma A.1 in [M]) prove that, in the assumptions of the lemma,

$$\begin{aligned} &\mu^{+,-}_{\bar R_1\cup R_2}(B^c) \leq \varepsilon(L) \\ &\mu^{+,-}_{\bar R_1\cup R_2}(A^c | \{x_\Gamma > 2\delta L + \frac{1}{2}\} \cap B) \leq \varepsilon(L) \end{aligned} \tag{4.16}$$

Therefore, by taking L large enough, it is enough to prove that

$$\mu^{+,-}_{\bar R_1\cup R_2}(x_\Gamma \leq 2\delta L + \frac{1}{2} | B) \leq c\delta \tag{4.17}$$

where c is a suitable numerical constant independent of δ and L, provided that the latter is large enough (depending on δ).
In the sequel we denote by P_n the Gibbs measure $\mu^{+,-}_{S_1\cup\ldots\cup S_{2n}}$ conditioned to the event B and we let

$$p_n \equiv P_n(x_\Gamma \leq 2\delta L + \frac{1}{2})$$

As a first step towards the proof of (4.17) we observe that, using the cluster expansion technique, for any open contour $\Gamma : J\partial_2 \to \partial_4$ with $d_\Gamma \leq \frac{\delta L}{2}$ one easily gets (see [DKS])

$$P_n(\Gamma \text{ is the open contour connecting } \partial_2 \text{ with } \partial_4) =$$

$$\frac{Z^{+,+}_{S_1\cup\ldots\cup S_{2n}}}{Z^{+,-}_{S_1\cup\ldots\cup S_{2n}}} e^{-2\beta|\Gamma| - \sum_{\substack{\Lambda\subset S_1\cup\ldots\cup S_{2n} \\ \Lambda\cap\Delta(\Gamma)\neq\emptyset}} \phi(\Lambda)} (1+\varepsilon(L)) \tag{4.18}$$

where the coefficients $\phi(\Lambda)$ are exponentially small in the diameter of Λ and invariant under vertical translations. Using (4.18) together with (4.16), it follows that (see e.g. the proof of lemma A.1 in [M])

$$\sup_{x,y\in I_2\cup\ldots\cup I_{2^n-1}} \frac{P_n(x_\Gamma = x)}{P_n(x_\Gamma = y)} \leq 1+\varepsilon(L) \tag{4.19}$$

that is the law of x_Γ under P_n is almost uniform outside the two intervals I_1 and I_{2^n}. In particular

$$\sup_{x\in I_2\cup\ldots I_{2^n-1}} P_n(x_\Gamma = x) \leq \frac{1+\varepsilon(L)}{(2^n-2)\delta L} \tag{4.20}$$

We now estimate p_N by induction on n. We will show that

$$p_n \leq p_{n-1}(\frac{1}{2}+2\frac{(1+\varepsilon(L))}{(2^n-2)}+\varepsilon(L))+\varepsilon(L) \quad \forall\, n\geq 3 \tag{4.21}$$

which, for L large enough depending on δ, implies that

$$p_N \leq \prod_{j=3}^{N}(\frac{1}{2}+2\frac{(1+\varepsilon(L))}{(2^j-2)}+\varepsilon(L))+N\varepsilon(L)\leq c\delta$$

for a suitable numerical constant c independent of δ and L.
In order to establish (4.21), let A_n^- be the event

$$A_n^- \;=\; \{\exists \text{ a minus } *\text{-chain in } S_{2^{n-1}+1} \text{ connecting } \partial_2 \text{ with } \partial_4\}$$

Then we write

$$p_n \leq P_n(x_\Gamma\in I_1\cup I_2 \mid A_n^-)P_n(A_n^-)+P_n(x_\Gamma\in I_1\cup I_2\cap (A_n^-)^c) \tag{4.22}$$

Using the Peierls argument one immediately shows that the second term in the r.h.s of (4.22) is bounded from above by $\varepsilon(L)$. Monotonicity and the definition of p_{n-1} imply that

$$P_n(x_\Gamma\in I_1\cup I_2 \mid A_n^-)\leq P_{n-1}(x_\Gamma\in I_1\cup I_2)\equiv p_{n-1}$$

so that the first term is bounded from above by $p_{n-1}P_n(A_n^-)$. Let us now estimate $P_n(A_n^-)$. We have

$$P_n(A_n^-) \;\leq$$

$$\leq P_n(\{x_\Gamma \in \bigcup_{i=1}^{2^{n-1}} I_i\}) + P_n(A_n^- \cap \{x_\Gamma \in \bigcup_{i=2^{n-1}+3}^{2^n} I_i\}) + P_n(x_\Gamma \in I_{2^{n-1}+1} \cup I_{2^{n-1}+2}) \leq$$

$$\leq \frac{1}{2} + \varepsilon(L) + 2\frac{(1+\varepsilon(L))}{(2^n-2)} \qquad \forall\, 3 \leq n \leq N \tag{4.23}$$

where we used the symmetry between the lower and upper half of the rectangle $S_1 \cup \ldots \cup S_{2^n}$ to get $P_n(x_\Gamma \in \bigcup_{i=1}^{2^{n-1}} I_i) = \frac{1}{2}$, monotonicity and the Peierls argument to get the $\varepsilon(L)$ term and (4.20) to get the last term. In conclusion

$$P_n(A_n^-) \;\leq \frac{1}{2} + 2\frac{(1+\varepsilon(L))}{(2^n-2)} + \varepsilon(L)$$

and (4.21) follows.

To prove the second part of the lemma we first observe that, by FKG, $\mu^{+,\sigma}_{\bar R_1 \cup R_2}(\bar S^+) \geq \mu^{+,-}_{\bar R_1 \cup R_2}(\bar S^+)$ and that the event $\bar S^+$ is contained in the event $B \cap \{x_\Gamma \in \cup_{i=\frac{3}{4}2^N+1}^{2^N-1} I_i\}$. Thanks to the previous results (see (4.16), (4.17) and (4.19)) the $\mu^{+,-}_{\bar R_1 \cup R_2}$ probability of this last event is greater or equal than

$$(1-\varepsilon(L))(1-c\delta)\frac{2^N}{4(2^N-2)(1+\varepsilon(L))} \geq \frac{1}{5}$$

for δ small enough and L sufficiently large. ■

4.4. Proof of Lemma 2.4

As in the proof of lemma 2.2 one can bound from above the quantity appearing in the statement of the lemma by

$$2|f|_\infty \sum_{x \in R_2 \setminus R_3} \int d\mu^+_{R_1}(\sigma_1)[\mu^{\sigma_1,+}_{\bar R_2}(\sigma_2(x)=+1) - \mu^{\sigma_1,+}_{R_2}(\sigma_2(x)=+1)] \tag{4.24}$$

where $\bar R_2 \equiv R_2 \setminus \{x \in R_2;\ dist(x, \partial_3) \leq \frac{L}{8}(1-\delta)\}$. Next we write:

$$\int d\mu^+_{R_1}(\sigma_1)\mu^{\sigma_1,+}_{\bar R_2}(\sigma_2(x)=+1) \leq$$

$$\leq \mu^+_{\bar R_2 \cup R_1}(\sigma(x)=+1) + 2\sum_{y \in \partial_{ext}(\bar R_2)\cap R_1} (\mu^+_{R_1}(\sigma(y)=+1) - \mu^+_{\bar R_2 \cup R_1}(\sigma(y)=+1)) \leq$$

$$\leq \mu^+_{\bar{R}_2\cup R_1}(\sigma(x)=+1)+\varepsilon(L) \tag{4.25}$$

and

$$\int d\mu^+_{R_1}(\sigma_1)\mu^{\sigma_1,+}_{R_2}(\sigma_2(x)=+1)\geq \mu^+_{R_2\cup R_1}(\sigma(x)=+1) \tag{4.26}$$

where we have used once more DLR, (1.4) and lemma 4.1.

The result now follows by plugging (4.25) and (4.26) into (4.24) and applying once more lemma 4.1 in order to estimate $\mu^+_{\bar{R}_2\cup R_1}(\sigma(x)=+1)-\mu^+_{R_2\cup R_1}(\sigma(x)=+1)$ ■

4.5 Proof of Lemma 2.5

Let $F(\sigma_2)\equiv\int d\mu^{\sigma_2}_{R_3}(\sigma_3)g(\sigma_3)$. Then we write

$$\begin{aligned}
&|\int d\mu^+_{R_1\cup R_2}(\sigma_2)F(\sigma_2)-\int d\mu_{\Lambda_L}(\sigma_2|m>0)F(\sigma_2)\,|\leq\\
&\leq|\int d\mu^+_{R_1\cup R_2}(\sigma_2)F(\sigma_2)\;-\;\int d\mu_{\Lambda_L}(\sigma_2|S^+_{R'})F(\sigma_2)\;|\;+\\
&+|\int d\mu_{\Lambda_L}(\sigma_2|S^+_{R'})F(\sigma_2)-\int d\mu_{\Lambda_L}(\sigma_2|m>0)F(\sigma_2)|
\end{aligned} \tag{4.27}$$

where $R'=\{x\in R_2\cap R_3: dist(x,\partial_1(R_3))\geq\delta L\}$ and

$$S^+_{R'}\;=\;\{\exists \text{a plus } *\text{-chain in } R' \text{ connecting } \partial_2 \text{ with } \partial_4\}$$

We now observe that, as in the proof of lemma 2.2, the projection on $\Omega_{\Lambda_L\setminus R_3}$ of $\mu^+_{R_1\cup R_2}$ is smaller than the projection over the same set of the measure $\mu_{\Lambda_L}(\sigma_2|S^+_{R'})$. Thus we can bound from above the first term in the r.h.s. of (4.27) by

$$\begin{aligned}
&2|g\,|_\infty\sum_{x\in V\setminus R_3}[\mu_{\Lambda_L}(\sigma(x)=+1\mid S^+_{R'})\;-\;\mu^+_{R_1\cup R_2}(\sigma(x)=+1)]\leq\\
&\leq 2|f\,|_\infty\sum_{x\in V\setminus R_3}[\mu^+_{R_1\cup R_2\setminus R'}(\sigma(x)=+1)\;-\;\mu^+_{R_1\cup R_2}(\sigma(x)=+1)]\leq\varepsilon(L)
\end{aligned} \tag{4.28}$$

where in the second inequality we used, as before, (1.4), lemma 4.1 and the fact that $|F\,|_\infty\leq|f\,|_\infty$. In order to estimate from above the second term in the r.h.s. of (4.27) we first need to recall the following basic fact about the measure $\mu_{\Lambda_L}(\sigma|m>0)$ (see [Sch]).

$$\mu_{\Lambda_L}(\exists \text{ a plus chain}\subset\Lambda_L\setminus\Lambda^\delta_L|m>0)\geq 1-\varepsilon(L) \tag{4.29}$$

where $\Lambda_L^\delta \equiv \{x \in \Lambda_L;\ dist(x, \partial\Lambda_L) \geq \delta L\}$. From (4.29) and the Peierls argument it easily follows that

$$\mu_{\Lambda_L}((S^+_{R'})^c | m > 0) \leq \varepsilon(L) \quad ; \quad |\mu_{\Lambda_L}(S^+_{R'}) - \frac{1}{2}| \ \leq \ \varepsilon(L) \tag{4.30}$$

By writing now

$$\int d\mu_{\Lambda_L}(\sigma_2 | S^+_{R'}) F(\sigma_2) = \frac{\int d\mu_{\Lambda_L}(\sigma_2 | m > 0) F(\sigma_2)}{2\mu_{\Lambda_L}(S^+_{R'})} - \frac{\int d\mu_{\Lambda_L}(\sigma_2; m > 0; (S^+_{R'})^c) F(\sigma_2)}{\mu_{\Lambda_L}(S^+_{R'})} +$$

$$+ \frac{\int d\mu_{\Lambda_L}(\sigma_2; m < 0; S^+_{R'}) F(\sigma_2)}{\mu_{\Lambda_L}(S^+_{R'})} \tag{4.31}$$

and using (4.30) we immediately get that the second term in the r.h.s. of (4.27) is bounded from above by $3\varepsilon(L)$. ■

4.6 Proof of Lemma 2.6

Let us consider the following two subsets of $\partial_{ext}(R_4) \cap \Lambda_L$:

$$\Lambda_1 \ = \ \{x \in \partial_{ext}(R_4) \cap \Lambda_L \ : \ x_2 \ \leq \ \frac{L}{2}(1 - 7\delta)\}$$

$$\Lambda_2 \ = \ \{x \in \partial_{ext}(R_4) \cap \Lambda_L \ : \ \frac{L}{2}(1 + 8\delta) \ \leq \ x_2 \ \leq \ L\}$$

and the associated local magnetizations $m_{\Lambda_1} = \frac{\sum_{x \in \Lambda_1} \sigma(x)}{|\Lambda_1|}$, $m_{\Lambda_2} = \frac{\sum_{x \in \Lambda_2} \sigma(x)}{|\Lambda_2|}$. Then we get

$$\int d\mu^\sigma_{R_1}(\sigma_1 | C^{(+,-)}) \int d\mu^\sigma_{R_3}(\sigma_2 | C^{(+,-)}) \mu^{\sigma_2}_{R_4}((S^+_v)^c) \ \leq$$

$$\mu^\sigma_{R_1}(m_{\Lambda_1} \leq (1 - \delta') | C^{(+,-)}) + \mu^\sigma_{R_3}(m_{\Lambda_2} \leq (1 - \delta') | C^{(+,-)}) +$$

$$+ \sup_{\substack{\sigma \\ m_{\Lambda_1}(\sigma) > (1-\delta') \\ m_{\Lambda_2}(\sigma) > (1-\delta')}} \mu^\sigma_{R_4}((S^+_v)^c) \tag{4.32}$$

where $\delta' = \frac{\delta}{20}$.

Notice that the third term in the r.h.s. of (4.32) is bounded from above by

$$e^{4\beta\delta'(|\Lambda_1| + |\Lambda_2|)} \mu^\tau_{R_4}((S^+_v)^c) \tag{4.33}$$

where

$$\tau(x) = +1 \qquad \forall x \ \in \Lambda_1 \cup \Lambda_2$$
$$\tau(x) = -1 \qquad \forall x \ \in \partial_{ext}(R_4) \cap \Lambda_L \setminus (\Lambda_1 \cup \Lambda_2)$$

Using the Peierls argument as in the proof of lemma 2.1, it is easy to check that

$$\mu^{\tau}_{R_4}((S^+_v)^c) \le \varepsilon(L) \tag{4.34}$$

Let us now estimate the first term in the r.h.s. of (4.32), the second one being identical. Using the definition of the event $C^{(+,-)}$ and (1.4) we immediately get that

$$\mu^{\sigma}_{R_1}(m_{\Lambda_1} \le (1-\delta')|C^{(+,-)}) \le \mu^{+}_{R_1}(m_{\Lambda_1} \le (1-\delta')) \le$$

$$\le \mu^{+}_{R_1}(m_{\Lambda'_1} \le (1-\frac{\delta'}{2})\frac{|\Lambda_1|}{|\Lambda'_1|}) \tag{4.35}$$

where $\Lambda'_1 = \{x \in \Lambda_1;\ x_2 \ge \frac{\delta' L}{4}(1-7\delta)\}$. Notice that the event $m_{\Lambda'_1} \le (1-\frac{\delta'}{2})\frac{|\Lambda_1|}{|\Lambda'_1|})$ depends only on the spins $\sigma(x)$ with $\mathrm{dist}(x,\partial R_1) \ge \frac{\delta' L}{4}(1-7\delta)$. Thus we can apply lemma 4.1 to get

$$\mu^{+}_{R_1}(m_{\Lambda'_1} \le (1-\frac{\delta'}{2})\frac{|\Lambda_1|}{|\Lambda'_1|})) \le \mu^{+}(m_{\Lambda'_1} \le (1-\frac{\delta'}{2})\frac{|\Lambda_1|}{|\Lambda'_1|}) + \varepsilon(L) \le 2\varepsilon(L) \tag{4.36}$$

where, in the last inequality, we have used Lemma 1 of [Sch].

4.7. Proof of Lemma 2.7

Without loss of generality we can assume that L is odd. Let

$$\partial_1^{left} = \{x \in \partial_{ext} R_2;\ 0 < x_1 \le \frac{L}{2},\ x_2 = \lfloor \frac{L(1+\delta)}{4} \rfloor\}$$
$$\partial_1^{right} = \{x \in \partial_{ext} R_2;\ \frac{L}{2} < x_1 \le L,\ x_2 = \lfloor \frac{L(1+\delta)}{4} \rfloor\}$$
$$\partial_3^{left} = \{x \in \partial_{ext} R_2;\ 0 < x_1 \le \frac{L}{2},\ x_2 = \lfloor \frac{L(3-\delta)}{4} \rfloor + 1\}$$
$$\partial_3^{right} = \{x \in \partial_{ext} R_2;\ \frac{L}{2} < x_1 \le L,\ x_2 = \lfloor \frac{L(3-\delta)}{4} \rfloor + 1\}$$

and let, for any $\alpha \in (0, \frac{1}{2})$, $A(\alpha)$ be the intersection of the following four events

$$A^l_1(\alpha) = \{\sigma; \sum_{x \in \partial_1^{left}} \sigma(x) \geq \frac{L}{2}(1-\alpha)\}$$

$$A^r_1(\alpha) = \{\sigma; \sum_{x \in \partial_1^{right}} \sigma(x) \leq -\frac{L}{2}(1-\alpha)\}$$

$$A^l_3(\alpha) = \{\sigma; \sum_{x \in \partial_3^{left}} \sigma(x) \leq -\frac{L}{2}(1-\alpha)\}$$

$$A^r_3(\alpha) = \{\sigma; \sum_{x \in \partial_3^{right}} \sigma(x) \geq \frac{L}{2}(1-\alpha)\}$$

Notice that if $\tau \in A(\alpha)$ then

$$\mu^{\tau}_{R_2}((S^{++} \cup S^{--})^c) \leq e^{8\beta\alpha L} \mu_{R_2}^{{}^{+}_{-},{}^{-}_{+}}((S^{++} \cup S^{--})^c)$$

where the boundary condition ${}^{+}_{-},{}^{-}_{+}$ means $+1$ on ∂_1^{left}, -1 on ∂_1^{right} and viceversa on ∂_3^{left}, ∂_3^{right}.

Thus we can estimate from above

$$\int d\mu^{\sigma}_{R_1}(\sigma_1 | C^{(+,-)}) \int d\mu^{\sigma}_{R_3}(\sigma_2 | C^{(-,+)}) \mu^{\sigma_2}_{R_2}((S^{++} \cup S^{--})^c)$$

by

$$\int d\mu^{\sigma}_{R_1}(\sigma_1 | C^{(+,-)}) \int d\mu^{\sigma}_{R_3}(\sigma_2 | C^{(-,+)}) (\sigma_2 \in A(\alpha)^c) +$$
$$+ e^{8\beta\alpha L} \mu_{R_2}^{{}^{+}_{-},{}^{-}_{+}}((S^{++} \cup S^{--})^c) \tag{4.37}$$

Let us now show that the first term in (4.37) is exponentially small in L provided that δ and β are, respectively, small and large enough depending on α. This in turn follows if we can prove, in the same range of the parameters, that

$$\mu^{\sigma}_{R_1}((A^l_1(\alpha))^c | C^{(+,-)}) \leq \varepsilon(L)$$
$$\mu^{\sigma}_{R_1}((A^r_1(\alpha))^c | C^{(+,-)}) \leq \varepsilon(L)$$
$$\mu^{\sigma}_{R_3}((A^l_3(\alpha))^c | C^{(-,+)}) \leq \varepsilon(L)$$
$$\mu^{\sigma}_{R_3}((A^l_3(\alpha))^c | C^{(-,+)}) \leq \varepsilon(L)$$

Let us consider only the first of the above inequalities, the others being similar.

If we denote by m_δ the (unormalized) magnetization

$$m_\delta = \sum_{\substack{x\in \\ \partial_1^{left}\cap\{x;\delta L\le x_1\le \frac{L}{2}(1-5\delta)\}}} \sigma(x)$$

we have

$$m_\delta - \frac{7}{2}\delta L \le \sum_{x\in\partial_1^{left}} \sigma(x)$$

so that

$$\mu^\sigma_{R_1}((A^l_1(\alpha))^c | C^{(+,-)}) \le \mu^\sigma_{R_1}(m_\delta \le \frac{L}{2}(1-\alpha+7\delta)|C^{(+,-)})$$

Notice that $\{m_\delta \le \frac{L}{2}(1-\alpha+7\delta)\}$ is a decreasing event. Thus, using the definition of the event $C^{(+,-)}$ and monotonicity (1.4), we have

$$\mu^\sigma_{R_1}(m_\delta \le \frac{L}{2}(1-\alpha+7\delta)|C^{(+,-)}) \le \mu^+_{R_1}(m_\delta \le \frac{L}{2}(1-\alpha+7\delta)) \le$$

$$\le \mu^+(m_\delta \le \frac{L}{2}(1-\alpha+7\delta)) + \varepsilon(L) \tag{4.38}$$

where μ^+ denotes the infinite volume plus phase. In the last inequality we have used lemma 4.1 and the fact that

$$dist(\partial_1^{left}\cap\{x;\delta L\le x_1\le \frac{L}{2}(1-5\delta)\}, \partial_{ext}R_1) \ge \delta L$$

It remains to show that the r.h.s of (4.38) is exponentially small in L for any α, any δ small enough and any β large enough depending on α. This is actually the content of Lemma 1 of [Sch].

Let us now examine the second term in the r.h.s of (4.37). We claim that, for β large enough independent of δ and α

$$\mu_{R_2}^{\frac{+}{-},\frac{-}{+}}((S^{++}\cup S^{--})^c) \le e^{-\beta\frac{L}{14}} \tag{4.39}$$

If we now take e.g. $\alpha = \frac{1}{140}$ we get, using (4.39), that also the second term in the r.h.s of (4.37) is exponentially small in L and the lemma follows.

In order to prove (4.39) we first observe that, because of the boundary conditions $\frac{+}{-}, \frac{-}{+}$, there exist two open contours, Γ_1 and Γ_2, such that

$$\Gamma_1 : \partial_1 \to \partial_{i_1} \quad ; \quad \Gamma_2 : \partial_3 \to \partial_{i_2} \qquad i_1, i_2 \in \{2,4\}$$

It is quite clear that for typical configurations the two contours Γ_1 and Γ_2 have length $\approx \frac{L}{2}$ and end in opposite lateral sides. It is therefore natural to introduce the following events

$$\begin{aligned} C_1 &= \{\sigma;\ \Delta(\Gamma_1) \subset \{x \in R_2 : dist(x, \partial_1) \ \le\ \frac{L}{12}(1-\delta)\}\} \\ C_2 &= \{\sigma;\ \Delta(\Gamma_2) \subset \{x \in R_2 : dist(x, \partial_3) \ \le\ \frac{L}{12}(1-\delta)\}\} \\ C &= C_1 \cap C_2 \cap \{\sigma;\ i_1 \neq i_2\} \end{aligned}$$

and write

$$\mu_{R_2}^{\overset{+}{-},\overset{-}{+}}((S^{++} \cup S^{--})^c) \le \mu_{R_2}^{\overset{+}{-},\overset{-}{+}}((S^{++} \cup S^{--})^c)|C) + \mu_{R_2}^{\overset{+}{-},\overset{-}{+}}(C^c) \tag{4.40}$$

In order to estimate the first term in (4.40) let us fix the two open contours Γ_1 and Γ_2 in such a way that the conditions specified by the event $C_1 \cap C_2$ are satisfied and, without loss of generality, let us assume that $i_1 = 4$ and $i_2 = 2$. Then we write

$$\begin{aligned} &\mu_{R_2}^{\overset{+}{-},\overset{-}{+}}((S^{++} \cup S^{--})^c)|\Gamma_1, \Gamma_2) \le \mu_{R_2}^{\overset{+}{-},\overset{-}{+}}((S^{++})^c)|\Gamma_1, \Gamma_2) \le \\ &\le \mu_{R_2}^{+}((S^{++})^c) \le \mu_{R_2}^{+}(\exists \text{ a contour } \Gamma;\ |\Gamma| \ge \frac{L}{12}(1-\delta)) \le e^{-2\beta \frac{L}{13}} \end{aligned} \tag{4.41}$$

for β large enough. In the above chain of inequalities we have used monotonicity (1.4) together with the hypothesis $i_1 = 4,\ i_2 = 2$ to replace the measure $\mu_{R_2}^{\overset{+}{-},\overset{-}{+}}(\cdot\,|\Gamma_1, \Gamma_2)$ with the measure $\mu_{R_2}^{+}$ and the standard Peierls argument to derive the final estimate.

We are left with the estimate of $\mu_{R_2}^{\overset{+}{-},\overset{-}{+}}(C^c)$. If we use estimate (4.9) we immediately get

$$\begin{aligned} &\mu_{R_2}^{\overset{+}{-},\overset{-}{+}}(\Delta(\Gamma_1) \not\subset \{x \in R_2 : dist(x, \partial_1) \ \le\ \frac{L}{12}(1-\delta)\}) \le \\ &\sum_{\substack{\Gamma \\ |\Gamma| \ge \frac{L}{2} + \frac{L}{12}(1-\delta)}} e^{-2\beta|\Gamma| + \beta L} \le e^{-2\beta \frac{L}{13}} \end{aligned} \tag{4.42}$$

and similarly for Γ_2. It remains to estimate the probability that $i_1 = i_2$. One easily realizes that $i_1 = i_2$ implies the existence of another open contour $\Gamma_3 : \partial_2 \to \partial_4$. By proceeding as in the derivation of (4.9) we get

$$\mu_{R_2}^{\overset{+}{-},\overset{-}{+}}(i_1 = i_2) \le 2 \sum_{\substack{\Gamma_1 : |\Gamma_1| \ge \frac{L}{2} \\ \Gamma_2 : |\Gamma_2| \ge \frac{L}{2} \\ \Gamma_3 : |\Gamma_3| \ge L}} e^{-2\beta(|\Gamma_1| + |\Gamma_2| + |\Gamma_3|) + 2\beta L} \le \varepsilon(L) \tag{4.43}$$

for β large enough.

If we combine together (4.41) and (4.43) we get (4.39) for any β large enough (independent of δ and α).

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References

[CM] F. Cesi and F. Martinelli:*On the layering transition of an SOS surface interacting with a wall II. Glauber Dynamics* Commun. Math. Phys. in press

[CGMS] F. Cesi, G. Guadagni, F. Martinelli and R.H. Schonmann:*On the 2D stochastic Ising model in the phase coexistence region close to the critical point.* Roma preprint 1995

[D] E. B. Davies:*Spectral properties of metastable Markov semigroups.* J. Funct. Anal **52** 315 (1983)

[DKS] R. Dobrushin, R. Koteckỳ, S. Shlosman:*Wulff Construction. A Global Shape From Local Interaction.* Translation of Math. Monographs, AMS **104** (1992)

[FKG] C. M. Fortuin, P. W. Kasteleyn, J. Ginibre:*Correlations Inequalities on Some Partially Ordered Sets.* Comm. Math. Phys. **22** 89 (1971)

[FV] M. I. Freidlin, A. D. Ventzel: *Random Perturbations of Dynamical Systems.* Springer Verlag (1984)

[HY] Y. Higuchi, N. Yoshida: *Slow relaxation of 2D-stochastic Ising models at low temperature.* preprint (1995)

[M] F. Martinelli: *On The Two Dimensional Dynamical Ising Model In The Phase Coexistence Region.* Journ. Stat. Phys. **76**, No. 5/6, 1179 (1994)

[Sch] R. H. Schonmann:*Second order large deviation estimates for ferromagnetic systems in the phase coexistence region.* Commun. Math. Phys. **112**, 409 (1987)

[Sh] S. Sholsman: *The Droplet in the Tube : A Case of Phase Transition in the Canonical Ensemble.* Comm. Math. Phys. **125** 81 (1989)

[YaV] H. T. Yau, R.S. Varadhan:*private communication*